CHAPTER 11

PROPERTIES OF TRIANGLES

11.1 INTRODUCTION

Here, we shall discuss the various properties of tringels.

11.1.1 Sine Formula

In any triangle ABC, the sides are proportional to the sines of the opposite angles, i.e., $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R$; R = circumradius of ΔABC.

11.1.2 Cosine Formula

In any triangle ABC, to find the cosine of an angle in terms of the sides.

$$\therefore \quad \cos A=\frac{b^2+c^2-a^2}{2bc};\ \cos B=\frac{a^2+c^2-b^2}{2ac};\ \cos C=\frac{a^2+b^2-c^2}{2ab}$$

11.1.3 Projection Formula

In any triangle ABC, $a = c\cos B + b\cos C$; $b = a\cos C + c\cos A$; $c = a\cos B + b\cos A$ the sine, cosine and Tangent of the half-anlges in terms of the sides:

(i) $\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$; $\sin\frac{B}{2}=\sqrt{\frac{(s-a)(s-c)}{ac}}$; $\sin\frac{}{2}\ \sqrt{\frac{(s-a)(s-b)}{ab}}$

(ii) $\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$; $\cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ac}}$; $\cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}$

(iii) $\tan\frac{A}{2}=\frac{\sin A/2}{\cos A/2}=\sqrt{\frac{(s-b)(s-c)}{bc}}\div\sqrt{\frac{s(s-a)}{bc}}$; $\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$; $\tan\frac{B}{2}=\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$

and $\tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

11.1.3.1 sin A in terms of the sides of the triangle

$$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} = 2\sqrt{\frac{(s-b)(s-c)}{bc}} \times \sqrt{\frac{s(s-a)}{bc}}$$

$$\Rightarrow \quad \sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} = \frac{2\Delta}{bc}. \text{ Similarly, } \sin B = \frac{2}{ca}\sqrt{s(s-a)(s-b)(s-c)} = -\frac{2\Delta}{ca}.$$

$$\sin C = \frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)} = \frac{2\Delta}{ab}\text{; } \Delta = \text{area of } \Delta\text{ ABC.}$$

11.2 NAPIER'S ANALOGY

In any triangle ABC, $\tan\frac{(A-B)}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$; $\tan\frac{(B-C)}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$; $\tan\frac{(C-A)}{2} = \frac{c-a}{c+a}\cot\frac{B}{2}$

11.2.1 Solution of Triangle

Case 1: When three sides of a triangle are given:

In this case, the following formulae are generally used

(i) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$ (ii) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$

(iii) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)\,(s-c)}{s(s-a)}}$ (iv) $\cos A = \frac{b^2+c^2-a^2}{2bc}$ etc.

Case 2: When two sides and the included angle of the triangle are given:

Let b, c and A be given, then 'a' can be found from the formula $a^2 = b^2 + c^2 - 2bc \cos A$

Now, angle B can be found from the formulae $\cos B = \frac{c^2+a^2-b^2}{2ac}$ or $\sin B = \frac{b\sin A}{a}$ and C from C = 180° – A – B.

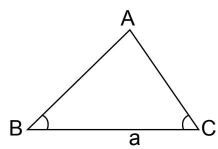
Another way to solve such triangle is, first, to find $\frac{B-C}{2}$ by using the formulae $\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot\frac{A}{2}$ and therefore by addition and subtraction B and C and the third side 'a' by cosine formula, $a^2 = b^2 + c^2 - 2bc \cos A$ or $a = \frac{b\sin A}{\sin B}$ or a = b cos C + c cos B.

Case 3: When two angles and the included side of a triangle are given.

Let angle B, C and side a be given. The angle A can be found from

A = 180° – B – C and the sides b and c from sine rule,

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ i.e., $b = \frac{a\sin B}{\sin A}$ and $c = \frac{a\sin C}{\sin A}$



Case 4: Ambiguous Case:

When two sides (say) a and b and the angle (say) A opposite to one side 'a' are given. There are following three possibilities.

(i) Either there is no such triangle.
(ii) One triangle.
(iii) Two triangles which have the same given elements.

We have, $\frac{b}{\sin B} = \frac{a}{\sin A} \Rightarrow \sin B = \frac{b\sin A}{a}$. ... (1)

Also, $c^2 - 2(b\cos A).c + b^2 - a^2 = 0$ ... (2)

gives $c = b\cos A \pm \sqrt{a^2 - b^2\sin^2 A}$ ... (3)

Now, the following cases may raise

(a) When $a < b \sin A \Rightarrow \sin B > 1$ form equation (1) or from equation (3), c is imaginary which is impossible. Hence, no triangle is possible.

(b) When $b \sin A = a \Rightarrow$ from equation (1),
$\sin B = 1 \Rightarrow B = 90°$ and from equation (3) $c = b \cos A$.
This value of c is admissible only when b cos A is positive, i.e., when the angle A is acute. In such a case $a < b$ ($b \sin A = a$) or $A < B$. Hence, only one definite triangle is possible.

Note:

In this case, $a = b$ is not possible since $A = B = 90°$ which is not possible. Since no triangle can have two right angles.

(c) When $b \sin A < a$ and $\sin B < 1$ from (4). In this case there are three possibilities:

(i) If $a = b$, then $A = B$ and from equation (3), we get $c = 2b \cos A$ or 0. Hence, in this case, we get only one triangle (since in this case, it is must that A and B are acute angles).

(ii) If $a < b$, then $A < B$. Therefore, A must be an acute angle.

$\therefore$ $b \cos A > 0$. Further, $a^2 < b^2$.

$\Rightarrow a^2 < b^2 (\cos^2 A + \sin^2 A)$

$\Rightarrow \sqrt{a^2 - b^2\sin^2 A} < b\cos A$

From equation (3) it is clear that both values of c are positive so we get two triangles such that

and $c_2 = b\cos A - \sqrt{a^2 - b^2 \sin A}$

It is also clear from equation (1) that there are two values of B which are supplementary.

(iii) If $a > b$, then $A > B$ also $a^2 - b^2 \sin^2 A > b^2 \cos^2 A$ or $\sqrt{a^2 - b^2\sin^2 A} > b\cos A$.

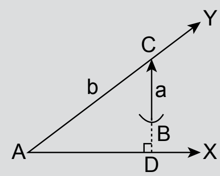
Hence one value of c is positive and other is negative for any value of angle A. Therefore, we get only one solution. Since, for given values of a, b and A, there is a doubt or ambiguity in the determination of the triangle. Hence, this case is called ambiguous case of the solution of triangles.

11.3 GEOMETRIC DISCUSSION

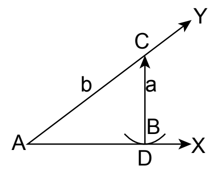
Let a, b and the angle A be given. Draw a line AX. At A, construct angle $\angle XAY = A$. Cut a segment AC = b from AY. Now, describe a circular arc with its centre C and radius a. Also draw CD perpendicular to AX.

$\therefore$ $CD = b \sin A$. The following cases may arise:

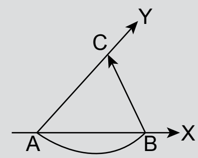
(a) If $a < b \sin A$, i.e., $a < CD$, then the circle will not meet AX and hence, there is no triangle satisfying the given condition



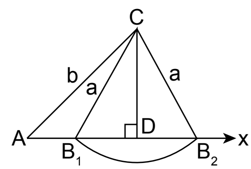
(b) If $a = b \sin A$, the circle will touch AX at D (or B) and only one right angled triangle is possible. In this case:
$B = 90°$ and $A < 90°$



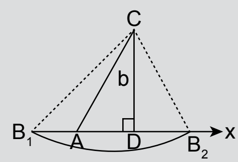
(c) If $a = b$, ($\angle A \neq 90°$) then the circle will cut AX at B and passes through A. Hence, here we get only one solution of given data (as shown in the figure).



(d) If $a > b \sin A$, then the circle will cut AX at two distinct points *(other than A)*. Let the point be B_1 and B_2.
Sub-case 1: If $b \sin A < a < b$, then both B_1 and B_2 are on the same side of A as shown in the figure and we get two distinct triangles ACB_1 and ACB_2.



Sub-case 2: If $a > b$, then the two points B_1 and B_2 are on the opposite sides of A and only one of the triangle ACB_1 or ACB_2 will satisfy the given data. If A is an acute angle, then ΔCAB_2 is the required triangle and if A is obtuse angle, then ΔAB_1C is the required triangle.



11.4 AREA OF TRIANGLE ABC

If Δ represents the area of a triangle ABC, then $\Delta = 1/2\ (BC.AD)$

$$= \frac{1}{2} a (c \sin B) \left(\text{as } \sin B = \frac{AD}{c} \right) = \frac{1}{2} ac \sin B \ \text{ Also } \sin C = \frac{AD}{b} \Rightarrow AD = b \sin C;$$

$$\therefore \quad \Delta = \frac{1}{2}.a.b \sin C; \text{ Similarly } \ \Delta = \frac{1}{2} bc \sin A .$$

$$\therefore \quad \Delta = \frac{1}{2} ab \sin C = \frac{1}{2} bc \sin A = \frac{1}{2} ca \sin B$$

(i) Area of a triangle in terms of sides (Hero's formula):

$$\Delta = \frac{1}{2} bc \sin A = \frac{1}{2} bc.2 \sin\frac{A}{2} \cos\frac{A}{2} = bc \sqrt{\frac{(s-b)(s-c)}{bc}} \sqrt{\frac{s(s-a)}{bc}}$$

$$\Rightarrow \quad \Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

(ii) Area of triangle in terms of one side and sine of three angles:

$$\Delta = \frac{1}{2}bc\sin A = \frac{1}{2}(k\sin B)(k\sin C)\sin A = \frac{1}{2}k^2\sin A\sin B\sin C$$

$$= \frac{1}{2}\left(\frac{a}{\sin A}\right)^2\sin A\sin B\sin C = \frac{a^2}{2}.\frac{\sin B\sin C}{\sin A}$$

Thus, $\Delta = \frac{a^2}{2}.\frac{\sin B\sin C}{\sin A} = \frac{b^2}{2}.\frac{\sin A\sin C}{\sin B} = \frac{c^2}{2}\frac{\sin A\sin B}{\sin C}$

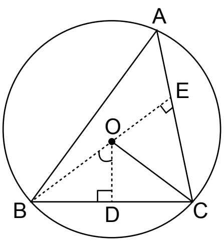
11.5 'M–N' THEOREM

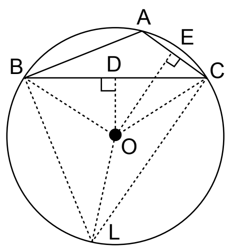
In any triangle ABC, if D is any point on the base BC such that BD : DC :: m : n, $\angle BAD = \alpha$, $\angle CAD = \beta$, $\angle CDA = \theta$, then $(m + n)\cot\theta = m\cot\alpha - n\cot\beta = n\cot B - m\cot C$.

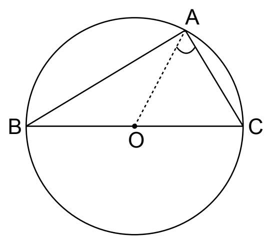
11.5.1 Some Definitions

11.5.1.1 Circumcircle

The circle which passes through the angular points of a triangle is called its circumscribing circle or more briefly circumcircle. The centre of this circle is called circumcentre. Generally, it is denoted by O and its radius always denoted by R. Another property of circum centre is that it is the point of concurrency of perpendicular bisectors of sides of a triangle.







11.5.1.2 Radius of circum circle 'R' of any triangle

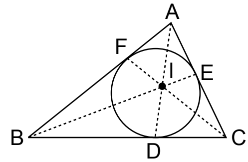
In ΔABC, we have $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

The circumradius may be expressed in terms of sides of the triangle.

$$R = \frac{a}{2\sin A} = \frac{abc}{2bc\sin A} = \frac{abc}{4\Delta} \qquad \left(\because \Delta = \frac{1}{2}bc\sin A\right); \text{Thus } R = \frac{abc}{4\Delta}$$

11.5.1.3 Incircle

The circle which can be inscribed within the triangle so as to touch each of the sides is called its in-**scribed circle** or more briefly its **incircle**. The centre of this circle is called **incentre**. It is denoted by I and its radius always denoted by r. **In-centre is the point of concurrency of internal angles bisectors of the triangle.**



Radius r of the incircle of triangle ABC

Since Δ = Area ΔIBC + ar(Δ ICA) + ar(Δ IAB)

$\Rightarrow$ $\Delta = (1/2)\ ar + (1/2)\ br + (1/2)\ cr = 1/2\ (a + b + c)r$

$\Rightarrow$ $\Delta = sr \Rightarrow r = \Delta/s;\ s = \dfrac{a+b+c}{2}$ = semi-perimeter

The radius of incircle in terms of sides and tangent of the half angle:

$$r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}$$

The radius of incircle in terms of one side and the functions of the half angles:

$$r = \frac{a\sin(B/2)\sin(A/2)}{\cos(A/2)} = \frac{b\sin(B/2)\sin(C/2)}{\cos(B/2)} = \frac{C\sin(A/2)\sin(B/2)}{\cos(C/2)}$$

since a = 2R sinA = 4R sinA/2 cosA/2

$\therefore$ r = 4R sinA/2 sin B/2 sin C/2

11.5.1.4 Escribed circles

The circle which touches the sides BC and two sides AB and AC (produced) of triangle ABC is called **escribed circle** opposite the angle A. The centre of escribed circle is called **ex-centre** and is denoted by I_1 or I_A and radius by r_1 or r_A.

Radii of escribed circles of a triangle: $r_1 = \dfrac{\Delta}{s-a}$, $r_2 = \dfrac{\Delta}{s-b}$, $r_3 = \dfrac{\Delta}{s-c}$

Radii of the Escribed circles in terms of sides and the tangents of half angle:

r_1 = s tan A/2; r_2 = s tan B/2; r_3 = s tan C/2

Radii of the escribed circles in terms of one side and function of half angles:

$$r_1 = \frac{a\cos(B/2)\cos(C/2)}{\cos(A/2)},\ r_2 = \frac{b\cos(C/2)\cos(A/2)}{\cos(B/2)},\ r_3 = \frac{c\cos(A/2)\cos(B/2)}{\cos(C/2)}$$

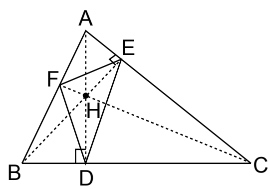
Now, Since a = 2R sin A = 4R sin A/2 cosA/2

$\Rightarrow$ r_1 = 4R sin A/2 cosB/2 cosC/2; r_2 = 4R cosA/2 sinB/2 cos C/2 and
r_3 = 4RcosA/2 cosB/2 sinC/2

11.6 ORTHOCENTRE AND PEDAL TRIANGLE

Let ABC be any triangle and let D, E, F be the feet of the perpendiculars from the angular points on the opposite sides of the triangle ABC, DEF is known as **Pedal Triangle** of ABC.

The three perpendiculars AD, BE and CF always meet in a single point H which is called the **orthocentre of triangle.**



11.6.1 Sides and Angles of the Pedal Triangle

$\angle FDE = 180° - 2A$; $\angle DEF = 180° - 2B$; $\angle DFE = 180° - 2C$

FD = b cos B; DE = c cos C; FE = a cos A

or FD = R sin 2B; DE = R sin 2C; FE = R sin 2A

11.6.1.1 Perimeter of pedal triangle

R(sin 2A + sin 2B + sin 2C) = 4R sinA sinB sinC

Note:

If the angle ACB of the given triangle is obtuse, the expressions 180° – 2C and c cosC are both negative and the values we have obtained, require some modification. In this case, the angles are 2A, 2B, 2C – 180° and the sides are a cosA, b cos B, – c cos C.

Distance of the orthocentre from the angular points of the triangle

AH = 2R cos A;

BH = 2R cos B ; CH = 2R cosC

11.6.1.2 Distances of the orthocentre from the sides of the triangle

HD = 2R cosB cosC; HE = 2R cosA cosC; HF = 2R cosA cosB

Cor. $\frac{AH}{HD} = \frac{2R\cos A}{2R\cos B\cos C} = \frac{\frac{\sin A}{\cos B\cos C}}{\sin A/\cos A} = \frac{\frac{\sin(B+C)}{\cos B\cos C}}{\tan A} = \frac{\tan B + \tan C}{\tan A}$

Area and Circum-radius of the Pedal Triangle

(a) Area of triangle = 1/2 (product of two sides)× (sin of included angle) = 1/2 (Rsin 2B) . (R Sin 2C).

$\sin(180° - 2A)$; $\frac{1}{2} = R^2 \sin 2A \sin 2B \sin 2C$

(b) Circumradius = $\frac{EF}{2\sin FDE} = \frac{R\sin 2A}{2\sin(180° - 2A)} = \frac{R}{2}$.

(c) The in-radius of the Pedal Triangle $DEF = \frac{\text{Area of }(\Delta DEF)}{\text{Semi Perimeter of }\Delta DEF}$

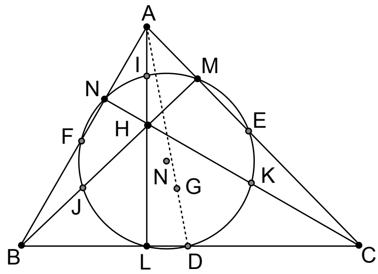
$$= \frac{\frac{1}{2}R^2 \sin 2A \sin 2B \sin 2C}{2R\sin A\sin B\sin C} = 2R\cos A\cos B\cos C$$

11.7 IN-CENTRE OF PEDAL TRIANGLE

Since, HD, HE and HF bisect the angles FDE, DEF and EFD respectively. So that H is the in-centre of the triangle DEF. Thus, the **orthocentre of a triangle is the in-centre of the pedal triangle**.

11.8 CIRCUMCIRCLE OF PEDAL TRIANGLE (NINE-POINT CIRCLE)

The circumcircle of pedal triangle for any ΔABC is called a nine-point circle.

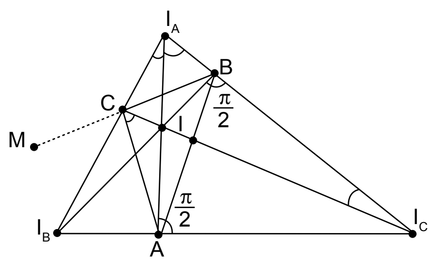


11.8.1 Properties of Nine-point Circle

1. If passes through nine points of triangle L, M, N (feet of altitudes) D, E, F; (mid points of sides) and midpoints of HA, HB, HC, where H is orthocentre of triangle ABC.
2. Its centre is called **nine-points centre** (N). It is the circumcentre of a pedal triangle.
3. Its radius is $R_9 = \frac{1}{2}R$.
4. O (orthocentre), N, G, C (circumcentre) are collinear.
   - N divides OC in ratio 1:1
   - G divides OC in ratio 2:1
5. If circumcentre of triangle be origin and centroid has coordinate (x, y), then coordinate of orthocentre = (3x, 3y); coordinate of nine point centre $= \left(\frac{3x}{2}, \frac{3y}{2}\right)$.

11.9 THE EX-CENTRAL TRIANGLE

Let ABC be a triangle and I be the centre of incircle. Let I_A, I_B, I_C be the centres of the escribed circles which are opposite to A, B and C respectively then I_A, I_B, I_C is called the **ex-central triangle** of Δ ABC. By geometry IC bisects the angle ACB and I_BC bisects the angle ACM.



$\angle ICI_B = \angle ACI + \angle ACI_B = \frac{1}{2} \angle ACB + \frac{1}{2}\angle ACM = \frac{1}{2} \angle (180°) = 90°$

Similarly, $\angle ICI_A = 90°$.

Hence, $I_A\,I_B$ is a straight line perpendicular to IC. Similarly, AI is perpendicular to the straight line I_BI_C and BI is perpendicular to the straight line $I_A\,I_C$.

Also, since IA and I_AA both bisect the angle BAC, hence A, I and I_A are collinear. Similarly, BII_B and CII_C are straight lines.

Hence I_A, $I_B\,I_C$ is a triangle, thus the triangle ABC is the pedal triangle of its ex-central triangle I_A, I_B, I_C The angles IBI_A and ICI_A are right angles, hence the points B, I, C, I_A are concyclic. Similarly, C, I, A, I_B and the points A, I, B, I_C are concyclic.

The lines AI_A, BI_B, CI_C meet at the incentre I, which is therefore, the orthocentre of the ex-central triangle I_A, $I_B\,I_C$.

Remarks:

1. *Each of the four points I, I_A, $I_B\,I_C$ is the orthocente of the triangle formed by joining the other three points.*
2. *The circumcentre, the centroid, the centre of the nine point circle and the orthocentre all lie on* a *straight line.*

11.10 CENTROID AND MEDIANS OF ANY TRIANGLE

In triangle ABC, the midpoint of sides BC, CA and AB are D, E and F respectively. The lines AD, BE and CF are called medians of the triangle ABC, the point of concurrency of three medians is called centroid. Generally, it is represented by G.

By geometry: $AG=\frac{2}{3}AD, BG=\frac{2}{3}BE$ and $CG=\frac{2}{3}CF$.

11.11 LENGTH OF MEDIANS

$$AD=\frac{1}{2}\sqrt{2b^2+2c^2-a^2}\ ,\ BE=\frac{1}{2}\sqrt{2c^2+2a^2-b^2}\ \text{ and }\ CF=\frac{1}{2}\sqrt{2a^2+2b^2-c^2}$$

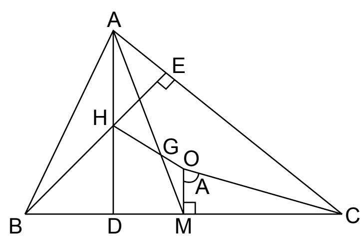
The angles that the median makes with sides

Let $\angle BAD = \beta$ and $\angle CAD = \gamma$, we have $\frac{\sin\gamma}{\sin C}=\frac{DC}{AD}=\frac{a}{2x}$ (Let AD = x);

$$\therefore\quad \sin\gamma=\frac{a\sin C}{2x}=\frac{a\sin C}{\sqrt{2b^2+2c^2-a^2}}\ ;\ \sin\beta=\frac{a\sin B}{\sqrt{2b^2+2c^2-a^2}}\ ;\ \text{Again; }\frac{\sin\theta}{\sin C}=\frac{AC}{AD}=\frac{b}{x}$$

$$\sin\theta=\frac{b\sin C}{x}=\frac{2b\sin C}{\sqrt{2b^2+2c^2-a^2}}$$

11.11.1 The Centroid Lies on the Line Joining the Circumcentre to the Orthocentre



Let O and H represent the circum-centre and orthocenter respectively. OM is perpendicular to BC. Let AM meets HO at G. The two triangles AHG and GMO are equiangular.

AH = 2R cosA and in ΔOMC, OM = RcosA

$$\Rightarrow\quad \frac{AH}{OM}=\frac{2R\cos A}{R\cos A}=2$$

Hence, by similar triangles $\frac{AG}{GM}=\frac{HG}{GO}=\frac{AH}{OM}=2$

$\Rightarrow$ G divides AM in the ratio 2 : 1

Clearly, G is the centroid of ΔABC and G divides HA in the ratio 2 : 1. Thus centroid lies on the line joining the orthocentre to the circum-centre and divides it in the ratio 2 :1.

The distance of the orthocentre from the circum-centre:

$$OH = R\sqrt{1-8\cos A\cos B\cos C}$$

The distance between the incentre and circumcentre:

$$OI = R\sqrt{1-8\sin B/2\sin C/2\sin A/2}$$

The distance of an ex-centre from the circum-centre

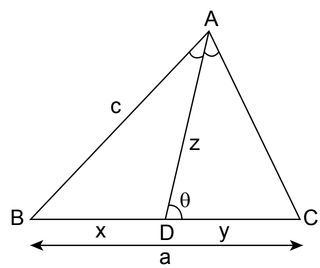
$$OI_1 = R\sqrt{1+8\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}},\ OI_2 = R\sqrt{1+8\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}},$$

$$OI_2 = R\sqrt{1+8\cos(A/2)\cos(B/2)\sin(C/2)}$$

11.11.1.1 The length of angle bisector and the angle that the bisector makes with the sides

Let AD be the bisector of angle A and x and y be the portions of base BC. From geometry $\frac{BD}{DC}=\frac{AB}{AC}$

or $\frac{x}{c}=\frac{y}{b}=\frac{x+y}{b+c}=\frac{a}{b+c}$



$\therefore\quad x=\frac{ac}{b+c}$ and $y=\frac{ab}{b+c}$ ..(i)

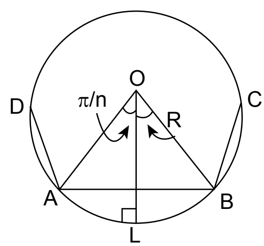
Further $\Delta ABC = \Delta ABD + \Delta ADC$

$\Rightarrow\ \frac{1}{2}bc\sin A=\frac{1}{2}cz\sin\frac{A}{2}+\frac{1}{2}bz\sin\frac{A}{2}$

$z=\left(\frac{bc}{b+c}\right)\frac{\sin A}{\sin A/2}=\left(\frac{2bc}{b+c}\right)=\cos A/2$ ..(ii);

Also $\theta = \angle BAD + B = A/2 + B$

The Perimeter and Area of a Regular Polygon of n-sides Inscribed in a circle of radius r

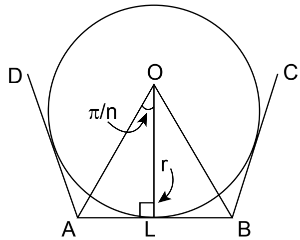


Perimeter of polygon = nAB = 2nR sin π/n

Area of polygon = n(Area of triangle AOB) = $\frac{nR^2}{2}\sin\frac{2\pi}{n}$

The Perimeter & Area of Regular Polygon of n-sides Circumscribed about a given circle of radius 'r'

Perimeter of Polygon = n AB = 2n AL = $2nOL\tan\frac{\pi}{n} = 2n\pi\tan\frac{\pi}{n}$



Area of Polygon = n(Area of triangle AOB) = $n\dfrac{(OL.AB)}{2} = nr^2 \tan\dfrac{\pi}{n}$

The Radii of the inscribed and circumscribing circles of a regular polygon having n sides each of length 'a'.

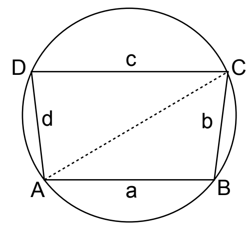
$$R = \frac{a}{2\sin \pi/n} = \frac{a}{2}\operatorname{cosec}\frac{\pi}{n}; \qquad r = \frac{a}{2\tan\pi/n} = \frac{a}{2}\cot\frac{\pi}{n}$$

11.12 RESULT RELATED TO CYCLIC QUADRILATRAL

(a) **Ptolemy's Theorem:** In a cyclic quadrilateral ABCD, AC.BD = AB.CD + BC.DA, i.e., the product of diagonals is equal to the sum of product of opposite sides.

(b) D = area of cyclic quadrilateral

$$= \frac{1}{2}(ab+cd)\sin B = \sqrt{(s-a)(s-b)(s-c)(s-d)}\ ; \text{ where } \frac{a+b+c+d}{2}.$$



(c) $AC = \sqrt{\dfrac{(ac+bd)(ad+bc)}{(ab+cd)}}$

(d) Circum-radius (R) of cyclic-quadrilateral

$$ABCD = \frac{AC}{2\sin B} = \frac{AC}{2\left(\dfrac{2\Delta}{ab+cd}\right)} = \frac{(ab+cd)AC}{4A}$$

$$\therefore\ R = \frac{1}{4}\sqrt{\frac{(ac+bd)(ad+bc)(ab+cd)}{(s-a)(s-b)(s-c)(s-d)}}$$

(e) $\cos B = \dfrac{a^2+b^2-c^2-d^2}{2(ab+cd)}$