7 Transform Theory

7.1 Laplace Transform

The Laplace transform method solve differential equations and corresponding initial and boundary value problems. The process of solution consists of three main steps:

1st step: The given "hard" problem is transformed into a "simple" equation **(subsidiary equation).**

2nd step: The subsidiary equation is solved by purely algebraic manipulations.

3rd step: The solution of the subsidiary equation is transformed back to obtain the solution of the given problem.

In this way Laplace transforms reduce the problem of solving a differential equation to an algebraic problem. This process is made easier by tables of functions and their transforms, whose role is similar to that of integral tables in calculus.

This switching from operations of calculus to algebraic operations on transforms is called **operational calculus,** a very important area of applied mathematics, and for the engineer, the Laplace transform method is practically the most important operation method. It is particularly useful in problems where the mechanical or electrical driving method. It is particularly useful in problems where the mechanical or electrical driving force has discontinuities, is impulsive or is a complicated periodic function, not merely a sine or cosine. Another operational method is the Fourier transform.

The Laplace transform also has the advantage that it solve initial value problems directly, without first determining a general solution. It also solves nonhomogeneous differential equations directly without first solving the corresponding homogeneous equation.

System of ODES and partial differential equations can also be treated by Laplace transforms.

7.2 Definition

Let $f(t)$ be a function of t defined for all positive values of t. Then the Laplace transforms of $f(t)$, denoted by $L\{f(t)\}$ is defined by

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)dt \qquad \text{... (i)}$$

provided that the integral exists, s is a parameter which may be a real or complex number.

$L\{f(t)\}$ being clearly a function of s is briefly written as $\bar{f}(s)$ or as $F(s)$.

i.e. $$L\{f(t)\} = \bar{f}(s),$$

which can also be written as $$f(t) = L^{-1}\{\bar{f}(s)\}$$

Then $f(t)$ is called the inverse Laplace transform of $\bar{f}(s)$. The symbol L. Which transforms $f(t)$ into $\bar{f}(s)$, is called the Laplace transformation operator.

Example:

If $$f(t) = 1$$

$$L[f(t)] = \int e^{-st}\cdot 1dt = \left[\frac{e^{-st}}{-s}\right]_0^{\infty} = \frac{e^{-\infty} - e^0}{-s} = \frac{1}{s}$$

Similarly Laplace transforms of other common functions can also be evaluated and is shown below:

7.3 Transforms of Elementary Functions

The direct application of the definition gives the following formulae:

1. $L(1) = \frac{1}{s}$ $(s > 0)$

2. $L(t^n) = \frac{n!}{s^{n+1}}$, when $n = 0, 1, 2, 3, \ldots$ $\left[\text{otherwise } \frac{\Gamma(n+1)}{s^{n+1}}\right]$

3. $L(e^{at}) = \frac{1}{s-a}$ $(s > a)$

4. $L(\sin at) = \frac{a}{s^2 + a^2}$ $(s < 0)$

5. $L(\cos at) = \frac{s}{s^2 + a^2}$ $(s > 0)$

6. $L(\sinh at) = \frac{a}{s^2 - a^2}$ $(s > |a|)$

7. $L(\cosh at) = \frac{s}{s^2 - a^2}$ $(s > |a|)$

7.4 Properties of Laplace Transforms

7.4.1 Linearity Property

If a, b, c be any constants and f, g, h any functions of t, then

$$L[af(t) + bg(t) - ch(t)] = a\,L\{f(t)\} + bL[g(t)] - cL\{h(t)\}$$

7.4.2 First Shifting Property

If $L\{f(t)\} = \bar{f}(s)$, then

$$L\{e^{at}f(t)\} = \bar{f}(s-a)$$

Application of this property leads us to the following useful results:

1. $L(e^{at}) = \frac{1}{s-a}$ $\left[\because L(1) = \frac{1}{s}\right]$

2. $L(e^{at}\,t^n) = \frac{n!}{(s-a)^{n+1}}$ (n is positive integer) $\left[\because L(t^n) = \frac{n!}{s^{n+1}}\right]$

3. $L(e^{at}\sin bt) = \frac{b}{(s-a)^2 + b^2}$ $\left[\because L(\sin bt) = \frac{b}{s^2 + b^2}\right]$

4. $L(e^{at}\cos bt) = \frac{s-a}{(s-a)^2 + b^2}$ $\left[\because L(\cos bt) = \frac{s}{s^2 + b^2}\right]$

5. $L(e^{at}\sinh bt) = \frac{b}{(s-a)^2 - b^2}$ $\left[\because L(\sinh bt) = \frac{b}{s^2 - b^2}\right]$

6. $L(e^{at}\cosh bt) = \frac{s-a}{(s-a)^2 - b^2}$ $\left[\because L(\cosh bt) = \frac{s}{s^2 - b^2}\right]$

   where in each case $s > a$.

4.3 Change of Scale Property

If $L\{f(t)\} = \bar{f}(s)$, then $L\{f(at)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$

Proof:

$$L\{f(at)\} = \int_0^\infty e^{-st} f(at)dt$$

$$= \int_0^\infty e^{-su/a} f(u).du/a$$

Put $at = u$
$\Rightarrow dt = du/a$

$$= \frac{1}{a}\int_0^\infty e^{-su/a} f(u)du = \frac{1}{a}\bar{f}(s/a).$$

4.4 Existence Conditions

$\int_0^\infty e^{-st} f(t)dt$ exists if $\int_0^\lambda e^{-st} f(t)dt$ can actually be evaluated and its limit as $\lambda \to \infty$ exists. Otherwise we ay use the following theorem:

If $f(t)$ is continuous and $\lim_{t\to\infty}\{e^{-at}f(t)\}$ is finite; then the Laplace transform of $f(t)$, i.e. $\int_0^\infty e^{-st} f(t)dt$ exists for a.

It should however, be noted that the above conditions are sufficient rather than necessary.

For example, $L(1/\sqrt{t})$ exists, though $1/\sqrt{t}$ is infinite at $t = 0$. Similarly a function $f(t)$ for which $\lim_{t\to\infty}\{e^{-at}f(t)\}$ inite and having a finite discontinuity will have a Laplace transform for $s > a$.

.5 Transforms of Derivatives

1. If $f'(t)$ be continuous and $L\{f(t)\} = \bar{f}(s)$, then

$$L\{f'(t)\} = s\bar{f}(s) - f(0)$$

2. If $f'(t)$ and its first $(n-1)$ derivatives be continuous, then

$$L\{f^n(t)\} = s^n\bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{n-1}(0)$$

.5.1 Differential Equations, Initial Value Problems

We shall now discuss how the Laplace transform method solved differential equations.

We begin with an initial value problem.

$$y'' + ay' + by = r(t),$$
$$y(0) = K_0,\ y'(0) = K_1 \qquad \ldots(i)$$

with constant a and b. Here (r) is the **input** (driving force) applied to the mechanical system and $y(t)$ is the put (response of the system). In Laplace's method we do three steps.

1st Step: Taking Laplace transform of LHS and RHS of 1 we get

$$L(y'') + aL(y') + bL(y) = L(r).$$

Now substituting $L(y') = sL(y) - f(0)$ and $L(y'') = s^2 L(y) - sf(0) - f'(0)$, we get

$[s^2L(y) - sy(0) - y'(0)] + a[sL(y) - y(0)] + by = L(r).$

Now writing $Y = L(y)$ and $R = L(r)$. This gives

$[s^2Y(s) - sy(0) - y'(0)] + a[sY(s) - y(0)] + by = R(s)$

This is called the subsidiary equation. Collecting Y-terms, we have

$$(s^2 + as + b)Y(s) = (s + a)\,y(0) + y'(0) + R(s).$$

2nd Step: We solve the subsidiary equation **algebraically** for Y. Division by $s^2 + as + b$ and use of the so-called **transfer function**

$$Q(s) = \frac{1}{s^2 + as + b}$$

gives the solution

$$Y(s) = [(s+a)y(0) + y'(0)]Q(s) + R(s)\,Q(s) \qquad \text{...(ii)}$$

If $y(0) = y'(0) = 0$, this is simply $Y = RQ$; thus Q is the quotient

$$Q = \frac{Y}{R} = \frac{L(\text{output})}{L(\text{input})}$$

and this explains the name of Q. Note that Q depends only on a and b, but does not depend on either $r(t)$ or on the initial conditions.

3rd Step. We reduce (ii) (usually by partial fractions, as in calculus) to a sum of terms whose inverse can be found from the table, so that the solution $y(t) = L^{-1}(Y)$ of (i) is obtained.

Example 1.

Initial problem: Explanation of the basic steps

Solve $\qquad y'' - y = t, \qquad y(0) = 1.\ y'(0) = 1.$

Solution:

1st Step.

By taking Laplace transform of LHS and RHS of $y'' - y = t$, we get the following subsidiary equation

$$s^2L(y) - sy(0) - y'(0) - L(y) = 1/s^2, \qquad \text{thus } (s^2 - 1)Y = s + 1 + 1/s^2.$$

where $Y = L(y)$

2nd Step.

The transfer function is $Q = 1/(s^2 - 1)$, and

$$Y = (s+1)Q + \frac{1}{s^2}Q = \frac{s+1}{s^2-1} + \frac{1}{s^2(s^2-1)} = \frac{1}{s-1} + \left(\frac{1}{s^2-1} - \frac{1}{s^2}\right)$$

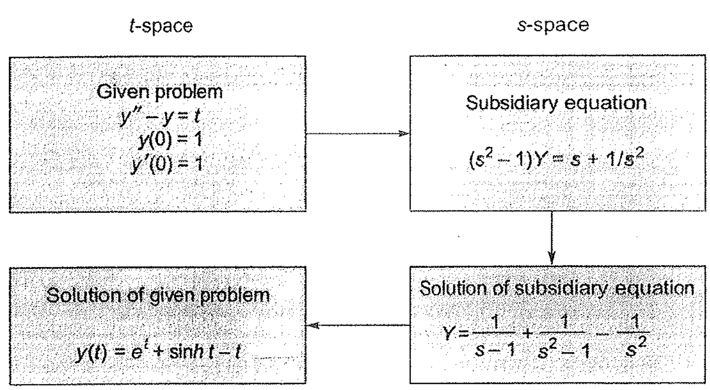
3rd Step.

From this expression for Y, we obtain the solution by inverse Laplace transform as follows

$$y(t) = L^{-1}(Y) = L^{-1}\left\{\frac{1}{s-1}\right\} + L^{-1}\left\{\frac{1}{s^2-1}\right\} - L^{-1}\left\{\frac{1}{s^2}\right\} = e^t + \sinh t - t.$$

$$= e^t + \frac{e^t - e^{-t}}{2} - t = \frac{3e^t - e^{-t} - 2t}{2}$$

The diagram in Fig. below summarizes our approach.



Laplace transform method

Comparison with the usual method

The problem can also be solved by the usual method without using Laplace transforms as shown below:

$$y'' - y = t, \qquad y(0) = 1, y'(0) = 1$$
$$(D^2 - 1)y = 0$$

Auxiliary equation

$$D^2 - 1 = 0$$
$$(D + 1)(D - 1) = 0$$
$$m_1 = 1 \text{ and } m_2 = -1$$

So complementary function is $y = c_1 e^t + c_2 e^{-t}$

Now particular integral

$$\text{P.I.} = \frac{1}{D^2 - 1}(t)$$
$$= -(1 + D^2 - D^4 \ldots)t = -t + 0 - 0 \cdots = -t$$

So complete solution is

$$y = c_1 e^t + c_2 e^{-t}$$
$$y' = c_1 e^t - c_2 e^{-t}$$

Putting initial conditions $y(0) = 1$ and $y'(0) = 1$, we get

$$c_1 + c_2 = 1 \text{ and } c_1 - c_2 = 2$$

$\Rightarrow$ $$c_1 = \frac{3}{2} \text{ and } c_2 = -\frac{1}{2}$$

So C.S. is $$y = \frac{3}{2}e^t - \frac{1}{2}e^{-t} - t = \frac{1}{2}(3e^t - e^{-t} - 2t)$$

Which is exactly the same solution as obtained by Laplace transform method.

Note: Laplace transform method has obtained the solution directly without any evaluation of constants c_1, c_2 etc.

7.4.6 Transforms of Integrals

If $L\{f(t)\} = \bar{f}(s)$, then $L\left\{\int_0^t f(u)du\right\} = \frac{1}{s}\bar{f}(s)$

7.4.7 Multiplication By t^n

If $L\{f(t)\} = f(s)$, then $$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[\bar{f}(s)], \text{ where } n = 1, 2, 3\ldots$$

7.4.8 Division By t

If $L\{f(t)\} = \bar{f}(s)$, then $$L\left\{\frac{1}{t}f(t)\right\} = \int_s^\infty \bar{f}(s)ds$$

provided the integral exists.

7.5 Evaluation of Integrals by Laplace Transforms

Example:

Evaluate

(a) $\int_0^\infty te^{-2t}\sin t\, dt$ (b) $\int_0^\infty \frac{\sin mt}{t}dt$ (c) $L\left\{\int_0^t \frac{e^t \sin t}{t}dt\right\}$

Solution:

(a)
$$\int_0^\infty te^{-2t}\sin t\,dt = \int_0^\infty e^{-st}(t\sin t)dt \text{ where } s = 2$$
$$= L(t\sin t), \text{ by definition.}$$
$$= (-1)\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2} = \frac{2\times 2}{(2^2+1)^2} = \frac{4}{25}$$

(b) Since,
$$L(\sin mt) = m/(s^2+m^2) = f(s), \text{ say}$$
$\therefore$
$$L\left(\frac{\sin mt}{t}\right) = \int_s^\infty f(s)ds = \int_s^\infty \frac{mds}{s^2+m^2} = \left|\tan^{-1}\frac{s}{m}\right|_s = \frac{\pi}{2} - \tan^{-1}\frac{s}{m}$$

Now since,
$$L\left(\frac{\sin mt}{t}\right) = \int_0^\infty e^{-st}\frac{\sin mt}{t}dt$$
$\therefore$
$$\int_0^\infty e^{-st}\frac{\sin mt}{t}dt = \frac{\pi}{2} - \tan^{-1}\frac{s}{m}$$

Now,
$$\underset{s\to 0}{\text{Lt}}\tan^{-1}(s/m) = 0 \text{ if } m > 0 \text{ or } \pi \text{ if } m < 0$$

Thus taking limits as $s \to 0$, we get
$$\int_0^\infty \frac{\sin mt}{t}dt = \frac{\pi}{2} \text{ if } m > 0 \text{ or } -\frac{\pi}{2} \text{ if } m < 0$$

(c) Since,
$$L\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{ds}{s^2+1} = \tan^{-1} s = \frac{\pi}{2} - \tan^{-1}s = \cot^{-1} s.$$
$\therefore$
$$L\left\{e^t\left(\frac{\sin t}{t}\right)\right\} = \cot^{-1}(s-1), \text{ by shifting property}$$

Thus,
$$L\left[\int_0^t\left\{e^t\left(\frac{\sin t}{t}\right)\right\}dt\right] = \frac{1}{s}\cot^{-t}(s-1)$$

Example:

Evaluate $\int_{-\infty}^{\infty} 12\cos 2\pi t\cdot\frac{\sin 4\pi t}{4\pi t}\cdot dt$

Solution:

Since function is even function so,
$$I = 2\int_{-\infty}^{\infty} 12\cos 2\pi t\cdot\frac{\sin 4\pi t}{4\pi t}\cdot dt$$
$$= \frac{3}{\pi}\int_0^\infty\left[\frac{\sin 6\pi t + \sin 2\pi t}{t}\right]dt$$

[**Note:** $2\cos C\sin D = \sin(C+D) + \sin(C-D)$]

$$= \frac{3}{\pi}\left[\int_0^\infty \frac{\sin 6\pi t}{t}dt + \int_0^\infty \frac{\sin 2\pi t}{t}dt\right] = \frac{3}{\pi}\left[\frac{\pi}{2}+\frac{\pi}{2}\right] = 3$$

7.6 Inverse Transforms – Method of Partial Fractions

Having found the Laplace Transforms of a few functions, let us now determine the inverse transforms of given functions of s. We have seen that $L\{(f(t)\}$ in each case, is a rational algebraic function. Hence to find the inverse transforms, we first express the given function of s into partial fractions which will, then, be recognizable as one of the following standard forms:

1. $L^{-1}\left[\frac{1}{s}\right]=1$

2. $L^{-2}\left[\frac{1}{s-a}\right]=e^{at}$

3. $L^{-1}\left[\frac{1}{s^n}\right]=\frac{t^{n-1}}{(n-1)!}$, $n = 1, 2, 3,...$

4. $L^{-1}\left[\frac{1}{(s-a)^n}\right]=\frac{e^{at}t^{n-1}}{(n-1)!}$

5. $L^{-1}\left(\frac{1}{s^2+a^2}\right)=\frac{1}{a}\sin at$

6. $L^{-1}\left(\frac{s}{s^2+a^2}\right)=\cos at$

7. $L^{-1}\left(\frac{1}{s^2-a^2}\right)=\frac{1}{a}\sinh at$

8. $L^{-1}\left(\frac{1}{s^2-a^2}\right)=\cosh at$

9. $L^{-1}\left[\frac{1}{(s-a)^2+b^2}\right]=\frac{1}{b}e^{at}\sin bt$

10. $L^{-1}\left[\frac{s-a}{(s-a)^2+b^2}\right]=e^{at}\cos bt$

11. $L^{-1}\left[\frac{s}{(s^2-a^2)^2}\right]=\frac{1}{2a}t\sin at$

12. $L^{-1}\left[\frac{1}{(s^2-a^2)^2}\right]=\frac{1}{2a^2}(\sin at - at\cos at)$

All these results need to be memorised. The results (1) to (10) follow at once from their corresponding results in transforms of elementary functions and properties of Laplace transforms. Results (11) and (12) can be proved.

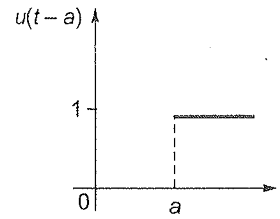
Note on Partial Fractions: To resolve a given fraction into partial fractions, we first factorise the denominator into real factors. These will be either linear or quadratic, and some factors repeated. We know from algebra that a proper fraction can be resolved into a sum of partial fractions such that

1. to a non-repeated linear factor $s - a$ in the denominator corresponds a partial fraction of the form $A/(s - a)$.
2. to a repeated linear factor $(s - a)^r$ in the denominator corresponds the sum of r partial fractions of the form $\frac{A_1}{s-a}+\frac{A_2}{(s-a)^2}+\frac{A_3}{(s-a)^3}+\ldots+\frac{A_r}{(s-a)^r}$.
3. to a non-repeated quadratic factor $(s^2 + as + b)$ in the denominator, corresponds a partial fraction of the form $\frac{As+B}{s^2+as+b}$.
4. to a repeated quadratic factor $(s^2 + as + b)^r$ in the denominator, corresponds the sum of r partial fractions of the form $\frac{A_1s+B_1}{s^2+as+b}+\frac{A_2s+B_2}{(s^2+as+b)^2}+\ldots+\frac{A_rs+B_r}{(s^2+as+b)^r}$.

   Then we have to determine the unknown constants A, A_1, B_1 etc.

In all other cases, equate the given fraction to a sum of suitable partial fractions in accordance with 1 to 4 above, having found the partial fractions corresponding to the non-repeated linear factors by the above rule. Then multiply both sides by the denominator of the given fraction and equate the coefficients of like powers of s or substitute convenient numerical values of s on both sides. Finally solve the simplest of the resulting equations to find the unknown constants.

7.7 Unit Step Function



At times, we come across such fractions of which the inverse transform cannot be determined from the formulae so far derived. In order to cover such cases, we introduce the unit step function (or Heaviside's unit function*).

Def. The unit step function $u(t-a)$ is defined as follows

$$u(t-a) = \begin{cases} 0 \text{ for } t < a \\ 1 \text{ for } t \geq a \end{cases}$$

where a is always positive.

7.7.1 Transform of Unit Function

$$L\{(u(t-a)\} = \int_0^\infty e^{-st} u(t-a) dt$$

$$= \int_0^a e^{-st} \cdot 0\, dt + \int_a^\infty e^{-st} \cdot 1 dt = 0 + \left| \frac{e^{-st}}{-s} \right|_a^\infty$$

Thus, $L\{u(t-a)\} = e^{-as/s}$.

The product $f(t)\, u(t-a) = \begin{cases} 0 & \text{for } t < 0 \\ f(t) & \text{for } t \geq a \end{cases}$

The function $f(t-a) \,.\, u(t-a)$ represents the graph $f(t)$ shifted through a distance a to the right and is of special importance.

7.8 Second Shifting Property

If $L\{f(t)\} = \bar{f}(s)$, then

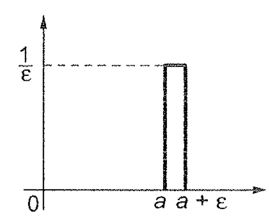
$$L\{f(t-a) \times u(t-a)\} = e^{-as}\bar{f}(s)$$

Proof: $L\{f(t-a) \times u(t-a)\} = \int_0^\infty e^{-st} f(t-a) u(t-a) dt$

$$= \int_0^a e^{-st} f(t-a)(0)\, dt + \int_a^\infty e^{-st} f(t-a) dt \qquad [\text{Put } t-a = u]$$

$$= \int_0^\infty e^{-s(u+a)} f(u) du = e^{-sa} \int_0^\infty e^{-su} f(u) du = e^{-as}\bar{f}(s)$$

7.9 Unit Impulse Function



The idea of a very large force acting for a very short time is of frequent occurrence in mechanics. To deal with such and similar ideas, we introduce the unit impulse function (also called Dirac delta function).

Thus unit impulse function is considered as the limiting form of the function (Figure)

$$\delta_s(t-a) = 1/\varepsilon, \qquad a \leq t \leq a + \varepsilon$$

$$= 0, \qquad \text{otherwise}$$

as $\varepsilon \to 0$. It is clear from figure that as $\varepsilon \to 0$, the height of the strip increases indefinitely and the width decreases in such a way that its area is always unity.

Thus the unit impulse function $\delta(t-a)$ is defined as follows:

$$\delta(t-a) = \infty \qquad \text{for } t = a$$

$$= 0 \qquad \text{for } t \neq a$$

such that $\int_0^\infty \delta(t-a) dt = 1 \qquad (a \geq 0)$

As an illustration, a load w_0 acting at the point $x = a$ of a beam may be considered as the limiting case of uniform loading w_0/ε per unit length over the portion of the beam between $x = a$ and $x = a + \varepsilon$. Thus

$$w(x) = w_0/\varepsilon \qquad a < x < a + \varepsilon,$$
$$= 0 \qquad \text{otherwise}$$

i.e. $$w(x) = w_0\delta(x + a)$$

7.9.1 Transform of Unit Impulse Function

If $f(t)$ be a function of t continuous at $t = 1$, then

$$\int_0^{\infty} f(t)\delta_0(t-a)\cdot dt = \int_a^{a+\varepsilon} f(t)\cdot\frac{1}{\varepsilon}dt = (a+\varepsilon-a)f(\eta)\cdot\frac{1}{\varepsilon} = f(\eta) \text{ where } a < \eta < a + \varepsilon.$$

by Mean value theorem for integrals.

As $\varepsilon \to 0$, we get $\int_0^{\infty} f(t)\delta(t-a)dt = f(a)$

In particular, putting $f(t) = e^{-st}$ in above integral

we have $$\int_0^{\infty} e^{-st}\,\delta(t-a)dt = e^{-as}$$

Now LHS is nothing but $L\{\delta(t-a)\}$

$\therefore$ $$L\{\delta(t-a)\} = e^{-as}$$

7.10 Periodic functions

If $f(t)$ is a periodic function with period T, i.e. $f(t + T) = f(t)$, then

$$L\{f(t)\} = \frac{\int_0^T e^{-st} f(t)dt}{1-e^{-sT}}$$

Example:

If $f(t) = \begin{cases} \sin t & 0 < t < \pi \\ 0 & \pi < t < 2\pi \end{cases}$, $f(t)$ is periodic function with time period 2π. Determine the Laplace transform of $f(t)$.

Solution:

Laplace transform of periodic function

$$L\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt = \frac{1}{1-e^{-2\pi s}}\int_0^{\pi} e^{-st}\sin t dt$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\frac{e^{-st}}{s^2+1}(-s\cdot\sin t - 1\cos t)\right]_0^{\pi}$$

$$= \frac{1}{1-e^{-2\pi}}\left[\frac{e^{-\pi s}}{s^2+1}+\frac{1}{s^2+1}\right] = \frac{(1+e^{-\pi s})}{(s^2+1)(1-e^{-\pi s})(1+e^{-\pi s})} = \frac{1}{(s^2+1)(1-e^{-\pi s})}$$

7.11 Fourier Transform

Fourier series is an approximation process where any general (periodic or aperiodic) signal is expressed as um of harmonically related sinusoids. It gives us frequency domain representation.

If the signal is periodic Fourier series represents the signal in the entire interval $(-\infty, \infty)$. i.e. Fourier series an be generalized for periodic signals only.

Definition: Suppose f is a piecewise continuous periodic function of period $2L$, then f has a Fourier series representation

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

Where the coefficients a's and b's are given by the Euler-Fourier formulas:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \quad n = 0, 1, 2, 3, \ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx, \quad n = 1, 2, 3, \ldots$$

7.12 Drichilet's Conditions

The sufficient condition for the convergence of a Fourier series are called Drichilet's conditions.

1. $f(x)$ is periodic, single valued and finite.
2. $f(x)$ has a finite number of finite discontinuities in any one period.
3. $f(x)$ has a finite number of maxima and minima.

7.12.1 Fourier Cosine and Sine Series

If f is an even periodic function of period $2L$, then its Fourier series contains only cosine (include, possibly, the constant term) terms. It will not have any sine term. That is, its Fourier series is of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

Its Fourier coefficients are determined by

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{m\pi x}{L}\,dx, \qquad n = 0, 1, 2, 3, \ldots$$

$$b_n = 0, \qquad n = 1, 2, 3, \ldots.$$

If f is an odd periodic function of period $2L$, then its Fourier series contains only sine terms. It will not have any cosine term. That is, its Fourier series is of the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

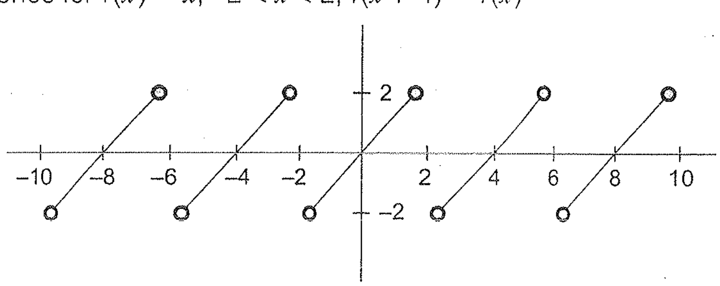
Its Fourier coefficients are determined by

$$a_n = 0, \qquad n = 0, 1, 2, 3, \ldots$$

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx, \qquad n = 0, 1, 2, 3, \ldots$$

Example:

Find a Fourier series for $f(x) = x$, $-2 < x < 2$, $f(x + 4) = f(x)$



Solution:

First note that $T = 2L = 4$, hence $L = 2$

The constant term is one half of a_0,

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx = \frac{1}{2}\int_{-2}^{2} x\,dx = \frac{1}{2}\frac{x^2}{2}\Big|_{-2}^{2} = \frac{1}{2}(2-2) = 0$$

The rest of the cosine coefficients, for n = 1, 2, 3, ... , are

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_{-2}^{2} x\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left(\frac{2x}{n\pi}\sin\frac{n\pi x}{2}\Big|_{-2}^{2} - \frac{2}{n\pi}\int_{-2}^{2}\sin\frac{n\pi x}{2}\,dx\right)$$

$$= \frac{1}{2}\left(\frac{2x}{n\pi}\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\Big|_{-2}^{2}\right)$$

$$= \frac{1}{2}\left[\left(0 + \frac{4}{n^2\pi^2}\cos(n\pi)\right) - \left(0 + \frac{4}{n^2\pi^2}\cos(-n\pi)\right)\right] = 0$$

Hence, there is no non-zero cosine coefficient for this function. That is, its Fourier series contains no cosine terms at all. (We shall see the significance of this fact a little later).

The sine coefficients, for n = 1, 2, 3, ... , are

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_{-2}^{2} x\sin\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left(\frac{-2x}{n\pi}\cos\frac{n\pi x}{2}\Big|_{-2}^{2} - \frac{-2}{n\pi}\int_{-2}^{2}\cos\frac{n\pi x}{2}\,dx\right)$$

$$= \frac{1}{2}\left(\frac{-2x}{n\pi}\cos\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\Big|_{-2}^{2}\right)$$

$$= \frac{1}{2}\left[\left(\frac{-4}{n\pi}\cos(n\pi) - 0\right) - \left(\frac{4}{n\pi}\cos(-n\pi) - 0\right)\right]$$

$$= \frac{-2}{n\pi}\left[(\cos(n\pi) + \cos(n\pi)\right] = \frac{-4}{n\pi}\cos(n\pi)$$

$$= \begin{cases} \dfrac{4}{n\pi}, & n = \text{odd} \\ \dfrac{-4}{n\pi}, & n = \text{even} \end{cases} = \frac{(1)^{n+1}4}{n\pi}$$

Therefore,

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{2}$$

Example:

Find a Fourier series for $f(x) = x$, $0 < x < 4$, $f(x + 4) = f(x)$. How will it be different from the series in the previous example?

Solution:

$$a_0 = \frac{1}{2}\int_0^4 x\,dx = \frac{1}{2}\frac{x^2}{2}\Big|_0^4 = \frac{1}{4}(8-0) = 4$$

For $n = 1, 2, 3, \ldots$

$$a_n = \frac{1}{2}\int_0^4 x\cos\frac{n\pi x}{2}dx = \frac{1}{2}\left(\frac{2x}{n\pi}\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\Big|_0^4\right)$$

$$= \frac{1}{2}\left[\left(0+\frac{4}{n^2\pi^2}\cos(2n\pi)\right)-\left(0+\frac{4}{n^2\pi^2}\cos(0)\right)\right] = 0$$

$$b_n = \frac{1}{2}\int_0^4 x\sin\frac{n\pi x}{2}dx = \frac{1}{2}\left(\frac{-2x}{n\pi}\cos\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\Big|_0^4\right)$$

$$= \frac{1}{2}\left[\left(\frac{-8}{n\pi}\cos(2n\pi - 0) - (0-0)\right)\right] = \frac{-4}{n\pi}$$

Consequently,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right) = 2 + \frac{-4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{2}$$

Comment: Just because a Fourier series could have infinitely many (non-zero) terms does not mean that it will always have that many terms. If a periodic function f can be expressed by finitely many terms normally found in a Fourier series, then the expression must be the Fourier series of f. (This is analogous to the fact that the Maclaurin's series of any polynomial function is just the polynomial itself, which is a sum of finitely many powers of x.)

Example: The Fourier series (period 2π) representing

$f(x) = 5 + \cos(4x) - \sin(5x)$ is just $f(x) = 5 + \cos(4x) - \sin(5x)$.

Example: The Fourier series (period 2π) representing $f(x) = 6\cos(x)\sin(x)$ is not exactly itself as given, since the product $\cos(x)$ is not a term in a Fourier series representation. However, we can use the double-angle formula of sine to obtain the result: $6\cos(x)\sin(x) = 3\sin(2x)$.

Consequently, the Fourier series is $f(x) = 3\sin(2x)$.

7.12.2 The Cosine and Sine Series Extensions

If f and f' are piecewise continuous functions defined on the interval $0 \le t \le L$, then f can be extended into an even periodic function, F, of period $2L$, such that $f(x) = F(x)$ on the interval $[0, L]$, and whose Fourier series is, therefore, a cosine series. Similarly, f can be extended into an odd periodic function of period $2L$, such that $f(x) = F(x)$ on the interval $(0, L)$, and whose Fourier series is, therefore, a sine series. The process that such extensions are obtained is often called cosine.sine half-range expansions.

Even (cosine series) extension of f(x)

Given $f(x)$ defined on $[0, L]$. Its even extension of period $2L$ is

$$F(x) = \begin{cases} f(x) & 0 \le x \le L \\ f(-x) & -L < x < 0 \end{cases} \qquad F(x+2L) = F(x)$$

Where,

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{L}, \qquad \text{such that}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx, \qquad n = 0, 1, 3, \ldots$$

$$b_n = 0, \qquad n = 1, 2, 3, \ldots$$

Odd (sine series) extension of $f(x)$

Given $f(x)$ defined on $[0, L]$. Its odd extension of period $2L$ is

$$F(x) = \begin{cases} f(x) & 0 < x < L \\ 0, & x = 0, L \\ -f(-x), & -L < x < 0 \end{cases} \qquad F(x+2L) = F(x)$$

Where,
$$F(x) = \sum_{n=1}^{\infty} b_n \cos\frac{n\pi x}{L}, \qquad \text{such that}$$

$$a_n = 0, \qquad n = 0, 1, 2, 3, \ldots$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx, \qquad n = 0, 1, 3, \ldots$$

Example:

Let $f(x) = x$, $0 \leq x < 2$. Find its cosine and sine series extensions of period 4.

Solution:

Cosine series:
$$f(x) = 1 - \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\cos\frac{(2n-1)\pi x}{2}$$

Sine series:
$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{2}$$

■■■■

Previous GATE and ESE Questions

Q.1 If L defines the Laplace Transform of a function, $L[\sin(at)]$ will be equal to

(a) $\frac{a}{s^2 - a^2}$ (b) $\frac{a}{s^2 + a^2}$

(c) $\frac{s}{s^2 + a^2}$ (d) $\frac{s}{s^2 - a^2}$

[CE, GATE-2003, 2 marks]

Q.2 Laplace transform of the function $\sin \omega t$ is

(a) $\frac{s}{s^2 + \omega^2}$ (b) $\frac{\omega}{s^2 + \omega^2}$

(c) $\frac{s}{s^2 - \omega^2}$ (d) $\frac{\omega}{s^2 - \omega^2}$

[ME, GATE-2003, 2 marks]

Q.3 A delayed unit step function is defined as

$u(t-a) = \begin{cases} 0, & \text{for } t < a \\ 1, & \text{for } t \geq a \end{cases}$. Its Laplace transform is

(a) $a.e^{-as}$ (b) $\frac{e^{-as}}{s}$

(c) $\frac{e^{as}}{s}$ (d) $\frac{e^{as}}{a}$

[ME, GATE-2004, 2 marks]

Q.4 A solution for the differential equation $\dot{x}(t) + 2x(t) = \delta(t)$ with initial condition $x(0^-) = 0$ is

(a) $e^{-2t}u(t)$ (b) $e^{2t}u(t)$

(c) $e^{-t}u(t)$ (d) $e^{t}u(t)$

[EC, GATE-2006, 1 mark]

Q.5 If $F(s)$ is the Laplace transform of function $f(t)$, then Laplace transform of $\int_0^t f(\tau)d\tau$ is

(a) $\frac{1}{s}F(s)$ (b) $\frac{1}{s}F(s) - f(0)$

(c) $sF(s) - f(0)$ (d) $\int F(s)\,ds$

[ME, GATE-2007, 2 marks]

Q.6 Evaluate $\int_0^\infty \frac{\sin t}{t} dt$

(a) π (b) $\frac{\pi}{2}$

(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{8}$

[CE, GATE-2007, 2 marks

Q.7 Laplace transform for the function $f(x) = \cosh(ax$ is

(a) $\frac{a}{s^2 - a^2}$ (b) $\frac{s}{s^2 - a^2}$

(c) $\frac{a}{s^2 + a^2}$ (d) $\frac{s}{s^2 + a^2}$

[CE, GATE-2009, 2 marks

Q.8 The inverse Laplace transform of $\frac{1}{(s^2 + s)}$ is

(a) $1 + e^t$ (b) $1 - e^t$

(c) $1 - e^{-t}$ (d) $1 + e^{-t}$

[ME, GATE-2009, 1 mark]

Q.9 The Laplace transform of a function $f(t)$ is $\frac{1}{s^2(s+1)}$. The function $f(t)$ is

(a) $t - 1 + e^{-1}$ (b) $t + 1 + e^{-1}$

(c) $-1 + e^{-1}$ (d) $2t + e^t$

[ME, GATE-2010, 2 marks]

Q.10 Given $L^{-1}\left[\frac{3s+1}{s^2 + 4s^2 + (K-3)s}\right]$. If $\lim_{t\to\infty} f(t) = 1$ then the value of K is

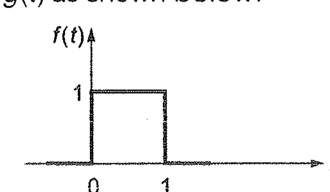
(a) 1 (b) 2

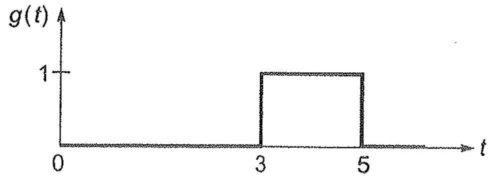
(c) 3 (d) 4

[EC, GATE-2010, 2 marks]

Common Data Questions 11 and 12

Given $f(t)$ and $g(t)$ as shown below:





Q.11 $g(t)$ can be expressed as

(a) $g(t) = f(2t-3)$ (b) $g(t) = f\left(\frac{t}{2}-3\right)$

(c) $g(t) = f\left(2t-\frac{3}{2}\right)$ (d) $g(t) = f\left(\frac{t}{2}-\frac{3}{2}\right)$

[EE, GATE-2010, 2 marks]

Q.12 The Laplace transform of $g(t)$ is

(a) $\frac{1}{s}\left(e^{3s}-e^{5s}\right)$ (b) $\frac{1}{s}\left(e^{-5s}-e^{-3s}\right)$

(c) $\frac{e^{-3s}}{s}\left(1-e^{-2s}\right)$ (d) $\frac{1}{s}\left(e^{5s}-e^{3s}\right)$

[EE, GATE-2010, 2 marks]

Q.13 The inverse Laplace transform of the function $F(s) = \frac{1}{s(s+1)}$ is given by

(a) $f(t) = \sin t$ (b) $f(t) = e^{-t}\sin t$

(c) $f(t) = e^{-t}$ (d) $f(t) = 1 - e^{-t}$

[ME, GATE-2012, 2 marks]

Q.14 Consider the differential equation

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + y(t) = \delta(t) \text{ with}$$

$$y(t)\Big|_{t=0} = -2 \text{ and } \frac{dy}{dt}\Big|_{t=0} = 0.$$

The numerical value of $\frac{dy}{dt}\Big|_{t=0}$ is

(a) – 2 (b) – 1

(c) 0 (d) 1

[EC, IN GATE-2012, 2 marks]

Q.15 The function $f(t)$ satisfies the differential equation $\frac{d^2f}{dt^2} + f = 0$ and the auxiliary conditions, $f(0) = 0$, $\frac{df}{dt}(0) = 4$. The Laplace transform of $f(t)$ is given by

(a) $\frac{2}{s+1}$ (b) $\frac{4}{s+1}$

(c) $\frac{4}{s^2+1}$ (d) $\frac{2}{s^2+1}$

[ME, GATE-2013, 2 Marks]

Q.16 Laplace transform of $\cos(\omega t)$ is $\frac{s}{s^2+\omega^2}$. The laplace transform of $e^{-2t}\cos(4t)$ is

(a) $\frac{s-2}{(s-2)^2+16}$ (b) $\frac{s+2}{(s-2)^2+16}$

(c) $\frac{s-2}{(s+2)^2+16}$ (d) $\frac{s+2}{(s+2)^2+16}$

[ME, GATE-2014 : 1 Mark]

Q.17 Let $X(s) = \frac{3s+5}{s^2+10s+21}$ be the Laplace Transform of a signal $x(t)$. Then, $x(0^+)$ is

(a) 0 (b) 3

(c) 5 (d) 21

[EE, GATE-2014 : 1 Mark]

Q.18 With initial values $y(0) = y'(0) = 1$, the solution of the differential equation $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$ at $x = 1$ is _____.

[EC, GATE-2014 : 2 Marks]

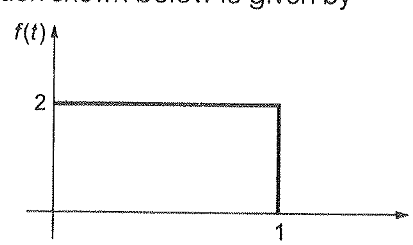
Q.19 The Laplace transform of e^{i5t} where $i = \sqrt{-1}$, is

(a) $\frac{s-5i}{s^2-25}$ (b) $\frac{s+5i}{s^2+25}$

(c) $\frac{s+5i}{s^2-25}$ (d) $\frac{s-5i}{s^2+25}$

[ME, GATE-2015 : 1 Mark]

Q.20 Laplace transform of the function $f(t)$ is given by $F(s) = L\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$. Laplace transform of the function shown below is given by



(a) $\frac{1-e^{-2s}}{s}$ (b) $\frac{1-e^{-s}}{2s}$

(c) $\frac{2-2e^{-s}}{s}$ (d) $\frac{1-2e^{-s}}{s}$

[ME, GATE-2015 : 2 Marks]

Q.21 If $f(t)$ is a function defined for all $t \geq 0$, its Laplace transform $F(s)$ is defined as

(a) $\int_0^{\infty} e^{st} f(t)\, dt$ (b) $\int_0^{\infty} e^{-st} f(t) dt$

(c) $\int_0^{\infty} e^{ist} f(t) dt$ (d) $\int_0^{\infty} e^{-ist} f(t) dt$

[ME, GATE-2016 : 1 Mark]

Q.22 Consider the function $f(x) = 2x^3 - 3x^2$ in the domain [–1, 2]. The global minimum of $f(x)$ is_____.

[ME, GATE-2016 : 2 Marks]

Q.23 Laplace transform of cos (ωt) is

(a) $\frac{s}{s^2+\omega^2}$ (b) $\frac{\omega}{s^2+\omega^2}$

(c) $\frac{s}{s^2-\omega^2}$ (d) $\frac{\omega}{s^2-\omega^2}$

[ME, GATE-2016 : 1 Mark]

Q.24 Solutions of Laplace equation having continuous second-order partial derivatives are called

(a) biharmonic functions
(b) harmonic functions
(c) conjugate harmonic functions
(d) error functions

[ME, GATE-2016 : 1 Mark]

Q.25 The Laplace Transform of $f(t) = e^{2t}\sin(5t)\, u(t)$ is

(a) $\frac{5}{s^2-4s+29}$ (b) $\frac{5}{s^2+5}$

(c) $\frac{s-2}{s^2-4s+29}$ (d) $\frac{5}{s+5}$

[EE, GATE-2016 : 1 Mark]

Q.26 Consider a causal LTI system characterized by differential equation $\frac{dy(t)}{dt} + \frac{1}{6}y(t) = 3x(t)$. The response of the system to the input $x(t) = 3e^{-t/3}$ $u(t)$, where $u(t)$ denotes the unit step function, is

(a) $9\, e^{-t/3}\, u(t)$
(b) $9\, e^{-t/6}\, u(t)$
(c) $9\, e^{-t/3}\, u(t) - 6e^{-t/6}\, u(t)$
(d) $54\, e^{-t/6}\, u(t) - 54\, e^{-t/3}\, u(t)$

[EE, GATE-2016 : 1 Mark]

Q.27 The Fourier series of the function,

$$f(x) = 0, \quad -\pi < x \leq 0$$
$$= \pi - x, \quad 0 < x < \pi$$

in the interval $[-\pi, \pi]$ is

$$f(x) = \frac{\pi}{4} + \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^3} + \ldots\right] + \left[\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \ldots\right]$$

The convergence of the above Fourier series at $x = 0$ gives

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ (b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$

(c) $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$ (d) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4}$

[CE, GATE-2016 : 1 Mark]

Q.28 The Laplace transform of te^t is

(a) $\frac{s}{(s+1)^2}$ (b) $\frac{1}{(s-1)^2}$

(c) $\frac{1}{(s+1)^2}$ (d) $\frac{s}{s-1}$

[ME, GATE-2017 : 1 Mark]

Q.29 For the function

$$f(x) = \begin{cases} -2, & -\pi < x < 0 \\ 2, & 0 < x < \pi \end{cases}$$

The value of a_n in the Fourier series expansion of $f(x)$ is

(a) 2 (b) 4
(c) 0 (d) –2

[ESE Prelims-2017]

Q.30 Given the Fourier series in $(-\pi, \pi)$ for $f(x) = x \cos x$, the value of a_0 will be

(a) $-\frac{2}{3}\pi^2$ (b) 0

(c) 2 (d) $\frac{(-1)^n 2n}{n^2-1}$

[EE, ESE-2017]

Q.31 The Fourier series expansion of the saw-toothed waveform $f(x) = x$ in $(-\pi, \pi)$ of period 2π gives the series, $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$

The sum is equal to

(a) $\frac{\pi}{2}$ (b) $\frac{\pi^2}{4}$

(c) $\frac{\pi^2}{16}$ (d) $\frac{\pi}{4}$ **[EE, ESE-2017]**

.32 The Laplace transform $F(s)$ of the exponential function. $f(t) = e^{at}$ when $t \geq 0$, where a is a constant and $(s - a) > 0$, is

(a) $\frac{1}{s+a}$ (b) $\frac{1}{s-a}$

(c) $\frac{1}{a-s}$ (d) ∞

[CE, GATE-2018 : 2 Marks]

.33 $F(s)$ is the Laplace transform of the function
$f(t) = 2t^2 e^{-t}$
F(1) is ______ (correct to two decimal places).

[ME, GATE-2018 : 2 Marks]

.34 The Fourier cosine series for an even function $f(x)$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$$

The value of the coefficient a_2 for the function $f(x) = \cos^2(x)$ in $[0, \pi]$ is

(a) –0.5 (b) 0.0

(c) 0.5 (d) 1.0

[ME, GATE-2018 : 1 Mark]

Q.35 The position of a particle $y(t)$ is described by the differential equation:

$$\frac{d^2y}{dt^2} = -\frac{dy}{dt} - \frac{5y}{4}$$

The initial conditions are $y(0) = 1$ and $\left.\frac{dy}{dt}\right|_{t=0} = 0$.

The position (accurate to two decimal places) of the particle at $t = \pi$ is ______.

[EC, GATE-2018 : 2 Marks]

Q.36 Let $f(x) = \begin{cases} -\pi, \text{ if } & -\pi < x \leq 0 \\ \pi, \text{ if } & 0 < x \leq \pi \end{cases}$ be a periodic function of period 2π. The coefficient of $\sin 5x$ in the Fourier series expansion of $f(x)$ in the interval $[-\pi, \pi]$ is

(a) $\frac{4}{5}$ (b) $\frac{5}{4}$

(c) $\frac{4}{3}$ (d) $\frac{3}{4}$

[ESE Prelims-2018]

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Answers Transform Theory

.	(b)	2.	(b)	3.	(d)	4.	(a)	5.	(a)	6.	(b)	7.	(b)	8.	(c)	9.	(a)
0.	(d)	11.	(d)	12.	(c)	13.	(d)	14.	(d)	15.	(c)	16.	(d)	17.	(b)	18.	(0.77)
9.	(b)	20.	(c)	21.	(b)	22.	(–5)	23.	(a)	24.	(b)	25.	(a)	26.	(d)	27.	(c)
8.	(b)	29.	(c)	30.	(b)	31.	(d)	32.	(b)	33.	(0.5)	34.	(c)	35.	(–0.21)	36.	(a)

(b)

$$L[\sin \omega t] = \frac{\omega}{s^2 + \omega^2}$$

(d)

$$L[U(t-a)] = \int_0^\infty e^{-St}\, U(t-a)dt$$

$$= \int_0^a e^{-st}\cdot 0\cdot dt + \int_0^\infty e^{-st}\cdot 1\cdot dt$$

$$= 0 + \int_a^\infty e^{-st}dt = \left[\frac{e^{-st}}{-s}\right]_a^\infty = \frac{e^{-as}}{s}$$

(a)

$\dot{x}(t) + 2x(t) = \delta(t)$

Taking L.T. on both sides

$sX(s) - x(0) + 2X(s) = 1$

$X(s)\,[s+2] = 1$

$$X(s) = \frac{1}{s+2}$$

$$x(t) = e^{-2t}u(t)$$

(a)

$$L\left[\int_0^t\int_0^t \ldots \int_0^t f(t)\, dt^n\right] = \frac{1}{s^n}F(s)$$

In this problem $n = 1$

So, $$L\left[\int_0^t f(\tau)d\tau\right] = \frac{1}{s}F(s)$$

(b)

Since,

$$L(\sin mt) = \frac{m}{(s^2+m^2)} = f(s), \text{ say.}$$

$$\therefore L\left(\frac{\sin mt}{t}\right) = \int_s^\infty f(s)\, ds = \int_s^\infty \frac{m ds}{s^2+m^2}$$

$$= \left|\tan^{-1}\frac{s}{m}\right|_s^\infty$$

or by Definition,

$$\int_0^\infty e^{-st}\frac{\sin mt}{t}dt = \frac{\pi}{2}\tan^{-1}\frac{s}{m}$$

Now $\underset{s\to 0}{Lt}\tan^{-1}\left(\frac{s}{m}\right) = 0$ if $m > 0$ or π if $m < 0$.

Thus taking limits as $s \to 0$, we get

$$\int_0^\infty \frac{\sin mt}{t}dt = \frac{\pi}{2} \text{ if } m > 0 \text{ or } -\frac{\pi}{2} \text{ if } m < 0.$$

In this problem $m = 1$ which is > 0 therefore the answer is $\frac{\pi}{2}$.

7. (b)

It is a standard result that

$$L(\cosh at) = \frac{s}{s^2 - a^2}$$

8. (c)

$$L^{-1}\left(\frac{1}{s^2+s}\right) = ?$$

$$\frac{1}{s^2+s} = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

$$L^{-1}\left(\frac{1}{s^2+s}\right) = L^{-1}\left(\frac{1}{s}\right) - -L^{-1}\left(\frac{1}{s+1}\right)$$

$= 1 - e^{-t}$ [Using standard formulae]

Standard formula:

$$L^{-1}\left(\frac{1}{s}\right) = 1$$

$$L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$$

$$L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$$

9. (a)

$$f(t) = L^{-1}\left[\frac{1}{s^2(s+1)}\right]$$

$$\frac{1}{s^2(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1}$$

$$\frac{1}{s^2(s+1)} = \frac{As(s+1) + B(s+1) + C(s^2)}{s^2(s+1)}$$

Matching coefficient of s^2, s and constant in numerator we get,

$A + C = 0$ ... (i)

$A + B = 0$ ... (ii)

$$B = 1 \quad \text{... (iii)}$$

Solving we get $A = -1$, $B = 1$, $C = 1$

So, $$f(t) = L^{-1}\left[\frac{-1}{s} + \frac{1}{s^2} + \frac{1}{s+1}\right]$$

$$= -1 + t + e^{-t} = t - 1 + e^{-t}$$

(d)

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

Given that,

$$F(s) = \left[\frac{3s+1}{s^3 + 4s^2 + (K-3)s}\right]$$

$$\lim_{t \to \infty} f(t) = 1$$

$$\Rightarrow \lim_{s \to 0} s\left[\frac{3s+1}{s^3 + 4s^2 + (K-3)s}\right] = 1$$

$$\Rightarrow \lim_{s \to 0}\left[\frac{3s+1}{s^2 + 4s + (K-3)}\right] = 1$$

$$\Rightarrow \frac{1}{K-3} = 1$$

$$\Rightarrow K - 3 = 1$$

$$\Rightarrow K = 4$$

d)

Ve need

$$g(3) = f(0) \text{ and } g(5) = f(1)$$

Only choice (d) satisfies both these conditions is seen below:

Choice (d) is

$$g(t) = f\left(\frac{t}{2} - \frac{3}{2}\right)$$

$$g(3) = f\left(\frac{3}{2} - \frac{3}{2}\right) = f(0)$$

ınd $$g(5) = f\left(\frac{5}{2} - \frac{3}{2}\right) = f(1)$$

c)

By definition of Laplace transform,

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$L\{f(t)\} = \int_0^3 e^{-st} f(t)\, dt + \int_3^5 e^{-st} f(t)\, dt + \int_5^\infty e^{-st} f(t)\, dt$$

$$= \int_0^3 e^{-st} .0. dt + \int_3^5 e^{-st} .1. dt + \int_5^\infty e^{-st} .0. dt$$

$$= \left[-\frac{e^{-st}}{s}\right]_3^5 = -\left[\frac{e^{-5s} - e^{-3s}}{s}\right]$$

$$= \frac{e^{-3s} - e^{-5s}}{s} = \frac{e^{-3s}}{s}\,[1 - e^{-2s}]$$

13. **(d)**

$$F(s) = \frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

$$= \frac{A(s+1) + B(s)}{s(s+1)}$$

$\Rightarrow A(s + 1) + B(s) = 1$

Put $s = 0$

$\Rightarrow$ $A = 1$

and $s = -1$

$\Rightarrow$ $B = -1$

So $$F(s) = \frac{1}{s} - \frac{1}{s+1}$$

Now $$f(t) = L^{-1}(F(s)) = e^{0t} - e^{-t}$$

$$f(t) = 1 - e^{-t}$$

14. **(d)**

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y(t) = \delta(t)$$

taking Laplace transform on both the sides we have

$$s^2Y(s) + 2s + 2sY(s) + 4 + Y(s) = 1$$

$$(s^2 + 2s + 1)\, Y(s) = -(2s + 3)$$

$$Y(s) = \frac{-(2s+3)}{(s+1)^2}$$

$$Y(s) = -\left[\frac{2}{(s+1)} + \frac{1}{(s+1)^2}\right]$$

$$\Rightarrow \quad Y(t) = -[2e^{-t} + te^{-t}]u(t)$$

$$\frac{dy}{dt} = -[-2e^{-t} + e^{-t} - te^{-t}]u(t)$$

$$\left.\frac{dy}{dt}\right|_{\text{at } t=0+} = -[-2 + 1 - 0]$$

$$\left.\frac{dy}{dt}\right|_{\text{at } t=0+} = 1$$

15. **(c)**

$$L\left\{\frac{d^2f}{dt^2} + f\right\} = 0$$

$$L\{f\} = F\{s\}$$

$$L\{f''\} = s^2F(s) - sf(0) - f'(s)$$

$$= s^2F(s) - 4$$

$s^2F(s) - 4 + F(s) = 0$

$(s^2 + 1) F(s) = 4$

$$F(s) = \frac{4}{s^2+1}$$

$$L\{f\} = \frac{4}{s^2+1}$$

16. (d)

$$L(e^{at}\cos bt) = \frac{s+2}{(s+2)^2+b^2}$$

$a = -2,\ b = 4$

$$\therefore\ L[e^{-2t}\cos(4t)] = \frac{s+a}{(s+a)^2+16}$$

17. (b)

Given, $X(s) = \left[\dfrac{3s+5}{s^2+10s+21}\right]$

Using initial value theorem,

$$x(0^+) = \lim_{s\to\infty}[sX(s)]$$

$$\therefore\quad x(0^+) = \lim_{s\to\infty}\left[\frac{s(3s+5)}{s^2+10s+21}\right]$$

$$= \lim_{s\to\infty}\left[\frac{3+\frac{5}{s}}{1+\frac{10}{s}+\frac{21}{s^2}}\right] = \frac{3}{1} = 3$$

18. Sol.

Given

$y(0) = y'(0) = 1$

$$\frac{d^2y}{dx^2} + \frac{4dy}{dx} + 4y = 0 \quad \text{...(i)}$$

Taking the Laplace transform of equation (i), we get

$s^2Y(s) - sY(0) - y'(0) + 4[sY(s) - y(0)] + 4Y(s) = 0$

$[s^2 + 4s + 4]Y(s) = sY(0)\,y'(0) + 4y(0)$

$[s^2 + 4s + 4]\,Y(s) = s + 1 + 4$

$$Y(s) = \frac{s+5}{(s^2+4s+4)} = \frac{(s+5)}{(s+2)^2}$$

$$= \frac{1}{(s+2)} + \frac{3}{(s+2)^2}$$

$y(x) = e^{-2x} + 3x\,e^{-2x}$

at $x = 1$, $y(x) = e^{-1} + 3e^{-2}$

$= 0.77$

19. (b)

$e^{i5t} = \cos 5t + i\sin 5t$

$$L\{e^{(i5t)}\} = \frac{s}{s^2+25} + \frac{5i}{s^2+25} = \frac{s+5i}{s^2+25}$$

20. (c)

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

$$= \int_0^1 2e^{-st}dt + \int_1^\infty 0\cdot e^{-st}dt$$

$$= 2\left[\frac{e^{-st}}{-s}\right]_0^1 = \frac{2}{-s}[e^{-s}-1]$$

$$= \frac{2(1-e^{-s})}{s} = \frac{2-2e^{-s}}{s}$$

21. (b)

$$L(f(t)) = \int_0^\infty e^{-st}f(t)dt$$

22. Sol.

$f(x) = 2x^3 - 3x^2$ in $[-1, 2]$

$f'(x) = 6x^2 - 6x$

$f'(x) = 0$

$6x^2 - 6x = 0$

$6x(x-1) = 0$

$x = 0, 1$

$f''(x) = 12x - 6$

$f''(0) = -6$ Max

$f''(1) = 6$ Min

x	$f(x)$	
$x = -1$	$f(-1) = -5$	G. Min.
$x = 2$	$f(2) = 4$	
$x = 0$	$f(0) = 0$	
$x = 1$	$f(1) = -1$	

G. Minima is –5 at $x = 1$.

23. (a)

$$L(\cos\omega t) = \frac{s}{s^2+\omega^2}$$

24. (b)

Solution of laplace equation having continuous Second order partial derivating

$\therefore\quad \nabla^2\phi = 0$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

$\therefore\quad \phi$ is harmonic function.

25. (a)

Laplace transform of sin 5t $u(t) \rightarrow \frac{5}{s^2+25}$

$$e^{2t}\sin 5t\, u(t) \rightarrow \frac{5}{(s-2)^2+25} = \frac{5}{s^2-4s+29}$$

26. (d)

The differential equation,

$$\frac{dy(t)}{dt} + \frac{1}{6}y(t) = 3\,x(t)$$

So, $sY(s) + \frac{1}{6}Y(s) = 3\,X(s)$

$$Y(s) = \frac{3\,X(s)}{\left(s+\frac{1}{6}\right)}$$

$$X(s) = \frac{9}{\left(s+\frac{1}{3}\right)}$$

So, $$Y(s) = \frac{9}{\left(s+\frac{1}{3}\right)\left(s+\frac{1}{6}\right)}$$

$$= \frac{54}{\left(s+\frac{1}{6}\right)} - \frac{54}{\left(s+\frac{1}{3}\right)}$$

So, $$y(t) = (54e^{-1/6t} - 54e^{-1/3t})\,u(t)$$

27. (c)

The function is $f(x) = 0$,

$-p < x \le 0$

$= p - x,\ 0 < x < \pi$

And Fourier series is

$$f(x) = \frac{\pi}{4} + \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right]$$
$$+\left[\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \ldots\right] \quad \ldots(i)$$

At $x = 0$, (a point of discontinuity), the fourier series converges to $\frac{1}{2}\left[f\left(0^{-1}\right) + f\left(0^{+}\right)\right]$,

where $f(0^-) = \lim_{x\to 0}(\pi - x) = \pi$

Hence, eq. (i), we get

$$\frac{\pi}{2} = \frac{\pi}{4} + \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \ldots\right]$$

$$\Rightarrow \quad \frac{1}{1} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \frac{\pi^2}{8}$$

28. (b)

$$f(t) = te^t$$

$$L(t) = \frac{1}{s^2}$$

By first shifting rule

$$L(te^t) = \frac{1}{(s-1)^2}$$

29. (c)

$$f(x) = \begin{cases} -2 & -\pi < x < 0 \\ 2 & 0 < x < \pi \end{cases}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos\left(\frac{n\pi x}{\pi}\right)dx$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx = \frac{2}{\pi}\int_0^{\pi} 2\cos nx\, dx$$

$$= \frac{4}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi} = \frac{4}{\pi}\left[\frac{\sin n\pi}{n} - \frac{\sin 0}{n}\right]$$

$$= \frac{4}{\pi}(0-0) = 0$$

Alternative:

Since function is odd function.

$\Rightarrow \quad a_n = 0$

30. (b)

$$f(x) = x\cos x \text{ in } (-\pi, \pi)$$
$$f(-x) = (-x)\cos(-x)$$
$$= -x\cos x$$
$$= -f(x)$$

$f(x) = x\cos x$ is odd function

$\therefore \quad a_0 = 0$

31. (d)

Given, $f(x) = x$

$f(-x) = -x$

i.e. $\therefore$ $f(x) = x$ is an odd function

Hence, $a_0 = 0$

$a_n = 0$

Therefore the Fourier series for the function $f(x)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \ldots(i)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\, dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right)-1\left(\frac{-\sin nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi\cos n\pi}{n}-(-\pi)\left(\frac{-\cos n\pi}{n}\right)\right]$$

$$= \frac{1}{n}[-2\cos n\pi] = \frac{2}{n}(-1)^{n+1}$$

i.e. $b_n = \frac{2}{n}(-1)^{n+1}$ ...(ii)

Substituting equation (ii) in equation (i) we get

$$\therefore \quad f(x) = \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin nx$$

$$x = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin nx}{n}$$

$$x = 2\left[\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\frac{\sin 4x}{4}\right]$$

put, $x = \frac{\pi}{2}$ on both sides

$$\frac{\pi}{2} = 2\left\{1-0-\frac{1}{3}+0+\frac{1}{5}+0-\frac{1}{7}+\ldots\right\}$$

$$\frac{\pi}{4} = \left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\right]$$

(b)

$$L(e^{at}) = \frac{1}{S-a}$$

$$L(a^{at}) = \int_0^{\infty}e^{-st}e^{at}dt = \int_0^{\infty}e^{-(s-a)t}dt$$

$$= \left.\frac{e^{-(s-a)t}}{-(s-a)}\right|_0^{\infty} = -\frac{1}{s-a}(0-1) = \frac{1}{s-a}$$

Sol.

$$L(t^2) = \frac{L^2}{S^3}$$

$$L(2t^2) = \frac{4}{S^3}$$

$$F(s) = L[e^{-t}\cdot 2t^2] = \frac{4}{(S+1)}$$

$$F(1) = \frac{4}{(1+1)^3} = \frac{4}{8} = 0.5$$

(c)

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$f(x) = \frac{1}{2}+\frac{\cos 2x}{2}$$

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cdot\cos nx$$

$$a_0 = 1$$
$$a_1 = 0,$$
$$a_2 = \frac{1}{2}$$

35. Sol.

$$\frac{d^2y}{dt^2}+\frac{dy}{dt}+\frac{5y}{4} = 0$$

$$y(0) = 1$$
$$y'(0) = 0$$

By applying Laplace transform,

$$s^2Y(s)-s(1)+sY(s)-1+\frac{5}{4}Y(s) = 0$$

$$Y(s) = \frac{s+1}{s^2+s+\frac{5}{4}} = \frac{s+1}{\left(s+\frac{1}{2}\right)^2+1}$$

$$= \frac{\left(s+\frac{1}{2}\right)}{\left(s+\frac{1}{2}\right)^2+1}+\frac{\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+1}$$

By taking inverse Laplace transform,

$$y(t) = e^{-t/2}\left[\cos(t)+\frac{1}{2}\sin(t)\right];\ t>0$$

At $t=\pi$,

$$y(t=\pi) = e^{-\pi/2}[(-1)+(0)]$$

$$= -e^{-\pi/2} = -0.2078 \simeq -0.21$$

36. (a)

$$a_5 = \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin 5x\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}-\pi\sin 5x dx+\int_0^{\pi}\pi\sin 5x dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left[\frac{-\cos 5x}{5}\right]_{-\pi}^{0}+\pi\left[\frac{-\cos 5x}{5}\right]_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\pi\left(\frac{-2}{5}\right)+\pi\left(\frac{2}{5}\right)\right] = \frac{1}{\pi}\left[\frac{4\pi}{5}\right] = \frac{4}{5}$$

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