CHAPTER 19

COMPLEX NUMBER

19.1 INTRODUCTION

While working with real numbers (ℝ) we would not find relations to equations, such as $x^2 + 9 = 0$ (??). So, to look forward we have to difine another set of number systems.

19.1.1 Imaginary Numbers (Non-real Numbers)

A number whose square is non-positive, is termed as an imaginary number, e.g., $\sqrt{-2}$ or $(1+\sqrt{-2})$.

Iota: Euler introduced the symbol i for the number $\sqrt{-1}$. It is known as iota (a Greek word for 'imaginary'). Thus, $\sqrt{-2} = \sqrt{2}i$ and $1+\sqrt{-2} = 1+\sqrt{2}i$ are imaginary numbers.

Remark:

(i) *Imaginary numbers do not follow the property of order, i.e., for z_1 and z_2 imaginary numbers we cannot say which one is greater. Since i is neither positive nor negative, nor zero.*

(ii) *Here non-possible does not imply negative, e.g., $1+\sqrt{-2}$ is also non-positive.*

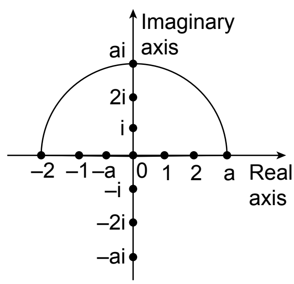
19.1.2 Purely Imaginary Numbers (I)

The number z whose square is non positive real number (negative or zero) is termed as purely imaginary number. For example, $\sqrt{-5}$, i.e., I = {z : z = ai; where a ∈ ℝ and i = $\sqrt{}$ }.

19.1.2.1 Geometrical representation of purely imaginary numbers

Single multiplication by i is equivalent to geometrical rotation of number by π/2 radians anti-clockwise.

Therefore, purely imaginary numbers are represented as points lying on y axis of argand plane. For example: z = ai is represented by point (0, a) on y axis as shown here:



Remarks:

1. *The plane formed by real and imaginary axes is called Argand/Gaussian/Complex Plane.*
2. *It should be kept in mind that any equation not having real roots does not necessarily posses imaginary roots. For example, the equation x + 5 = x + 7 is neither satisfied by real numbers nor is satisfied by imaginary numbers.*

19.1.3 Properties of Iota

1. $i^0 = 1, i^2 = -1, i^3 = -i, i^4 = 1$
2. Periodic properties of i ; $i^{4n} = 1, i^{4n+1} = i, i^{4n+2} = -1, i^{4n+3} = -i \ \forall \ n \in \mathbb{Z}$
3. $i^{-1} = -i$
4. Sum of four consecutive power terms of i is zero, that is, $i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0 \ \forall \ n \in \mathbb{Z}$.
5. For any two real numbers a and b; $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ is true only when atleast one of a and b is non-negative real number, i.e., both a and b are non-negative.

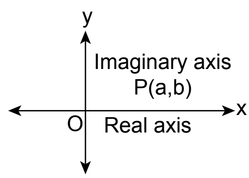
19.2 COMPLEX NUMBER

A number z resulting as a sum of a purely real number x and a purely imaginary number iy is called a complex number, i.e., a number of the form z = x + iy where $x, y \in \mathbb{R}$ and $i = \sqrt{-1}$ is called a complex number. Here x is called real part and y is called imaginary part of the complex number and they are expressed as Re(z) = x, Im (z) = y. A complex z = x + iy number may also be defined as an ordered pair of real numbers and may be denoted by the symbol (x, y).

The set of complex numbers is denoted by $\mathbb{C}$ and is given by = {z : z = x + iy; where $x, y \in \mathbb{R}$ and $i = \sqrt{-1}$ }.

19.3 ARGAND PLANE

Any complex number, z = a + ib, can be written as an ordered pair (a, b) which can be represented on a plane by the point P(a, b) (known as affix of point P) as shown in the figure. This plane is called Argand plane, complex plane or the Gaussian plane.



19.3.1 Representation of Complex Numbers

Complex numbers can be represented by following forms:

1. **Cartesian form (rectangular form):** A complex number, z = x + iy, can be represented by the point P having coordinate (x, y).
2. **Vector form (Algebraic form):** Every complex number z is regarded as a position vector $(\overrightarrow{OP})$ which is sum of two position vectors: Purely real vector x $(\overrightarrow{OA})$ and purely imaginary vector iy $(\overrightarrow{OB})$.

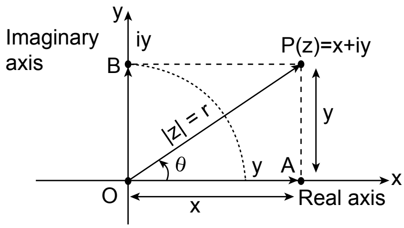
   $\because \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \overrightarrow{OA} + \overrightarrow{OB} \ \Rightarrow \ z = x + iy$

   Modulus of z: Distance of point P from the origin is called modulus of complex number z and is denoted by |z|. It is length of vector $(\overrightarrow{OP})$. It is distance of P(z) from origin.

$\therefore |z| = |\overrightarrow{OP}| = \sqrt{x^2+y^2} = \sqrt{(\text{Re}(z))^2 + (\text{Im}(z))^2}$

Argument of z: Argument of z is the angle made by $\overrightarrow{OP}$ with the positive direction of real axis. Also known as amplitude z and is denoted by amp (z).

Arg(z) = θ; where $\tan\theta = \frac{y}{x}$, θ lies in the quadrant in which complex number z lies.



Note:

The principal arguments θ ∈ (–π, π].

3. **Polar form (amplitude modulus form):** In ΔOAP : OP = |z| = r
   ⇒ OA = x = r cosθ and AP = y = r sinθ ⇒ z = x + iy = r (cosθ + i sinθ) = r cisθ

Remark

cis θ is unimodular complex number and acts as unit vector in the direction of θ where θ is arg z.

4. **Euler form (Exponential form):** Euler represented complex number z as an exponential function of its argument θ (radians) and described here. As we know that using Taylor's series expansion cos θ and sinθ can be expanded in terms of polynomial in θ as given below:

$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots\ldots$ and $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots\ldots$

$\therefore$ (cosθ + isinθ) = $1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \ldots\ldots$ to $\infty = e^{i\theta}$ ⇒ z = x + iy = r (cosθ + i sinθ) = $re^{i\theta}$

Advantages of using Euler form:

- ❑ Convenient for division and multiplication of complex numbers.
- ❑ Suitable for exponential, logarithmic and irrational functions involving complex numbers.

19.3.1.1 Inter-conversion from polar/trigonometric to algebraic form

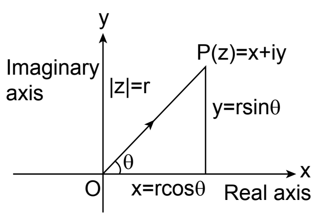
(i) **Algebraic form to polar form:** Given z = x + iy, then

$r = \sqrt{x^2+y^2}$; $\cos\theta = \frac{x}{r}$; $\sin\theta = \frac{y}{r}$ gives θ = ϕ (say)

In polar form $z = \sqrt{x^2+y^2}(\cos\phi + i\sin\phi)$

(ii) **Polar form to algebraic form:** Given z = r(cosθ + isinθ) = rcosθ + i(rsinθ)

⇒ z = x + iy; where x = rcosθ and y = rsinθ



19.3.2 Properties of Complex Numbers

(i) **Equality:** Two complex numbers z_1 and z_2 are equal only when their real and imaginary parts are respectively equal, i.e., Re(z_1) = Re(z_2) and I(z_1) = I(z_2) or $|z_1| = |z_2|$ and arg (z_1) = arg (z_2).

Remarks:

Students must note that x, y ∈ ℝ and x, y ≠ 0. If x + y = 0 ⇒ x = − y is correct, but x + iy = 0 ⇒ x = − iy is incorrect (unless both x and y are zero)

Hence, a real number cannot be equal to the imaginary number, unless both are zero.

(ii) **Inequality:** Inequality in complex number is not defined because 'i' is neither positive, zero nor negative. So 4 + 3i > 1 + 2i or i < 0 or i > 0 is meaningless.

(iii) If **Re(z) = 0** then z is **purely imaginary** and if **Im (z) = 0**, then z is **purely real**.

(iv) z = 0 ⇒ Re(z) = Im (z) = 0, therefore the complex number 0 is purely real and purely imaginary or both.

(v) If z = x + iy, then iz = −y + ix ⇒ Re(iz) = − Im(z) and Im(iz) = Re(z).

(vi) Conjugate of complex number: z = x + i y is denoted as $\overline{z}$ = (x − iy), i.e., a complex number with same real part as of z and negative imaginary part as that of z.

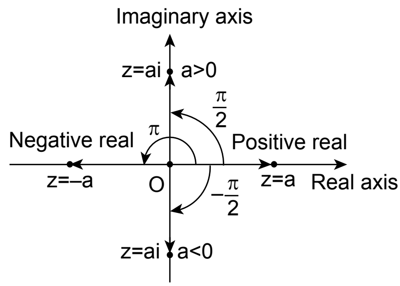
(vii) If z is purely real positive ⇒ Arg(z) = 0.

(viii) If z is purely real negative ⇒ Arg(z) = π.

(ix) If z is purely imaginary with positive imaginary part ⇒ Arg(z) = π/2.

(x) If z is purely imaginary with negative imaginary part ⇒ Arg(z) = −π/2.

(xi) Arg(0) is not defined.



19.3.2.1 *Binary operations defined on set of complex numbers*

Binary operation on set of complex number is a function from set of complex numbers to itself. That is, if z_1, z_2 ∈ C and * is a binary operation on the set of complex numbers then $z_1 * z_2$ ∈ C. Following binary operations are defined on set of complex numbers.

Addition of two complex numbers: Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2 \Rightarrow z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$

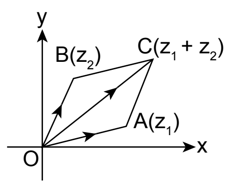
$= (x_1 + x_2) + i\,(y_1 + y_2)$; i.e., $z_1 + z_2 = [R(z_1) + R(z_2)] + i[I(z_1) + I(z_2)] \in C$.

19.3.2.2 *Geometric representation*

Consider two complex numbers $z_1 = (x_1 + iy_1)$ and $z_2 = (x_2 + iy_2)$ represented by vector $z_1 = \overrightarrow{OA}$; z $\overrightarrow{OB}$ as shown in figure.

Then by parallelogram law of vector addition $z_1 + z_2 = \overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$.

Hence C represents the affix of $z_1 + z_2$.



Notes:

In ΔOAC [Since sum of two sides of a Δ is always greater than the third side] ∴ OA + AC ≥ OC

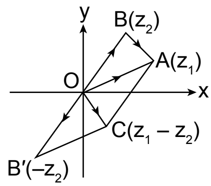
⇒ $|\overrightarrow{OA}| + |\overrightarrow{OB}| \geq |\overrightarrow{OC}|$

⇒ $|z_1| + |z_2| \geq |z_1 + z_2|$ *This is called triangle inequality. Also considering OAB; OA + OB ≥ AB*

⇒ $|\overrightarrow{OA}| + |\overrightarrow{OB}| \geq |\overrightarrow{BA}| \Rightarrow |z_1| + |z_2| \geq |z_1 - z_2|$

Subtraction of two complex numbers: Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$; then $z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2)$ $= (x_1 - x_2) + i\,(y_1 - y_2)$ i.e., $\mathbf{z_1 - z_2 = [R(z_1) - R(z_2)] + i[I(z_1) - I(z_2)] \in C}$.

19.3.2.3 Geometric representation



Using vector representation again, we have $\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = z_1 - z_2 = \overrightarrow{OC}$.

Hence, the other diagonal of the parallelogram represents the difference vector of z_1 and z_2.

Notes:

1. *While $\overrightarrow{BA}$ represents the free vector corresponding to $z_1 - z_2$, $\overrightarrow{OC}$ represents the position vector of $z_1 - z_2$.*

$\Rightarrow$ *C is affix of complex number $z_1 - z_2$.*

2. *In a triangle, the difference of two sides is always less than the third side.*

$\Rightarrow$ $\left\| \overrightarrow{OB} \right| - \left| \overrightarrow{OA} \right\| \le \left| \overrightarrow{AB} \right| \Rightarrow ||z_2| - |z_1|| \le |z_2 + z_1|$

3. *Triangle Inequality:* $||z_1| - |z_2|| \le |z_1 \pm z_2| \le |z_1| + |z_2|$

Multiplication of two complex numbers: Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then $z_1.z_2 = (x_1 + iy_1).(x_2 + iy_2) = [R(z_1).R(z_2) - I(z_1).I(z_2)] + i[R(z_2).I(z_1) + R(z_1).I(z_2)] \in C$.

Geometric representation: Let A and B are two points in the complex plane respectively, affixes of z_1 and z_2; where $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$; $z_1.z_2 = r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$.

19.3.3 Result

The product rule can be generalized to n complex numbers. Let $z_n = r_n(\cos\theta_n + i\sin\theta_n)$, where $n = 1, 2, ...$

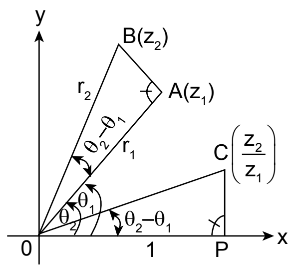
Now, $|z_1.\ z_2...z_n| = r_1r_2...r_n = |z_1|\ |z_2|.....|z_n|$ and $\arg(z_1\ z_2...z_n) = \theta_1 + \theta_2 + ... + \theta_n = \arg z_1 + \arg z_2 + ... + \arg z_n$.

As special case $\arg z^n = n \arg z$.

Division of two complex numbers: Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2 \Rightarrow z_1/z_2 = (x_1 + iy_1)/(x_2 + iy_2)$

$$= \frac{(x_1.x_2 + y_1.y_2)}{(x_2^2 + y_2^2)} + \frac{i(x_2.y_1 - x_1.y_2)}{(x_2^2 + y_2^2)} \in C.$$

Geometric representation: Let A and B are two points in the complex plane which are affixes of z_1 and z_2 respectively, where $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.



Then, we get $\frac{z_2}{z_1} = \frac{r_2}{r_1}[\cos(\theta_2 - \theta_1) + i\sin(\theta_2 - \theta_1)]$.

Notes:

1. *If θ_1 and θ_2 are principal values of argument of z_1 and z_2, then $\theta_1 + \theta_2$ may not necessarily be the principal value of argument of $z_1.\ z_2$ and $\theta_1 - \theta_2$ may not necessarily be principal value of argument of z_1/z_2. To make this argument as principal value, add or subtract $2n\pi$ where n is such an integer, which makes the argument as principal value.*

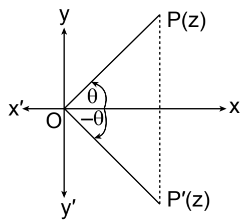
2. *Note that angle α between two vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ is $\alpha = \theta_2 - \theta_1$, $\alpha = \arg z_2 - \arg z_1$.*

19.4 ALGEBRAIC STRUCTURE OF SET OF COMPLEX NUMBERS

(i) Complex numbers obey closure law (for addition subtraction and multiplication), commutative law (for addition and multiplication) associative law (for addition and multiplication), existence of additive and multiplicative identitiy and inverse.

(ii) **Existence of conjugate element:** Every complex number z = x + iy has unique **conjugate** denoted as x – iy.

19.4.1 Conjugate of a Complex Number



Conjugate of a complex number z = x + iy is defined as $\overline{z} = x - iy$. It is mirror image of z in real axis as mirror shown in the figure given here:

Let z = r (cosθ + isinθ) ⇒ $\overline{z} = r(\cos\theta - i\sin\theta) = r[\cos(-\theta) + i\sin(-\theta)]$

⇒ $\overline{z}$ has its affix point having magnitude r and argument (–θ).

19.4.2 Properties of Conjugate of a Complex Number

1. $R(\overline{z}) = R(z), I(\overline{z}) = -I(z)$

2. $z.\overline{z} = |\overline{z}|^2 = |z|^2 = (R(z))^2 + (I(z))^2$

3. $(\overline{\overline{z}}) = z, (\overline{\overline{\overline{z}}}) = \overline{z}$ and so on.

4. $|z| = |\overline{z}|$ and – Agr z = Arg $\overline{z}$

5. If $z = \overline{z}$, i.e., arg z = arg $\overline{z}$ ⇒ z is purely real.

6. If $\overline{z}$ = – z, i.e., arg (–z) = arg($\overline{z}$) ⇒ z is purely imaginary

7. $R(z) = \dfrac{z + \overline{z}}{2} = x = R(\overline{z})$; $Im(z) = \dfrac{z - \overline{z}}{2i} = y = -Im(\overline{z})$

8. $\cos\theta = \left(\dfrac{e^{i\theta} + e^{-i\theta}}{2}\right)$; $\sin\theta = \left(\dfrac{e^{i\theta} - e^{-i\theta}}{2i}\right)$

9. $\overline{(z_1 \pm z_2 \pm z_3 \pm ... \pm z_n)} = \overline{z}_1 \pm \overline{z}_2 \pm \overline{z}_3 \pm ... \pm \overline{z}_n$

10. $\overline{(z_1.z_2.z_3...z_n)} = (\overline{z}_1).(\overline{z}_2).(\overline{z}_3).....(\overline{z}_n)$

11. $\overline{(z_1/z_2)} = \dfrac{(\overline{z}_1)}{(\overline{z}_2)}$

12. $\overline{(z^n)} = (\overline{z})^n$

13. If ω = f(z), then $\overline{\omega} = f(\overline{z})$, where f(z) is algebraic polynomial.

14. $z_1\overline{z}_2 + z_2\overline{z}_1 = 2R(\overline{z}_2 z_1)$

15. $|z_1 + z_2| = \sqrt{|z_1|^2 + |z_2|^2 + 2\,Re(z_1 z_2)}$

16. $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

19.4.3 Modulus of a Complex Number

Modulus of a complex number, z = x + iy, is denoted by |z|. If point p(x, y) represents the complex number z on Argand's plane, then $|z| = OP = \sqrt{x^2 + y^2}$ = distance between origin and point $P = \sqrt{[R(z)]^2 + [I(z)]^2}$.

19.4.3.1 *Properties of modulus of complex numbers*

1. Modulus of a complex numbers is distance of complex number from the origin and hence, is non-negative and $|z| \geq 0 \Rightarrow |z| = 0$ iff $z = 0$ and $|z| > 0$ iff $z \neq 0$.
2. $-|z| \leq \text{Re}(z) \leq |z|$ and $-|z| \leq \text{Im}(z) \leq |z|$
3. $|z| = |\overline{z}| = |-z| = |-\overline{z}|$
4. $z\overline{z} = |z|^2$
5. $|z_1 z_2| = |z_1||z_2|$. In general $|z_1 z_2 z_3 ......z_n| = |z_1||z_2||z_3|......|z_n|$.
6. $(z_2 \neq 0)$
7. **Triangle inequality:** $|z_1 \pm z_2| \leq |z_1| + |z_2|$. In general $|z_1 \pm z_2 \pm z_3 ......\pm z_n| \leq |z_1| \pm |z_2| \pm |z_3| \pm ......\pm |z_n|$
8. Similarly $|z_1 \pm z_2| \geq |z_1| - |z_2|$.
9. $|z^n| = |z|^n$
10. $||z_1| - |z_2|| \leq |z_1 \pm z_2| \leq |z_1| + |z_2|$. Thus, $|z_1| + |z_2|$ is the greatest possible value of $|z_1 \pm z_2|$ and $||z_1| - |z_2||$ is the least possible value of $|z_1 \pm z_2|$.
11. $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm (z_1\overline{z}_2 + \overline{z}_1 z_2)$ or $|z_1|^2 + |z_2|^2 \pm 2\text{Re}(z_1\overline{z}_2)$ or $|z_1|^2 + |z_2|^2 \pm 2|z_1||z_2| \cos(\theta_1 - \theta_2)$
12. $|z_1\overline{z}_2 + \overline{z}_1 z_2|^2 = 2|z_1||z_2|\cos(\theta_1 - \theta_2)$; where $\theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$.
13. $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \dfrac{z_1}{z_2}$ is purely imaginary
14. $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\{|z_1|^2 + |z_2|^2\}$
15. $|az_1 + bz_2|^2 + |bz_1 - az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2)$ where $a, b \in R$.
16. Unimodular: If z is unimodular, then $|z| = 1$. Now, if $f(z)$ is a unimodular, then it can always be expressed as $f(z) = \cos\theta + i\sin\theta$, $\theta \in \mathbb{R}$.

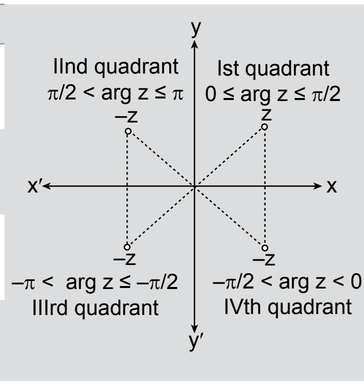
19.4.3.2 *Argument and principal argument of complex number*

Argument of z (arg z) is also known as amp(z) is angle which the radius vector $\overrightarrow{OP}$ makes with positive direction of real axis.

Principle Argument: In general, argument of a complex number is not unique, if θ is the argument, then $2n\pi + \theta$ is also the argument of the complex number where $n = 0, \pm 1, \pm 2, ...$ Hence, we define principle value of argument θ, which satisfies the condition $-\pi < \theta \leq \pi$. Hence, Principle value of arg(z) is taken as an angle lying in $(-\pi, \pi]$. It is denoted by Arg(z). Thus, $\arg(z) = \text{Arg}(z) \pm 2k\pi;\ k \in \mathbb{Z}$.

A complex number z, given as $(x + iy)$, lies in different quadrant depending upon the sign of x and y. Based on the quadrantal location of the complex number its principle argument are given as follows.

Sign of *x* and *y*	Location of *z*	Principal Argument
x > 0, y > 0	Ist quadrant	$\theta = \alpha = \tan^{-1}\left\|\frac{y}{x}\right\|$
x < 0, y > 0	IInd quadrant	$\theta = (\pi - \alpha) = \pi - \tan^{-1}\left\|\frac{y}{x}\right\|$
x < 0, y < 0	IIIrd quadrant	$\theta = -\pi + \tan^{-1}\left\|\frac{y}{x}\right\|$
x > 0, y < 0	IVth quadrant	$\theta = -\alpha = -\tan^{-1}\left\|\frac{y}{x}\right\|$



19.4.3.3 *Caution*

An a usual mistake is to take the argument of z = x + iy as $\tan^{-1}$ (y/x) is irrespective of the value of x and y.

- Remember that $\tan^{-1}$ (y/x) lies in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

  Whereas the principal value of argument of z (Arg(z)) lies in the interval (–π, π].

$$\text{Thus, if } z = x + iy, \text{ then } \operatorname{Arg}(z) = \begin{cases} \tan^{-1}(y/x) & \text{if } x > 0, y \geq 0 \\ \tan^{-1}(y/x) + \pi & \text{if } x < 0, y \geq 0 \\ \tan^{-1}(y/x) - \pi & \text{if } x < 0, y < 0 \\ \pi/2 & \text{if } x = 0, y > 0 \\ -\pi/2 & \text{if } x = 0, y < 0 \\ \text{Not defined} & \text{for } x = 0, y = 0 \end{cases}$$

19.4.3.4 *Properties of argument of complex number*

1. arg $(z_1.z_2)$ = arg z_1 + arg z_2
2. arg(z^n) = n (argz)
3. $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$
4. arg(z) = 0 ⇔ complex number z is purely real and positive.
5. arg(z) = π ⇔ complex number z is purely real and negative.
6. arg(z) = π/2 ⇔ complex number z is purely imaginary with positive Im(z).
7. arg(z) = – π/2 ⇔ complex number z is purely imaginary with negative Im(z).
8. arg(z) = not defined ⇔ z = 0.
9. arg(z) = π/4 ⇔ z = (1 + i) or (x + xi), etc. for (x > 0).

Properties of Principal Arguments: (Principal argument of complex number is denoted by arg (z))

1. If $z_k = r_k(\cos\theta_k + i\sin\theta_k) = r_k e^{i\theta_k}$ are number of complex numbers then $\operatorname{Arg}\left(\prod_{k=1}^{n} z_k\right) = \sum_{k=1}^{n} \operatorname{Arg} z_k \pm 2k\pi$,

   where k ∈ ℤ choose k suitably such that principal Arg of the resultant number lies in principal range.

2. $\text{Arg}\left(\frac{z}{\bar{z}}\right) = 2\text{Arg}(z)$
3. $\text{Arg}(z^n) = n\ \text{Arg}\ z \pm 2k\pi$
4. $\text{Arg}(-z) = -\pi + \text{Arg}\ z$ or $\pi + \text{Arg}\ z$ respectively, when Arg z > 0 or < 0
5. $\text{Arg}(1/z) = -\text{Arg}\ z$

Method of Solving Complecs Equations

Let the given equation be f(z) = g(z). To solve this equation, we have the following four methods.

Method 1: Put z = x + iy in the given equation and equate the real and imaginary parts of both sides and solve to find x and y; hence z = x + iy.

Method 2: Put z = r(cosθ + isinθ) and equate the real and imaginary parts of both sides; solve to get r and θ; hence z.

Method 3: Take conjugate of both sides of given equations. Thus, we get two equations. f(z) = g(z) ...... (1) and $f(\bar{z}) = g(\bar{z})$ ....(2)

Adding and Subtracting the above two equations, we get two new equations, solving then we get z.

Method 4: **Geometrical Solution:** From the given equation, we follow the geometry of complex number z and find its locus.

19.4.4 Square Roots of a Complex Number

Square roots of z = a + ib are given by $\pm\left[\sqrt{\frac{|z|+a}{2}} + i\sqrt{\frac{|z|-a}{2}}\right]$; b > 0 and $\pm\left[\sqrt{\frac{|z|+a}{2}} - i\sqrt{\frac{|z|-a}{2}}\right]$; b < 0

19.4.4.1 *Shortcut method*

Step 1: Consider $\frac{\text{Im}(z_0)}{2} = \frac{b}{2}$.

Step 2: Factorize b/2 into factors x; y:$x^2 - y^2 = \text{Re}(z_0) = a$.

Step 3: Therefore, $a + ib = (x + iy)^2$.

⇒ $\sqrt{a+ib} = \pm(x+iy)$, e.g., $\sqrt{8-15i}$; a = 8, b = −15 < 0

⇒ $\frac{b}{2} = -\frac{15}{2}$ = x.y such that $x^2 - y^2 = 8$ ⇒ $x = \frac{5}{\sqrt{2}}; y = -\frac{3}{\sqrt{2}}$

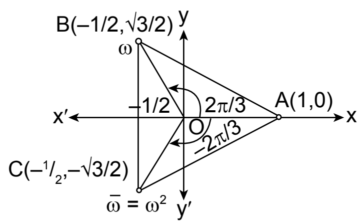
⇒ $\sqrt{8-15i} = \pm\left(\frac{5}{\sqrt{2}} - \frac{3i}{\sqrt{2}}\right) = \pm\frac{1}{\sqrt{2}}(5-3i)$

19.4.4.2 *Cube root of unity*

Let $\sqrt[3]{1}$ = cube root of unity

⇒ $x^3 = 1$; where $\omega = \frac{-1+\sqrt{3}i}{2}$ and $\omega^2 = \frac{-1-\sqrt{3}i}{2}$

∴ Cube roots of unity are 1, ω, ω^2 and ω, ω^2 are called the imaginary cube roots of unity.



19.4.4.3 *Properties of cube root of unity*

P(1): $|\omega| = |\omega^2| = 1$

P(2): $\overline{\omega} = \omega^2$

P(3): $\omega^3 = 1$

P(4): $\omega^{3n} = 1$; $\omega^{3n+1} = \omega$ and $\omega^{3n+2} = \omega^2\ \forall n \in \mathbb{Z}$

P(5): Sum of cube roots of unity is 0. That is, $1 + \omega + \omega^2 = 0$.

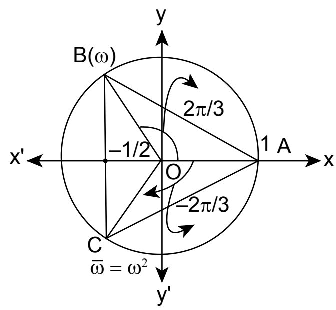
Remarks:

$\because\ \overline{\omega} = \omega^2$ $\quad\because\ 1 + \omega + \overline{\omega} = 0$

$\because\ \overline{\omega} = \omega^2$ and $\omega = \omega.1 = \omega.\omega^3 = \omega^4 = (\omega^2)^2 = (\overline{\omega})^2$

$\because\ 1 + \omega + \omega^2 = 1 + (\overline{\omega})^2 + \overline{\omega}$ $\quad\because\ 1 + \overline{\omega} + (\overline{\omega})^2 = 0$

P(6): $1 + \omega^n + \omega^{2n} = \begin{cases} 3; \text{when n is multiple of 3} \\ 0; \text{when n is not a multiple of 3} \end{cases}$



P(7): 1, ω, ω² are the vertices of an equilateral Δ having each side = √3.

P(8): Circumcentre of Δ ABC with vertices as cube roots of unity lies at origin and the radius of circumcircle is 1 unit Clearly, OA = OB = OC = 1.

Remark:

From the above properties, clearly cube roots of unity are the vertices of an equilateral Δ having each side = √3, and circumscribed by circle of unit radius and having its centre at origin.

P(9): $\arg(\omega) = \arg\left(-\frac{1}{2} + \frac{\sqrt{3}i}{2}\right) = \frac{\pi}{3}$; $\arg(\omega^2) = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = \frac{4\pi}{3}$.

P(10): Any complex number a + ib for which $(a:b) = \frac{1}{\sqrt{3}}$ or $\sqrt{3}:1$ can always be expressed in terms of i, w, w².

$$\text{e.g., } 1 + i\sqrt{3} = -2\omega^2,\quad \sqrt{3} + i = \frac{i}{2}(1 + i\sqrt{3}) = 2i\left(\frac{1 + i\sqrt{3}}{2i}\right) = \frac{2}{i}\left(\frac{-1 + i\sqrt{3}}{2}\right) = \frac{2\omega}{i}$$

19.4.4.4 *Important relation involving complex cubic roots of unity*

(a) $x^2 + x + 1 = (x - \omega)(x - \omega^2)$

(b) $x^2 - x + 1 = (x + \omega)(x + \omega^2)$

(c) $x^2 + xy + y^2 = (x - y\omega)(x - y\omega^2)$

(d) $x^2 - xy + y^2 = (x + y\omega)(x + y\omega^2)$

(e) $x^2 + y^2 = (x + iy)(x - iy)$

(f) $x^3 + y^3 = (x + y)(x + y\omega)(x + y\omega^2)$

(g) $x^3 - y^3 = (x - y)(x - y\omega)(x - y\omega^2)$

(h) $x^2 + y^2 + z^2 - xy - yz - zx = (x + y\omega + z\omega^2)(x + y\omega^2 + z\omega)$

(i) $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + y\omega + z\omega^2)(x + \omega^2 y + \omega z)$

19.5 DE MOIVER'S THEOREM

This theorem states that:

(i) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, if n is an rational number.

(ii) $(\cos\theta + i\sin\theta)^{1/n} = [\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)]^{1/n}$

$(\because$ period of $\sin\theta$ and $\cos\theta$ is $2\pi) = \cos\dfrac{(2k\pi + \theta)}{n} + i\sin\dfrac{(2k\pi + \theta)}{n}$, $k = 0, 1, 2, \ldots, n - 1$.

19.5.1 n^th Root of Unity

Let x be an nth root of unity, then $x = (1)^{\frac{1}{n}} = (\cos 0 + i\sin 0)^{\frac{1}{n}} = \cos\left(\dfrac{2r\pi + 0}{n}\right) + i\sin\left(\dfrac{2r\pi + 0}{n}\right), r = 0,$ 1, 2,....., n – 1.

$= \cos\left(\dfrac{2r\pi + 0}{n}\right) + i\sin\left(\dfrac{2r\pi + 0}{n}\right), r = 0, e^{\frac{i2r\pi}{n}}; r = 0$, 1,2,..........n – 1= $1, e^{i\frac{2\pi}{n}} e^{i\frac{4\pi}{n}}, \ldots e^{i\frac{2(n-1)\pi}{n}} = 1, \alpha, \alpha^2, \ldots$ α^{n-1}; where $\alpha = e^{i\frac{2\pi}{n}}$.

19.5.2 Properties of n^th Root of Unity

P(1): nth roots of unity form a G.P

P(2): $1 + \alpha + \alpha^2 + \ldots + \alpha^{n-1} = 0$

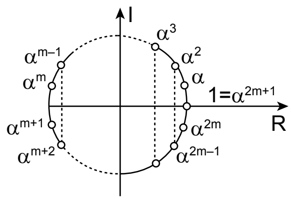
P(3): $1. \alpha.\alpha^2.\alpha^{n-1} = (-1)^{n-1}$

P(4): nth roots of unity are vertices of n-sided regular polygon circumscribed by a unit circle having its centre at the origin.

Case (i): When n is odd

Let n = 2m + 1, m is some positive integers, then only one root is real, that is 1 and remaining 2m roots are non real complex conjugates.

The 2m non-real roots are $(\alpha, \alpha^{2m}), (\alpha^2, \alpha^{2m-1}), (\alpha^3, \alpha^{2m-2}) \ldots (\alpha^m, \alpha^{m+1})$, where the ordered pairs are $(z, \overline{z})$, i.e., non-real roots and their conjugates and $\alpha = e^{i\frac{2\pi}{n}}$.



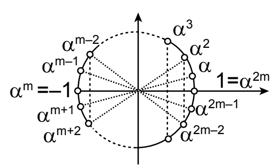
Note:

The nth roots given as ordered pairs are conjugate and reciprocal of each other.

$$\left\{\because\ \alpha^{-1} = \frac{1}{\alpha} = \frac{\alpha^{2m+1}}{\alpha} = \alpha^{2m} = \alpha^{2m+1} = 1; \overline{\alpha^m} = \left(\frac{1}{\alpha}\right)^m = \frac{1}{\alpha^m} = \frac{\alpha^{2m+1}}{\alpha^m} = \alpha^{m+1}\right\}$$

Case (ii): When n is even:

Let n = 2m, $\alpha = \text{cis}\,\dfrac{2\pi}{n} = \text{cis}\left(\dfrac{\pi}{m}\right)$; except 1 and –1, other roots are non-real complex conjugate pairs.



Note:

The nth roots arranged vertically below are conjugate and reciprocal of each other and diagonally (passing through origin) are negative of each other.

19.5.2.1 n^{th} root of a complex number $\sqrt[n]{z}$

Let, $z = r \text{ cis } \theta$, $z^{1/n} = (r^{1/n})\,(\text{cis}(2k\pi + \theta))^{1/n} = (r^{1/n}) \text{ cis}\left(\frac{2k\pi}{n} + \frac{\theta}{n}\right)$, where $r^{1/n}$ is positive n^{th} root of r.

$= (r^{1/n}) \text{ cis } \frac{2k\pi}{n}.\text{cis}\left(\frac{\theta}{n}\right)$; where $\text{cis}\frac{2k\pi}{n}$, is the nth root of unity, k = 0, 1, 2,......, n – 1.

19.5.2.2 To find logarithm of a complex number

Consider z = x + iy, {converting 'x + iy' into Euler's form, such that θ = principal value of argument of z}, then we get $\log_e (x + iy) = \log_e (|z|e^{i\theta})$

$\Rightarrow$ $\log_e(x + iy) = \log_e |z| + \log_e e^{i\theta} \Rightarrow \log_e (x + iy) = \log_e |z| + i\theta$
In general, $\log_e(x + iy) = \log_e|z| + i(\theta + 2n\pi)$; $n \in \mathbb{Z}$

To find $(x + iy)^{(a+ib)}$, i.e., $(z_1)^{z_2}$

Let $u + iv = (x + iy)^{(a+ib)}$

$\Rightarrow$ $\ell n\,(u + iv) = (a + ib)\,\ell n\,(x + iy) \Rightarrow (u + iv) = e^{(a + ib)\,\ell n\,(x + iy)}$; now solve for u and v by expressing (x + iy) in polar form.

For example, $x = (i)^i \Rightarrow \ell nx = i\ell ni = i\ell n\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = i\ell n(e^{i\pi/2}) = i^2\frac{\pi}{2}\ell ne$

$\Rightarrow$ $\ell nx = -\frac{\pi}{2} \Rightarrow x = e^{-\frac{\pi}{2}}$. Thus, $(i)^i = e^{-\pi/2}$.

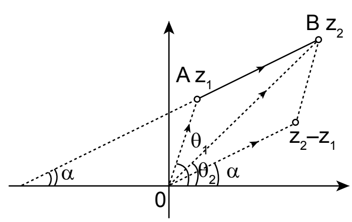
Alternatively, $(i)^i = e^{\ell n(i)^i} = e^{i\ell ni} = e^{i\ell n\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)} = e^{i\ell n(e^{\frac{i\pi}{2}})} = e^{i.i\frac{\pi}{2}} = e^{-\frac{\pi}{2}}$

19.6 GEOMETRY OF COMPLEX NUMBER

19.6.1 Line Segment in Argand's Plane

Any line segment joining the complex numbers z_1 and z_2 (directed towards z_2) represents a complex number given by $z_2 - z_1$. Since every complex number has magnitude and direction, therefore $z_2 - z_1$ also.

$|z_2 - z_1|$ represents the length of line segment BA, i.e., the distance between z_1 and z_2 and $\arg(z_2 - z_1)$ represents the angle which line segment AB (on producing) makes with positive direction of real axis.

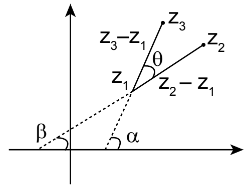


19.6.1.1 Angle between to lines segments (Roation theorm or coni's theorem)

Consider three complex numbers z_1, z_2 and z_3, such that the angle between line segments joining z_1 to z_2 and z_3 to z_1 is equal to θ.

Then $\theta = \alpha - \beta = \text{Arg}(z_3 - z_1) - \text{Arg}(z_2 - z_1) = \text{Arg}\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \text{Arg}\left(\frac{\text{Post-rotation vector}}{\text{Pre-rotation vector}}\right)$

$\Rightarrow \quad \text{Arg}\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \theta = \text{Arg}(\rho e^{i\theta})$



$\Rightarrow \quad (z_3 - z_1) = (z_2 - z_1)\, \rho\, e^{i\theta}$, where $\rho = \left|\frac{z_3 - z_1}{z_2 - z_1}\right|$. If $z_1 = 0$.

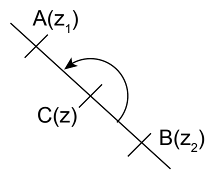
$\Rightarrow \quad z_3 = \rho z_2\, e^{i\theta}$, $\arg(z_3/z_2)$ is an angle through which z_2 is to be rotated to coincide it with z_3.

If a complex number $(z_2 - z_1)$ is multiplied by another complex number $re^{i\theta}$, then the complex number $(z_2 - z_1)$ gets rotated by the argument (θ) of multiplying complex number in anti-clockwise direction (It is called **Rotation Theorem or Coni's Theorem**).

19.6.2 Application of the Rotation Theorem

(i) **Section Formula:** Let us rotate the line BC about the point C, so that it becomes parallel to the line CA. The corresponding equation of rotation will be $\frac{z_1 - z}{z_2 - z} = \frac{|z_1 - z|}{|z_2 - z|} . e^{i\pi} = \frac{m}{n}(-1)$

$\Rightarrow \quad nz_1 - nz = -mz_2 + mz \Rightarrow z = \frac{nz_1 + mz_2}{m + n}$



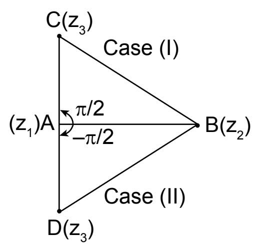
Similarly, if C(z) divides the segment AB, externally in the ratio of m : n, then $z = \frac{nz_1 - mz_2}{m - n}$.

In the specific case, if C(z) is the mid point of AB then $z = \frac{z_1 + z_2}{2}$.

(ii) **Condition for Collinearity:** If there are three real numbers (other than 0) l, m and n, such that $lz_1 + mz_2 + nz_3 = 0$ and $l + m + n = 0$, then complex numbers z_1, z_2 and z_3 will be collinear.

(iii) **To find the conditions for perpendicularity of two straight lines:** Condition that $\angle A$ of ΔABC where $A(z_1)$ $B(z_2)$ $C(z_3)$ is right angle, and can be obtained by applying Rotation Theorem at A.

$$\text{Arg}\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \frac{\pi}{2}, -\frac{\pi}{2} \quad .....(i)$$



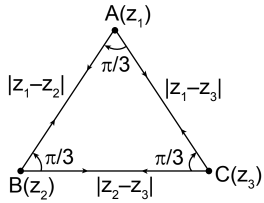
$\Rightarrow \quad \left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \rho e^{\pm\frac{i\pi}{2}} = \pm\rho i;\ \rho = \left|\frac{z_3 - z_1}{z_2 - z_1}\right| \Rightarrow R\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = 0$

$\Rightarrow \quad \frac{z_3 - z_1}{z_2 - z_1} + \frac{\overline{z}_3 - \overline{z}_1}{\overline{z}_2 - \overline{z}_1} = 0 \Rightarrow |z_2 - z_3|^2 = |z_3 - z_1|^2 + |z_2 - z_1|^2$

If ABC is **right-angled isosceles triangle** with AB = AC.

$\Rightarrow \quad \rho = 1 \Rightarrow \frac{z_3 - z_1}{z_2 - z_1} = \pm i$

(iv) **Conditions for ΔABC to be an equilateral triangle:** Let the ΔABC where $A(z_1)$ $B(z_2)$ $C(z_3)$ be an equilateral triangle.



The following conditions hold:

(i) $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$.

(ii) $\text{Arg}\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \pm\frac{\pi}{3}$ and $|z_3 - z_1| = |z_2 - z_1|$.

(Applying the rotation theorem at A and knowing CA = BA.)

(iii) $\text{Arg}\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \text{Arg}\left(\frac{z_1 - z_2}{z_3 - z_2}\right) = \frac{\pi}{3}$. (Applying rotation theorem at A and B.)

(iv) $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$

(v) $\frac{z_1 - z_2}{z_3 - z_2} = e^{i\frac{\pi}{3}} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$

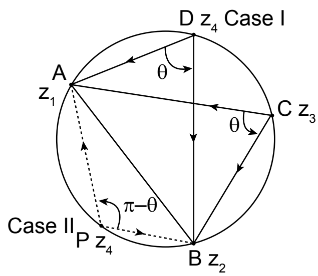
(vi) $\frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} + \frac{1}{z_1 + z_2} = 0$

(vii) Conditions for four points to be concyclic or condition for points z_1, z_2, z_3, z_4 to represent a cyclic quadrilateral:
If points $A(z_1)$, $B(z_2)$, $C(z_3)$, $D(z_4)$ are con-cyclic, then the following two cases may occur:

Case I: If z_3 and z_4 lies on same segment with respect to the chord joining z_1 and z_2.

$$\text{Arg}\left(\frac{z_2 - z_4}{z_1 - z_4}\right) - \text{Arg}\left(\frac{z_2 - z_3}{z_1 - z_3}\right) = 0 \quad \Rightarrow \quad \text{Arg}\Bigg(\underbrace{\frac{z_2 - z_4}{z_1 - z_4}.\frac{z_1 - z_3}{z_2 - z_3}}_{w}\Bigg) = 0$$

⇒ w is real and positive or $I_m(\omega) = 0$ and $\text{Re}(\omega) > 0$.



Case II: If z_3 and z_4 lie on opposite segment of circle with respect to chord joining z_1 and z_2

$$\text{Arg}\left(\frac{z_2 - z_3}{z_1 - z_3}\right) + \text{Arg}\left(\frac{z_1 - z_4}{z_2 - z_4}\right) = \pi$$

⇒ Arg (1/w) = π ⇒ Arg (w) = −π
So the principal argument of w = π

⇒ ω is negative real number, or $\text{Im}(\omega) = 0$ and $\text{Re}(\omega) < 0$

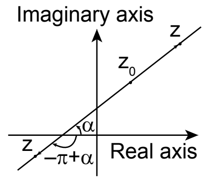
Conclusion! Four complex numbers z_1, z_2, z_3, z_4 to be concyclic.

$$\text{Arg}\left(\underbrace{\frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)}}_{w}\right) = 0 \text{ or } \pi \Rightarrow w \text{ is purely real } I(w) = 0 \Rightarrow \overline{w} = w.$$

19.6.3 Loci in Argand Plane

A(1): Straight line in Argand plane: A line through z_0 making angle α with the positive real axis.
$\text{Arg}(z - z_0) = \alpha$ or $-\pi + \alpha$.

- The given equation excludes the point z_0.
- Arg $(z - z_0) = \alpha$ represents the right-ward ray.
- Arg $(z - z_0) = -\pi + \alpha$ represents the left-ward ray.



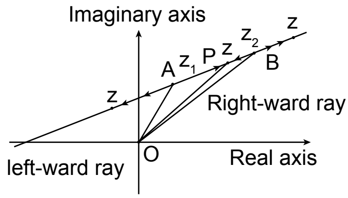
A(2): Line through points A($\mathbf{z_1}$) and B($\mathbf{z_2}$): Consider a straight line passing through A(z_1) and B(z_2) taking a variable point P(z) on it.

∵ for each position of P, $\overrightarrow{AP}$ is collinear with $\overrightarrow{AB}$.

⇒ $\overrightarrow{AP} = \lambda\overrightarrow{AB}$ ⇒ $\overrightarrow{AP} = \lambda(z_2 - z_1)$

∵ $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$; $z = z_1 + \lambda(z_2 - z_1)$; $z = z_1(1 - \lambda) + \lambda z_2$

19.6.3.1 Conclusion

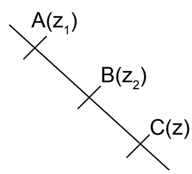


1. if $z = xz_1 + yz_2$; $x + y = 1$ and x and $y \in \mathbb{R}$, then z, z_1, z_2 are collinear.
2. Equation represents line segment AB if $\lambda \in [0, 1]$.
3. Right-ward ray through B, if $\lambda \in (1, \infty)$.
4. Left-ward ray through A, if $\lambda \in (-\infty, 0)$.

(i) Equation of straight line with the help of coordinate geometry:

Writing $x = \dfrac{z+\bar{z}}{2}$, $y = \dfrac{z-\bar{z}}{2i}$, etc., in $\dfrac{y-y_1}{y_2-y_1} = \dfrac{x-x_1}{x_2-x_1}$ and re-arranging the terms, we find that the equation of the line through z_1 and z_2 is given by $\dfrac{z-z_1}{z_2-z_1} = \dfrac{\bar{z}-\bar{z}_1}{\bar{z}_2-\bar{z}_1}$ or $\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0.$

(ii) Equation of a straight line with the help of rotation formula:
Let A(z_1) and B(z_2) be any two points lying on any line and we have to obtain the equation of this line. For this purpose, let us take any point C(z) lying on this line. Since Arg $\left(\dfrac{z-z_1}{z_2-z_1}\right) = 0$ or π.

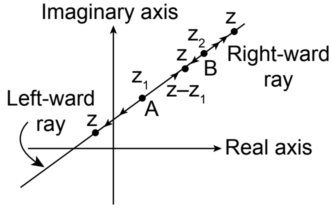


$$\frac{z-z_1}{z_2-z_1} = \frac{\bar{z}-\bar{z}_1}{\bar{z}_2-\bar{z}_1} \qquad \dots \text{(i)}$$

This is the equation of the line that passes through A(z_1) and B(z_2). After rearranging the terms, it can also be put in the following form $\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0.$

(iii) Line segment AB: The equation of the line segment AB is given as Arg$\left(\dfrac{z-z_1}{z-z_2}\right) = \pi$.

(iv) Equation of two rays excluding the line segment AB: Arg$\left(\dfrac{z-z_1}{z-z_2}\right) = \pi$.



(v) **Complete line except z_1 and z_2:** (general equation of line):

The equation is given as $\text{Arg}\left(\dfrac{z-z_1}{z-z_2}\right)=0,\pi$, i.e., $I\left(\dfrac{z-z_1}{z-z_2}\right)=0$

$\Rightarrow \quad \dfrac{z-z_1}{z-z_2}=\dfrac{\overline{z}-\overline{z}_1}{\overline{z}-\overline{z}_2}$ $\qquad \Rightarrow \quad z\overline{z}-\overline{z}_2z-z_1\overline{z}+z_1\overline{z}_2 = z\overline{z}-\overline{z}_1z-z_2\overline{z}+z_2\overline{z}_1$

$\Rightarrow \quad (\overline{z}_1-\overline{z}_2)z+(z_2-z_1)\overline{z}+z_1\overline{z}_2-z_2\overline{z}_1=0$ $\quad \Rightarrow \quad \dfrac{(\overline{z}_1-\overline{z}_2)}{2i}z+\dfrac{(z_2-z_1)}{2i}\overline{z}+I(z_1\overline{z}_2)=0$

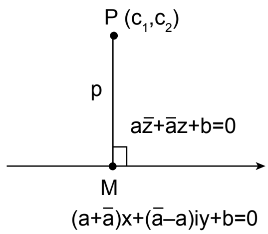
$\Rightarrow \quad a\overline{z}+\overline{a}z+b=0$; where $\qquad \Rightarrow$ where $a=\dfrac{z_2-z_1}{2i}$ and $\overline{a}=\dfrac{\overline{z}_2-\overline{z}_1}{-2i}=\dfrac{\overline{z}_1-\overline{z}_2}{2i}$

Remark:

Two points $P(z_1)$ and $Q(z_2)$ lie on the same side or opposite side of the line $\overline{a}z+a\overline{z}+b$ accordingly, as $\overline{a}z_1+a\overline{z}_1+b$ and $\overline{a}z_2+a\overline{z}_2+b$ have the same sign or opposite sign.

19.7 THEOREM

Perpendicular distance of P(c) (where $c = c_1 + ic_2$) from the straight line is given by $p=\dfrac{|a\overline{c}+\overline{a}c+b|}{2|a|}$.



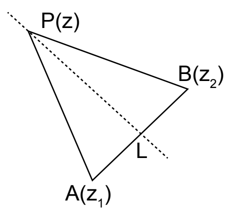
- **Slope of a given line:** Let the given line be $z\overline{a}+\overline{z}a+b=0$. Replacing z by x + iy, we get $(x+iy)\overline{a}+(x-iy)a+b=0$

  $\Rightarrow \quad (a+\overline{a})x+iy(\overline{a}-a)+b=0$

  It's slope is = $\dfrac{a+\overline{a}}{i(a-\overline{a})}=\dfrac{2\,\text{Re}(a)}{2i^2\text{lm}(a)}=-\dfrac{\text{Re}(a)}{\text{lm}(a)}$

- **Equation of a line parallel to a given line:** Equation of a line, parallel to the line $z\overline{a}+\overline{z}a+b=0$, is $z\overline{a}+\overline{z}a+\lambda=0$ (where λ is a real number).
- **Equation of a line perpendicular to a given line:** Equation of a line perpendicular to the line $z\overline{a}+\overline{z}a+b=0$ is $z\overline{a}-\overline{z}a+i\lambda=0$ (where λ is a real number).
- **Equation of perpendicular bisector:**
  Consider a line segment joining $A(z_1)$ and $B(z_2)$.
  Let the line 'L' be it's perpendicular bisector.
  If P(z) be any point on the 'L', then we have:
  PA = PB $\Rightarrow |z-z_1|=|z-z_2|$

  $\Rightarrow \quad z(\overline{z}_2-\overline{z}_1)+\overline{z}(z_2-z_1)+z_1\overline{z}_1-z_2\overline{z}_2=0$



19.8 COMPLEX SLOPE OF THE LINE

If z_1 and z_2 are two unequal complex numbers represented by points P and Q, then $\dfrac{z_1 - z_2}{\overline{z}_1 - \overline{z}_2}$ is called the complex slope of the line joining z_1 and z_2 (i.e., line PQ). It is denoted by w. Thus, $w = \dfrac{z_1 - z_2}{\overline{z}_1 - \overline{z}_2}$.

Notes:

1. *The equation of line PQ is* $z - z_1 = w(\overline{z} - \overline{z}_1)$. *Clearly,* $|w| = \dfrac{|z_1 - z_2|}{|\overline{z}_1 - \overline{z}_2|} = \dfrac{|z_1 - z_2|}{|z_1 - z_2|} = 1$.
2. *The two lines having complex slopes* w_1 *and* w_2 *are parallel, if and only if,* $w_1 = w_2$.
3. *Two lines with complex slopes* ω_1 *and* ω_2 *are perpendicular if* $\omega_1 + \omega_2 = 0$.

19.8.1 Circle in Argand Plane

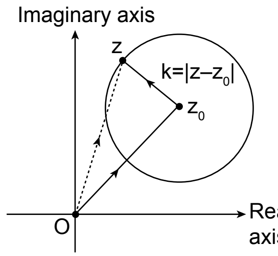
A(1): Centre radius form:

The equation of circule with z_0 as centre and a positive real number k as radius as given as $|z - z_0| = k$

$\Rightarrow \quad |z - z_0|^2 = k^2$

$\Rightarrow \quad (z - z_0)(\overline{z} - \overline{z}_0) = k^2 \Rightarrow z\overline{z} - z_0\overline{z} - \overline{z}_0 z + |z_0|^2 - k^2 = 0$ ......(1)

If $z_0 = 0$, then $|z| = K$



A(2): General Equation of Circle:

Referring to equation (1), thus we can say:

$z\overline{z} + \overline{a}z + a\overline{z} + b = 0$ ....(2)

where a is a complex constant and $b \in \mathbb{R}$ represents a general circle.

Comparing (2) with (1), we note that **centre = –a and radius =** $\sqrt{|a|^2 - b}$

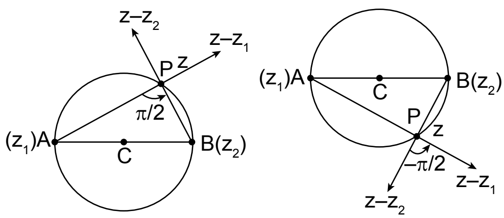
A(3): Diametric Form of Circle:

As we know that diameter of any circle subtends right angle at any point on the circumference. Equation of circle with $A(z_1)$ and $B(z_2)$ as end points of diameter.

$$\text{Arg}\left(\frac{z - z_2}{z - z_1}\right) = \begin{cases} \dfrac{\pi}{2} & \text{Case I} \\ -\dfrac{\pi}{2} & \text{Case II} \end{cases} \Rightarrow \frac{z - z_2}{z - z_1} = \pm ki; \text{ where } k = \left|\frac{z - z_2}{z - z_1}\right| \Rightarrow \frac{z - z_2}{z - z_1} = -\frac{\overline{z} - \overline{z}_2}{\overline{z} - \overline{z}_1}$$

$\Rightarrow \quad (z - z_1)(\overline{z} - \overline{z}_2) + (z - z_2)(\overline{z} - \overline{z}_1) = 0$; further $\dfrac{z - z_2}{z - z_1} + \dfrac{\overline{z} - \overline{z}_2}{\overline{z} - \overline{z}_1} = 0$ is diametric form.

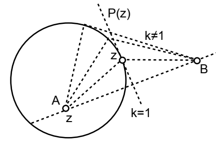
$\Rightarrow \quad |z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$



Case I   Case II

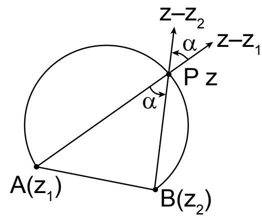
19.9 APPOLONEOUS CIRCLE

If $\left|\frac{z-z_1}{z-z_2}\right| = k$, i.e., $|z - z_1| = k\,|z - z_2|$. Then equation represents apploloneous circle of A (z_1) B(z_2) with respect to ratio k, when k = 1, this gives $|z - z_1| = |z - z_2|$ which is straight line, i.e., perpendicular bisector of line segment joining z_1 to z_2.



19.10 EQUATION OF CIRCULAR ARC

As per the figure; equation of circular arc at which chord AB, (where A(z_1) and B(z_2)) subtends angle α is given as $\text{Arg}\left(\frac{z-z_2}{z-z_1}\right) = \alpha$.



Case I: If $0 < \alpha < \pi/2$ or $-\pi/2 < \alpha < 0$ (**Major arc of circle**)

Case II: $\alpha = \pm\frac{\pi}{2}$ (**Semicircular arc**)

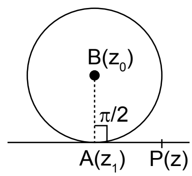
Case III: $\alpha \in \left(-\pi, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$ (**Minor arc of circle**)

Case IV: $\alpha = 0$ (**Major arc of ∞ radius**)

Case V: $\alpha = \pi$ (**Minor arc of ∞ radius**)

19.10.1 Equation of Tangent to a Given Circle

Let $|z - z_0| = r$ be the given circle and we have to obtain the tangent at A(z_1). Let us take any point P(z) on the tangent line at A(z_1).



Clearly $\angle PAB = \pi/2$; $\arg\left(\frac{z-z_1}{z_0-z_1}\right) = \pm\frac{\pi}{2}$

$\Rightarrow$ $\frac{z-z_1}{z_0-z_1}$ is purely imaginary

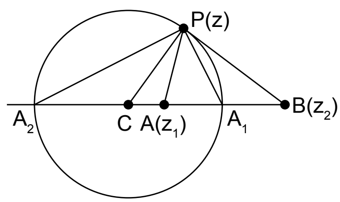
$\Rightarrow$ $z(\bar{z}_0 - \bar{z}_1) + \bar{z}(z_0 - z_1) + 2|z_1|^2 - z_1\bar{z}_0 - \bar{z}_1 z_0 = 0$

In particular if given circle is $|z| = r$, equation of the tangent at $z = z_1$ would be $z\bar{z}_1 + \bar{z}z_1 = 2|z_1|^2 = 2r^2$.

If $\left|\frac{z-z_1}{z-z_2}\right| = \lambda$ ($\lambda \in R^+$, $\lambda \neq 1$); where z_1 and z_2 are given complex numbers and z is a arbitrary complex number, then z would lie on a circle.

19.10.2 Explanation

Let A(z_1) and B(z_2) be two given complex numbers and P(z) be any arbitrary complex number. Let PA_1 and PA_2 be internal and external bisectors of angle $\angle APB$ respectively. Clearly, $\angle A_2PA_1 = \pi/2$.



Now, $\frac{AP}{BP} = \frac{|z-z_1|}{|z-z_2|} = \left|\frac{z-z_1}{z-z_2}\right| = \lambda$ (say)

Thus, points A_1 and A_2 would divide AB in the ratio of $\lambda : 1$ internally and externally respectively. Hence P(z) would be lying on a circle with A_1A_2 being it's diameter. Note: If we take 'C' to be the mid-point of A_2A_1, it can be easily prove that CA . CB = $(CA_1)^2$, i.e., $|z_1 - z_0||z_2 - z_0| = r^2$, where the point C is denoted by z_0 and r is the radius of the circle.

Notes:

(i) *If we take 'C' to be the mid-point of A_2A_1, it can be easily proved that CA . CB = $(CA_1)^2$, i.e., $|z_1 - z_0||z_2 - z_0| = r^2$, where the point C is denoted by z_0 and r is the radius of the circle.*

(ii) *If $\lambda = 1 \Rightarrow |z - z_1| = |z - z_2|$ hence P(z) would lie on the right bisector of the line $A(z_1)$ and $B(z_2)$. Note that in this case z_1 and z_2 are the mirror images of each other with respect to the right bisector.*

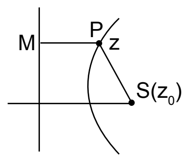
19.10.3 Equation of Parabola

Equation of parabola with directrix $a\bar{z} + \bar{a}z + b = 0$ and focus z_0 is given as SP = PM

$$|z - z_0| = \frac{|a\bar{z} + \bar{a}z + b|}{2|a|}$$

$$\Rightarrow \quad 4|z - z_0|^2|a|^2 = |a\bar{z} + \bar{a}z + b|^2 \quad \Rightarrow \quad 4a\bar{a}(z - z_0)(\bar{z} - \bar{z}_0) = (a\bar{z} + \bar{a}z + b)^2$$

$$\Rightarrow \quad 4a\bar{a}(z\bar{z} - z\bar{z}_0 - z_0\bar{z} + z_0\bar{z}_0) = (a\bar{z} + \bar{a}z + b)^2$$



19.10.4 Equation of Ellipse

Ellipse is locus of point P(z), such that sum of its distances from two fixed points $A(z_1)$ and $B(z_2)$ (i.e., foci of ellipse) remains constant (2a).

$\Rightarrow$ PA + PB = 2a $\Rightarrow |z - z_1| + |z - z_2| = 2a$; where 2a is length of major axis.

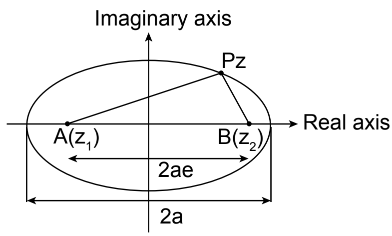
Case I: If $2a > |z_1 - z_2| = AB$ (**Locus is ellipse**)

Case II: $2a = |z_1 - z_2|$ (**Locus is segment AB**)

Case III: $2a < |z_1 - z_2|$ (**No locus**)

Case IV: If $|z - z_1| + |z - z_2| > 2a : 2a > |z_1 - z_2|$ (**Exterior of ellipse**)

Case V: If $|z - z_1| + |z - z_2| < 2a : 2a > |z_1 - z_2|$ (**Interior of ellipse**)



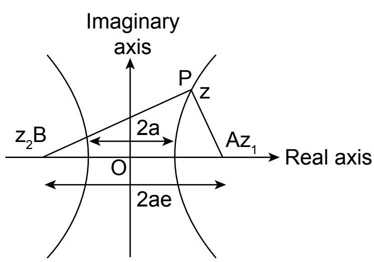
19.11 EQUATION OF HYPERBOLA

Hyperbola is locus of point P(z), such that difference of its distances from two fixed point $A(z_1)$ and $B(z_2)$ (foci of hyperbola) remains constant (2a).

$\Rightarrow$ PA – PB = 2a

$\Rightarrow ||z - z_1| - |z - z_2|| = 2a$; where 2a is length of major axis.

Case I: If $2a < |z_1 - z_2| = AB$ (**locus is branch of hyperbola**).



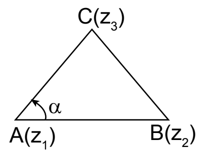
Case II: $2a = |z_1 - z_2|$ (**Locus is union of two rays**)

Case III: $2a > |z_1 - z_2|$ (**No locus**)

Case IV: If $||z - z_1| - |z - z_2|| > 2a : 2a < |z_1 - z_2|$ (**Exterior of hyperbola**)

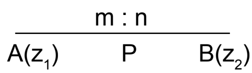
Case V: If $|z - z_1| - |z - z_2| < 2a : 2a < |z_1 - z_2|$ (**Interior of hyperbola**)

19.12 SOME IMPOTANT FACTS



A (1): If A, B, C are the vertices of a triangle represented by complex numbers z_1, z_2, z_3, respectively, in anti-clockwise sense and $\Delta BAC = \alpha$, then $\frac{z_3 - z_1}{|z_3 - z_1|} = \frac{z_2 - z_1}{|z_2 - z_1|}.e^{i\alpha}$.

A(2): If z_1 and z_2 are two complex numbers representing the points A and B, then the point on AB which divides line segment AB in ratio m : n is given by $\frac{nz_1 + mz_2}{m + n}$.



A(3): If a, b, c are three real numbers not all simultaneously zero, such that $az_1 + bz_2 + cz_3 = 0$ and $a + b + c = 0$ then z_1, z_2, z_3 will be collinear.

A(4): If z_1, z_2, z_3 represent the vertices A,B,C of ΔABC, then:

(i) **Centroid** of $\Delta ABC = \frac{z_1 + z_2 + z_3}{3}$

(ii) **In centre** of $\Delta ABC = \frac{az_1 + bz_2 + cz_3}{a + b + c}$

(iii) **Orthocentre** of $\Delta ABC = \frac{(a\sec A)z_1 + (b\sec B)z_2 + (c\sec C)z_3}{(a\sec A) + (b\sec B) + (c\sec C)} = \frac{(z_1\tan A + z_2\tan B + z_3\tan C)}{\tan A + \tan B + \tan C}$

(iv) **Circumcentre** of $\Delta ABC = \frac{z_1\sin 2A + z_2\sin 2B + z_3\sin 2C}{\sin 2A + \sin 2B + \sin C}$

(v) If z_1, z_2, z_3 are the vertices of an equilateral triangle, then the circumcentre z_0 may be given as $z^2_1 + z_2^2 + z_2^3 = 3z_0^2$.

(vi) If z_1, z_2, z_3 are the vertices of an isosceles triangle, right angled at z_2, then $z^2_1 + z_2^2 + z^2_3 = 2z_2(z_1 + z_3)$.

(vii) If z_1, z_2, z_3 are the vertices of right-angled isosceles triangle then $(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$.

(viii) Area of triangle formed by the points z_1, z_2 and z_3 is $\left|\frac{1}{4i}\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}\right|$.

19.12.1 Dot and Cross Product

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers i.e., (vectors). The dot product (also called the scalar product) of z_1 and z_2 is defined by $z_1 . z_2 = |z_1|\,|z_2| \cos\theta = x_1x_2 + y_1y_2 = \text{Re}\{\bar{z}_1 z_2\} = \frac{1}{2}\{\bar{z}_1 z_2 + z_1\bar{z}_2\}$.

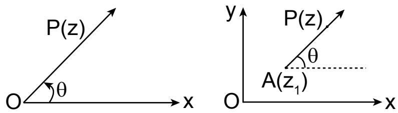
Where θ is the angle between z_1 and z_2 which lies between 0 and π.

If vectors z_1, z_2 are perpendicular then $z_1.z_2 = 0 \Rightarrow \frac{z_1}{\overline{z}_1}+\frac{z_2}{\overline{z}_2}=0$, i.e., Sum of complex slopes = 0.

The cross product of z_1 and z_2 is defined by $z_1.z_2 = |z_1|\,|z_2| \sin\theta = x_1y_2 - y_1x_2 =$ $\text{Im}\{\overline{z}_1 z_2\} = \frac{}{2i}\{\overline{z}_1 z_2 - z_1\overline{z}_2\}$.

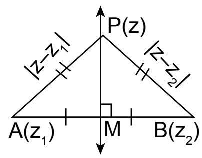
If vectors z_1, z_2 are parallel then $z_1 .\, z_2 = 0 \Rightarrow \frac{z_1}{\overline{z}_1}=\frac{z_2}{\overline{z}_2}$, i.e., complex slopes are equal.

A(5): amp(z) = θ represents a ray emanating from the origin and inclined at an angle θ with the positive direction of x-axis.



Also $\arg(z - z_1) = \theta$ represents the ray originating from $A(z_1)$ inclined at an angle θ with positive direction of x-axis as shown in the above diagram.

A(6): $|z - z_1| = |z - z_2|$ represents **perpendicular bisector** of line segment joining the points $A(z_1)$ and $B(z_2)$ as shown here:



A(7): The equation of a line passing through the points $A(z_1)$ and $B(z_2)$ can be expressed in determinant form as $\begin{vmatrix} z & \overline{z} & 1 \\ z_1 & \overline{z}_1 & 1 \\ z_2 & \overline{z}_2 & 1 \end{vmatrix} = 0$; it is also the condition for three points z_1, z_2, z_3 (when z is replaced by z_3) to be collinear.

A(8): Reflection Points for a Straight Lines:

Two given points, P and Q are the reflection points of a given straight line if the given line is the right bisector of the segment PQ. Note that the two points denoted by the complex number z_1 and z_2 will be the reflection points for the straight line $\overline{\alpha}z + \alpha\overline{z} + r = 0$ if and only if, $\overline{\alpha}z_1 + \alpha\overline{z}_2 + r = 0$, where r is real and α is non-zero constant.

19.12.2 Inverse Points w.r.t. a Circle

Two points, P and Q are said to be inverse w.r.t. a circle with centre O and radius ρ, if

(i) The point O, P, Q are collinear and P, Q are on the same side of O.
(ii) OP, OQ = ρ^2.

Note:

That the two points z_1 and z_2 will be the inverse point w.r.t. the circle $z\overline{z} + \overline{\alpha}z + \alpha\overline{z} + r = 0$, if and only if $z_1\overline{z}_2 + \overline{\alpha}z_1 + \alpha\overline{z}_2 + r = 0$.

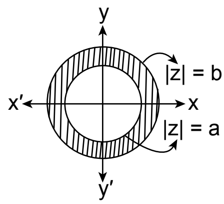
19.12.3 Ptolemys Theorem's

It states that the product of the length of the diagonal of a convex quadrilateral in scribed in a circle is equal to the sum of the products of lengths of the two pairs of its opposite sides, i.e., $|z_1 - z_3||z_2 - z_4| = |z_1 - z_2|$ $|z_3 - z_4| + |z_1 - z_4|\,|z_2 - z_3|$.

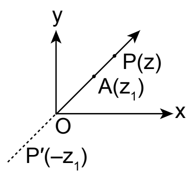
A(8): $|z - z_1| = a$ represents circle of radius a and having centre at z_1.
$|z - z_1| < a$ represents **interior** of the given circle.
$|z - z_1| > a$ represents **exterior** of the given circle.

A(9): The equation $|z - z_1|_2 + |z - z_2|^2 = k$, will represent a circle if $k \geq 1/2\ |z_1 - z_2|^2$.

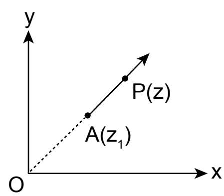
A(10): $\mathbf{a < |z| < b}$ represents points lying inside the circular annulus bounded by circles having radii a and b and having their centres at origin as shown below:



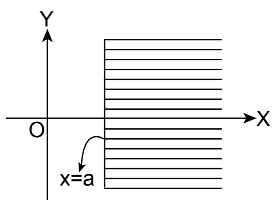
A(11): $|z + z_1| = |z| + |z_1|$ represents the ray originating from origin and passing through the point $A(z_1)$ as shown below: $|z + z_1| = PP' = PO + OP' = |z| + OA = |z| + |z_1|$ $(\because OP' = OA)$



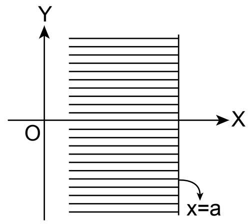
A(12): $\mathbf{|z - z_1| = |z| - |z_1|}$ represents a ray originating from $A(z_1)$, but not passing through the origin as shown below: $|z - z_1| = OP - OA = |z| - |z_1|$.



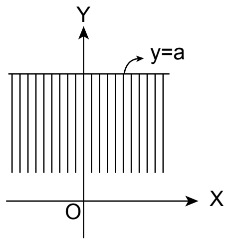
A(13): $\mathbf{Re(z) \geq a}$ represents the half-plane to the right of straight line, x = a, including the line itself as shown below:



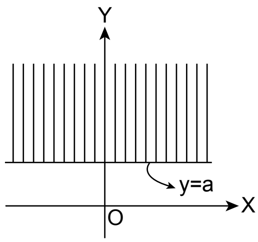
$\mathbf{Re(z) \leq a}$ represents the half-plane to the left of straight line, x = a, including the line itself as shown here:



Im(z) ≤ a represents the half-plane below the straight line, y = a, including the line itself as shown below:

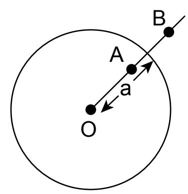


Im(z) ≥ a represents the half-plane above the straight line, y = a, including the line itself as shown below:



A(13): Inverse points w.r.t. a circle

Two points A and B are said to be inverse w.r.t. a circle with its centre 'O' and radius a, if:



(i) The points O, A, B are collinear and on the same side of O, and

(ii) OA.OB = a^2.

Remark

Two points, z_1 and z_2, will be the inverse points w.r.t. the circle $z\overline{z} + \overline{\beta}z + \beta\overline{z} + r = 0$, if and only if, $z_1\overline{z}_2 + \overline{\beta}z_1 + \beta\overline{z}_2 + r = 0$.

A(14): If λ is a positive real constant, and z satisfies $\left|\frac{z - z_1}{z - z_2}\right| = \lambda$, then the point z describes a circle of which A, B are inverse points; unless λ = 1, in which case z describes the perpendicular bisector of AB.

A(15): To convert an equation in cartesian to complex form put $x = \frac{z + \overline{z}}{2}$ and $y = \frac{z - \overline{z}}{2i}$ and to convert an equation complex form to Cartesian form put z = x + iy and $\overline{z} = x - iy$.