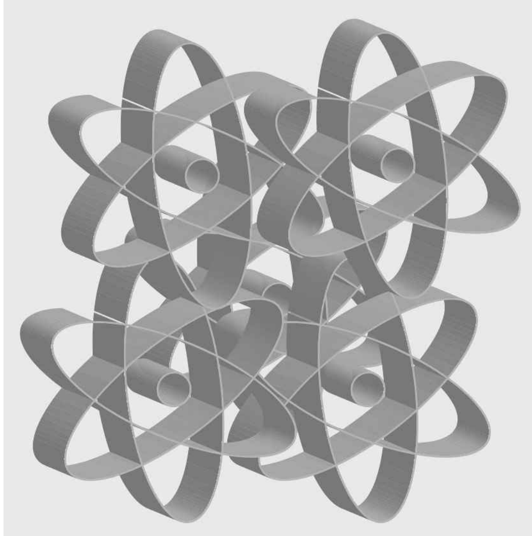
CHAPTER 11

Remainder and Factor Theorems



INTRODUCTION

A real valued function f(x) of the form $a_0x^n + a_1x^{n-1} + ... + a_n$, ($a_0 \neq 0$) is called as a polynomial of degree n, where n is a non-negative integer. Here $a_0, a_1, ..., a_n$ are the coefficients of various powers of x.

Example

(i) $4x^6 + 5x^5 + x^4 + x^2 - 1$ is a polynomial in x of degree 6.

(ii) $2x^3 + x^2 + 1$ is a polynomial in x of degree 3.

NOTE: A constant is considered to be a polynomial of zero degree.

In earlier classes we have learnt the different operations on polynomials like addition, subtraction, multiplication and division. Here we shall learn two important theorems on polynomials.

Remainder theorem

If p(x) is any polynomial and 'a' is any real number, then the remainder when p(x) is divided by (x – a) is given by p(a).

Proof

Let q(x) and r(x) be the quotient and the remainder respectively when p(x) is divided by x – a.

∴ By division algorithm

Dividend = quotient × divisor + remainder

i.e., $p(x) = q(x)(x - a) + r(x)$

If x = a, then

$p(a) = q(a)(a - a) + r(a) \Rightarrow r(a) = p(a)$

i.e., $p(x) = (x - a)q(x) + p(a)$

Thus the remainder is p(a)

NOTE:

1. If p(a) = 0, we say that 'a' is a zero of the polynomial p(x).
2. If p(x) is a polynomial and 'a' is a zero of p(x), then p(x) = (x – a) q(x).
3. If p(x) is divided by ax + b, then the remainder is given by $p\left(\frac{-b}{a}\right)$
4. If p(x) is divided by ax – b, then the remainder is given by $p\left(\frac{b}{a}\right)$

Example

Find the remainder when the polynomial $p(z) = z^3 - 3z + 2$ is divided by z – 2

Solution

Given $p(z) = z^3 - 3z + 2$

The remainder when p(z) is divided by z – 2 is given by p(2).

Now, $p(2) = (2)^3 - 3(2) + 2$

$= 8 - 6 + 2 = 4$

Hence, when p(z) is divided by z – 2 the remainder is 4.

Factor theorem

If p(x) is a polynomial of degree n (≥ 1) and a be any real number such that p(a) = 0, then (x – a) is a factor of p(x).

Proof

Let q(x) be the quotient and (x – k) (k∈R) be a factor of p(x)

Given p(a) = 0

∴ By division algorithm

Dividend = quotient × divisor + remainder

p(x) = q(x) (x – k) + p(a)

⇒ p(x) = q(x) (x – k) (∵ p(a) = 0)

Therefore (x – k) is a factor of f(x), which is possible only if f(k) = 0

Hence (x – a) is a factor of p(x) (∵ p(a) = 0)

NOTE:

1. If p(–a) = 0, then (x + a) is a factor of p(x).
2. If $p\left(\frac{-b}{a}\right) = 0$, then (ax + b) is a factor of p(x).
3. If $p\left(\frac{b}{a}\right) = 0$, then (ax – b) is a factor of p(x).
4. If sum of all the coefficients of a polynomial is zero, then (x – 1) is one of its factors.
5. If sum of the coefficients of odd powers of x is equal to the sum of the coefficients of even powers of x, then one of the factors of the polynomial is (x + 1).

Examples

(i) Determine whether $x - 3$ is a factor of $f(x) = x^2 - 5x + 6$
Given $f(x) = x^2 - 5x + 6$
Now $f(3) = (3)^2 - 5\ (3) + 6$
$= 9 - 15 + 6 .= 0 \Rightarrow f(3) = 0$
Hence, by factor theorem we can say that $(x - 3)$ is a factor of $f(x)$

(ii) Determine whether $(x - 1)$ is a factor of $x^3 - 6x^2 + 11x - 6$
Let $f(x) = x^3 - 6x^2 + 11x - 6$
Now $f(1) = (1)^3 - 6\ (1)^2 + 11\ (1) - 6$
$= 1 - 6 + 11 - 6 = 0 \Rightarrow f(1) = 0$
Hence, by factor theorem we can say that $(x - 1)$ is a factor of $f(x)$

Factorization of polynomials using factor theorem

(i) Factorize $x^2 (y - z) + y^2 (z - x) + z^2 (x - y)$
Let us assume the given expression as a polynomial in x, say f (x)
$f(x) = x^2 (y - z) + y^2 (z - x) + z^2 (x - y)$
Now put $x = y$ in the given expression
$\Rightarrow f(y) = y^2 (y - z) + y^2 (z - y) + z^2 (y - y)$
$= y^3 - zy^2 + y^2 z - y^3 + 0 = 0 \Rightarrow f(y) = 0$
$\Rightarrow x - y$ is a factor of the given expression

Similarly if we consider the given expression as a polynomial in y we get $y - z$ is a factor of the given expression and we also get $z - x$ is a factor of the expression when we consider it as an expression in z.
Let $x^2 (y - z) + y^2 (z - x) + z^2 (x - y) = k(x - y)(y - z)(z - x)$
For $x = 0, y = 1$ and $z = 2$, we get
$0^2 (1 - 2) + 1^2 (2 - 0) + 2^2 (0 - 1) = k(0 - 1)(1 - 2)(2 - 0)$
$\Rightarrow -2 = -2k \Rightarrow k = 1$
$\therefore$ the factors of the given expression are $x - y$, $y - z$ and $z - x$

(ii) Use factor theorem to factorize $x^3 + y^3 + z^3 - 3xyz$
Given expression is $x^3 + y^3 + z^3 - 3xyz$
Consider the expression as a polynomial in variable x say f(x)
i.e., $f(x) = x^3 + y^3 + z^3 - 3xyz$
Now $f[-(y + z)] = [-(y + z)]^3 + y^3 + z^3 - 3[-(y + z)]\, yz$
$= -(y + z)^3 + y^3 + z^3 + 3yz(y + z)$
$= -(y + z)^3 + (y + z)^3 = 0 \Rightarrow f[-(y + z)] = 0$
$\Rightarrow$ According to factor theorem $x - [-(y + z)]$ i.e., $x + y + z$ is a factor of $x^3 + y^3 + z^3 - 3xyz$

Now using the long division method we get the other factor as

$x^2 + y^2 + z^2 - xy - yz - zx$
$\therefore x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

Horner's process for synthetic division of polynomials

When a polynomial $f(x) = p_0 x^n + p_1 x^{n-1} + \text{- - -} + p_{n-1} x + p_n$ is divided by a binomial $x - \alpha$, let the quotient be Q (x) and remainder be r.

We can find quotient Q(x) and remainder r by using Horner's synthetic division process as explained below.

α	p_0	p_1	p_2	---- p_{n-1}	p_n	--- I st row
left (corner)		$q_0\alpha$	$q_1\alpha$	---- $q_{n-2}\alpha$	$q_{n-1}\alpha$	--- II nd row
	q_0	q_1	q_2	---- q_{n-1}	r	--- III rd row

Step 1: Write all the coefficients $p_0, p_1, p_2, - - - - - , p_n$ of the given polynomial f(x) in the order of descending powers of x as in the first row. When any term in f(x) (as seen with descending powers of x) is missing we write zero for its coefficient.

Step 2: Divide the polynomial f(x) by $(x - \alpha)$ by writing α in the left corner as shown above $(x - \alpha = 0 \Rightarrow x = \alpha)$

Step 3: Write the first term of the third row as $q_0 = p_0$ then multiply q_0 by α to get $q_0\,\alpha$ and write it under p_1, as the first element of the second row

Step 4: Add $q_0\,\alpha$ to p_1 to get q_1, the second element of the third row

Step 5: Again multiply q_1 with α to get $q_1\alpha$ and write $q_1\,\alpha$ under p_2 and add $q_1\,\alpha$ to p_2 to get q_2 which is the third element of the third row

Step 6: Continue this process till we obtain q_{n-1} in the third row. Multiply q_{n-1} with α and write $q_{n-1}\,\alpha$ under p_n and add $q_{n-1}\,\alpha$ to p_n to get r in third row as shown above

In the above process the elements of the third row i.e., $q_0, q_1, q_2, - - - , q_{n-1}$ are the coefficients of the quotient Q(x) in the same order of descending powers starting with x^{n-1}

$\therefore Q(x) = q_0x^{n-1} + q_1x^{n-2} + - - - - + q_{n-2}\,x + q_{n-1}$

and the remainder is r i.e., the last element of the third row

Note: If the remainder r = 0 then α is one of the roots of f(x) = 0 or $x - \alpha$ is a factor of f(x)

Example

Factorize $x^4 - 10x^2 + 9$

Let $p(x) = x^4 - 10x^2 + 9$

Here sum of coefficients = 0, and also

sum of coefficients even powers of x = sum of coefficients of odd powers of x

$\therefore$ (x – 1) and (x + 1) are the factors of p(x).

Multiplier of x – 1 is 1 and x + 1 is –1

$\therefore$ The quotient is $x^2 - 9$

Hence $p(x) = (x - 1)(x + 1)(x^2 - 9)$

$\Rightarrow p(x) = (x - 1)(x + 1)(x - 3)(x + 3)$

1	1	0	–10	0	9
	0	1	1	–9	–9
–1	1	1	–9	–9	0
	0	–1	0	9	
	1	0	–9	0	

Problems based on factor and remainder theorems

Examples

1. Find the value of a if $ax^3 - (a + 1)x^2 + 3x - 5a$ is divisible by (x –2).

Solution

Let $p(x) = ax^3 - (a + 1)x^2 + 3x - 5a$

If p(x) is divisible by (x – 2), then its remainder is zero i.e., p(2) = 0

$\Rightarrow a(2)^3 - (a + 1)(2)^2 + 3(2) - 5a = 0$

$\Rightarrow 8a - 4a - 4 + 6 - 5a = 0$

$\Rightarrow -a + 2 = 0$

$\Rightarrow a = 2$

$\therefore$ The required value of a is 2.

2. If the polynomial $x^3 + ax^2 - bx - 30$ is exactly divisible by $x^2 - 2x - 15$. Find a and b and also the third factor.

Solution

Let $p(x) = x^3 + ax^2 - bx - 30$

Given p(x) is exactly divisible by $x^2 - 2x - 15$ i.e., $(x - 5)(x + 3)$

$\Rightarrow$ p(x) is divisible by $(x + 3)$ and $(x - 5)$

$\therefore p(-3) = 0$ and $p(5) = 0$

Consider $p(-3) = 0$

$\Rightarrow (-3)^3 + a(-3)^2 - b(-3) - 30 = 0$

$\Rightarrow -27 + 9a + 3b - 30 = 0$

$\Rightarrow 9a + 3b - 57 = 0$

$\Rightarrow 3a + b - 19 = 0 \rightarrow (1)$

Now consider $p(5) = 0$

i.e., $5^3 + a(5)^2 - b(5) - 30 = 0$

$\Rightarrow 125 + 25a - 5b - 30 = 0$

$\Rightarrow 25a - 5b + 95 = 0$

$\Rightarrow 5a - b + 19 = 0 \rightarrow (2)$

Adding (1) and (2), we get

$8a = 0$

$\Rightarrow a = 0$

Substituting a in (1), we get $b = 19$

$\therefore$ The required values of a and b are 0 and 19 respectively

$\Rightarrow p(x) = x^3 + 0(x^2) - 19x - 30$

i.e., $p(x) = x^3 - 19x - 30$

Thus, the third factor is $x + 2$.

–3	1	0	–19	–30
	0	–3	9	30
5	1	–3	–10	0
	0	5	10	
	1	2	0	

3. Find the linear polynomial in x which when divided by $(x - 3)$ leaves 6 as remainder and is exactly divisible by $(x + 3)$.

Solution

Let the linear polynomial be $p(x) = ax + b$

Given $p(3) = 6$ and $p(-3) = 0$

$\Rightarrow a(3) + b = 6$ and $a(-3) + b = 0$

$\Rightarrow 3a + b = 6 \rightarrow (1)$ and $-3a + b = 0 \rightarrow (2)$

Adding (1) and (2),

$2b = 6 \Rightarrow b = 3$

Substituting the value of b in (1), we get $a = 1$

$\therefore$ The required linear polynomial is $x + 3$.

4. A quadratic polynomial in x leaves remainders as 4 and 7 respectively when divided by $(x + 1)$ and $(x - 2)$. Also it is exactly divisible by $(x - 1)$. Find the quadratic polynomial.

Solution

Let the quadratic polynomial be $p(x) = ax^2 + bx + c$

Given $p(-1) = 4$, $p(2) = 7$ and $p(1) = 0$

$p(-1) = a(-1)^2 + b(-1) + c = 4$

$\Rightarrow a - b + c = 4 \rightarrow (1)$

Now $p(1) = 0$ and $p(2) = 7$

$\therefore\ a(1)^2 + b(1) + c = 0$ and

$a(2)^2 + b(2) + c = 7$

$\Rightarrow a + b + c = 0 \rightarrow (2)$

$4a + 2b + c = 7 \rightarrow (3)$

Subtracting (2) from (1), we have

$2b = -4$

$\Rightarrow b = -2$. Subtracting (2) from (3), we have

$3a + b = -7$

$\Rightarrow 3a - 2 = 7\ (\because b = -2)$

$\Rightarrow 3a = 9 \Rightarrow a = 3$

Substituting the values of a and b in (1), we get $c = -1$

Hence, the required quadratic polynomial is $3x^2 - 2x - 1$

5. Find a common factor of the quadratic polynomials $3x^2 - x - 10$ and $2x^2 - x - 6$.

Solution

Consider $p(x) = 3x^2 - x - 10$ and $q(x) = 2x^2 - x - 6$

Let $(x - k)$ be a common factor of $p(x)$ and $q(x)$

$\therefore\ p(k) = q(k) = 0$

$\Rightarrow 3k^2 - k - 10 = 2k^2 - k - 6$

$\Rightarrow k^2 - 4 = 0$

$\Rightarrow k^2 = 4$

$\Rightarrow k = \pm 2$

$\therefore$ The required common factor is $(x - 2)$ or $(x + 2)$.

6. Find the remainder when x^{999} is divided by $x^2 - 4x + 3$

Solution

Let $q(x)$ and $mx + n$ be the quotient and the remainder respectively when x^{999} is divided by $x^2 - 4x + 3$.

$\therefore\ x^{999} = (x^2 - 4x + 3)\ q(x) + mx + n$
If x = 1,
$1^{999} = (1 - 4 + 3)\ q(x) + m(1) + n$
$\Rightarrow 1 = 0 \times q(x) + m + n$
$\Rightarrow m + n = 1 \rightarrow (1)$
If x = 3,
$3^{999} = (3^2 - 4(3) + 3)\ q(x) + 3m + n$
$\Rightarrow 3^{999} = 0 \times q(x) + 3m + n$
$\Rightarrow 3m + n = 3^{999} \rightarrow (2)$

Subtracting (1) from (2) we get
$2m = 3^{999} - 1$
$m = \frac{1}{2}(3^{999} - 1)$
Substituting m in (1), we have

$$n = 1 - \frac{1}{2}(3^{999} - 1) = 1 - \frac{1}{2}3^{999} + \frac{1}{2} = \frac{3}{2} - \frac{1}{2}3^{999}$$

$$n = \frac{3}{2}\ (1 - 3^{998})$$

$\therefore$ The required remainder is $\frac{1}{2}(3^{999} - 1)\ x + \frac{3}{2}(1 - 3^{998})$.

7. Find the remainder when x^5 is divided by $x^3 - 4x$.

Solution

Let q(x) be the quotient and $\ell x^2 + mx + n$ be the remainder when x^5 is divided by $x^3 - 4x$ i.e., $x(x - 2)\ (x + 2)$
$\therefore\ x^5 = (x^3 - 4x)\ q(x) + \ell x^2 + mx + n$
Put x = 0
$\Rightarrow 0 = 0 \times q(x) + \ell(0) + m(0) + n$
$\Rightarrow n = 0$

Put x = 2
$\Rightarrow 2^5 = (8 - 8)\ q(x) + \ell(2)^2 + m(2) + n$
$\Rightarrow 32 = 4\ell + 2m + n$
$\Rightarrow 4\ell + 2m = 32\ (\because\ n = 0)$
$\Rightarrow 2\ell + m = 16$ ----- (1)

Put x = –2
$(-2)^5 = (-8 + 8)\ q(x) + \ell(-2)^2 + m\ (-2) + n$
$\Rightarrow -32 = 4\ell - 2m + n$
$\Rightarrow 4\ell - 2m = -32\ (\because\ n = 0)$

$\Rightarrow 2\ \ell - m = -16 \dashrightarrow (2)$
Adding (1) and (2),

$4\ell = 0$

$\Rightarrow \ell = 0$

Substituting ℓ in (1), we get

$2(0) + m = 16$

$\Rightarrow m = 16$

$\therefore$ The required remainder is $0(x^2) + 16x + 0$ i.e., $16x$

test your concepts

Very short answer type questions

1. Let $f(x) = a_0x^n + a_1x^{n-1} + \ldots\ldots\ldots\ldots + a_n\ (a_0 \neq 0)$ be a polynomial of degree n. If $x + 1$ is one of its factors, then ______.
2. If a polynomial f(x) is divided by $(x + a)$, then the remainder obtained is ______.
3. If $a - b$ is a factor of $a^n - b^n$, then n is ______.
4. If $f(x) = x^3 + 2$ is divided by $x + 2$, then the remainder obtained is ______.
5. The condition for which $ax^2 + bx + a$ is exactly divisible by $x - a$ is ______.
6. If $x + 1$ is a factor of $x^m + 1$, then m is ______.
7. The remainder when $f(x) = x^3 + 5x^2 + 2x + 3$ is divided by x is ______.
8. The remainder when $(x - a)^2 + (x - b)^2$ is divided by x is ______.
9. The remainder when $x^6 - 4x^5 + 8x^4 - 7x^3 + 3x^2 + 2x - 7$ is divided by $x - 1$ is ______.
10. For two odd numbers x and y, if $x^3 + y^3$ is divisible by 2^k, $k \in N$, then $x + y$ is divisible by 2^k. [True/False]
11. One of the factors of $2x^{17} + 3x^{15} + 7x^{23}$ is ______ $(x^{17} / x^{15} / x^{23})$
12. If $(x - 2)^2$ is the factor of an expression of the form $x^3 + bx + c$, then the other factor is ______.
13. What should be added to $3x^3 + 5x^2 - 6x + 3$ to make it exactly divisible by $x - 1$?
14. The remainder when $2x^6 - 5x^3 - 3$ is divided by $x^3 + 1$ is ______.
15. The remainder when f(x) is divided by g(x) is $f\left(-\frac{3}{2}\right)$, then g(x) is necessarily $2x + 3$. [True/False]
16. Find the remainder when the polynomial $x^2 + 13x + 11$ is divided by $x - 1$.
17. Find the value of the polynomial $a^2 - \frac{1}{6}a + \frac{3}{2}$ when $a = \frac{1}{2}$.
18. The polynomial $7x^2 - 11x + a$ when divided by $x + 1$ leaves a remainder of 8. Then find the value of 'a'.
19. If $x + 2$ is a factor of f(x) and $f(x) = x^3 + 4x^2 + kx - 6$, then find the value of k.

20. Find the values of a if $x^3 - 5x(a - 1) - 3(x + 1) + 5a$ is divisible by $x - a$.

21. Find the value of a if $x - a$ is a factor of the polynomial $x^5 - ax^4 + x^3 - ax^2 + 2x + 3a - 2$.

22. Find the remainder when $x^3 + 3px + q$ is divided by $(x^2 - a^2)$ without actual division.

23. The remainder obtained when $x^2 + 3x + 1$ is divided by $(x - 5)$ is ______.

24. If the polynomial $3x^4 - 11x^2 + 6x + k$ is divided by $x - 3$, it leaves a remainder 7. Then the value of k is ______.

25. $(7x - 1)$ is a factor of $7x^3 + 6x^2 - 15x + 2$ (True/False)

26. If $ax^2 + bx + c$ is exactly divisible by $2x - 3$, then the relation between a, b and c is _________.

27. If $x^2 + 5x + 6$ is a factor of $x^3 + 9x^2 + 26x + 24$, then find the remaining factor.

28. If $(2x - 1)$ is a factor of $2x^2 + px - 2$, then the other factor is ______.

29. The expression $x^{m^n} - 1$ is divisible by $x + 1$, only if M is (even/odd) ___________.

30. If $x + m$ is one of the factors of the polynomial $x^2 + mx - m + 4$, then the value of m is ______.

Short answer type questions

31. For what values of m and n is $2x^4 - 11x^3 + mx + n$ is divisible by $x^2 - 1$?

32. Find a linear polynomial which when divided by $(2x + 1)$ and $(3x + 2)$ leaves remainders 3 and 4 respectively.

33. Prove that $x^m + 1$ is a factor of $x^{mn} - 1$ if n is even.

34. The remainders of a polynomial f(x) in x are 10 and 15 respectively when f(x) is divided by $(x - 3)$ and $(x - 4)$. Find the remainder when f(x) is divided by $(x - 3)(x - 4)$.

35. If x^{555} is divided by $x^2 - 4x + 3$, then find its remainder.

36. If $(x^2 - 1)$ is a factor of $ax^3 - bx^2 - cx + d$, then find the relation between a and c.

37. When $x^4 - 3x^3 + 4x^2 + p$ is divided by $(x - 2)$, the remainder is zero. Find the value of p.

38. Find the common factors of the expressions $a_1 x^2 + b_1 x + c_1$ and $a_2 x^2 + b_2 x + c_1$ where $c_1 \neq 0$.

39. If $(x - 3)$ is a factor of $x^2 + q$ (where $q \in Q$), then find the remainder when $(x^2 + q)$ is divided by $(x - 2)$.

40. If $p + q$ is a factor of the polynomial $p^n - q^n$, then n is

41. The expression $x^{4005} + y^{4005}$ is divisible by_______.

42. The value of a for which $x - 7$ is a factor of $x^2 + 11x - 2a$ is ______.

43. If a polynomial f(x) is divided by $(x - 3)$ and $(x - 4)$ it leaves remainders as 7 and 12 respectively, then find the remainder when f(x) is divided by $(x - 3)(x - 4)$.

44. Find the remainder when $5x^4 - 11x^2 + 6$ is divided by $5x^2 - 6$.

45. If $f(x - 2) = 2x^2 - 3x + 4$, then find the remainder when f(x) is divided by $(x - 1)$.

Essay type questions

46. Factorize $x^4 - 2x^3 - 9x^2 + 2x + 8$ using remainder theorem.

47. Find the remainder when x^{29} is divided by $x^2 - 2x - 3$.

48. If $x^2 - 2x - 1$ is a factor of $px^3 + qx^2 + 1$, (where p, q are integers) then find the value of p + q.

49. If $x^2 - x + 1$ is a factor of $x^4 + ax^2 + b$, then the values of a and b are respectively ______.

50. If $\ell x^2 + mx + n$ is exactly divisible by (x – 1) and (x + 1) and leaves a remainder 1 when divided by x + 2, then find m and n.

CONCEPT APPLICATION

Concept Application Level—1

1. The value of a for which the polynomial $y^3 + ay^2 - 2y + a + 4$ in y has (y + a) as one of its factors is ______.

(1) $\frac{-3}{4}$ (2) $\frac{4}{3}$ (3) $\frac{3}{4}$ (4) $\frac{-4}{3}$

2. If the expression $2x^3 - 7x^2 + 5x - 3$ leaves a remainder of 5k – 2 when divided by x + 1, then find the value of k.

(1) 3 (2) – 3 (3) 5 (4) – 5

3. Find the remainder when $x^{2003} + y^{6009}$ is divided by $x + y^3$.

(1) y^{4006} (2) 1

(3) 0 (4) Cannot be determined

4. Find the remainder when $x^6 - 7x^3 + 8$ is divided by $x^3 - 2$.

(1) – 2 (2) 2 (3) 7 (4) 1

5. If both the expressions $x^{1248} - 1$ and $x^{672} - 1$, are divisible by $x^n - 1$, then the greatest integer value of n is ___________.

(1) 48 (2) 96 (3) 54 (4) 112

6. When $x^2 - 7x + 2$ is divided by x – 8, then the remainder is ______.

(1) 122 (2) 4 (3) 45 (4) 10

7. If $ax^2 + bx + c$ is exactly divisible by 4x + 5, then

(1) 25a – 5b + 16 c = 0. (2) 25a + 20 b + 16 c = 0.

(3) 25a – 20 b – 16 c = 0. (4) 25 a – 20 b + 16 c = 0.

8. The expression $2x^3 + 3x^2 - 5x + p$ when divided by x + 2 leaves a remainder of 3p + 2. Find p.

(1) –2 (2) 1 (3) 0 (4) 2

9. $3x - 4$ is a factor of ______.

(1) $18x^4 - 3x^3 - 28x^2 - 3x + 4$ (2) $3x^4 - 10x^3 - 7x^2 + 38x - 24$

(3) $9x^4 - 6x^3 + 5x^2 - 15$ (4) $9x^4 + 36x^3 + 17x^2 - 38x - 24$

10. Which of the following is a factor of $5x^{20} + 7x^{15} + x^9$?

(1) x^{20} (2) x^{15} (3) x^9 (4) x^{24}

11. If $(x + 3)^2$ is a factor of $f(x) = ex^3 + kx + 6$, then find the remainder obtained when f(x) is divided by $x - 6$.

(1) 1 (2) 0 (3) 5 (4) 4

12. The expression $x^{mn} + 1$ is divisible by $x + 1$, only if

(1) n is odd. (2) m is odd. (3) both m and n are even. (4) Cannot say

13. If both the expressions $x^{1215} - 1$and $x^{945} - 1$, are divisible by $x^n - 1$, then the greatest integer value of n is ______.

(1) 135 (2) 270 (3) 945 (4) None of these

14. If $(x - 2)$ is a factor of $x^2 + bx + 1$ (where $b \in Q$), then find the remainder when $(x^2 + bx + 1)$ is divided by $2x + 3$.

(1) 7 (2) 8 (3) 1 (4) 0

15. When $x^3 + 3x^2 + 4x + a$ is divided by $(x + 2)$, the remainder is zero. Find the value of a.

(1) 4 (2) 6 (3) –8 (4) –12

16. If $(x + 1)$ and $(x - 1)$ are the factors of $ax^3 + bx^2 + cx + d$, then which of the following is true?

(1) $a + b = 0$ (2) $b + c = 0$ (3) $b + d = 0$ (4) None of these

17. Find the remainder when x^5 is divided by $x^2 - 9$.

(1) $81x$ (2) $81x + 10$ (3) $3^5x + 3^4$ (4) None of these

18. The remainder when $x^{45} + x^{25} + x^{14} + x^9 + x$ divided by $x^2 - 1$ is ______.

(1) $4x - 1$ (2) $4x + 2$ (3) $4x + 1$ (4) $4x - 2$

19. For what values of a and b is the expression $x^4 + 4x^3 + ax^2 - bx + 3$ a multiple of $x^2 - 1$?

(1) $a = 1, b = 7$ (2) $a = 4, b = -4$ (3) $a = 3, b = -5$ (4) $a = -4, b = 4$

20. When the polynomial $p(x) = ax^2 + bx + c$ is divided by $(x - 1)$ and $(x + 1)$, the remainders obtained are 6 and 10 respectively. If the value of p(x) is 5 at $x = 0$, then the value of $5a - 2b + 5c$ is ______.

(1) 40 (2) 44 (3) 21 (4) 42

21. If $p - q$ is a factor of the polynomial $p^n - q^n$, then n is ______.

(1) a prime number (2) an odd number (3) an even number (4) All the above

22. When the polynomial $f(x) = ax^2 + bx + c$ is divided by x, $x - 2$ and $x + 3$, remainders obtained are 7, 9 and 49 respectively. Find the value of $3a + 5b + 2c$.

(1) –2 (2) 2 (3) 5 (4) –5

23. If $f(x + 1) = 2x^2 + 7x + 5$, then one of the factors of f(x) is ______.

(1) $2x + 3$ (2) $2x^2 + 3$ (3) $3x + 2$ (4) None of these

24. If (x – p) and (x – q) are the factors of $x^2 + px + q$, then the values of p and q are respectively ______.

(1) 1, –2
(2) 2, –3
(3) $\frac{-1}{3}, \frac{-2}{3}$
(4) None of these

25. Let $f\left(x - \frac{1}{x}\right) = x^2 + \frac{1}{x^2}$, find the remainder when f(x) is divided by x – 3.

(1) $\frac{82}{9}$
(2) $\frac{8}{3}$
(3) 10
(4) 11

26. If $(x - 2)^2$ is a factor of $f(x) = x^3 + px + q$, then find the remainder when f(x) is divided by x – 1.

(1) 4
(2) –4
(3) –5
(4) 5

27. A quadratic polynomial in x leaves remainders 4, 4 and 0 respectively when divided by (x – 1), (x – 2) and (x – 3). Find the quadratic polynomial.

(1) $-2x^2 + 6x + 3$
(2) $-2x^2 + 6x$
(3) $-2x^2 + 6x + 5$
(4) $-2x^2 + 6x - 5$

28. If $f(x + 3) = x^2 + x - 6$, then one of the factors of f(x) is ______.

(1) x – 3
(2) x – 4
(3) x – 5
(4) x – 6

29. If $(x - 1)^2$ is a factor of $f(x) = x^3 + bx + c$, then find the remainder when f(x) is divided by (x – 2).

(1) 2
(2) –3
(3) 4
(4) –4

30. For what values of m and n, the expression $2x^2 - (m + n)x + 2n$ is exactly divisible by (x – 1) and (x – 2)?

(1) m = 5, n = 2
(2) m = 3, n = 4
(3) m = 4, n = 2
(4) m = 2, n = 4

Concept Application Level—2

31. The ratio of the remainders when the expression $x^2 + bx + c$ is divided by (x – 3) and (x – 2) respectively is 4 : 5. Find b and c, if (x – 1) is a factor of the given expression.

(1) $b = \frac{-11}{3}, c = \frac{14}{3}$
(2) $b = \frac{-14}{3}, c = \frac{11}{3}$
(3) $b = \frac{14}{3}, c = \frac{-11}{3}$
(4) None of these

32. If the polynomials $f(x) = x^2 + 9x + k$ and $g(x) = x^2 + 10x + \ell$ have a common factor, then $(k - \ell)^2$ is equal to ______.

(1) $9\ell - 10k$
(2) $10\ell - 9k$
(3) Both (1) and (2)
(4) None of these

33. When f(x) is divided by (x – 2), the quotient is Q(x) and the remainder is zero. And when f(x) is divided by [Q(x) – 1], the quotient is (x – 2) and the remainder is R(x). Find the remainder R(x).

(1) x + 2
(2) – x + 2
(3) x – 2
(4) Cannot be determined

34. Find the values of m and n, if (x – m) and (x – n) are the factors of the expression $x^2 + mx - n$.

(1) m = – 1, n = – 2
(2) m = 0, n = 1
(3) $m = \frac{-1}{2}, n = \frac{1}{2}$
(4) m = –1, n = 2

35. Let $f\left(x+\frac{1}{x}\right)=x^2+\frac{1}{x^2}$, find the remainder when f(x) is divided by 2x + 1.

(1) $\frac{-7}{4}$ (2) $\frac{9}{4}$ (3) $\frac{-9}{4}$ (4) $\frac{11}{4}$

36. A polynomial f(x) leaves remainders 10 and 14 respectively when divided by (x – 3) and (x – 5). Find the remainder when f(x) is divided by (x – 3) (x – 5).

(1) 2x + 6 (2) 2x – 4 (3) 2x + 4 (4) 2x – 6

37. If f(x + 3) = x^2 – 7x + 2, then find the remainder when f(x) is divided by (x + 1).

(1) 8 (2) – 4 (3) 20 (4) 46

38. A polynomial f(x) when divided by (x – 5) and (x – 7) leaves remainders 6 and 16 respectively. Find the remainder when f(x) is divided by (x – 5) (x – 7).

(1) 5x + 7 (2) 5x – 7 (3) 5x + 19 (4) 5x – 19

39. A polynomial p(x) leaves remainders 75 and 15 respectively, when divided by (x – 1) and (x + 2). Then the remainder when f(x) is divided by (x – 1) (x + 2) is ______.

(1) 5(4x + 11) (2) 5(4x – 11) (3) 5(3x + 11) (4) 5(3x – 11)

40. The leading coefficient of a polynomial f(x) of degree 3 is 2006. Suppose that f(1) = 5, f(2) = 7 and f(3) = 9. Then find f(x).

(1) 2006 (x – 1) (x – 2) (x – 3) + 2x + 3
(2) 2006 (x – 1) (x – 2) (x – 3) + 2x + 1
(3) 2006 (x – 1) (x – 2) (x – 3) + 2x – 1
(4) 2006 (x – 2) (x – 3) (x – 1) – (2x – 3)

41. The ratio of the remainders when the expression x^2 + ax + b is divided by (x – 2) and (x – 1) respectively is 4 : 3. Find a and b if (x + 1) is a factor of the expression.

(1) 9, –10 (2) –9, 10 (3) 9, 10 (4) –9, –10

42. If x^3 – ax^2 + bx – 6 is exactly divisible by x^2 – 5x + 6, then $\frac{a}{b}$ is ______.

(1) $\frac{6}{11}$ (2) $\frac{-6}{11}$ (3) $\frac{1}{3}$ (4) $-\frac{1}{3}$

43. If f(x) = x^2 + 5x + a and g(x) = x^2 + 6x + b have a common factor, then which of the following is true?

(1) $(a-b)^2 + 5(a-b) + b = 0$
(2) $(a+b)^2 + 5(a+b) + a = 0$
(3) $(a+b)^2 + 6(a+b) + b = 0$
(4) $(a-b)^2 + 6(a-b) + b = 0$

44. If $ax^4 + bx^3 + cx^2 + dx$ is exactly divisible by x^2 – 4, then $\frac{a}{c}$ is ______.

(1) $\frac{1}{4}$ (2) $\frac{-1}{4}$ (3) $\frac{-1}{8}$ (4) $\frac{1}{8}$

45. If x^2 + x + 1 is a factor of $x^4 + ax^2 + b$, then the values of a and b respectively are

(1) 2, 4 (2) 2, 1 (3) 1, 1 (4) None of these

Concept Application Level—3

46. Find the remainder when x^{33} is divided by $x^2 - 3x - 4$.

(1) $\left(\frac{4^{33}-1}{5}\right)x + \left(\frac{4^{33}-4}{5}\right)$ (2) $\left(\frac{4^{33}+1}{5}\right)x + \left(\frac{4^{33}-4}{5}\right)$

(3) $\left(\frac{4^{33}-4}{5}\right)x + \left(\frac{4^{33}+1}{5}\right)$ (4) $\left(\frac{4^{33}+4}{5}\right)x + \left(\frac{4^{33}-1}{5}\right)$

47. If $6x^2 - 3x - 1$ is a factor of $ax^3 + bx - 1$ (where a, b are integers), then find the value of b.

(1) 1 (2) 3 (3) –5 (4) –7

48. If the polynomials $f(x) = x^2 + 6x + p$ and $g(x) = x^2 + 7x + q$ have a common factor, then which of the following is true?

(1) $p^2 + q^2 + 2pq + 6p - 7q = 0$ (2) $p^2 + q^2 - 2pq + 7p - 6q = 0$

(3) $p^2 + q^2 - 2pq + 6p - 7q = 0$ (4) $p^2 + q^2 + 2pq + 7p - 6q = 0$

49. A polynomial of degree 2 in x, when divided by (x + 1), (x + 2) and (x + 3), leaves remainders 1, 4 and 3 respectively. Find the polynomial.

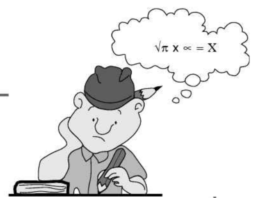
(1) $\frac{1}{2}(x^2 + 9x + 6)$ (2) $\frac{1}{2}(x^2 - 9x + 6)$ (3) $\frac{-1}{2}(x^2 - 9x + 6)$ (4) $\frac{-1}{2}(x^2 + 9x + 6)$

50. When a third degree polynomial f(x) is divided by (x – 3), the quotient is Q(x) and the remainder is zero. Also when f(x) is divided by [Q(x) + x + 1], the quotient is (x – 4) and remainder is R(x). Find the remainder R(x).

(1) $Q(x) + 3x + 4 + x^2$ (2) $Q(x) + 4x + 4 - x^2$

(3) $Q(x) + 3x + 4 - x^2$ (4) Cannot be determined

KEY



Very short answer type questions

1. $a_1 + a_3 + a_5 + \ldots\ldots\ldots = a_0 + a_2 + a_4 + \ldots\ldots\ldots\ldots$

2. f (–a)

3. $n \in N$

4. – 6

5. $a = 0$ or $a^2 + b + 1 = 0$

6. odd

7. 3

8. $a^2 + b^2$

9. – 4

10. True

11. x^{15}

12. x + 4

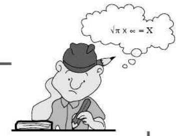
13. –5

14. 4

15. False

16. 25

17. $\frac{5}{3}$



18. -10

19. 1

20. 1 and 3

21. $\frac{2}{5}$

22. $(a^2 + 3p)x + q$.

23. 41

24. -155

25. True

26. $9a+6b+4c = 0$

27. $(x + 4)$.

28. $x + 2$

29. even number

30. 4

Short answer type questions

31. $m = 11$ and $n = -2$

32. $-6x$

34. $5(x - 1)$

35. $\frac{1}{2}(3^{555} - 1)x + \frac{3}{2}(1 - 3^{554})$

36. $a = c$

37. -8

38. $\left(x + \frac{b_1 - b_2}{a_1 - a_2}\right)$

39. -5

40. 42

41. $x + y$

42. 63

43. $5x - 8$

44. 0

45. 13

Essay type questions

46. $(x - 1)(x + 1)(x + 2)(x - 4)$.

47. $\left(\frac{3^{29} + 1}{4}\right)x + \left(\frac{3^{29} - 3}{4}\right)$

48. -3

49. $1, 1$

50. $m = 0, n = -1/3$

key points for selected questions

Very short answer type questions

16. Put $x = 1$ in $x^2 + 13x + 11$, the result obtained is the required remainder.

17. Substitute $a = \frac{1}{2}$ in $a^2 - \frac{a}{6} + \frac{3}{2}$ and then simplify.

18. (i) Let $f(x) = 7x^2 - 11x + a$.
 (ii) Use $f(-1) = 8$ and solve for a.

19. Use, $f(-2) = 0$ and solve for k.

20. (i) Consider given polynomial as f(x)
 (ii) Use, $f(a) = 0$ and solve for a

21. (i) Consider the given polynomial as f(x).
 (ii) Use, $f(a) = 0$ and solve for a.

22. (i) Let $f(x) = x^3 + 3px + q$ and divisor is $x^2 - a^2$.
 (ii) By division rule, $f(x) = Q(x)(x^2 - a^2) + (\ell x + n)$ where $(\ell x + n)$ is the required remainder.
 (iii) Put $x = a$ and $x = -a$, and frame the equations.
 (iv) Solve the equations to get p and q.

23. Use remainder theorem

24. Use remainder theorem

25. Use factor theorem

26. Use factor theorem

27. (i) Let $f(x) = x^3 + 9x^2 + 26x + 24$.
 (ii) Factor is $(x^2 + 5x + 6)$ i.e., $(x + 2)(x + 3)$.

(iii) By using Horner's method, get coefficients of the quotient when f(x) is divided by (x + 2).
(iv) Again get coefficients of the new quotient when the previous quotient is divided by (x + 3).
(v) If the coefficients are a and b, then the remaining factor is (ax + b).

28. Find p using factor theorem

29. Use factor theorem

30. Use factor theorem

Short answer type questions

31. (i) Let $f(x) = 2x^4 - 11x^3 + mx + n$.
(ii) Divisor is $x^2 - 1$ i.e., $(x + 1)(x - 1)$.
(iii) Use, $f(1) = 0$ and $f(-1) = 0$ and frame equations in m and n.
(iv) Then solve the equations for m and n.

32. (i) Let $f(x) = ax + b$ and divisors are $(2x + 1)$ and $(3x + 2)$.
(ii) By the remainder theorem, $f\left(-\frac{1}{2}\right) = 3$, $f\left(-\frac{2}{3}\right) = 4$.
(iii) Solve two equations for a and b.

33. (i) Let $f(x) = (x^m)^n - 1$.
(ii) As divisor is $x^m - 1$, remainder is $f(1) = 0$.

34. (i) Let $f(x) = x^2 + ax + b$
(ii) Given $f(3) = 10$ and $f(4) = 15$
(iii) Then assume f(x) as $f(x) = Q(x)(x - 3)(x - 4) + (\ell x + n)$
(iv) Put x = 3 and x = 4, then get two equations in ℓ and n.
(v) Then solve the equations.

35. (i) Let $f(x) = x^{555}$
(ii) By division rule, $f(x) = Q(x)(x^2 - 4x + 3) + (ax + b)$.
(iii) Factorize $(x^2 - 4x + 3)$.
(iv) Then substitute the zeroes of the factors in the equation which is mentioned in step (ii).
(v) Then solve two equations for a and b.

38. (i) Let $f(x) = a_1x^2 + b_1x + c_1$ and $g(x) = a_2x^2 + b_2x + c_1$.
(ii) Let (x – k) be the common factor of f(x) and g(x).
(iii) Then equate f(k) and g(k) to get k.

40. Use remainder theorem.

41. $x^n + y^n$ is divisible by x + y, if n is odd.

42. Use factor theorem

44. Use remainder theorem

Essay type questions

46. (i) Let $f(x) = x^4 - 9x^2 + 2x + 8$
(ii) As sum of the coefficients of f(x) is zero, (x – 1) in a factor of f(x)
(iii) As sum of the coefficient even powers of x is equal to the odd power of x., (x + 1) is also a factor of f(x).
(iv) Then apply Horner's method to get the remaining factors.

49. Use factor theorem.

50. (i) $f(1) = 0$, $f(-1) = 0$ and $f(-2) = 1$.
(ii) Solve for ℓ, m and n.

KEY

Concept Application Level—1,2,3

1. 4	**2.** 2	**9.** 1	**10.** 3
3. 3	**4.** 1	**11.** 2	**12.** 2
5. 2	**6.** 4	**13.** 1	**14.** 1
7. 4	**8.** 4	**15.** 1	**16.** 3
		17. 1	**18.** 3

19. 4	**20.** 2
21. 3	**22.** 1
23. 1	**24.** 1
25. 4	**26.** 4
27. 2	**28.** 3
29. 3	**30.** 3
31. 2	**32.** 1
33. 3	**34.** 4
35. 1	**36.** 3
37. 4	**38.** 1
39. 1	**40.** 1
41. 4	**42.** 1
43. 4	**44.** 2
45. 3	**46.** 2
47. 3	**48.** 2
49. 4	**50.** 3

Concept Application Level—1,2,3

Key points for select questions

1. Use factor theorem.

2. Use remainder theorem.

3. Use remainder theorem.

4. Use remainder theorem.

5. The greatest possible value of n is the HCF of 1278 and 672.

6. Use remainder theorem.

7. Use factor theorem.

8. Use remainder theorem

9. Use factor theorem.

10. $5x^{20} + 7x^{15} + x^9 = x^9(5x^{11} + 7x^6 + 1)$

11. Since the coefficient of x^2 is zero, the sum of the roots is zero.

12. Use factor theorem.

13. Largest possible value of n is the HCF of 1215 and 945.

17. Use division algorithm.

18. Use division algorithm.

19. $(x + 1)$ and $(x - 1)$ are the factors of the given expression.

20. $P(1) = 6, P(-1) = 10$ and $P(0) = 5$.

21. Use division algorithm.

22. $f(0) = 7, f(2) = 9$ and $f(-3) = 49$.

23. Put $x = x - 1$ in $f(x + 1)$ to get $f(x)$.
(i) Write $2x^2 + 7x + 5$ in terms of $x + 1$.
(ii) Replace $x + 1$ by x.
(iii) Apply remainder theorem.

24. (i) $x^2 + px + q = (x - p)(x - q)$.
(ii) Compare the terms in L.H.S and R.H.S

25. (i) $f\left(x - \frac{1}{x}\right) = \left(x - \frac{1}{x}\right)^2 + 2$.
(ii) Replace $\left(x - \frac{1}{x}\right)$ with x.
(iii) Use remainder theorem to obtain remainder.

26. (i) Since the coefficient of x^2 is 0, the sum of the roots is '0'. ⇒ Third root is – 4.
(ii) Apply remainder theorem for $f(x) = (x - 2)^2 (x + 4)$.

27. (i) Let $f(x) = ax^2 + bx + c$. $f(1) = 4; f(2) = 4; f(3) = 0$
(ii) Solve for a, b, and c.

28. (i) Put $x = x - 3$ in $f(x + 3)$ to get $f(x)$.
(ii) Apply factor theorem.

29. (i) Coefficient of x^2 is 0, therefore sum of roots is 0.
∴ Third root = –2.
(ii) Apply factor theorem.
(iii) To obtain the remainder, use the remainder theorem.

30. (i) Take the given polynomial as f(x).
(ii) $f(1) = 0, f(2) = 0$.

31. $\frac{f(3)}{f(2)} = \frac{4}{5}$ and $f(1) = 0$.

32. (i) Let the common factor be $x - a$ and find f(a), and g(a).
(ii) Obtain the value of a in terms of k and ℓ.

33. Dividend = Divisor × Quotient + Remainder.

34. (i) $x^2 + mx - n = (x - m)(x - n)$.
(ii) Equate the corresponding terms.

35. (i) $f\left(x + \frac{1}{x}\right) = \left(x + \frac{1}{x}\right)^2 - 2.$

(ii) Replace $x + \frac{1}{x}$ by x.

(iii) Put x = 1/2.

36. (i) f(3) = 10, f(5) = 14

(ii) Dividend = Divisor × Quotient + Remainder.

39. (i) f(1) = 75, f(–2) = 15.

(ii) Dividend = Divisor × Quotient + Remainder.

40. Verify from the options whether f(1) = 5, f(2) = 7 and f(3) = 9 by using remainder theorem.

41. $\frac{f(2)}{f(1)} = \frac{4}{0}$ and f(–1) = 0.

42. (i) $x^2 - 5x + 6 = (x - 2)(x - 3)$

(ii) f(2) = 0, f(3) = 0.

43. (i) Let the common factor be (x – a), then f(a) = g(a), obtain value of 'a'.

(ii) Substitute value of 'a' in f(x).

44. f(2) = 0 and f(–2)= 0.

45. $x^4 + x^2 + 1 = (x^2 - x + 1)(x^2 + x + 1).$

48. (i) Let the common factor be (x – a), then make f(a) = g(a), and get the value of 'a'.

(ii) Substitute value of 'a' in f(x).

49. Let $f(x) = ax^2 + bx + c$, given f(–1) = 1, f(–2) = 4 and f(–3) = 3.

50. Dividend = Divisor × Quotient + Remainder.