CHAPTER 15

CIRCLE AND FAMILY OF CIRCLE

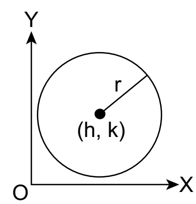
15.1 INTRODUCTION

A circle is the most regular object, we know. Each point on a circle's circumference is equidistant from its centre. The shape and symmetry of circle has been fascinating mathematicians since ages.

15.2 DEFINITON OF CIRCLE

A circle is the locus of a point moving in a plane so that its distance from a fixed point remains constant. The fixed point is called **centre** of the circle and the constant distance is called the **radius** of the circle.

15.2.1 Equation of a Circle in Various Forms



Centre–radius form: Equation of a circle with Centre at (h, k) and radius 'r' is $(x - h)^2 + (y - k)^2 = r^2$.

- **Standard Form:** When centre is (0, 0) and radius is 'a' then the standard form becomes $x^2 + y^2 = a^2$.

15.2.2 General Equation

The equation $x^2 + y^2 + 2gx + 2fy + c = 0$ is called general equation of circle in canonical form. Comparing with equation, $x^2 + y^2 - 2\alpha x - 2\beta y + \alpha^2 + \beta^2 - r^2 = 0$. The equation $x^2 + y^2 + 2gx + 2fy + c = 0$ can also be written as $(x+g)^2 + (y+f)^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2$.

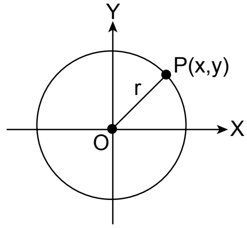
Hence, centre $\equiv (-g, -f)$, i.e., $\left(-\frac{1}{2}\text{ coefficient of } x; -\frac{1}{2}\text{ coefficient of } y\right)$

and radius $\equiv \sqrt{g^2 + f^2 - c}$

- $g^2 + f^2 - c > 0 \Rightarrow$ real circle with positive radius.
- $g^2 + f^2 - c = 0 \Rightarrow$ represent a point circle.
- $g^2 + f^2 - c < 0 \Rightarrow$ represent an imaginary.

Note:

A general equation of second degree non-homogenous is
$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ *in x, y represents a circle, if*

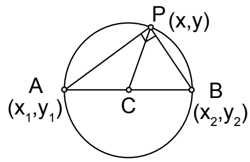


- *Coefficients of* x^2 = *coefficients of* y^2 *i.e.,* $a = b \neq 0$
- *Coefficient of xy is zero, i.e.,* $h = 0$.
- $g^2 + f^2 - c \leq 0$
  The general equation may be of the form $Ax^2 + Ay^2 + 2Gx + 2Fy + c = 0$ *represent a equation of circle.*

  Centre $= \left(-\frac{G}{A}, -\frac{F}{A}\right)$ *and radius* $= \frac{1}{A}\sqrt{G^2 + F^2 - AC}$

15.2.3 Diametric Form

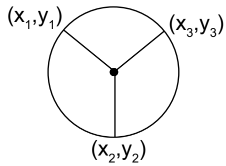
If (x_1, y_1) and (x_2, y_2) are the extremities of one of the diameter of a circle, then its equation is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.



15.2.4 Equation of Circle Thorugh Three Points

The equation of circle through three non-collinear points:

$$A(x_1, y_1), B(x_2, y_2), C(x_3, y_3) \text{ is } \begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^3 & x_3 & y_3 & 1 \end{vmatrix} = 0$$



15.2.5 The Carametric Coordinates of any Point on the Circle

Parametric Equation of Circle: When both x and y coordinates of the point on the circle are expressed as a function of single parameter, e.g., t or θ, etc., then the equation is called parametric equation of circle.

Case 1: Standard Equation: $x^2 + y^2 = r^2$ parametric equation: $x = r\cos\theta$ and $y = r\sin\theta$

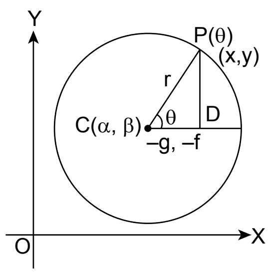
By restricting the values of parameter, we can express the part of curve (the arc of circle, line segment, etc.) very conveniently; which is not as easy in case of Cartesian equation of curve.

$\theta \in [0, 2\pi)$ full circle; $\theta \in (0, \pi)$ upper semicircle
$\theta \in (\pi, 2\pi)$ lower semicircle; $\theta \in (\alpha, \beta)$ circular arc

Case 2: General equation $(x - \alpha)^2 + (y - \beta)^2 = r^2$ parametric equation $x = \alpha + r\cos\theta$ and $y = \beta + r\sin\theta$

$\frac{x-\alpha}{\cos\theta} = \frac{y-\beta}{\sin\theta} = r$, where θ is parameter and constant represents circle.

$\frac{x-\alpha}{\cos\theta} = \frac{y-\beta}{\sin\theta} = r$, where r is parameter and θ is constant represents straight line.

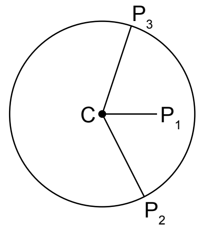


- Parametric coordinates of any point on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ are $x = -g + \sqrt{g^2 + f^2 - c}\cos\theta$
  $y = -f + \sqrt{g^2 + f^2 - c}\sin\theta$ $\{(-g, -f)\}$ is the centre and $\sqrt{g^2 + f^2 - c}$ is the radius of the circle.

15.2.6 Position of a Point with Respect to a Circle

Point P(x_1, y_1) lies inside, on or outside the circle.

$S = x^2 + y^2 + 2gx + 2fy + c = 0$, accordingly as $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ is < 0, = 0 or > 0 respectively



$\Rightarrow \quad \sqrt{(x_1+g)^2+(y_1+f)^2} \Leftrightarrow \sqrt{g^2+f^2-c}$

$\Rightarrow \quad (x_1 + g)^2 + (y_1 + f)^2 \Leftrightarrow g^2 + f^2 - c$

$\Rightarrow \quad x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c \Leftrightarrow 0$ or $S_1 \Leftrightarrow 0$; where

$S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

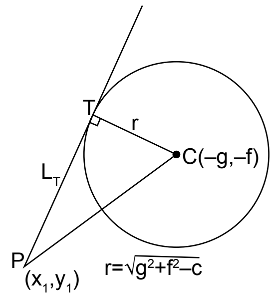
So, $S_1 > 0 \Rightarrow (x_1, y_1)$ is outside the circle

$S_1 = 0 \quad \Rightarrow (x_1, y_1)$ is on the circle

$S_1 < 0 \quad \Rightarrow (x_1, y_1)$ is inside the circle

- **Length of tangent from point P to the circle**
  $S = x^2 + y^2 + 2gx + 2fy + c = 0$
- $L_T = PT = \sqrt{PC^2 - r^2} = \sqrt{(x_1+g)^2+(y_1+f)^2-(g^2+f^2-c)}$

  $= \sqrt{(x_1^2+y_1^2+2gx_1+2fy_1+c} = \sqrt{S_1}$
- If S_1 is called power of point P w.r.t circle S = 0
  $\sqrt{S_1}$= length of tangent drawn from P to circle.
  - If P lies outside S_1 then is + ve $\Rightarrow$two tangents drawn.
  - If P lies on circle $S_1 = 0 \Rightarrow$ only one tangent
  - If P lies inside circle $S_1 < 0 \Rightarrow$no (imaginary) tangent



15.2.7 Position of a Line with Respect to a Circle

Let L = 0 be a line and S = 0 be a circle, if 'r' be the radius of a circle and p be the length of perpendicular from the centre of circle on the line, then if:

$p > r \Rightarrow$ Line is outside the circle

$p = r \Rightarrow$ Line touches circle

$p < r \Rightarrow$ Line is the chord of circle

$p = 0 \Rightarrow$ Line is diameter of circle

Notes:

(i) Length of the intercept made by the circle on the line is $2\sqrt{r^2 - p^2}$.

(ii) The length of the intercept made by the line $y = mx + c$ *with the circle* $x^2 + y^2 = a^2$ *is* $2\sqrt{\dfrac{a^2(1+m^2)-c^2}{1+m^2}}$.

15.2.7.1 *Condition for Tangency*

(i) The line y = mx + c is tangent to the circle $x^2 + y^2 = a^2$, if and only if, $c^2 = a^2(1 + m^2)$. If it is tangent, then the point of contact is given by $\left(\dfrac{-ma^2}{c}, \dfrac{a^2}{c}\right)$.

(ii) The line, lx + my + n = 0, is tangent to the circle $x^2 + y^2 = a^2$, if and only if, $n^2 = a^2 (l^2 + m^2)$. If it is tangent, then point of contact is given by $\left(\dfrac{-la^2}{n}, \dfrac{-ma^2}{n}\right)$.

Note:

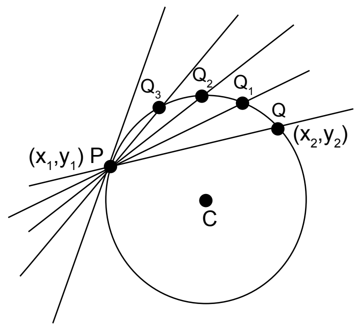
$y = mx \pm a\sqrt{1+m^2}$ $\forall\ m \in \mathbb{R}$ is called family of tangents or tangent in term of slope. In case, the slope of tangent is given or tangents passing from a given point are to be obtained this formula can be applied.

15.3 EQUATION OF TANGENT AND NORMAL

15.3.1 Tangents

Tangent line to a circle at a point $P(x_1, y_1)$ is defined as a limiting case of a chord PQ where Q is (x_2, y_2) such that $Q \to P$. As $Q \to P$ i.e., $x_2 \to x_1$ and $y_2 \to y_1$.

Then chord $PQ \to$ tangent at $P \Rightarrow$ Slope of chord $PQ \to$ slope of tangent at P.



$$\Rightarrow \quad m_t = \lim_{\substack{x_2 \to x_1 \\ y_2 \to y_1}} \left(\frac{y_2 - y_1}{x_2 - x_1}\right) = \lim_{\substack{x_2 \to x_1 \\ y_2 \to y_1}} -\left(\frac{x_1 + x_2}{y_1 + y_2}\right) = -\frac{x_1}{y_1}$$

$$\therefore \quad y - y_1 = -\frac{x_1}{y_1}(x - x_1) \qquad \Rightarrow T = xx_1 + yy_1 - a^2 = 0$$

$$\because \quad x_1^2 + y_1^2 = a^2 \qquad \text{....(1)}$$

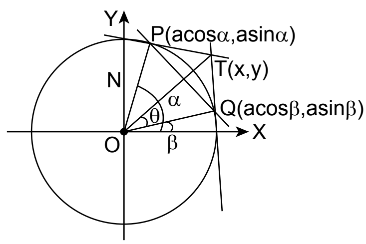
$$x_2^2 + y_2^2 = a^2 \qquad \text{....(2)}$$

$$\Rightarrow \quad (x_2^2 - x_1^2) = -(y_2^2 - y_1^2) \qquad \Rightarrow \frac{y_2 - y_1}{x_2 - x_1} = -\left(\frac{x_1 + x_2}{y_1 + y_2}\right)$$

- ❑ If the equations of the circle are given in general form then the equation of tangent to $S = x^2 + y^2 + 2gx + 2fy + c = 0$ at a point (x_1, y_1) is $T = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

15.3.2 Parametric Form

Equation of tangent to circle $x^2 + y^2 = a^2$ at $(a \cos \alpha, a \sin \alpha)$ is $x \cos \alpha + y \sin \alpha = a$

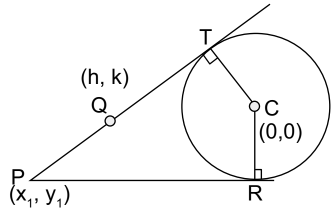


- ❑ Point of intersection of the tangent drawn to the circle $x^2 + y^2 = a^2$ at the point $P(\alpha)$ and $Q(\beta)$ is

$$x = \frac{a\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}; \quad y = \frac{a\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}$$

15.3.3 Pair of Tangents

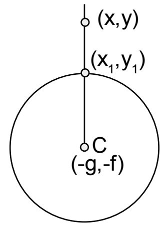
Combined equation of the pair of tangents drawn from an external point 'P' to a given circle is $SS_1 = T^2$, $S_1 = x_1^2 + y_1^2 - a^2$ and $T \equiv xx_1 + yy_1 - a^2 = 0$.



15.3.4 Normals

Normal is defined as a line perpendicular to the tangent line to the circle at the point of tangency $P(x_1, y_1)$.

If the equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$



$\Rightarrow$ slope of the normal $m = \dfrac{y_1 + f}{x_1 + g}$

$\Rightarrow$ Equation of normal $(y - y_1) = \dfrac{y_1 + f}{x_1 + g}(x - x_1)$.

Equation of normal in determinant form is given by $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ -g & -f & 1 \end{vmatrix} = 0.$

- ❑ **Director Circle:** The locus of point of intersection of two perpendicular tangents is called the director circle. The director circle of the circle $x^2 + y^2 = a^2$ is $x^2 + y^2 = 2a^2$.
- ❑ **Diameter of a circle:** The locus of middle points of a system of parallel chords of a circle is called the diameter of a circle. The diameter of the circle $x^2 + y^2 = r^2$ corresponding to the system of parallel chords $y = mx + c$ is $x + my = 0$.

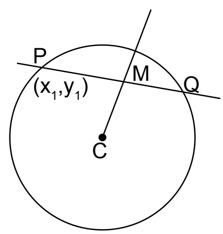
Notes:

(i) Every diameter passes through the centre of the circle.

(ii) A diameter is perpendicular to the system of parallel chords.

15.3.5 Equation of Chord with Mid-point as (x_1,y_1)

Slope of chord = $-\dfrac{x_1}{y_1} \Rightarrow$ equation of chord: $(y - y_1) = -\dfrac{x_1}{y_1}(x - x_1)$



$\Rightarrow \quad yy_1 - y_1^2 = -xx_1 + x_1^2 \qquad \Rightarrow \quad yy_1 + xx_1 = x_1^2 + y_1^2$

$\Rightarrow \quad xx_1 + yy_1 - a^2 = x_1^2 + y_1^2 - a^2$, i.e., $T = S_1$.

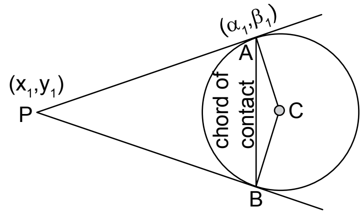
- ❑ For any conic section the equation of chord whose mid point is (x_1, y_1) is given by $T = S_1$.

15.4 CHORD OF CONTACT

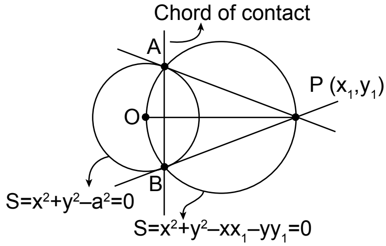
From a point $P(x_1, y_1)$, exterior to a circle, two tangents can be drawn to the circle. Let these tangents be PA and PB. Then the line segment AB is a chord of the circle and is called chord of contact of $P(x_1, y_1)$ with respect to the circle.

If $S = 0$ is the circle, then equation of the chord of contact of $P(x_1, y_1)$ w.r.t. the circle $S = 0$ is $T = 0$.

Equation of locus through intersection of $S = 0$ and $S' = 0$ is $S + \lambda S' = 0$, i.e., $(x^2 + y^2 - a^2) + \lambda(x^2 + y^2 - xx_1 - yy_1) = 0$.

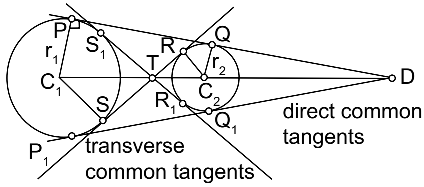


For $\lambda = -1$, the curve becomes $x x_1 + y y_1 = a^2$.



15.4.1 Relative Position of Two Circles

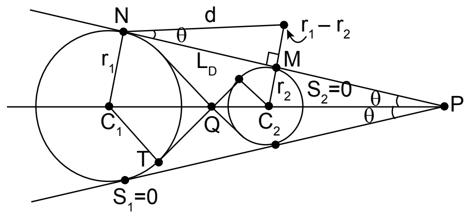
$S_1 = x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $S_2 = x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$.



Case 1: Two circle lies outside each other:

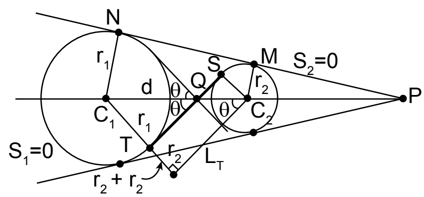
- Distance between centres: $d > r_1 + r_2$.
- Four common tangent (two direct, two transverse)
- PQ divides C_1C_2 in ratio $r_1 : r_2$ externally/internally.
- Equation of direct common tangent $\underbrace{y - \beta = m(x - \alpha), \text{where P is}(\alpha, \beta)}_{\text{Two values of m can be obtained from condition that this line touches both the circles } S_1 = 0 \text{ and } S_2 = 0}$

  Similarly, we get equation of T.C.T.

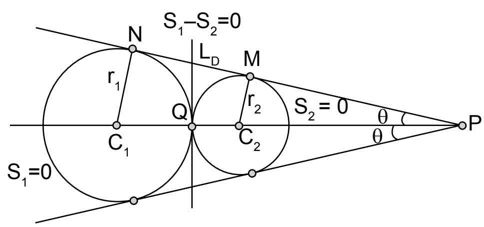


- **Direct Common Tangent:** Length of direct common tangent is defined as distance between point of contacts i.e., $L_D = MN = \sqrt{d^2 - (r_1 - r_2)^2}$.

  Angle between D.C.T. = $2\theta = 2\sin^{-1}\left(\frac{|r_1 - r_2|}{d}\right)$



- **Transverse Common Tangent:** Length of transverse common tangent is defined as distance between point of contacts i.e., S and T $L_T = ST = \sqrt{d^2 - (r_1 + r_2)^2}$. Angle between T.C.T. = $2\alpha = 2\sin^{-1}\left(\frac{r_1 + r_2}{d}\right)$



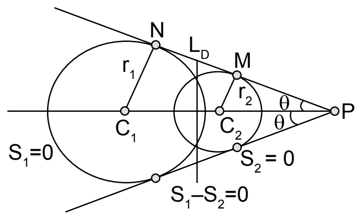
Case 2: Two circles touch each other externally. $C_1C_2 = d = r_1 + r_2$.

- Three common tangent (two DCT and one TCT).
- Equation of D.C.T. (obtained as in case I).
- Equation of T.C.T. is $S_1 - S_2 = 0$.

- Direct Common Tangent: $L_D = \sqrt{(r_1+r_2)^2 - (r_1-r_2)^2} = 2\sqrt{r_1 r_2}$. Angle between D.C.T. = 2θ $= 2\sin^{-1}\left(\dfrac{r_1-r_2}{r_1+r_2}\right)$.
- Transverse Common Tangent: $L_T = \sqrt{(r_1+r_2)^2 - (r_1+r_2)^2} = 0$. Angle between T.C.T. = 2α $= 2\sin^{-1}\left(\dfrac{r_1-r_2}{r_1+r_2}\right) = \pi$.

Case 3: Two Circles intersect each other: $|r_1 - r_2| < C_1C_2 < r_1 + r_2$.

- Two common tangent (two DCT and no TCT)
- Equation of common chord is $S_1 - S_2 = 0$
- $P\left(\dfrac{r_2g_1 - r_1g_2}{r_1-r_2}, \dfrac{r_2f_1 - r_1f_2}{r_1-r_2}\right)$
- Equation of D.C.T.
  $\underbrace{y-\beta = m(x-\alpha),\text{ where P is }(\alpha,\beta)}_{\text{Two values of m can be obtained from condition that this line touches both the circles } S_1=0 \text{ and } S_2=0}$



- Direct Common Tangent: $L_D = MN = \sqrt{d^2 - (r_1-r_2)^2}$
- Angle between D.C.T.: $2\theta = 2\sin^{-1}\left(\dfrac{|r_1-r_2|}{d}\right)$

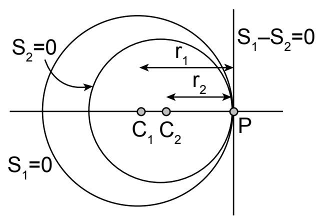
Case IV: Two Circles touch each other internally: $C_1C_2 = |r_1 - r_2|$

- Two direct common tangents
- Equation of D.C.T.: $S_1 - S_2 = 0$

15.4.2 Direct Common Tangent

$$L_D = \sqrt{d^2 - (r_1-r_2)^2} = \sqrt{(r_1-r_2)^2 - (r_1-r_2)^2} = 0.$$

Angle between D.C.T. = 2θ = $2\sin^{-1}\left(\dfrac{r_1-r_2}{r_1-r_2}\right) = \pi$



Case V: If $0 < C_1C_2 = d < |r_1 - r_2|$, then the circle lies completely inside other.

- **Angle of Intersection:** Angle of intersection (q) between two curve is defined as angle between their tangents at their point of intersection, which is same as angle between their normals at the point of intersection.

  $\because\ \cos\theta = \dfrac{r_1^2 + r_2^2 - d^2}{2r_1r_2} \Rightarrow \theta = \cos^{-1}\left(\dfrac{r_1^2 + r_2^2 - d^2}{2r_1r_2}\right)$

- **Orthogonal Intersection:** If the angle of intersection is $\pi/2$, then it is called as orthogonal intersection. Condition of orthogonality of the above two circles is:

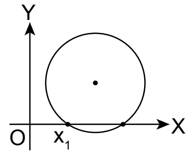
  : $r_1^2 + r_2^2 = d^2 \Rightarrow g_1^2 + f_2^2 - c_1 + g_2^2 + f_2^2 - c_2 = (g_2 - g_1)^2 + (f_2 - f_1)^2$

  $\Rightarrow 2(g_1g_2 + f_1f_2) = c_1 + c_2$

15.5 INTERCEPT MADE ON COORDINATE AXES BY THE CIRCLE

The intercept made by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$.

Let circle intersect x-axis at two points, $(x_1, 0)$ and $(x_2, 0)$, then x_1, x_2 are roots $x^2 + 2gx + c = 0$.



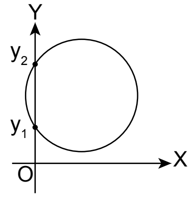
$\therefore$ Length of x-intercept = $|x_1 - x_2| = 2\sqrt{g^2 - c}$.

Similarly, length of y-intercept = $|y_1 - y_2| = 2\sqrt{f^2 - c}$.

$\therefore$ Conditions that given circle touches:

(i) x-axis is $g^2 = c$

(ii) y-axis is $f^2 = c$



Notes:

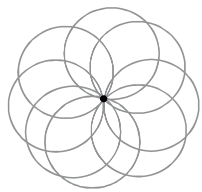
Circle $x^2 + y^2 + 2gx + 2fy + c = 0$ cuts:

(i) x-axis in two real, coincident or imaginary points according as $g^2 >, =, < c$.

(ii) y-axis in two real, coincident or imaginary points according as $f^2 > = < c$.

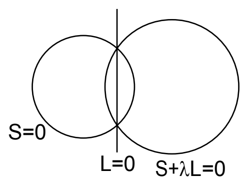
15.6 FAMILY OF CIRCLES

General Equation of Circle, $x^2 + y^2 + 2gx + 2fy + c = 0$ contains three unknown parameters (effective). Therefore three conditions are necessary in order to determine a circle uniquely and if only two conditions are given, then the obtained equation contains a parameter and it is described as family of circle.

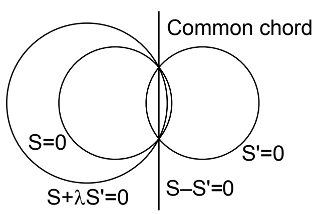


Following are the ways of expressing some known family of circles:

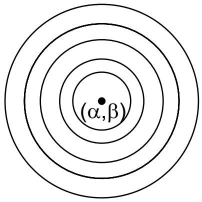
1. **Equation of circle through intersection of a circle S = 0 and a line L = 0**; $S + \lambda L = 0$.



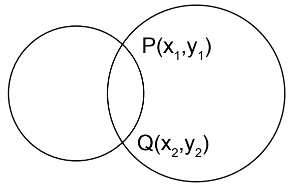
2. **Equation of family of circle passing through intersection of two circles** $S_1 = 0$ and $S_2 = 0$ is given as $S_1 + \lambda(S_1 - S_2) = 0$.



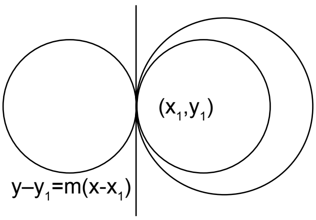
3. **Family of concentric circles:** The family of circles with the same centre and different radii is called a family of concentric circles $\mathbf{(x-a)^2 + (y-b)^2 = r^2}$ where (a, b) is the fixed point and r is a parameter.



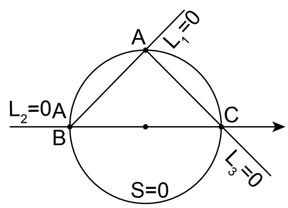
4. **Equation of any circle passes through two points (x_1,y_1) and (x_2,y_2)** $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+\lambda\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0$



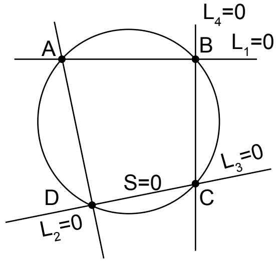
5. **Equation of family of circle touching the line with slope m at the point (x_1,y_1) is** $(x-x_1)^2+(y-y_1)^2+\lambda\{(y-y_1)-m(x-x_1)\}=0$ and if m is infinite, the family of circle is $(x-x_1)^2+(y-y_1)^2+\lambda(x-x_1)=0$ where λ is a parameter.



6. **Equation of circle circumscribing a triangle with sides** $L_1=0$, $L_2=0$ and $L_3=0$ is $L_1L_2+\lambda L_2L_3+\mu L_3L_1=0$ where, λ, μ is obtained by applying the condition that coefficient x^2 = coefficient y^2 and coefficient of xy = 0



7. **Family of conic circumscribing a quadrilateral with sides $L_1=0$, $L_2=0$, $L_3=0$ and $L_4=0$ taken in order** is $L_1L_3+\lambda L_2L_4=0$ and condition of concyclic ness and equation of possible circumcircle can be obtained by applying the condition that coefficient of x^2 = coefficient of y^2 and coefficient xy = 0 and analyzing the outcome mathematically.



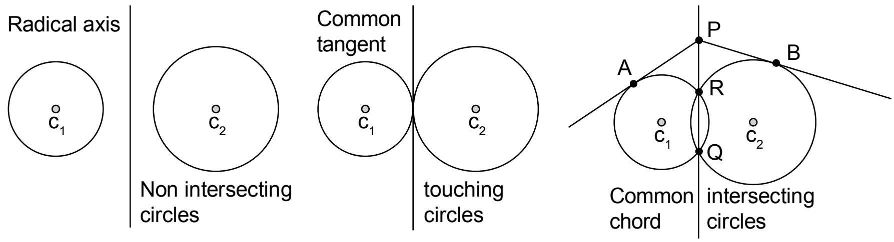
15.7 RADICAL AXES AND RADICAL CENTRE

Radical axis of S = 0 and S′ = 0 is the locus of the point from which the tangents drawn to the two circles are of equal lengths. Its equation is given by S – S′ = 0 (only if coefficients of x^2, y^2 in both circles are same.)

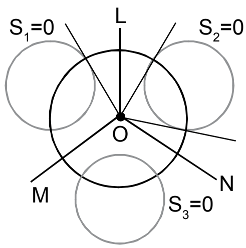
Remarks:

(i) *If the circles S = 0 and S′ = 0 intersect each other, then their common chord and their radical axis coincide. Thus they have the same eqn. S – S′ = 0*

(ii) *If two circles touch each other then their radical axis coincides with their common tangent at their point of contact. The equation is again S – S′ = 0.*



- **Radical Centre:** The common point of intersection of the radical axes of three circles taken two at a time is called the radical centre of the three circles.



Tips and Tricks:

1. *If two circles do not intersect ($c_1c_2 > r_1 + r_2$) then they have two transverse and two direct common tangents.*
2. *If two circles intersect ($c_1c_2 < r_1 + r_2$) then they have two direct tangents only.*
3. *If two circles touch externally ($c_1c_2 = r_1 + r_2$) then they have one transverse and two direct common tangents.*
4. *If two circles touch internally ($c_1c_2 = r_1 - r_2$) then they have only one common tangent.*
5. *If the point P lies outside the circle, then the polar and the chord of contact of this point P are same straight line.*
6. *If the point P lies on the circle, then the polar and the tangent to the circle at P are same straight line*
7. *The coordinates of the pole of the line $lx + my + n = 0$ with respect to the circel $x^2 + y^2 = a^2$ are*
8. *If (x_1, y_1) is the pole of the line $lx + my + n = 0$ w.r.t. the circle $x^2 + y^2 + 2gx + 2fy + c = .$, then where r is the radius of the circle.*