16 Limits, Continuity and Differentiability

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Calculus traditionally begins with a look at limits and continuity. In many courses, this is the only time students will see discontinuous functions with removable discontinuities and most won't see jump discontinuities elsewhere. The concept of limit is basic to all of calculus. Lack of understanding of the concept and lack of an adequate definition hampered the development of mathematics from the time of the cheek until long after the time of Leibnitz and Newton. Even later, Simon Antoine Jean Lhnilier (1750-1840) claimed that a variable quantity at all stages has a certain property, and then its limit has the same property. But, the limit of a sequence of rational number can be an irrational number, demonstrate the falsity of Lhnilier's claim. Later on the precise and abstract definition of limit of function not only brought the confusion to an end but also enabled us to discuss simultaneously a number of concepts like continuity and differentiability. To develop calculus for functions of one variable, we needed to make sense of the concept of a limit, which we needed to understand continuous functions and to define the derivative. Limits involving functions of two variables can be considerably more difficult to deal with; fortunately, most of the functions we encounter are fairly easy to understand.

Now, you must be careful when finding the derivative, because not every function has one. Most functions are differentiable, which means that a derivative exists at every point on the function. Some functions, however, are not completely differentiable. Many students interpret the term derivative incorrectly. What is the derivative of a function? What do we mean when we say things like "the derivative of is 2x"? In the simplest terms, the derivative is the slope of the tangent to the functions. If then the derivative (slope of tangent) at will have the value 2, the derivative (slope of tangent) at will have the value 4, and the derivative (slope of tangent) at any point x will have the value 2x. This is what we mean by saying that the derivative of is 2x. Essentially, the derivative of is a new function and plugging in an x-value in gives us the slope of the tangent to the curve of f at that particular x-value.

Note

We strongly urge you to remember this significance of the terms derivative and differentiation. It is suggested that you practice proving a few differentiation formulae from first principles, so that the connection between derivatives and slopes of tangents is clear in your mind (that is the reason that the proofs of the product, chain and quotient rule are all given in the theory – they are important!). In addition, whenever you carry out the process of differentiation, don't think of it as a mechanical rule with no basis – think of it as an activity to find the slopes of tangents to the given function at various points.

Differentiation: Once again, the term differentiation is frequently not understood properly. In simple terms, differentiation is the process by which you find the derivative of a function:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Or $$f'(x)=\lim_{h\to 0}\frac{f(x-h)-f(x)}{-h}$$

Thus, for example, when we say something line 'differentiate $f(x)=x^2$ to get $2x$', we are essentially talking about carrying out the process to find the derivative of $f(x)$ at any x-value, which comes out to be $2x$. The various rules we have made for differentiation-from the simpler standard formulae of differentiation to the product, quotient and chain rule – all have the same basis and meaning – finding the slope of the tangent the curve at any x-value. The derivative of f obtained at any point x is dependent on the value of x, and so the derivative is itself a function of x.

Existence of limit

- $\lim_{h\to 0} f(a+h)$ is the right-hand limit of $f(x)$ at $x=a$ and it is denoted by $f(a+0)$ or $f(a+)$ or $\lim_{x\to a+0} f(x)$.

  $\lim_{h\to 0} f(a-h)$ is the left-hand limit of $f(x)$ at $x=a$ and it is denoted by $f(a-0)$ or $f(a-)$ or $\lim_{x\to a-0} f(x)$.

- $\lim_{x \to a} f(x)$ exists if $f(a+0) = f(a-0)$ and the value of $\lim_{x \to a} f(x)$ is equal to the common value of $f(a+0)$ and $f(a-0)$.

Continuity of a function at a point

- A function $f(x)$ is a continuous at $x = a$ if $f(a+0) = f(a-0) = f(a)$,

i.e., $\lim_{h \to 0} f(a+h) = \lim_{h \to 0} f(a-h) = f(a)$.

If any two of the above three are unequal then $f(x)$ is discontinuous at $x = a$.

(i) If $\lim_{x \to a} f(x)$ does not exist then $f(x)$ cannot be continuous at $x = a$.

(ii) If $x = a$ is the left-end point of the domain of definition then x cannot tend to a from the left and so the question of getting $f(a-0)$ does not arise. In this case $f(x)$ will be continuous at $x = a$ if $f(a+0) = f(a)$. Similarly, if $x = a$ is the right-end point of the domain of definition then $f(x)$ will be continuous at $x = a$ provided $f(a-0) = f(a)$.

Differentiability of a function at a point

- Right-hand derivative of $f(x)$ at $x = a$, denoted by $f'(a+0)$ or $f'(a+)$, is the $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.
- Left-hand derivative of $f(x)$ at $x = a$, denoted by $f'(a-0)$ or $f'(a-)$, is the $\lim_{h \to 0} \frac{f(a+h) - f(a)}{-h}$
- A function $f(x)$ is said to be differentiable (finitely) at $x = a$ if $f'(a+0) = f'(a-0) =$ finite,

  i.e., $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h} =$ finite and the common limit is called the derivative of $f(x)$ at $x = a$, denoted by $f'(a)$.

  Clearly, $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$, $\{x \to a$ from the left as well as from the right$\}$.

Relation between continuity and differentiability

- $f(x)$ is differentiable (finitely) at $x = a$

  $\Rightarrow$ $f(x)$ is continuous at $x = a$
- $f(x)$ is not continuous at $x = a$

  $\Rightarrow$ $f(x)$ is not differentiable (finitely) at $x = a$.

TIP: While examining the continuity and differentiability of a function $f(x)$ at a point $x = a$, if you start with the differentiability and find that $f(x)$ is differentiable then you can conclude that the function is also continuous. But if you find $f(x)$ is not differentiable at $x = a$, you will also have to check the continuity separately.

Instead, if you start with continuity and find that the function is not continuous then you can conclude that the function is also non-differentiable. But if you find $f(x)$ is continuous, you will also have to check the differentiability separately.

Continuity and differentiability of some standard functions

- Polynomial functions (i.e., $a_0x^n + a_1x^{n-1} + ... + a_n$), $\sin x$, $\cos x$ and e^x are continuous and differentiable at all points of the set R of real numbers.
- $\log_e x$ is continuous and differentiable at all points of $(0, +\infty)$.
- $\tan x$ and $\sec x$ are continuous and differentiable at all points of $R - \left\{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, ...\right\}$.
- $\cot x$ and $\operatorname{cosec} x$ are continuous and differentiable at all points of $R - \{0, \pi, 2\pi, 3\pi, ...\}$.
- $|x - a|$ is continuous everywhere and differentiable at all points except at $x = a$.
- $[x]$ is continuous and differentiable everywhere except at $x =$ an integer.

Algebraic property of continuity and differentiability

- If $f(x)$ and $\phi(x)$ are both continuous (or differentiable) at $x = a$ then $f(x) \pm \phi(x)$ and $f(x) \times \phi(x)$ are continuous (or differentiable) at $x = a$.
- If $f(x)$ is continuous (or differentiable) and $\phi(x)$ is discontinuous (or non-differentiable) then $f(x) \pm \phi(x)$ and $f(x) \times \phi(x)$ are discontinuous (or non-differentiable) at $x = a$.

Continuity and differentiability in an interval

- $f(x)$ is continuous in an interval if it is continuous at each point of the interval.
- $f(x)$ is differentiable in an interval if it is differentiable at each point of the interval.

  Method of examining continuity and differentiability in an interval:

  (i) Detect all the pieces of the intervals of definition of the function.

  (ii) Apply continuity and differentiability of standard functions in each of the intervals.

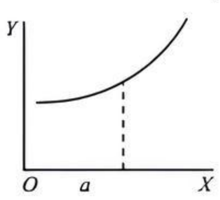
  (iii) Examine the continuity and differentiability at each turning point of definition of the function.

Some properties of a function continuous over a closed interval

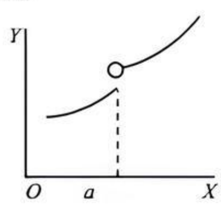
- If $f(x)$ is continuous in the closed interval $[a, b]$ then the value of $f(x)$ at all points in $[a, b]$ will lie between two fixed real numbers.
- In the interval $[a, b]$, $f(x)$ has a greatest value and a least value $f(x)$ will obtain all the values lying between min $f(x)$ and max $f(x)$ and max $f(x)$ for points in $[a, b]$ the range of $f(x) = [\min\ f(x), \max\ f(x)]$ when the domain of $f(x) = [a, b]$.

Rough sketch of a function in an interval

- If a function $f(x)$ is continuous at $x = a$, the graph of $f(x)$ at the corresponding point $\{a, f(a)\}$ will not be broken. But if $f(x)$ is discontinuous at $x = a$, the graph will be broken at the corresponding point.

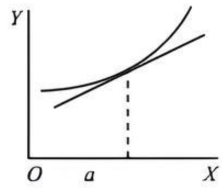


When continuous at $x = a$

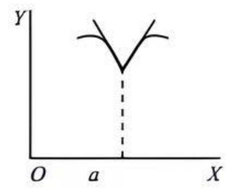


When discontinuous at $x = a$

- If a function $f(x)$ is differentiable at $x = a$, the graph of $f(x)$ will be such that there is only one tangent to the graph at the corresponding point. But if $f(x)$ is non-differentiable at $x = a$, there will not be unique tangent at the corresponding point of the graph.



(When differentiable at $x = a$)



(When non-differentiable at $x = a$)

Method of drawing a rough sketch of a function in an interval:

(i) Take the different pieces of intervals of definition of the function and draw graph in each of the intervals.

(ii) Examine the continuity and differentiability at each of the turning points of definition.

Let $x = a$ be a turning point. Let $f(x)$ be defined in $[b, a]$ and $(a, c]$. If $f(x)$ is continuous at a then the graphs for $f(x)$ in $[b, a]$ and $(a, c]$ will be joined at $x = a$ as in. If $f(x)$ is discontinuous at $x = a$, draw the graph of $f(x)$ in $[b, a]$ and the graph of $f(x)$ in $(a, c]$ with o at the beginning of the graph in $(a, c]$ as in. If $f(x)$ is continuous at $x = a$ but not differentiable at $x = a$ then the graph of $f(x)$ at $x = a$ will be as shown in.

Indeterminate forms of value

- The indeterminate forms of values of a function at a point are $\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, \infty - \infty; 1^{\infty}, 0^{0}, \infty^{0}$

Standard limits

- $\lim\limits_{x \to \pm\infty} \frac{1}{x} = 0$
- $\lim\limits_{x \to 0} \frac{1}{|x|} = \infty$
- $\lim\limits_{x \to 0} |x|^n = 0$, where $n > 0$
- $\lim\limits_{n \to \infty} x^n = 0$ if $|x| < 1$
- $\lim\limits_{n \to \infty} |x|^n = \infty$ if $|x| > 1$
- $\lim\limits_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$
- $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$, where x is in radian measure
- $\lim\limits_{x \to 0} \cos x = 1$
- $\lim\limits_{x \to 0} \frac{\tan x}{x} = 1$, where x is in radian measure

- $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$
- $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a$
- $\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x = e^a$
- $\lim_{x \to 0} (1 + ax)^{1/x} = e^a$

Properties of limits

- $\lim_{x \to a} \{f(x) \pm \phi(x)\} = \lim_{x \to a} f(x) \pm \lim_{x \to a} \phi(x)$
- $\lim_{x \to a} \{f(x) \times \phi(x)\} = \lim_{x \to a} f(x) \times \lim_{x \to a} \phi(x)$
- $\lim_{x \to a} \frac{f(x)}{\phi(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} \phi(x)}$
- $\lim_{x \to a} f(x) = e^{\lim_{x \to a} \log_e f(x)}$
- $\lim_{x \to a} \{f(x)\}^{\phi(x)} = \left\{\lim_{x \to a} f(x)\right\}^{\lim_{x \to a} \phi(x)}$
- $\lim_{x \to a} f\{\phi(x)\} = f\left(\lim_{x \to a} \phi(x)\right)$ if $f(x)$ is continuous

L'Hospital's rule

- If $f(a) = 0$ and $\phi(a) = 0$ then $\lim_{x \to a} \frac{f(x)}{\phi(x)} = \lim_{x \to a} \frac{f'(x)}{\phi'(x)}$
- If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} \phi(x) = \infty$, then

  $\lim_{x \to a} \frac{f(x)}{\phi(x)} = \lim_{x \to a} \frac{f'(x)}{\phi'(x)}$

Illustration

Illustration 1: If α is a repeated root of $ax^2 + bx + c = 0$, then $\lim_{x \to \alpha} \frac{\tan(ax^2 + bx + c)}{(x - \alpha)^2}$ is?

Solution: $\lim_{x \to \alpha} \frac{\tan(ax^2 + bx + c)}{(x - \alpha)^2} \left(\frac{0}{0} \text{form as } a\alpha^2 + b\alpha + c = 0\right)$

$$= \lim_{x \to a} \frac{(2ax + b)\sec^2(ax^2 + bx + c)}{2(x - \alpha)}$$

$$\left(\frac{0}{0} \text{form as } \alpha \text{ being a repeated root of } \alpha x^2 + bx + c = 0, 2a\alpha + b = 0\right)$$

$$\lim_{x \to a} \frac{2a\sec^2(ax^2 + bx + c) + (2ax + b)^2 + 2\sec^2(ax^2 + bx + c)\tan(ax^2 + bx + c)}{2}$$

Illustration 2: $\lim_{x \to 0} \frac{e^x + e^{-x} + 2\cos x - 4}{x^4}$ is equal to?

Solution: $= \lim_{x \to 0} \frac{e^x + e^{-x} + 2\cos x - 4}{x^4} \left(\frac{0}{0} \text{form}\right)$

$$= \lim_{x \to 0} \frac{e^x - e^{-x} - 2\sin x}{4x^3} \quad \left(\frac{0}{0} \text{form}\right)$$

$$= \lim_{x \to 0} \frac{e^x - e^{-x} - 2\cos x}{12x^2} \quad \left(\frac{0}{0} \text{form}\right)$$

$$= \lim_{x \to 0} \frac{e^x - e^{-x} - 2\sin x}{24x} \quad \left(\frac{0}{0} \text{form}\right)$$

Illustration 3: If $\lim_{x \to 0} \frac{729^x - 243^x + 81^x + 9^x + 3^x - 1}{x^4} = k(\log 3)^3$, then $k = ?$

Solution: Required limit

$$= \lim_{x \to 0} \frac{243^x(3^x - 1) - 9^x(3^{2x} - 1) + (3^x - 1)}{x^3}$$

$$= \lim_{x \to 0} \frac{(3^x - 1)\{(243)^x - (27)^x - 9^x + 1\}}{x^3}$$

$$= \lim_{x \to 0} \frac{(3^x - 1)\{(243)^x - (27)^x - 9^x + 1\}}{x^3}$$

$$= \lim_{x \to 0} \frac{(3^x - 1)}{x} \frac{(9^x - 1)}{x} \frac{(27^x - 1)}{x}$$

$$= \log 3 \cdot \log 9 \cdot \log 27 = \log 3.\ 2\log 3.3\log 3$$

Illustration 4: If $f(9) = 9$ and $f'(9) = 1$, then $\lim_{x \to 9} \frac{3 - \sqrt{f(x)}}{3 - \sqrt{x}}$ is equal to?

Solution: $\lim_{x \to 9} \frac{3 - \sqrt{f(x)}}{3 - \sqrt{x}} \quad \left(\frac{0}{0} \text{form}\right)$

$$\lim_{x \to 9} \frac{0 - \frac{1}{2\sqrt{f(x)}}.f'(x)}{0 - \frac{1}{2\sqrt{x}}} \text{ [Using L' Hospital's Rule]}$$

$$= \lim_{x \to 9} \frac{\sqrt{x}}{\sqrt{f(x)}} \cdot f'(x) = \frac{3}{3} \cdot f'(9) = 1.$$

Illustration 5: $\lim_{x \to 1} \frac{\sqrt[3]{x^2} - 2\sqrt[3]{x} + 1}{(x - 1)^2}$ is equal to?

Solution: $\lim_{x \to 1} \frac{\sqrt[3]{x^2} - 2\sqrt[3]{x} + 1}{(x - 1)^2}$

$$\lim_{x \to 1} \frac{y^2 - 2y + 1}{(y^3 - 1)^2} \text{ [Putting } \sqrt[3]{x} = y, \text{ as } x \to 1, y \to 1]$$

$$\lim_{y \to 1} \frac{(y - 1)^2}{(y - 1)^2(y^2 + y + 1)^2} = \lim_{y \to 1} \frac{1}{(y^2 + y + 1)^2} = \frac{1}{9}.$$

Illustration 6: $\lim\limits_{x\to-1}\left(\dfrac{x^4+x^2+x+1}{x^2-x+1}\right)^{\frac{1-\cos(x-1)}{(x+1)^2}}$ is equal to?

Solution: $\lim\limits_{x\to-1}\left(\dfrac{x^4+x^2+x+1}{x^2-x+1}\right)^{\frac{1-\cos(x-1)}{(x+1)^2}}$

$$=\lim_{x\to-1}\left(\frac{x^4+x^2+x+1}{x^2-x+1}\right)^{\frac{2\sin^2\left(\frac{x+1}{2}\right)}{(x+1)^2}}$$

$$=\lim_{x\to-1}\left(\frac{x^4+x^2+x+1}{x^2-x+1}\right)^{\frac{1}{2}\left(\frac{\sin\left(\frac{x+1}{2}\right)}{\left(\frac{x+1}{2}\right)}\right)^2}=\left(\frac{2}{3}\right)^{1/2}$$

Illustration 7: If $f(x)=\begin{cases}\dfrac{\sin[x]}{[x]}, & [x]\neq 0\\ 0\quad, & [x]=0\end{cases}$, where $[x]$ denotes the greatest integer $\le x$, then $\lim\limits_{x\to0}f(x)$ equals?

Solution: $=\lim\limits_{h\to0}f(0-h)=\lim\limits_{h\to0}\dfrac{\sin[-h]}{[-h]}=\lim\limits_{h\to0}\dfrac{\sin(-1)}{(-1)}=\sin 1.$

$$\lim_{h\to0}f(0+h)=\lim_{h\to0}\frac{\sin[h]}{[h]}=1$$

$[\therefore h\to0\Rightarrow(h)\to0]$

$\therefore\quad \lim\limits_{h\to0}f(x)$ does not exist.

Illustration 8: The value of $\lim\limits_{x\to3}\left(\log_a\dfrac{x-3}{\sqrt{x+6}-3}\right)$ is?

Solution: $\lim\limits_{x\to3}\left[\log_a\dfrac{x-3}{\sqrt{x+6-3}}\right]$

$$=\lim_{x\to3}\left[\log_a\frac{(x-3)(\sqrt{x+6+3})}{(x-3)}\right]$$

$$=\lim_{x\to3}\log_a(\sqrt{x+6}+3)=\log_a 6$$

Illustration 9: $\lim\limits_{x\to2}\dfrac{2^x-x^2}{x^x-2^2}$ is equal to?

Solution: $\lim\limits_{x\to2}\dfrac{2^x-x^2}{x^x-2^2}\qquad\left(\dfrac{0}{0}\text{form}\right)$

$$\lim_{x\to2}\frac{2^x\log2-2x}{x^x(1+\log x)}=\frac{4\log2-4}{4(1+\log2)}=\frac{\log2-1}{\log2+1}.$$

Illustration 10: $\lim\limits_{x\to0}\left(\operatorname{cosec}^3x\cdot\cot x-2\cot^3x\cdot\operatorname{cosec}x+\dfrac{\cot^4x}{\sec x}\right)$ is equal?

Solution: $\lim\limits_{x\to0}\left(\operatorname{cosec}^3x\cdot\cot x-2\cot^3x\cdot\operatorname{cosec}x+\dfrac{\cot^4x}{\sec x}\right)$

$$=\lim_{x\to0}\left(\frac{\cos x}{\sin^4x}-\frac{2\cos^3x}{\sin^4x}+\frac{\cos^5x}{\sin^4x}\right)$$

$$=\lim_{x\to0}\frac{\cos x(1-\cos^2x)^2}{\sin^4x}=\lim_{x\to0}\cos x=1$$

Illustration 11: The value of $\lim\limits_{x\to\infty}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right)$ is?

Solution: $\lim\limits_{x\to\infty}\left[\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right]$

$$=\lim_{x\to\infty}\frac{x+\sqrt{x+\sqrt{x-x}}}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}}$$

$$=\lim_{x\to\infty}\frac{\sqrt{x+\sqrt{x}}}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}}$$

$$=\lim_{x\to\infty}\frac{\sqrt{x}\left(1+\frac{1}{\sqrt{x}}\right)^{1/2}}{\sqrt{x}\left[\left(1+\frac{1}{\sqrt{x}}\sqrt{1+\frac{1}{\sqrt{x}}}\right)^{1/2}+1\right]}$$

$$=\frac{1}{1+1}=\frac{1}{2}$$

Illustration 12: $\lim\limits_{x\to\infty}\left(\dfrac{1}{1\cdot2}+\dfrac{1}{2\cdot3}+\dfrac{1}{3\cdot4}+...+\dfrac{1}{n(n+1)}\right)$ is equal to?

Solution: $\lim\limits_{x\to\infty}\left[\dfrac{1}{1\cdot2}+\dfrac{1}{2\cdot3}+\dfrac{1}{3\cdot4}+...+\dfrac{1}{n(n+1)}\right]$

$$=\lim_{x\to\infty}\left[\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+...+\left(\frac{1}{n}-\frac{1}{n+1}\right)\right]$$

$$=\lim_{x\to\infty}\left[1-\frac{1}{n+1}\right]=1-0=1.$$

Illustration 13: $\lim\limits_{x\to0}\dfrac{\log(1+x+x^2)+\log(1-x+x^2)}{\sec x-\cos x}$ is equal to?

Solution: $\lim\limits_{x\to0}\dfrac{\log(1+x+x^2)+\log(1-x+x^2)}{\sec x-\cos x}$

$$=\lim_{x\to0}\frac{\log[(1+x^2)^2-x^2]}{(1-\cos^2x)/\cos x}$$

$$=\lim_{x\to0}\frac{\log(1+x^2+x^4)}{\sin x\tan x}\qquad\left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to0}\frac{\log[1+x^2(1+x^2)]}{x^2(1+x^2)}\cdot x^2(1+x^2)\cdot\frac{1}{\frac{\sin x}{x}\cdot\frac{\tan x}{x}\cdot x^2}$$

$$=1\cdot\left[\text{as}\lim_{x\to0}\frac{\log(1+x)}{x}=1\right]$$