Determinants

Determinants of Matrices upto Order 3

Key Concepts

- To every square matrix $A = [a_{ij}]$ of order n, we can associate a number (real or complex) called the determinant of the square matrix A.
- The determinant of a matrix A is denoted by $|A|$ or det A or Δ.
- If M is the set of square matrices, K is the set of numbers (real or complex) and $f: M \to K$ is defined by $f(A) = k$, where $A \in M$ and $k \in K$, then $f(A)$ is called the determinant of A.

Calculation of Determinants

- Let $A = [a]$ be the matrix of order 1. Accordingly, the determinant of A is defined to be equal to a.
- Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a matrix of order 2 × 2. Accordingly, the determinant of A is defined as det A = $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\, a_{22} - a_{21}\, a_{12}$

To understand how to find the determinant of a 2 × 2 matrix with the help of an example,

To understand how to find the determinant of a 3 × 3 matrix,

- The determinant of a matrix A can be obtained by expanding along any other row or any other column also.
- For easier calculation, we expand the determinant along the row or column that contains the maximum number of zeroes.
- If $A = kB$, where A and B are square matrices of order n, then $|A| = k^n\, |B|$, where n = 1, 2, 3.

Solved Examples

Example 1

Evaluate $\begin{vmatrix} -5 & 4 & 2 \\ 3 & 1 & 6 \\ 2 & 5 & 3 \end{vmatrix}$.

Solution:

$$\Delta = \begin{vmatrix} -5 & 4 & 2 \\ 3 & 1 & 6 \\ 2 & 5 & 3 \end{vmatrix}$$

= −5 (1 × 3 − 5 × 6) − 4 (3 × 3 − 2 × 6) + 2 (3 × 5 − 2 × 1)

= −5 (3 − 30) − 4 (9 − 12) + 2(15 − 2)

= −5(−27) − 4(−3) + 2(13)

= 135 + 12 + 26

= 173

Example 2

Find the value of a if $\begin{vmatrix} a & -3 \\ 2 & 4 \end{vmatrix} = \begin{vmatrix} 1 & -a \\ 2 & 5 \end{vmatrix}$.

Solution:

We have

$$\begin{vmatrix} a & -3 \\ 2 & 4 \end{vmatrix} = \begin{vmatrix} 1 & -a \\ 2 & 5 \end{vmatrix}$$

⇒ a × 4 − 2 × (−3) = 1 × 5 −2 × (−a)

⇒ 4a + 6 = 5 + 2a

⇒ 4a − 2a = 5 − 6

⇒ 2a = −1

⇒ a = $-\frac{1}{2}$

Example 3

Find the equation which x satisfies if $\begin{vmatrix} x & x+1 \\ 2 & -3x \end{vmatrix} = 4$.

Solution:

We have

$$\begin{vmatrix} x & x+1 \\ 2 & -3x \end{vmatrix} = 4$$

$\Rightarrow x \times (-3x) - 2 \times (x + 1) = 4$

$\Rightarrow -3x^2 - 2x - 2 = 4$

$\Rightarrow -3x^2 - 2x - 6 = 0$

$\Rightarrow 3x^2 + 2x + 6 = 0$

This is the required equation that *x* satisfies.

Properties of Determinants

- The value of the determinant remains unchanged if its rows and columns are interchanged.
- The interchange of two rows or columns can be symbolically written as $C_i \leftrightarrow R_i$, where R_i is the i^{th} row and C_i is the i^{th} column.
- For example, $\begin{vmatrix} -9 & 8 & 1 \\ -5 & 5 & 3 \\ -1 & 2 & 4 \end{vmatrix} = \begin{vmatrix} -9 & -5 & -1 \\ 8 & 5 & 2 \\ 1 & 3 & 4 \end{vmatrix}$
- If *A* is a square matrix, then det (A) = det (A′) i.e., $|A| = |A'|$
- If any two rows (or columns) of a determinant are interchanged, then sign of determinant changes.
- The interchange of two rows is written as $R_i \leftrightarrow R_j$ and the interchange of two columns is written as $C_i \leftrightarrow C_j$.
- For example,
$$\begin{vmatrix} 1 & 1 & -2 \\ 8 & 6 & -3 \\ 5 & 4 & -9 \end{vmatrix} = -\begin{vmatrix} 5 & 4 & -9 \\ 8 & 6 & -3 \\ 1 & 1 & -2 \end{vmatrix}, \text{ by applying } R_1 \leftrightarrow R_3$$

- If any two rows (or columns) of a determinant are identical or proportional, then the value of the determinant is zero.

For example, $\begin{vmatrix} 4 & -9 & 11 \\ 12 & 3 & 13 \\ 8 & -18 & 22 \end{vmatrix} = 2\begin{vmatrix} 4 & -9 & 11 \\ 12 & 3 & 13 \\ 4 & -9 & 11 \end{vmatrix} = 0$

- If each element of a row (or a column) of determinant is multiplied by a constant α, then its determinant value gets multiplied by α. For example:

Consider $A = \begin{vmatrix} -6 & 2 & 8 \\ 3 & 1 & 2 \\ -1 & 4 & 7 \end{vmatrix}$

Now, $|A| = 64$

$$|3A| = \begin{vmatrix} -6 & 2 & 8 \\ 9 & 3 & 6 \\ -1 & 4 & 7 \end{vmatrix} = 192 = 3 \times 64 = 3|A|$$

- If some or all elements of a row or a column in a determinant can be expressed as the sum of two (or more) elements, then the determinant can be expressed as the sum of two (or more) determinants.
For example,

- If to each element of any row or column of a determinant, the equimultiples of corresponding elements of other row or column are added, then the value of the determinant remains unchanged.

- In other words, the value of the determinant remains same, if we apply the operation $R_i \rightarrow R_i + kR_j$ or $C_i \rightarrow C_i + kC_j$. That is,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + \lambda a_3 & b_1 + \lambda b_3 & c_1 + \lambda c_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Solved Examples

Example 1: Evaluate: $\begin{vmatrix} 270 & 255 & 234 \\ 255 & 230 & 213 \\ 234 & 213 & 186 \end{vmatrix}$

Solution:

$$\Delta = \begin{vmatrix} 270 & 255 & 234 \\ 255 & 230 & 213 \\ 234 & 213 & 186 \end{vmatrix}$$

By applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$, we obtain

$$\Delta = \begin{vmatrix} 36 & 21 & 234 \\ 42 & 17 & 213 \\ 48 & 27 & 186 \end{vmatrix}$$

By applying $C_1 \to C_1 - 2C_2$ and $C_3 \to C_3 - 10C_2$, we obtain

$$\Delta = \begin{vmatrix} -6 & 21 & 24 \\ 8 & 17 & 43 \\ -6 & 27 & -84 \end{vmatrix}$$

By applying $R_3 \to R_3 - R_1$ and $R_2 \to R_2 + \frac{4}{3}R_1$, we obtain

$$\Delta = \begin{vmatrix} -6 & 21 & 24 \\ 0 & 45 & 75 \\ 0 & 6 & -108 \end{vmatrix} = -54\begin{vmatrix} 1 & 7 & 8 \\ 0 & 15 & 25 \\ 0 & 2 & -36 \end{vmatrix} = -54(-540-50) = 31860$$

Example 2: Prove that: $\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2 \\ 10x+8y & 8x & 3 \end{vmatrix} = x^2$

Solution:

$$\Delta = \begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2 \\ 10x+8y & 8x & 3 \end{vmatrix}$$

$$= \begin{vmatrix} x & x & x \\ 5x & 4x & 2 \\ 10x & 8x & 3 \end{vmatrix} + \begin{vmatrix} y & x & x \\ 4y & 4x & 2 \\ 8y & 8x & 3 \end{vmatrix}$$

$$= \begin{vmatrix} x & x & x \\ 5x & 4x & 2 \\ 10x & 8x & 3 \end{vmatrix} + xy\begin{vmatrix} 1 & 1 & x \\ 4 & 4 & 2 \\ 8 & 8 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} x & x & x \\ 5x & 4x & 2 \\ 10x & 8x & 3 \end{vmatrix} + 0$$

$$= \begin{vmatrix} x & x & x \\ 5x & 4x & 2 \\ 10x & 8x & 3 \end{vmatrix}$$

By applying $R_3 \to R_3 - 2R_2$, we obtain

$$\Delta = \begin{vmatrix} x & x & x \\ 5x & 4x & 2 \\ 0 & 0 & -1 \end{vmatrix}$$

By applying $R_2 \to R_2 - 4R_1$, we obtain

$$\Delta = \begin{vmatrix} x & x & x \\ x & 0 & 2-4x \\ 0 & 0 & -1 \end{vmatrix}$$

By expanding along R_3, we obtain

$$\Delta = (-1)\times(0-x^2) = x^2$$

Hence proved.

Area of Triangle

Area of Triangle

- The area of a triangle whose vertices are (x_1, y_1), (x_2, y_2), and (x_3, y_3) is given by the determinant:

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

- The area of a triangle is always positive, therefore, the absolute value of the above mentioned determinant is always taken.
- If the area of a triangle is given, then both positive and negative values of the determinant are used for calculation.
- The area of the triangle formed by three collinear points is zero.

Solved Examples

Example 1: Using determinants, find the equation of the line joining the points $\left(\frac{1}{2}, \frac{1}{3}\right)$ and (−4, 5).

Solution:

Let the given points be A $\left(\frac{1}{2}, \frac{1}{3}\right)$ and B (−4, 5).

Let P(x, y) be the point lying on the line AB. Then, the points A, P, and B will be collinear and hence, the area of the triangle formed by joining these three points is zero.

$$\therefore \frac{1}{2}\begin{vmatrix} \frac{1}{2} & \frac{1}{3} & 1 \\ -4 & 5 & 1 \\ x & y & 1 \end{vmatrix} = 0$$

$$\Rightarrow \frac{1}{2}\begin{vmatrix} 5 & 1 \\ y & 1 \end{vmatrix} - \frac{1}{3}\begin{vmatrix} -4 & 1 \\ x & 1 \end{vmatrix} + 1\begin{vmatrix} -4 & 5 \\ x & y \end{vmatrix} = 0$$

$$\Rightarrow \frac{1}{2}(5-y) - \frac{1}{3}(-4-x) + 1(-4y-5x) = 0$$

$$\Rightarrow \frac{5}{2}-\frac{1}{2}y+\frac{1}{3}x+\frac{4}{3}-4y-5x=0$$

$$\Rightarrow x\left(\frac{1}{3}-5\right)+y\left(-\frac{1}{2}-4\right)+\left(\frac{5}{2}+\frac{4}{3}\right)=0$$

$$\Rightarrow -\frac{14x}{3}-\frac{9y}{2}+\frac{23}{6}=0$$

$$\Rightarrow \frac{14}{3}x+\frac{9y}{2}-\frac{23}{6}=0$$

$28x + 27y - 23 = 0$

This is the required equation of the line joining the two given points.

Example 2: Find the value of k, if the area of the triangle formed by joining the vertices (5, 5), (k, 1), and (3, −1) is 16 square units.

Solution:

The area of the triangle formed by joining the vertices (5, 5), (k, 1), and (3, −1) is given by,

$$\frac{1}{2}\begin{vmatrix} 5 & 5 & 1 \\ k & 1 & 1 \\ 3 & -1 & 1 \end{vmatrix}$$

$$=\frac{1}{2}\left[5(1+1)-5(k-3)+1(-k-3)\right]$$

$$=\frac{1}{2}\left[5(2)-5(k-3)+1(-k-3)\right]$$

$$=\frac{1}{2}\left[10-5k+15-k-3\right]$$

$$=\frac{1}{2}\left[-6k+22\right]=-3k+11$$

It is given that the area of the triangle is 16 square units. Therefore, we have

$-3k + 11 = \pm 16$

$\Rightarrow -3k = 16 - 11$ or $-3k = -16 - 11$

$\Rightarrow -3k = 5$ or $-3k = -27$

$\Rightarrow k = -\frac{5}{3}$ or $k = 9$

Thus, the value of k is $-\frac{5}{3}$ or 9.

Minors and Cofactors

Minors

- Minor of an element a_{ij} of a determinant is the determinant obtained by deleting its i^{th} row and j^{th} column in which the element a_{ij} lies. Minor of an element a_{ij} is denoted by M_{ij}.
- Minor of an element of a determinant of order n ($n \geq 2$) is a determinant of order ($n - 1$).
- For example, the minor of the element a_{22} in the determinant

$$\begin{vmatrix} 10 & 2 & -8 \\ 11 & 21 & 6 \\ 3 & 9 & 5 \end{vmatrix}$$ is given by, $$M_{22} = \begin{vmatrix} 10 & -8 \\ 3 & 5 \end{vmatrix} = 10 \times 5 - 3 \times (-8) = 50 + 24 = 74$$

Cofactors

- Cofactor of an element a_{ij}, denoted by A_{ij}, is defined by $A_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is the minor of a_{ij}.

For example, the cofactor of the element a_{23} in the determinant

$$\begin{vmatrix} -1 & 2 & -1 \\ 3 & 5 & 4 \\ 5 & -2 & -3 \end{vmatrix}$$ is given by,

$$A_{23} = (-1)^{2+3} = M_{23} = -1[2 - 10] = -1 \times -8 = 8$$

- The value of a determinant is equal to the sum of the product of elements of any row (or column) with their corresponding cofactors. That is,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$$

- If elements of a row (or column) are multiplied with cofactors of any other row (or column), then their sum is zero. That is, $a_{12}A_{13} + a_{22} A_{23} + a_{32} A_{33} = 0$
- To understand how to find the minor and cofactor of an element in a better way.

Solved Examples

Example 1: Find the minors and cofactors of the elements of the determinant: $\begin{vmatrix} x & x+y \\ 2x+y & -x \end{vmatrix}$

Solution:

The given determinant is $\begin{vmatrix} x & x+y \\ 2x+y & -x \end{vmatrix}$.

$M_{11} = -x$ $A_{11} = (-1)^{1+1} (-x) = -x$

$M_{12} = 2x + y$ $A_{12} = (-1)^{1+2} (2x + y) = -(2x + y)$

$M_{21} = x + y$ $A_{21} = (-1)^{2+1} (x + y) = -(x + y)$

$M_{22} = x$ $A_{22} = (-1)^{2+2} (x) = x$

Example 2: Using cofactors of elements of second column, evaluate: $\begin{vmatrix} 2 & 1 & 6 \\ -3 & 4 & 9 \\ 5 & 8 & -2 \end{vmatrix}$

Solution:

The given determinant is $\Delta = \begin{vmatrix} 2 & 1 & 6 \\ -3 & 4 & 9 \\ 5 & 8 & -2 \end{vmatrix}$

$M_{12} = 6 - 45 = -39$ $A_{12} = (-1)^{1+2} M_{12} = -1 \times -39 = 39$

$M_{22} = -4 - 30 = -34$ $A_{22} = (-1)^{2+2} M_{22} = -34$

$M_{32} = 18 + 18 = 36$ $A_{32} = (-1)^{3+2} M_{32} = (-1) \times 36 = -36$

We know that the value of a determinant is equal to the sum of the product of elements of any column (or row) with their corresponding cofactors.

$\therefore \Delta = a_{12} A_{12} + a_{22} A_{22} + a_{32} A_{32}$

$= 1 \times 39 + 4 \times (-34) + 8 \times (-36)$

$= 39 - 136 - 288$

$= -385$

Thus, the value of the given determinant is −385.

Adjoint and Inverse of a Matrix

Adjoint of a Matrix

- The adjoint of a square matrix $A = [a_{ij}]_{n \times n}$ is defined as the transpose of the matrix $[A_{ij}]_{n \times n}$, where A_{ij} is the cofactor of the element a_{ij}. Adjoint of the matrix A is denoted by *adj A*.
- For the matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, the adjoint is given by:

  $adj\ A = \text{Transpose of } \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

- For a square matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ of order 2, the *adj A* can be obtained by interchanging a_{11} and a_{22} and by changing the signs of a_{12} and a_{21}.

  $adj\ A = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

- If A is a square matrix of order n, then

  $A\,(adj\ A) = (adj\ A)\,A = |A|\,I$, where I is the identity matrix of order n.

- A square matrix A is said to be singular if $|A| = 0$, otherwise A is said to be non-singular (i.e., if $|A| \neq 0$).
- If A and B are non-singular matrices of the same order, then AB and BA are also non-singular matrices of the same order.

- The determinant of the product of matrices is equal to the product of their respective determinants. That is, $|AB| = |A|.|B|$, where A and B are square matrices of the same order.

- If A is a non-singular matrix of order n, then $|adj\ A| = |A|^{n-1}$

Inverse of a Matrix

- A square matrix A is invertible if and only if A is a non-singular matrix. If A is an invertible matrix of order n, then there exists a square matrix B of order n, such that $AB = BA = I$, where I is the identity matrix of order n. Here, matrix B is called the inverse of matrix A and vice-versa (written as $A^{-1} = B$ and $B^{-1} = A$).

  It can be noted that $(A^{-1})^{-1} = A$

  The matrix $\begin{bmatrix} 4 & 8 \\ 3 & 6 \end{bmatrix}$ is not invertible, since $\begin{vmatrix} 4 & 8 \\ 3 & 6 \end{vmatrix} = 24 - 24 = 0$.

- The inverse of an invertible square matrix A is given by $A^{-1} = \frac{1}{|A|} \cdot (adj\ A)$

- If A and B are two square matrices of the same order, then $(AB)^{-1} = B^{-1} A^{-1}$

Solved Examples

Example 1:

If $A = \begin{bmatrix} 12 & 4 \\ 10 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -5 \\ 6 & -7 \end{bmatrix}$, then verify that $(AB)^{-1} = B^{-1} A^{-1}$

Solution:

We have, $AB = \begin{bmatrix} 12 & 4 \\ 10 & -2 \end{bmatrix}\begin{bmatrix} 3 & -5 \\ 6 & -7 \end{bmatrix} = \begin{bmatrix} 36+24 & -60-28 \\ 30-12 & -50+14 \end{bmatrix} = \begin{bmatrix} 60 & -88 \\ 18 & -36 \end{bmatrix}$

Now, $|AB| = 60 \times (-36) - 18 \times (-88) = -2160 + 1584 = -576 \neq 0$

Therefore, inverse of AB exists.

$adj\ (AB) = \begin{bmatrix} -36 & 88 \\ -18 & 60 \end{bmatrix}$

$\therefore (AB)^{-1} = \frac{1}{|AB|} adj\,(AB) = \frac{-1}{576}\begin{bmatrix} -36 & 88 \\ -18 & 60 \end{bmatrix}$

Now, $|A| = 12 \times (-2) - 10 \times 4 = -24 - 40 = -64 \neq 0$

$|B| = 3 \times (-7) - 6 \times (-5) = -21 + 30 = 9 \neq 0$

$\therefore A^{-1}$ and B^{-1} exists.

$A^{-1} = \frac{1}{|A|} adj\, A = \frac{1}{-64}\begin{bmatrix} -2 & -4 \\ -10 & 12 \end{bmatrix}$

$B^{-1} = \frac{1}{|B|} adj\, B = \frac{1}{9}\begin{bmatrix} -7 & 5 \\ -6 & 3 \end{bmatrix}$

$\therefore B^{-1} A^{-1} = \frac{-1}{64 \times 9}\begin{bmatrix} -7 & 5 \\ -6 & 3 \end{bmatrix}\begin{bmatrix} -2 & -4 \\ -10 & 12 \end{bmatrix} = \frac{-1}{576}\begin{bmatrix} 14-50 & 28+60 \\ 12-30 & 24+36 \end{bmatrix} = \frac{-1}{576}\begin{bmatrix} -36 & 88 \\ -18 & 60 \end{bmatrix}$

Hence, $(AB)^{-1} = B^{-1} A^{-1}$

Example 2:

If $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & 1 \\ -2 & 2 & -3 \end{bmatrix}$, then find A^{-1}.

Solution:

For the given matrix A, we have $|A| = 1(3 - 2) - 2(-9 + 2) + 1(6 - 2) = 1 + 14 + 4 = 19$

Now, $A_{11} = 1$, $A_{12} = 7$, $A_{13} = 4$,

$A_{21} = 8$, $A_{22} = -1$, $A_{23} = -6$,

$A_{31} = 3$, $A_{32} = 2$, $A_{33} = -7$

$\therefore adj\, A = \begin{bmatrix} 1 & 8 & 3 \\ 7 & -1 & 2 \\ 4 & -6 & -7 \end{bmatrix}$

Thus, $A^{-1} = \frac{1}{|A|} adj\ A = \frac{1}{19}\begin{bmatrix} 1 & 8 & 3 \\ 7 & -1 & 2 \\ 4 & -6 & -7 \end{bmatrix}$

Consistency of System of Linear Equations

- A system of equations is said to be consistent, if its solution (one or more) exists.
- A system of equations is said to be inconsistent, if its solution does not exist.
- A system of linear equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

can be written as $AX = B$, where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

For the system of equations $AX = B$, we have the following possibilities:

- If $|A| \neq 0$, i.e. if A is non-singular, then there exists a unique solution.
- If $|A| = 0$, we calculate $(adj\ A)\ B$.

  If $(adj\ A)\ B \neq O$, (O being zero matrix), then solution does not exist and the system of equations is inconsistent.

  If $(adj\ A)\ B = O$, then system may be either consistent or inconsistent depending on whether the system has either infinitely many solutions or no solution.

- To understand how to check whether the given system of equations is consistent or inconsistent.

Solved Examples

Example 1:

Determine whether the solution of the following system of equations exists or not.

$12x - 9y - 7 = 0$

$-4x + 3y + 11 = 0$

Solution:

The given system of linear equations is

$12x - 9y - 7 = 0$ or $12x - 9y = 7$

$-4x + 3y + 11 = 0$ or $-4x + 3y = -11$

This system can be written as $AX = B$, where

$$A = \begin{bmatrix} 12 & -9 \\ -4 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, \text{and } B = \begin{bmatrix} 7 \\ -11 \end{bmatrix}$$

Now, $|A| = 12 \times 3 - (-9) \times (-4) = 36 - 36 = 0$

$\therefore A$ is a singular matrix.

Now, we calculate $(adj\ A)\ B$.

$$adj\ A = \begin{bmatrix} 3 & 9 \\ 4 & 12 \end{bmatrix}$$

$$\therefore (adj\ A)\ B = \begin{bmatrix} 3 & 9 \\ 4 & 12 \end{bmatrix} \begin{bmatrix} 7 \\ -11 \end{bmatrix} = \begin{bmatrix} -78 \\ -104 \end{bmatrix} \neq O$$

Hence, the given system of equations is inconsistent.

Example 2:

Examine the consistency of the system of following equations:

$2x - y + 3z = 0$

$5x + 2y + z = 7$

$x - 6y - 2z = 9$

Solution:

The given system of linear equations can be written as $AX = B$, where

$$A=\begin{bmatrix}2 & -1 & 3\\ 5 & 2 & 1\\ 1 & -6 & -2\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}, \text{and } B=\begin{bmatrix}0\\ 7\\ 9\end{bmatrix}$$

We have,

$$|A| = 2[-4+6]+1[-10-1]+3[-30-2] = 2\times 2+1\times(-11)+3\times(-32) = 4-11-96$$
$$= 4-107 = -103 \neq 0$$

$\therefore |A| \neq 0$ i.e., A is a non-singular matrix.

Hence, the given system of equations is consistent.

Solution of System of Linear Equations

- $a_1x+b_1y+c_1z=d_1$
  $a_2x+b_2y+c_2z=d_2$
  $a_3x+b_3y+c_3z=d_3$

  A system of above linear equations, which can also be written in the form of $AX = B$, can be solved by using matrix method.

- If A is non-singular, i.e. if $|A| \neq 0$, then its inverse exists and the unique solution of the system of equations $AX = B$ is given by $X = A^{-1}B$.

- If A is singular, i.e. $|A| = 0$, then in this case $(adjA)\cdot B$ is calculated and the following cases arise:

- If $(adjA)\cdot B \neq 0$ (0, being null matrix), then the solution of the system of equations AX =B does not exist.

- If $(adjA)\cdot B = 0$, then the system of equations may either have infinitely many solutions or no solution.

Solved Examples

Example 1: Solve the following system of equations by matrix method.

$4x - 3y = 17$

$-7x - 8y = 10$

Solution:

The given system of linear equations can be written in the form $AX = B$, where

$$A = \begin{bmatrix} 4 & -3 \\ -7 & -8 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, \text{ and } B = \begin{bmatrix} 17 \\ 10 \end{bmatrix}$$

Now, $|A| = 4\times(-8)-(-3)\times(-7) = -32-21 = -53 \neq 0$

Hence, A is a non-singular matrix. Therefore, the given system of linear equations has a unique solution which is given by,

$X = A^{-1}B$

$$A^{-1} = \frac{1}{|A|}\cdot(adjA) = \frac{1}{-53}\begin{bmatrix} -8 & 3 \\ 7 & 4 \end{bmatrix}$$

$$\therefore X = \begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{53}\begin{bmatrix} -8 & 3 \\ 7 & 4 \end{bmatrix}\begin{bmatrix} 17 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{53}\begin{bmatrix} -8\times17+3\times10 \\ 7\times17+4\times10 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{53}\begin{bmatrix} -136+30 \\ 119+40 \end{bmatrix} = -\frac{1}{53}\begin{bmatrix} -106 \\ 159 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Thus, the solution of the given system of linear equations is $x = 2$ and $y = -3$.

Example 2: The cost of 2 burgers, 1 packet of French fries, and 1 cold drink is Rs 90. The cost of 1 burger, 2 packets of French fries, and 1 cold drink is Rs 105. The cost of 3 burgers, 1 packet of French fries, and 2 cold drinks is Rs 125. Find the cost of one burger, one packet of French fries, and 1 cold drink by matrix method.

Solution:

Let the respective cost of one burger, one packet of French fries, and one cold drink be x, y, and z.

According to the given information, we obtain

$2x + y + z = 90$

$x + 2y + z = 105$

$3x + y + 2z = 125$

This system can be written as $AX = B$, where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \text{ and } B = \begin{bmatrix} 90 \\ 105 \\ 125 \end{bmatrix}$$

$$|A| = 2(4-1) - 1(2-3) + 1(1-6) = 6 + 1 - 5 = 7 - 5 = 2 \neq 0$$

Now,

$$A_{11} = 4 - 1 = 3,\ A_{12} = -(2-3) = 1,\ A_{13} = 1 - 6 = -5$$
$$A_{21} = -(2-1) = -1,\ A_{22} = 4 - 3 = 1,\ A_{23} = -(2-3) = 1$$
$$A_{31} = 1 - 2 = -1,\ A_{32} = -(2-1) = -1,\ A_{33} = 4 - 1 = 3$$

$$adjA = \begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ -5 & 1 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}(adjA) = \frac{1}{2}\begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ -5 & 1 & 3 \end{bmatrix}$$

Now, $X = A^{-1}B$

Therefore, we obtain

$$X=\begin{bmatrix}x\\y\\z\end{bmatrix}=\frac{1}{2}\begin{bmatrix}3 & -1 & -1\\1 & 1 & -1\\-5 & 1 & 3\end{bmatrix}\begin{bmatrix}90\\105\\125\end{bmatrix}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\frac{1}{2}\begin{bmatrix}270-105-125\\90+105-125\\-450+105+375\end{bmatrix}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\frac{1}{2}\begin{bmatrix}40\\70\\30\end{bmatrix}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}20\\35\\15\end{bmatrix}$$

$\therefore x = 20, y = 35, z = 15$

Thus, the respective costs of one burger, one packet of French fries, and one cold drink are Rs 20, Rs 35, and Rs 15.