Important Questions for Class 12

Maths

Chapter 6 - Applications of Derivatives

Very Short Answer Questions **1 Mark**

1. The side of a square is increasing at the rate of $0.2\,cm/sec$. Find the rate of increase of perimeter of the square.

Ans: It is given that the side of a square is increasing at the rate of $0.2\,cm/sec$.

Let us consider the edge of the given cube be $x\,cm$ at any instant.

According to the question,

The rate of side of the square increasing is,

$$\frac{dx}{dt} = 0.2\,cm/sec\ldots...(i)$$

Therefore the perimeter of the square at any time t will be,

$$P = 4x\,cm$$

By applying derivative with respect to time on both sides, we get

$$\Rightarrow \frac{dP}{dt} = \frac{d(4x)}{dt}$$

$$\Rightarrow \frac{dP}{dt} = 4\frac{dx}{dt}$$

$$\Rightarrow \frac{dP}{dt} = 4 \times 0.2 = 0.8\,cm/sec$$

Hence from equation (i). The rate at which the perimeter of the square will increase is 0.8 cm/sec.

2. The radius of the circle is increasing at the rate of $0.7\,\text{cm/sec}$. What is the rate of increase of its circumference?

Ans: It is given that the radius of a circle is increasing at the rate of $0.7\,\text{cm/sec}$.

Let us consider that the radius of the given circle be r cm at any instant.

According to the question,

The rate of radius of a circle is increasing as,

$$\frac{dr}{dt} = 0.7\,\text{cm/sec} \quad \text{...(i)}$$

Now the circumference of the circle at any time t will be,

$C = 2\pi\text{cm}$

By applying derivative with respect to time on both sides, we get

$$\Rightarrow \frac{dC}{dt} = \frac{d(2\pi r)}{dt}$$

$$\Rightarrow \frac{dC}{dt} = 2\pi\frac{dr}{dt}$$

$$\Rightarrow \frac{dC}{dt} = 2\pi \times 0.7 = 1.4\pi\,\text{cm/sec}$$

From the equation (i). We can conclude that the rate at which the circumference of the circle will be increasing is $1.4\pi\,\text{cm/sec}$

3. If the radius of a soap bubble is increasing at the rate of $\frac{1}{2}\text{cm/sec}$. At what rate its volume is increasing when the radius is 1cm.

Ans: It is given that the radius of an air bubble is increasing at the rate of $0.5\,\text{cm/sec}$.

Let us consider that the radius of the given air bubble be r cm and let V be the volume of the air bubble at any instant.

According to the question,

The rate at which the radius of the bubble is increasing is,

$$\frac{dr}{dt} = 0.5\,\text{cm/sec} \quad \ldots \text{(i)}$$

The volume of the bubble, i.e., volume of sphere is $V = \frac{4}{3}\pi r^3$

By applying derivative with respect to time on both sides,

$$\Rightarrow \frac{dV}{dt} = \frac{d\left(\frac{4}{3}\pi r^3\right)}{dt}$$

$$\Rightarrow \frac{dV}{dt} = \frac{4}{3}\pi\frac{d\left(r^3\right)}{dt}$$

$$\Rightarrow \frac{dV}{dt} = \frac{4}{3}\pi \times 3r^2\frac{dr}{dt}$$

$$\Rightarrow \frac{dV}{dt} = 4\pi r^2 \times 0.5 \quad \ldots\text{(ii)}$$

When the radius is 1cm,

The above equation becomes

$$\Rightarrow \frac{dV}{dt} = 4\pi \times (1)^2 \times 0.5$$

$$\Rightarrow \frac{dV}{dt} = 2\pi\,\text{cm}^3/\text{sec}$$

Hence the volume of air bubble is increasing at the rate of $2\pi\text{cm}^3/\text{sec}$.

4. A stone is dropped into a quiet lake and waves move in circles at a speed of $4\text{cm}/\text{sec}$. At the instant when the radius of the circular wave is 10cm, how fast is the enclosed area increasing?

Ans: It is given that when a stone is dropped into a quiet lake and waves are formed which moves in circles at a speed of $4\text{cm}/\text{sec}$.

Let us consider that,

r be the radius of the circle and A be the area of the circle.

When a stone is dropped into the lake, waves are formed which moves in circle at speed of $4\,\text{cm}/\sec$.

Thus, we can say that the radius of the circle increases at a rate of ,

$$\frac{dr}{dt} = 4\,\text{cm/sec}$$

Area of the circle is πr^2, therefore

$$\Rightarrow \frac{dA}{dt} = \frac{d\left(\pi r^2\right)}{dt}$$

$$\Rightarrow \frac{dA}{dt} = \pi\frac{d\left(r^2\right)}{dt}$$

$$\Rightarrow \frac{dA}{dt} = \pi \times 2r\frac{dr}{dt}$$

$$\Rightarrow \frac{dA}{dt} = 2\pi r \times 4 \ldots\ldots \text{(ii)}$$

Hence, when the radius of the circular wave is $10\,\text{cm}$, the above equation becomes

$$\Rightarrow \frac{dA}{dt} = 2\pi \times 10 \times 4$$

$$\Rightarrow \frac{dA}{dt} = 80\pi\,\text{cm}^2/\text{sec}$$

Thus, the enclosed area is increasing at the rate of $80\pi\,\text{cm}^2/\text{sec}$.

5. The total revenue in rupees received from the sale of x units of a product is given by, $R(x) = 13x^2 + 26x + 15$. Find the marginal revenue when $x = 7$.

Ans: Marginal revenue is the rate of change of total revenue with respect to the number of units sold.

Let us consider 'MR' be the marginal revenue, therefore

$$MR = \frac{dR}{dx}$$

It is given that,

Total revenue, i.e., $R(x) = 13x^2 + 26x + 15 \quad \ldots(1)$

We need to find marginal revenue when $x = 7$

i.e., MR when $x = 7$

$$\Rightarrow MR = \frac{d(R(x))}{dx}$$

$$\Rightarrow MR = \frac{d\left(13x^2 + 26x + 15\right)}{dx}$$

$$\Rightarrow MR = \frac{d\left(13x^2\right)}{dx} + \frac{d(26x)}{dx} + \frac{d(15)}{dx}$$

$$\Rightarrow MR = 13\frac{d\left(x^2\right)}{dx} + 26\frac{d(x)}{dx} + 0$$

$$\Rightarrow MR = 13 \times 2x + 26$$

$$\Rightarrow MR = 26x + 26$$

$$\Rightarrow MR = 26(x + 1)$$

Taking $x = 7$, we get

$$\Rightarrow MR = 26\left(7 + 1\right)$$

$$\Rightarrow MR = 26 \times 8$$

$$\Rightarrow MR = 208$$

Therefore, the required marginal revenue is $\text{Rs}\,208$.

6. Find the maximum and minimum values of function $f(x) = \sin 2x + 5$.

Ans: Given function is,

$$f(x) = \sin 2x + 5$$

We know that,

$-1 \le \sin\theta \le 1, \ \forall\theta \in R$

$-1 \le \sin 2x \le 1$

Adding 5 on both sides,

$-1+5 \le \sin 2x + 5 \le 1+5$

$4 \le \sin 2x + 5 \le 6$

Therefore,

Max value of $f(x) = \sin 2x + 5$ will be 6 and,

Min value of $f(x) = \sin 2x + 5$ will be 4.

7. Find the maximum and minimum values (if any) of the function $f(x) = -|x-1| + 7 \forall x \in R$

Ans: Given equation is $f(x) = -|x+1| + 3$

$|x+1| > 0$

$\Rightarrow -|x+1| < 0$

Maximum value of $g(x) =$ maximum value of $-|x+1| + 7$

$\Rightarrow 0 + 7 = 7$

Maximum value of $f(x) = 3$

There is no minimum value of $f(x)$.

8. Find the value of a for which the function $f(x) = x^2 - 2ax + 6, x > 0$ is strictly increasing.

Ans: Given function is $f(x) = x^2 - 2ax + 6, x > 0$

It will be strictly increasing when $f'(x) > 0$.

$f'(x) = 2x - 2a > 0$

$\Rightarrow 2(x - a) > 0$

$\Rightarrow x - a > 0$

$\Rightarrow a < x$

But $x > 0$

Therefore, maximum possible value of a is 0 and all other values of *a* will be less than 0.

Hence, we get $a \leq 0$.

9. Write the interval for which the function $\mathbf{f(x) = \cos x, 0 \leq x \leq 2\pi}$ is decreasing.

Ans: The given function is $f(x) = \cos x, 0 \leq x \leq 2\pi$.

It will be a strictly decreasing function when $f'(x) < 0$.

Differentiating w.r.t. x, we get

$f'(x) = -\sin x$

Now,

$f'(x) < 0$

$\Rightarrow -\sin x < 0$

$\Rightarrow \sin x > 0 \qquad \text{i.e.,} (0, \pi)$

Hence, the given function is decreasing in $(0, \pi)$.

10. What is the interval on which the function $\mathbf{f(x) = \dfrac{\log x}{x}, x \in (0, \infty)}$ is increasing?

Ans: The given function is $f(x) = \dfrac{\log x}{x}, x \in (0, \infty)$.

It will be a strictly increasing function when $f'(x) > 0$.

$$f(x) = \frac{\log x}{x}$$

Therefore,

$$f^{'}(x) = \frac{1}{x^2} - \frac{\log x}{x^2}$$
$$f^{'}(x) = \frac{1 - \log x}{x^2}$$
$$\because f^{'}(x) > 0$$
$$\Rightarrow \frac{1 - \log x^2}{x^2} > 0$$
$$\Rightarrow 1 - \log x > 0$$
$$\Rightarrow 1 > \log x$$
$$\Rightarrow e > x$$

Therefore, $f(x)$ is increasing in the interval $(0, e)$.

11. For which values of x, the functions $y = x^4 - \frac{4}{3}x^3$ is increasing?

Ans: The given function is $y = x^4 - \frac{4}{3}x^3$

It will be a strictly increasing function when $f'(x) > 0$.

$f'(x) > 0$ and,

$$f'(x) = 4x^3 - 4x^2$$
$$f'(x) = 4x^2(x - 1)$$
$$4x^2(x - 1) > 0$$

Now,

$$\frac{dy}{dx} = 0 \Rightarrow x = 0, x = 1$$

Since $f'(x) < 0 \forall x \in (-\infty, 0) \cup (0,1)$ and f is continuous in $(-\infty, 0]$ and $[0,1]$. Therefore f is decreasing in $(-\infty, 1]$ and f is increasing in $[1, \infty)$.

Here f is strictly decreasing in $(-\infty, 0) \cup (0,1)$ and is strictly increasing in $(1, \infty)$

12. Write the interval for which the function $f(x) = \frac{1}{x}$ is strictly decreasing.

Ans: The given equation is

$$f(x) = \frac{1}{x}.$$

It will be a strictly decreasing function when $f'(x) < 0$.

$$f(x) = x + \frac{1}{x}$$

$$\Rightarrow f'(x) = 1 - \frac{1}{x^2}$$

$$\Rightarrow f'(x) = \frac{x^2 - 1}{x^2}$$

$$\Rightarrow f'(x) = 0$$

$$\Rightarrow \frac{x^2 - 1}{x^2} = 0$$

$$\Rightarrow x^2 - 1 = 0$$

$$\Rightarrow x = \pm 1$$

The intervals are $(-\infty, -1), (-1, 1), (1, \infty)$

$$f'(0) < 0$$

$\therefore$ Strictly decreasing in $(-1, 1)$

13. Find the sub-interval of the interval $(0, \pi/2)$ in which the function $f(x) = \sin 3x$ is increasing.

Ans: The given function is $f(x) = \sin 3x$

On differentiating the above function with respect to x, we get,

$f'(x) = 3\cos 3x$

$f(x)$ will be increasing, when $f'(x) > 0$

Given that $x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow 3x \in \left(0, \frac{3\pi}{2}\right)$

Cosine function is positive in the first quadrant and negative in the second quadrant.

case 1:

When $3x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow \cos 3x > 0$

$\Rightarrow 3\cos 3x > 0$

$\Rightarrow f'(x) > 0 \text{ for } 0 < 3x < \frac{\pi}{2}$

$\Rightarrow f'(x) > 0 \text{ for } 0 < x < \frac{\pi}{6}$

$\therefore f(x)$ is increasing in the interval $\left(0, \frac{\pi}{6}\right)$

case 2:

When $3x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

$\Rightarrow \cos 3x < 0$

$\Rightarrow 3\cos 3x < 0$

$\Rightarrow f'(x) < 0$ for $\frac{\pi}{2} < 3x < \frac{3\pi}{2}$

$\Rightarrow f'(x) < 0$ for $\frac{\pi}{6} < x < \frac{\pi}{2}$

$\therefore f(x)$ is decreasing in the interval $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$

14. Without using derivatives, find the maximum and minimum value of $y = |3\sin x + 1|$.

Ans: The given function is $y = |3\sin x + 1|$

Maximum and minimum values of $\sin x = \{-1, 1\}$ respectively.

Therefore, the value of the given function will be maximum and minimum at only these points.

Taking $\sin x = -1$

$y = |3 \times (-1) + 1| => 2$

Now, put $\sin x = 1$

$y = |3 \times 1 + 1| => 4$

This maximum and minimum values of the given function are 4 and 2 respectively.

15. If $f(x) = ax + \cos x$ is strictly increasing on R, find a.

Ans: It is given that the function $f(x) = ax + \cos x$ is strictly increasing on R

Here function, $f(x) = ax + \cos x$

Differentiating $f(x)$ with respect to x we get,

$f'(x) = a + (-\sin x) = a - \sin x$

for strictly increasing, $f'(x) > 0$

Therefore,

$a - \sin x > 0$ it will be correct for all real value of x only when $a \in (-1,1)$

Hence the value of a belongs to $(-1,1)$.

16. Write the interval in which the function $f(x) = x^9 + 3x^7 + 64$ is increasing.

Ans: The given function is $f(x) = x^9 + 3x^7 + 64$.

For it to be a increasing function $f'(x) > 0$

On differentiating both sides with respect to *x*, we get

$$f(x) = x^9 + 3x^7 + 64$$
$$\Rightarrow f'(x) = 9x^8 + 21x^6$$
$$\Rightarrow f'(x) = 3x^6(3x^2 + 7)$$

$\because$ function is increasing.

$3x^6(3x^2 + 7) > 0$

$\Rightarrow$ function is increasing on R.

17. What is the slope of the tangent to the curve $f(x) = x^3 - 5x + 3$ at the point whose x co-ordinate is 2?

Ans: The given equation of the curve is $f = x^3 - 5x + 3 \quad \ldots(1)$

When $x = 2$,

$$y = 2^3 - 5.2 + 3$$
$$y = 8 - 10 + 3$$
$$y = 1$$

Therefore, the point on the curve is $(2,1)$.

Differentiating equation (1) with respect to x, we get

$$\frac{dy}{dx}=3x^2-5$$

Slope of tangent $\frac{dy}{dx}$

Since $x=2$,

$$\Rightarrow 3.2^2-5$$
$$\Rightarrow 12-5$$
$$\Rightarrow 7$$

Hence the slope of tangent is 7.

18. At what point on the curve $y=x^2$ does the tangent make an angle of 45° with positive direction of the x-axis?

Ans: The given equation of the curve is $y=x^2$

Differentiating the above with respect to x,

$$\Rightarrow \frac{dy}{dx}=2x^{2-1}$$
$$\Rightarrow \frac{dy}{dx}=2x \;...(1)$$

So,

$$\frac{dy}{dx}= \text{ The slope of tangent } =\tan\theta$$

The tangent makes an angle of 45° with x-axis

$$\frac{dy}{dx}=\tan 45^\circ=1\ldots(2)$$

Because the $\tan 45^\circ=1$

From the equation (1) & (2), we get

$\Rightarrow 2x = 1$

$\Rightarrow x = \dfrac{1}{2}$

Substitute $x = \dfrac{1}{2}$ in $y = x^2$

$\Rightarrow y = \left(\dfrac{1}{2}\right)^2$

$\Rightarrow y = \dfrac{1}{4}$

Hence, the required point is $\left(\dfrac{1}{2}, \dfrac{1}{4}\right)$.

19. Find the point on the curve $y = 3x^2 - 12x + 9$ at which the tangent is parallel to x-axis.

Ans: The given equation of the curve is $y = 3x^2 - 12x + 9$.

Differentiating the above equation with respect to *x*, we get

$\dfrac{dy}{dx} = 6x - 12$

$m = 6x - 12$

$\dfrac{dy}{dx} =$ The slope of tangent $= \tan\theta$

If the tangent is parallel to x-axis.

$m = 0$

$\Rightarrow 6x - 12 = 0$

$\Rightarrow x = 2$

When $x = 2$, then

$y = 3.2^2 - 12.2 + 9$
$y = 12 - 24 + 9$
$y = -3$

Hence, the required point $(x, y) = (2, -3)$.

20. What is the slope of the normal to the curve $y = 5x^2 - 4\sin x$ at $x = 0$.

Ans: The given equation of the curve is $y = 5x^2 - 4\sin x$.

Differentiating the above equation with respect to *x*, we get

$$\frac{dy}{dx} = 10x - 4\cos x$$

$$\frac{dy}{dx} = \text{ The slope of tangent } = \tan\theta$$

Thus, slope of tangent at $x = 0$ is,

$\Rightarrow 10 \times 0 - 4\cos 0$
$\Rightarrow 0 - 4 = 4$

Hence, slope of normal at the same point is,

$\because m_1 \times m_2 = -1$
$\Rightarrow 4 \times m_2 = -1$
$\Rightarrow m_2 = \frac{-1}{4}$

21. Find the point on the curve $y = 3x^2 + 4$ at which the tangent is perpendicular to the line with slope $-\frac{1}{6}$

Ans: Given,

The curve $y = 3x^2 + 4$ and the Slope of the tangent is $\frac{-1}{6}$

$y = 3x^2 + 4$

Differentiating the above w.r.t x

$$\Rightarrow \frac{dy}{dx} = 2 \times 3x^{2-1} + 0$$
$$\Rightarrow \frac{dy}{dx} = 6x \quad \ldots (1)$$

Since, tangent Is perpendicular to the line,

$$\therefore \text{ The Slope of the normal } = \frac{-1}{\text{The Slope of the tangent}}$$

i.e., $\frac{-1}{6} = \frac{-1}{6x}$

$$\Rightarrow \frac{1}{6} = \frac{1}{6x}$$

$$\Rightarrow x = 1$$

Substituting $x = 1$ in $y = 3x^2 + 4$

$$\Rightarrow y = 3(1)^2 + 4$$
$$\Rightarrow y = 3 + 4$$
$$\Rightarrow y = 7$$

Thus, the required point is $(1, 7)$.

22. Find the point on the curve $y = x^2$ where the slope of the tangent is equal to the $y-$ co-ordinate.

Ans: Given, equation of curve is $y = x^3$

On differentiating both sides with respect to x, we get

$$\frac{dy}{dx}=3x^2$$

It is given that Slope of tangent $=y$-coordinate of the point

or $\frac{dy}{dx}=y$

or $3x^2=y \quad \left[\because \frac{dy}{dx}=3x^2\right]$

or $3x^2=x^3 \quad \left[\because y=x^3\right]$

or $3x^2-x^3=0$ or $x^2(3-x)=0$

or Either $x^2=0$ or $3-x=0$

$\therefore x=0,3$

Now, on putting $x=0$ and 3 in Eq. (i), we get

$y=(0)^3=0$ [at $x=0$]

$y=(3)^3=27$[at $x=3$]

Hence, the required points are $(0,0)$ and $(3,27)$.

23. If the curves $y=2e^x$ and $y=ae^{-x}$ intersect orthogonally (cut at right angles), what is the value of a ?

Ans: The given equation of the curves are $y=2e^x$ and $y=ae^{-x}$.

On differentiating both equations with respect to *x*, we get

$y=2e^x$

$$\frac{dy}{dx}=m_1=2e^x$$

$y = ae^{-x}$

$\frac{dy}{dx} = m_2 = -ae^{-x}$

two curves cut orthogonally when $m_1 \times m_2 = -1$

$2e^x \times \left(-ae^{-x}\right) = -1$ (from above)

or $-2a = -1$

or $a = \frac{1}{2}$

So the condition being orthogonally is when $a = \frac{1}{2}$.

24. Find the slope of the normal to the curve $y = 8x^2 - 3$ at $x = \frac{1}{4}$.

Ans: The given equation of the curve is $y = 8x^2 - 3$**.**

On differentiating both sides with respect to *x*, we get

$\frac{dy}{dx} = 16x$

$\frac{dy}{dx} =$ The slope of tangent $= \tan\theta$

So, slope of tangent at $\left(\frac{1}{4}, 0\right)$.

$\left(\frac{dy}{dx}\right)_{x=\frac{1}{4}} = 16 \times \frac{1}{4} = 4$

Let,

$m_1 = 4$.

Since, normal and tangent are perpendicular to each other.

$\Rightarrow m_1 \times m_2 = -1$
$\Rightarrow 4 \times m_2 = -1$
$\Rightarrow m_2 = \frac{-1}{4}$

Hence slope of normal at the same point is $\frac{-1}{4}$.

25. Find the rate of change of the total surface area of a cylinder of radius r and height h with respect to radius when height is equal to the radius of the base of cylinder.

Ans: As we know that total surface area of cylinder $= S = 2\pi r^2 + 2\pi rh \ldots(1)$

Given,

Height is equal to the radius of the base of cylinder.

On differentiating equation (1) with respect to *r*, we get

$$\frac{dS}{dr} = 4\pi r + 2\pi h$$

Here,

S = surface area of cylinder and r = radius of cylinder.

Since, Height is equal to the radius of the base of cylinder, i.e., $h = r$.

Thus, rate of change of total surface area of cylinder when the radius is varying given by,

$\Rightarrow 4\pi r + 2\pi h \ [h = r]$
$\Rightarrow 4\pi r + 2\pi r$
$\Rightarrow 6\pi r$

26. Find the rate of change of the area of a circle with respect to its radius. How fast is the area changing w.r.t. its radius when its radius is 3cm?

Ans: The area of a circle (A) with radius (r) is given by,

$$A = \pi r^2$$

Now, the rate of change of the area with respect to its radius is given by,

$$\frac{dA}{dr} = \frac{d}{dr}\left(\pi r^2\right) = 2\pi r$$

At $r = 3\text{cm}$

$$\frac{dA}{dr} = 2\pi(3) = 6\pi$$

27. For the curve $y = (2x+1)^3$ find the rate of change of slope at $x = 1$.

Ans: The equation of the curve is $y = (2x+1)^3$

On differentiating with respect to x, we get

$$y' = 3(2x+1)^2 \cdot 2$$
$$y' = 6(2x+1)^2$$

Rate of change of slope at $x = 1$

$$y' = 6(2(1)+1)^2$$
$$= 6(2+1)^2$$
$$= 6(9)$$
$$= 54$$

28. Find the slope of the normal to the curve

$$\mathbf{x = 1 - a\sin\theta \; ; \; y = b\cos^2\theta \text{ at } \theta = \frac{\pi}{2}}$$

Ans: Given,

$x = 1 - a\sin\theta$ and $y = b\cos^2\theta$.

On differentiating both sides with respect to θ, we get

$$\therefore \frac{dx}{d\theta} = -a\cos\theta$$

and $\dfrac{dy}{d\theta} = 2b\cos\theta(-\sin\theta) = -2b\sin\theta\cos\theta$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-2b\sin\theta\cos\theta}{-a\cos\theta} = \frac{2b}{a}\sin\theta$$

Therefore, the slope of the tangent at $\theta = \dfrac{\pi}{2}$ is given by,

$$\left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{4}} = \frac{2b}{a}$$

Hence, the slope of the normal at $\theta = \dfrac{\pi}{2}$ is given by,

$$-\frac{1}{\text{slope of the tangent at } \theta = \frac{\pi}{4}} = \frac{-1}{\frac{2b}{a}} = -\frac{a}{2b}$$

29. If a manufacturer's total cost function is $C(x) = 1000 + 40x + x^2$, where x is the output, find the marginal cost for producing 20 units.

Ans: The given cost function is $C(x) = 1000 + 40x + x^2$.

Marginal cost is the rate of change of total cost with respect to output (unit produced)

Let MC be the marginal cost,

$\therefore MC = \frac{dC}{dx}$

We need to find the marginal cost when 20 units are produced, i.e.,

$MC = \frac{dC}{dx}$ at $x = 3$

Now,

$MC = \frac{dC}{dx}$

$MC = \frac{1000 + 40x + x^2}{dx}$

$MC = 0 + 40 + 2x$

$MC = 2x + 40$

We need MC at $x = 20$.

On putting $x = 20$.

$MC = 2 \times 20 + 40$

$MC = 40 + 40$

$MC = 80$

Hence, the required marginal cost is Rs. 80.

30. Find 'a' for which $f(x) = a(x + \sin x)$ is strictly increasing on R.

Ans: The given equation is $f(x) = a(x + \sin x)$.

On differentiating both sides with respect to *x*, we get

$f(x) = a(x + \sin x)$

$f'(x) = a(1 + \cos x)$

For $f(x)$ to be increasing, we must have

$f'(x) > 0$

$\Rightarrow a(1 + \cos x) > 0 \ldots (1)$

We know,

$-1 \le \cos x \le 1, \forall x \in R$

$\Rightarrow 0 \le (1+\cos x) \le 2, \forall x \in R$

$\therefore a > 0$ [From eq $\cdot$(1)]

$\Rightarrow a \in (0,\infty)$ or $a > 0.$

Short Answer Questions **4 Marks**

31. A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the y co-ordinate is changing 8 times as fast as the x coordinate.

Ans: The given equation of the curve is $6y = x^3 + 2$.

Differentiating both sides with respect to *t*,

$$6\frac{dy}{dt} = 3x^2\frac{dx}{dt} + 0$$

$$\Rightarrow 2\frac{dy}{dt} = x^2\frac{dx}{dt}$$

It is given that, $\left(\frac{dy}{dt} = 8\frac{dx}{dt}\right)$,

Therefore, we have,

$$2\left(8\frac{dx}{dt}\right) = x^2\frac{dx}{dt}$$

$$\Rightarrow 16\frac{dx}{dt} = x^2\frac{dx}{dt}$$

$$\Rightarrow \left(x^2 - 16\right)\frac{dx}{dt} = 0$$

$$\Rightarrow x^2 - 16 = 0$$

$$\Rightarrow x = \pm 4$$

When $x=4$,

$$\Rightarrow y=\frac{4^3+2}{6}$$

$$\Rightarrow y=\frac{66}{6}=11$$

when $x=(-4)$,

$$\Rightarrow y=\frac{(-4)^3+2}{6}$$

$$\Rightarrow y=-\frac{62}{6}=-\frac{31}{3}$$

Hence, the points required on the curve are $(4,11)$ and $\left(-4,\frac{-31}{3}\right)$.

32. A ladder 5 metres long is leaning against a wall. The bottom of the ladder is pulled along the ground away from the wall at the rate of $2\,cm/sec$. How fast is its height on the wall decreasing when the foot of the ladder is 4 metres away from the wall?

Ans: Consider *y* m be the height of the wall at which the ladder touches and let the foot of the ladder be *x* m away from the wall.

Then, by Pythagoras theorem, we have:

$x^2+y^2=25$ [Length of the ladder $=5\,m$]

$$\Rightarrow y=\sqrt{25-x^2}$$

Then, the rate of change of height (y) with respect to time (t) is given by, $\frac{dy}{dt}=\frac{-x}{\sqrt{25-x^2}}\cdot\frac{dx}{dt}$

It is given that $\frac{dx}{dt}=2\,cm/s$

$\therefore \frac{dy}{dt} = \frac{-2x}{\sqrt{25-x^2}}$

Now, when $x = 4\,m$, we have:

$$\frac{dy}{dt} = \frac{-2\times 4}{\sqrt{25-4^2}} = -\frac{8}{3}$$

Hence, the height of the ladder on the wall is decreasing at the rate of $-\frac{8}{3}\,cm/s$.

33. A balloon which always remain spherical is being inflated by pumping in 900 cubic cm of a gas per second. Find the rate at which the radius of the balloon increases when the radius is $15\,cm$.

Ans: Consider the volume of the sphere be *V* with radius *r*.

Its volume is given by $V = \frac{4}{3}\pi r^3$.

Therefore, rate of change of volume *V* with respect to time *t* is given by,

$$\Rightarrow \frac{dV}{dt} = \frac{dV}{dr}.\frac{dr}{dt}$$
$$\Rightarrow \frac{dV}{dt} = 4\pi r^2.\frac{dr}{dt}$$

It is given that $\frac{dV}{dt} = 900\ cm^3/s$.

$$\therefore 900 = 4\pi r^2.\frac{dr}{dt}$$
$$\Rightarrow \frac{dr}{dt} = \frac{900}{4\pi r^2}$$
$$\Rightarrow \frac{dr}{dt} = \frac{225}{\pi r^2}$$

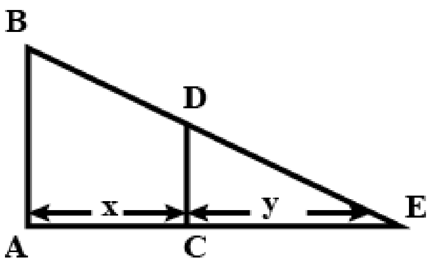
Hence, when radius is 15 cm.

$$\Rightarrow \frac{dr}{dt}=\frac{225}{\pi(15)^2}$$

$$\Rightarrow \frac{dr}{dt}=\frac{1}{\pi}\text{ cm/s}$$

34. A man 2 meters high walks at a uniform speed of $5\,km/hr$ away from a lamp post 6 metres high. Find the rate at which the length of his shadow increases.

Ans: Consider AB be the lamp post.



Let at any time *t*, the man CD be at distance *x* meters from the lamp post and 9 meters he length of the shadow.

$$\frac{dx}{dt}=5\,km/hr$$

Clearly ΔABC and ΔCDE are similar,

$$\Rightarrow \frac{AB}{CD}=\frac{AE}{CE}$$

$$\Rightarrow \frac{6}{2}=\frac{x+y}{y}$$

$$\Rightarrow 3y=x+y$$

$$\Rightarrow 2y=x$$

On differentiating both sides with respect to *x*, we get

$$\Rightarrow 2\frac{dy}{dt}=\frac{dx}{dt}$$

$$\Rightarrow 2\frac{dy}{dt} = \frac{5\times1000}{60}$$

$$\Rightarrow \frac{dy}{dt} = \frac{5\times100}{12} = \frac{500}{12} m/\min$$

$$\Rightarrow \frac{dy}{dt} = \frac{5}{2} km/h$$

35. Water is running out of a conical funnel at the rate of $5 cm^3/sec$. If the radius of the base of the funnel is $10 cm$ and altitude is $20 cm$, find the rate at which the water level is dropping when it is $5 cm$ from the top.

Ans: Given,

Water is running out at the rate of $5 cm^3/sec$, therefore

Rate in decrease in volume $\frac{dV}{dt} = 5 cm^3/sec$

Radius of the base of the conical funnel = *r* = 10 cm.

Altitude of the conical funnel = *h* = 5 cm.

The volume of water present in the conical flask at any time is given by

$$V = \frac{1}{3}\pi r^2 h \cdots (1)$$

When the flask is full, i.e., *r* = 10 cm and *h* = 5 cm. The ratio of $r:h$ remains same all through the cone.

so, $\frac{r}{h} = \frac{1}{2}$

$$r = \frac{h}{2}$$

Putting it in (1)

$$\Rightarrow V=\frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$$

$$\Rightarrow V=\frac{1}{12}\pi h^3$$

Differentiating both sides with respect to *t*,

$$\Rightarrow \frac{dV}{dt}=\frac{1}{12}\pi 3h^2\frac{dh}{dt}$$

$$\Rightarrow 5=\frac{\pi}{4}h^2\frac{dh}{dt}$$

when $h=5cm$

$$\frac{20}{\pi}=5^2\frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt}=\frac{4}{5\pi}\ cm/sec$$

36. The length x of a rectangle is decreasing at the rate of $5cm/sec$ and the width y is increasing as the rate of $4cm/sec$ when $x=8cm$ and $y=6cm$. Find the rate of change of

(a) Perimeter

Ans: Given,

Length (x) is decreasing at the rate of $5cm/sec$

Width (y) is increasing at the rate of $4cm/sec$

$$\Rightarrow \frac{dx}{dt}=-5cm/sec \text{ and } \frac{dy}{dt}=4cm/sec$$

Therefore, the perimeter (P) of the rectangle $=2(x+y)$

$$\Rightarrow \frac{dP}{dt}=2\left(\frac{dx}{dt}+\frac{dy}{dt}\right)$$

$$\Rightarrow \frac{dP}{dt} = 2(-5+4) = -2 \text{ cm/sec}$$

Hence, the perimeter is decreasing at the rate of -2 cm/sec

(b) Area of the rectangle.

Ans: Length (x) is decreasing at the rate of 5cm/sec

Width (y) is increasing at the rate of 4cm/sec

$$\Rightarrow \frac{dx}{dt} = -5\text{cm/sec} \text{ and } \frac{dy}{dt} = 4\text{cm/sec}$$

Therefore, the area (A) of the rectangle $= x \times y$

$$\therefore \left(\frac{dA}{dt}\right)_{x=8,y=6} = \left(y\frac{dx}{dt} + x\frac{dy}{dt}\right)_{x=8,y=6}$$

$$\Rightarrow \left(\frac{dA}{dt}\right)_{x=8,y=6} = 6(-5) + 8(4) = 2\text{cm}^2/\text{sec}$$

Hence, the area is increasing at the rate of $2\text{cm}^2/\text{sec}$.

37. Sand is pouring from a pipe at the rate of 12c.c / sec. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when height is 4cm ?

Ans: It is given that,

Sand is pouring from a pipe at the rate of 12c.c / sec.

Let us consider,

$r =$ radius; $h =$ height ; $V =$ Volume of sand cone, $t =$ time

Height of the cone $= \frac{1}{6}$ of the radius of the base.

$h = \frac{1}{6}r$ or $r = 6h$

Given,

$h = 4\,cm; \frac{dV}{dt} = 12\,cm^3/s$

Since the falling sand forms a cone therefore volume of the cone $= V = \frac{1}{3}\pi r^2 h$

Substituting the value of *r* in the above formula, we get

$\Rightarrow \frac{1}{3}\pi(6h)^2 h$

$\Rightarrow \frac{36h^3\pi}{3}$

$\Rightarrow 12\pi h^3$

Differentiating with respect to *t*,

$\therefore \frac{dV}{dt} = 12\pi \cdot 3h^2 \frac{dh}{dt}$

$\Rightarrow 12 = 12\pi \cdot 3(4)^2 \frac{dh}{dt}$

$\therefore \frac{dh}{dt} = \frac{1}{48\pi} cm/s$

38. The area of an expanding rectangle is increasing at the rate of $48\,cm^2/sec$ The length of the rectangle is always equal to the square of the breadth. At what rate is the length increasing at the instant when the breadth is $4.5\,cm$?

Ans: It is given that the expanding rectangle is increasing at the rate of $48\,cm^2/sec$.

Let us consider the length of rectangle be *l* and breadth be *b*.

Area of the rectangle,

$A = l \times b$

Differentiating both sides with respect to time (*t*), we get

$$\Rightarrow \frac{dA}{dt} = \frac{d[l \times b]}{dt}$$

Using product rule of differentiation given as: $-\frac{d(u.v)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$, we get,

$$\Rightarrow \frac{dA}{dt} = l\frac{db}{dt} + b\frac{dl}{dt}$$

On substituting $\frac{dA}{dt} = 48$, we get,

$$\Rightarrow b\frac{dl}{dt} + l\frac{db}{dt} = 48 \quad ...(1)$$

It is also given that the length of the rectangle is always equal to the square of its breadth. Therefore,

$$\Rightarrow l = b^2$$

Differentiating both sides with respect to time (t), we get,

$$\Rightarrow \frac{dl}{dt} = 2b\frac{db}{dt}$$

$$\Rightarrow \frac{db}{dt} = \frac{1}{2b}\frac{dl}{dt}$$

So, substituting the value of I and $\frac{db}{dt}$ in equation (1), we get,

$$\Rightarrow b\frac{dl}{dt} + b^2 \times \frac{1}{2b}\frac{dl}{dt} = 48$$

$$\Rightarrow \frac{3b}{2}\left(\frac{dl}{dt}\right) = 48$$

$$\Rightarrow \frac{dl}{dt} = \frac{32}{b}$$

In order to find the rate of increase in length when the breadth of the rectangle is 4.5cm.

Substitute $b = 4.5$ in the above relation, we get,

$$\Rightarrow \frac{dl}{dt} = \frac{32}{4.5}$$

$$\Rightarrow \frac{dl}{dt} = \frac{32 \times 10}{45}$$

$$\Rightarrow \frac{dl}{dt} = \frac{64}{9} = 7.11\text{cm/sec}$$

Hence, the length of the rectangle is increasing at the rate of 7.11cm/sec.

39. Find a point on the curve $y = (x-3)^2$ where the tangent is parallel to the line joining the points $(4,1)$ and $(3,0)$.

Ans: It is given that the points are $(3,0)$ and $(4,1)$.

Equation of the curve is $y = (x-3)^2$.

Let us consider $y = f(x)$

Then,

$f(x) = (x-3)^2, x \in [3,4]$

Since $f(x)$ is a polynomial function $f(x)$ is continuous in [3, 4]

On differentiating the above equation, we get

$f'(x) = 2(x-3)$ which exists in $(3,4)$

Therefore, it is also differentiable in $(3,4)$

Hence both the conditions of Lagrange's Mean value theorem is satisfied.

$f(3)=0$

$f(4)=1$

$f'(c)=2(c-3)$

$\therefore 2c-6=\dfrac{1-0}{4-3}$

$\Rightarrow 2c-6=1n$

$\Rightarrow 2c=7$

$\therefore c=\dfrac{7}{2}$

$\dfrac{7}{2}\in(3,4)$

Since c is the x-coordinate of that point at which the tangent is parallel to the chord joining the points $(3,0)$ and $(4,1)$.

On putting $x=\dfrac{7}{2}$ in equation (1) we get

$$y=\left(\frac{7}{2}-3\right)^2=\frac{1}{4}$$

Hence the required points are $\left(\dfrac{7}{2},\dfrac{1}{4}\right)$.

40. Find the equation of all lines having slope zero which are tangents to the curve $\mathbf{y=\dfrac{1}{x^2-2x+3}}$.

Ans: It is given that the equation of the curve is $y=\dfrac{1}{x^2-2x+3}$

The slope of the tangent to the given curve at any point (x,y) is given by,

$$\frac{dy}{dx}=\frac{-(2x-2)}{\left(x^2-2x+3\right)^3}$$

$$\frac{dy}{dx}=\frac{-2(x-1)}{\left(x^2-2x+3\right)^3}$$

If the slope of the tangent is 0, i.e., $m=0$, then we have:

$$\frac{-2(x-1)}{\left(x^2-2x+3\right)^3}=0$$

When $x=1$ and $y=\frac{1}{1-2+3}=\frac{1}{2}$

The equation of the tangent through $\left(1,\frac{1}{2}\right)$ is given by,

$$y-y_1=m\left(x-x_1\right) \quad \left[\text{one point form}\right]$$

$$y-\frac{1}{2}=0(x-1)$$

$$\Rightarrow y=\frac{1}{2}$$

41. Prove that the curves $x=y^2$ and $xy=k$ cut at right angles if $8k^2=1$.

Ans: The given equation of the curves are,

$$x=y^2 \quad ...(i)$$
$$xy=k \quad ...(ii)$$

Let the curves cut each other at (a,b), then

From equation, we get

$$a=b^2 \quad \text{and} \quad ab=k$$
$$x=y^2 \quad \text{and} \quad xy=k$$

On differentiating the above equations, we get

$$1 = 2y\frac{dy}{dx}; \quad x\frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx} = \frac{1}{2y}; \quad \frac{dy}{dx} = \frac{-y}{x}$$

Slope at $a, b = \frac{1}{2b}$ and slope at $a, b = \frac{-b}{a}$.

$$m_1 = \frac{1}{2b}; \quad m_2 = \frac{-b}{a}$$

Since curves cut at right angles, therefore,

$$\Rightarrow m_1 m_2 = -1$$

$$\Rightarrow \frac{1}{2b}\frac{-b}{a} = -1$$

$$\Rightarrow 2a = 1$$

$$a = b^2 \text{ and } ab = k \quad (\text{from above})$$

$$\Rightarrow b^2 \cdot b = k$$

$$\Rightarrow b^{3/2} = k$$

$$\Rightarrow b = k^{2/3}$$

Substituting in equations (1), we get

$$\Rightarrow 2 \cdot b^2 = 1$$

$$\Rightarrow 2.k^{2/3} = 1$$

$$\therefore 8k^2 = 1$$

42. Find the equation of the normal at the point $\left(am^2, am^3\right)$ for the curve $ay^2 = x^3$.

Ans: It is given that the equation of curve is $ay^2 = x^3$.

On differentiating the above equation with respect to x, we get

$$\Rightarrow 2ay\frac{dy}{dx} = 3x^2$$
$$\Rightarrow \frac{dy}{dx} = \frac{3x^2}{2ay}$$

Slope of the tangent to the curve at $\left(am^2, am^3\right)$ is,

$$\left(\frac{dy}{dx}\right)_{\left(am^2, am^3\right)} = \frac{3\left(am^2\right)^2}{2a\left(am^3\right)}$$
$$\left(\frac{dy}{dx}\right)_{\left(am^2, am^3\right)} = \frac{3a^2m^4}{2a^2m^3} = \frac{3m}{2}$$

Let us consider $m_1 = \frac{3m}{2}$.

Since, tangent and normal are perpendicular to each other, Therefore,

$m_1 \times m_2 = -1$

Let the slope of normal be m_2.

Slope of normal at $\left(am^2, am^3\right)$,

$$\Rightarrow m_1 \times m_2 = -1$$
$$\Rightarrow \frac{3m}{2} \times m_2 = -1$$
$$\Rightarrow \frac{-1}{\text{slope of the tangent at } \left(am^2, am^3\right)} = \frac{-2}{3m}$$

Equation of the normal at $\left(am^2, am^3\right)$ is,

$$\Rightarrow y - am^3 = \frac{-2}{3m}\left(x - am^2\right)$$
$$\Rightarrow 3my - 3am^4 = -2x + 2am^2$$
$$\Rightarrow 2x + 3my - am^2\left(2 + 3m^2\right) = 0$$
$$\Rightarrow 2x + 3my = am^2\left(2 + 3m^2\right)$$

43. Show that the curves $4x = y^2$ and $4xy = k$ cut as right angles if $k^2 = 512$

Ans: The given equation of the curves are,

$\Rightarrow 4x = y^2 \quad ...(i)$

$\Rightarrow 4xy = k \quad ...(ii)$

We have to prove that two curves cut at right angles if $k^2 = 512$.

Now,

$4x = y^2$

Differentiating both sides with respect to *x*, we get

$$\Rightarrow 4 = 2y \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{y}$$

$$\Rightarrow m_1 = \frac{2}{y} \qquad \ldots(iii)$$

$4xy = k$

Differentiating both sides with respect to *x*, we get

$$\Rightarrow 4\left(1 \times y + x\frac{dy}{dx}\right) = 0$$

$$\Rightarrow y + x\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y}{x}$$

$$\Rightarrow m_2 = \frac{-y}{x} \qquad ...(iv)$$

Since the two curves intersect orthogonally.

$\therefore m_1 \cdot m_2 = -1$

$$\Rightarrow \frac{2}{y} \times \frac{-y}{x} = -1 \qquad \text{[From(iii) and (iv)]}$$

$$\Rightarrow \frac{-2}{x} = -1$$

$$\Rightarrow x = 2$$

Now,

$$\Rightarrow 4xy = k$$

$$\Rightarrow \left(y^2\right)y = k \qquad \text{[Since ,} 4x = y\text{]}$$

$$\Rightarrow y^3 = k$$

$$\Rightarrow y = k^{\frac{1}{3}}$$

On substituting $y = k^{\frac{1}{3}}$ in equation (i), we get,

$$\Rightarrow 4x = \left(k^{\frac{1}{3}}\right)^2$$

$$\Rightarrow 4 \times 2 = k^{\frac{2}{3}}$$

$$\Rightarrow 8 = k^{\frac{2}{3}}$$

$$\Rightarrow k^2 = (8)^3 \qquad \text{[Taking cube on both sides]}$$

$$\Rightarrow k^2 = 512$$

44. Find the equation of the tangent to the curve $y = \sqrt{3x - 2}$ which is parallel to the line $4x - y + 5 = 0$.

Ans: The equation of the given curve is $y = \sqrt{3x - 2}$.

Let the slope of the tangent to the given curve at any point (x, y) is given by,

On differentiating the above equation with respect to *x*, we get

$$\frac{dy}{dx}=\frac{3}{2\sqrt{3x-2}}$$

The equation of the given line is $4x-2y+5=0$.

$$4x-2y+5=0$$

$$\therefore \quad y=2x+\frac{5}{2} \quad \text{(which is of the form } y=mx+c\text{)}$$

$\therefore$ Slope of the line $=2$

Now, the tangent to the given curve can be parallel to the line $4x-2y-5=0$ only if the slope of the tangent is equal to the slope of the line.

$$\Rightarrow \frac{3}{2\sqrt{3x-2}}=2$$

$$\Rightarrow \sqrt{3x-2}=\frac{3}{4}$$

$$\Rightarrow 3x-2=\frac{9}{16}$$

$$\Rightarrow 3x=\frac{9}{16}+2=\frac{41}{16}$$

$$\Rightarrow x=\frac{41}{48}$$

When $x=\frac{41}{48}$ then

$$y=\sqrt{3\left(\frac{41}{48}\right)-2}$$

$$y=\sqrt{\frac{41}{16}-2}$$

$$y=\sqrt{\frac{41-32}{16}}$$

$y=\sqrt{\frac{9}{16}}$

$y=\frac{3}{4}$

Equation of tangent at point $\left(\frac{41}{48},\frac{3}{4}\right)$ is given by $\left(y-y_1\right)=m\left(x-x_1\right)$.

$$y-\frac{3}{4}=2\left(x-\frac{41}{48}\right)$$

$$\Rightarrow \frac{4y-3}{4}=2\left(\frac{48x-41}{48}\right)$$

$$\Rightarrow 4y-3=\frac{48x-41}{6}$$

$$\Rightarrow 24y-18=48x-41$$

$$\Rightarrow 48x-24y=23$$

Hence, the equation of the required tangent is $48x-24y=23$.

45. Find the equation of the tangent to the curve $\sqrt{x}+\sqrt{y}=a$ at the point $\left(\frac{a^2}{4},\frac{a^2}{4}\right)$.

Ans: The given equation of the curve is $\sqrt{x}+\sqrt{y}=a$.

As to find the slope of the tangent of the given curve, differentiate with respect to x

$$\Rightarrow \frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\left(\frac{dy}{dx}\right)=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{\sqrt{x}}{\sqrt{y}}$$

At $\left(\frac{a^2}{4}, \frac{a^2}{4}\right)$ slope m is -1

Equation of the tangent is given by $y - y_1 = m(x - x_1)$

$$\Rightarrow y - \frac{a^2}{4} = -1\left(x - \frac{a^2}{4}\right)$$

$$\Rightarrow x + y = \frac{a^2}{2}$$

$$\Rightarrow 2x + 2y = a^2$$

46. Find the points on the curve $4y = x^3$ where slope of the tangent is $\frac{16}{3}$.

Ans: The equation of the given curve is $y = x^3$.

To get the slope of the given curve, differentiate both the sides with respect to *x*.

On differentiating the above equation with respect to *x*, we get

$$\frac{dy}{dx} = 3x^2$$

When the slope of the tangent is equal to the y-coordinate of the point,

then

$$\frac{dy}{dx} = y = 3x^2$$

On substituting $y = x^3$ in the above equation.

$$\Rightarrow 3x^2 = x^3$$

$$\Rightarrow x^2(x-3) = 0$$

$$\Rightarrow x = 0, x = 3$$

When $x = 0$, then $y = 0$ $\quad (\because y = x^3)$

When $x=3$, then $y=3(3)^2=27$.

Hence, the required points are $(0,0)$ and $(3,27)$.

47. Show that $\frac{x}{a}+\frac{y}{b}=1$ touches the curve $y=be^{-x/a}$ at the point where the curve crosses the y-axis.

Ans: The abscissa of the point where the curves intersects the axis of *y* i.e., $x=0$

$\therefore y=b\cdot e^{-0/a}=b\left[\because e^0=1\right]$

Therefore the slope point of intersection on the curve with *y*- axis is $(0,b)$.

Now slope of the given line at $(0,b)$ is given by

$$\Rightarrow \frac{1}{a}\cdot 1+\frac{1}{b}\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-1}{a}\cdot b$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{a}b$$

$$\Rightarrow \frac{dy}{dx}=\frac{-b}{a}=m_1\text{ [say]}$$

Also the slope of the curve at $(0,b)$ is,

$$\Rightarrow \frac{dy}{dx}=be^{-x/a}\frac{-1}{a}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-b}{a}e^{-x/a}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(0,b)}=\frac{-b}{a}e^{-0}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(0,b)}=\frac{-b}{a}=m_2\text{ [say]}$$

since $m_1 = m_2 = \frac{-b}{a}$

From above we can conclude that the line touches the curves at the point where the curves intersects the axis of *y*.

48. Find the equation of the tangent to the curve given by $x = a\sin^3 t$, $y = b\cos^3 t$ at a point where $t = \frac{\pi}{2}$.

Ans: The given equation of the curves are $x - a\sin^3 t$ and $y = b\cos^3 t$.

On differentiating the above equations with respect to *t*, we get,

$$\frac{dx}{dt} = 3a\sin^2 t\cos t \text{ and}$$
$$\frac{dy}{dt} = -3b\cos^2 t\sin t$$

So,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\Rightarrow \frac{-3b\cos^2 t\sin t}{3a^2\sin^2 t\cos t}$$
$$\Rightarrow -\frac{b\cos t}{a\sin t}$$

When $t = \frac{\pi}{2}$, then

$$\left(\frac{dy}{dx}\right)_{t=\frac{\pi}{2}} = -\frac{b\cos\frac{\pi}{2}}{a\sin\frac{\pi}{2}} = 0$$

So, when $t = \frac{\pi}{2}$ then $x = a$ and $y = 0$.

So, at $t=\frac{\pi}{2}$ or at $(a,0)$, equation of tangent of given curve

$y-0=0(x-a)$ or $y=0$.

49. Find the intervals in which the function $\mathbf{f(x)=\log(1+x)-\frac{x}{1+x}, x>-1}$ is increasing or decreasing.

Ans: Given function is,

$$f(x)=\log(1+x)-\frac{x}{1+x}$$

We need to find intervals at which $f(x)$ is increasing or decreasing.

$$f(x)=\log(1+x)-\frac{x}{1+x}$$

On differentiating the above equation with respect to *x*, we get

$$\Rightarrow f'(x)=\frac{1}{1+x}-\left[\frac{(1)(1+x)-x(0+1)}{(1+x)^2}\right]$$

$$\Rightarrow \frac{1}{1+x}-\left[\frac{1+x-x}{(1+x)^2}\right]$$

$$\Rightarrow \frac{1}{1+x}-\frac{1}{(1+x)^2}$$

$$\Rightarrow \frac{1+x-1}{(1+x)^2}=\frac{x}{(1+x)^2}$$

For $f(x)$ to be increasing, $f'(x)>0$

$$\frac{x}{(1+x)^2}>0$$

$$x<0\left[\because (1+x)^2>0, \text{ Domain: } (-1,\infty)\right]$$

$x \in (-1,0)$ for $f(x)$ to be decreasing

$\therefore f(x)$ is increasing on $(0,\infty)$ and decreasing on $(-1,0)$.

50. Find the intervals in which the function $f(x) = x^3 - 12x^2 + 36x + 17$ is

(a) Increasing

Ans: The given function is $f(x) = x^3 - 12x^2 + 36x + 17$

On factorising,

$$f(x) = 3x^2 - 24x + 36$$
$$f(x) = 3(x-6)(x-2)$$

For $f(x)$ to be increasing, we must have $f'(x) > 0$.

$$\Rightarrow 3(x-6)(x-2) > 0$$
$$\Rightarrow x < 2 \text{ or } x > 6$$
$$\Rightarrow x \in (-\infty, 2) \cup (6, \infty)$$

So, $f(x)$ is increasing on $(-\infty, 2) \cup (6, \infty)$.

(b) Decreasing.

Ans: The given function is $f(x) = x^3 - 12x^2 + 36x + 17$

On factorising,

$$f(x) = 3x^2 - 24x + 36$$
$$f(x) = 3(x-6)(x-2)$$

For $f(x)$ to be decreasing, we must have $f'(x) < 0$.

$$\Rightarrow 3(x-2)(x-6) < 0$$
$$\Rightarrow 2 < x < 6$$

So, $f(x)$ is decreasing on $(2,6)$.

51. Prove that the function $f\left(x\right)=x^2-x+1$ is neither increasing nor decreasing in $\left[0,1\right]$.

Ans: The given function is $f(x)=x^2-x+1$.

On differentiating the above function with respect to *x*, we get

$f'(x)=2x-1$

$f'(x)>0, \forall x \in\left(\frac{1}{2},1\right)$ $[\because f'(x)>0 \Rightarrow$ strictly increasing$]$

$f'(x)<0, \forall x \in\left(0,\frac{1}{2}\right)$ $[\because f'(x)<0 \Rightarrow$ strictly decreasing$]$

clearly, from above we can conclude that,

$f(x)$ is strictly increasing in the interval $\left(\frac{1}{2},1\right)$

$f(x)$ is strictly decreasing in the interval $\left(0,\frac{1}{2}\right)$

$\therefore f(x)$ is neither increasing nor decreasing on the whole interval $(0,1)$.

52. Find the intervals on which the function $f(x)=\frac{x}{x^2+1}$ is decreasing.

Ans: The given function is $f(x)=\frac{x}{x^2+1}$.

On differentiating the above equation with respect to *x*, we get

$$f'\left(x\right)=\frac{\left(x^2+1\right).1-x.2x}{\left(x^2+1\right)^2}$$

$$f'\left(x\right)=\frac{x^2+1-2x^2}{\left(x^2+1\right)^2}$$

$$f'(x)=\frac{1-x^2}{\left(1+x^2\right)^2}$$

$$f'(x)=\frac{(1-x)(1+x)}{\left(1+x^2\right)^2}$$

When $x>1$,

$1+x>0$ and $1-x<0$

$\therefore (1+x)(1-x)<0$

Hence, $f'(x)<0$

When $x<-1$,

$1+x<0$ and $1-x>0$

$\therefore (1+x)(1-x)<0$

Hence, $f'(x)<0$

Therefore, the interval of *x* when $f(x)=\frac{x}{x^2+1}$ is decreasing is $(-\infty,-1]\cup[1,\infty)$

53. Prove that $f(x)=\frac{x^3}{3}-x^2+9x, x\in[1,2]$ is strictly increasing. Hence find the minimum value of $f(x)$.

Ans: The given function is $f(x)=\frac{x^3}{3}-x^2+9x, x\in[1,2]$

On differentiating the above equation with respect to *x*, we get

$$f'(x)=\frac{1}{3}3x^2-2x+9$$

$$=x^2-2x+9$$

To prove that the given function is strictly increasing.

$f'(x)$ should be greater than equal to zero

$x^2 - 2x + 9 \geq 0$

Putting $x = 1$, we get

$x^2 - 2x + 9 \geq 0$

$1 - 2 + 9 \geq 0$

$8 \geq 0$

$\because f'(x)$ is greater than equal to zero, the given function is strictly increasing

To find the minimum value of $f(x)$,put $x = 1$

$f(x) = \frac{1}{3} - 1 + 9$

$= \frac{1}{3} + 8$

$= \frac{25}{3}$

54. Find the intervals in which the function $f(x) = \sin^4 x + \cos^4 x, 0 \leq x \leq \frac{\pi}{2}$ is increasing or decreasing.

Ans: The given function is $f(x) = \sin^4 x + \cos^4 x, 0 \leq x \leq \frac{\pi}{2}$.

On differentiating the above function with respect to *x*, we get

$f(x) = \sin^4 x + \cos^4 x$ or,

$f'(x) = 4\sin^3 x \cos x - 4\cos^3 x \sin x$

$f'(x) = -4\sin x \cos x\left[-\sin^2 x + \cos^2 x\right]$

$f'(x) = -2\sin 2x \cos 2x$

$f'(x) = -\sin 4x$

On equating,

$f'(x) = 0$ or $-\sin 4x = 0$

or $4x = 0, \pi, 2\pi, \ldots\ldots\ldots$

or $\quad x = 0, \frac{\pi}{4}, \frac{\pi}{2}$

Sub-intervals are $\left[0, \frac{\pi}{4}\right], \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

Or $f'(x) < 0$ in $\left[0, \frac{\pi}{4}\right]$

or $f(x)$ is decreasing in $\left[0, \frac{\pi}{4}\right]$ and,

$f'(x) > 0$ in $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

$\therefore f'(x)$ is increasing in $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$.

55. Find the least value of 'a' such that the function $f(x) = x^2 + ax + 1$ is strictly increasing on $(1, 2)$.

Ans: The given function is $f(x) = x^2 + ax + 1$.

On differentiating the above function with respect to *x*, we get

$\therefore f'(x) = 2x + a$

Now, function f will be increasing in $(1, 2)$, if $f'(x) > 0$ in $(1, 2)$.

$\Rightarrow 2x + a > 0$

$\Rightarrow 2x > -a$

$\Rightarrow x > \frac{-a}{2}$

Therefore, we have to find the least value of *a* such that

$\Rightarrow x > \frac{-a}{2}$, when $x \in (1,2)$

Hence, the least value of *a* for f to be increasing on $(1,2)$ is given by,

$\frac{-a}{2} = 1$

$\Rightarrow a = -2$

Hence, the required value of a is -2.

56. Find the interval in which the function $f(x) = 5x^{\frac{3}{2}} - 3x^{\frac{5}{2}}, x > 0$ is strictly decreasing.

Ans: The given function is $f(x) = 5x^{\frac{3}{2}} - 3x^{\frac{5}{2}}, x > 0$

Let f be a differentiable real function defined on an open interval (a,b).

If $f'(x) < 0$ for all $x \in (a,b)$, then $f(x)$ is decreasing on (a,b)

Here we have,

$f(x) = 5x^{\frac{3}{2}} - 3x^{\frac{5}{2}}, x > 0$

On differentiating the above function with respect to *x*, we get

$\Rightarrow f'(x) = \frac{d}{dx}\left(5x^{\frac{3}{2}} - 3x^{\frac{5}{2}}\right)$

$\Rightarrow f'(x) = \frac{15}{2}x^{\frac{1}{2}} - \frac{15}{2}x^{\frac{3}{2}}$

$\Rightarrow f'(x) = \frac{15}{2}x^{\frac{1}{2}}(1-x)$

For $f(x)$ lets find critical point, we must have

$\Rightarrow f'(x) = 0$

$$\Rightarrow \frac{15}{2}x^{\frac{1}{2}}(1-x)=0$$

$$\Rightarrow x^{\frac{1}{2}}(1-x)=0$$
$$\Rightarrow x=0,1$$

clearly, $f'(x)>0$ if $0<x<1$

and $f'(x)<0$ if $x>1$

Thus, $f(x)$ increases on $(0,1)$ and $f(x)$ is decreasing on interval $x\in(1,\infty)$.

57. Show that the function $f(x)=\tan^{-1}(\sin x+\cos x)$, is strictly increasing on the interval $\left(0,\frac{\pi}{4}\right)$.

Ans: The given function is $f(x)=\tan^{-1}(\sin x+\cos x)$

We need to show that the given function is increasing in $\left(0,\frac{\pi}{4}\right)$.

Applying first derivative with respect to x, we get

$$f'(x)=\frac{d\left(\tan^{-1}(\sin x+\cos x)\right)}{dx}$$

Applying the differentiation rule for $\tan^{-1}$, we get

$$f'(x)=\frac{1}{(\sin x+\cos x)^2+1}\cdot\frac{d(\sin x+\cos x)}{dx}$$

Applying the sum rule of differentiation, we get

$$f'(x)=\frac{1}{(\sin x+\cos x)^2+1}\left[\frac{d(\sin x)}{dx}+\frac{d(\cos x)}{dx}\right]$$

But the derivative of $\sin x=\cos x$ and that of $\cos x=-\sin x$, so

$$f'(x) = \frac{1}{(\sin x + \cos x)^2 + 1}[\cos x + (-\sin x)]$$

Expanding $(\sin x + \cos x)^2$, we get

$$f'(x) = \frac{\cos x - \sin x}{\sin^2 x + \cos^2 x + 2\sin x \cos x + 1}$$

But $\sin^2 x + \cos^2 x = 1$ and $2\sin x \cos x = \sin 2x$, so the above equation becomes,

$$f'(x) = \frac{\cos x - \sin x}{1 + \sin 2x + 1}$$

$$f'(x) = \frac{\cos x - \sin x}{\sin 2x + 2}$$

Now for $f(x)$ to be decreasing function we need $f'(x) \geq 0$,

$$\Rightarrow \frac{\cos x - \sin x}{\sin 2x + 2} \geq 0$$

$$\Rightarrow \cos x - \sin x \geq 0$$

$$\Rightarrow \cos x \geq \sin x$$

But this is possible only if $x \in \left(0, \frac{\pi}{4}\right)$

Hence the given function is increasing function in $\left(0, \frac{\pi}{4}\right)$.

58. Show that the function $f(x) = \cos\left(2x + \frac{\pi}{4}\right)$ is strictly increasing on $\left(\frac{3\pi}{8}, \frac{7\pi}{8}\right)$.

Ans: The given function is $f(x) = \cos\left(2x + \frac{\pi}{4}\right)$

On differentiating the above function with respect to *x*, we get

$$\frac{dy}{dx} = -2\sin\left(2x + \frac{\pi}{4}\right)$$

Since, $\frac{3\pi}{8} < x < \frac{7\pi}{8}$

$\Rightarrow \frac{3\pi}{4} < 2x < \frac{7\pi}{4}$ (Multiplying 2 on all sides)

$\Rightarrow \pi < 2x + \frac{\pi}{4} < 2\pi$ (adding $\frac{\pi}{4}$ on all sides)

i.e., 3^{rd} and 4^{th} quadrant.

So, $\sin\left(2x + \frac{\pi}{4}\right)$ is negative

$\Rightarrow \frac{dy}{dx}$ is positive.

Hence, $f(x)$ is increasing in the given interval.

59. Show that the function $f(x) = \frac{\sin x}{x}$ is strictly decreasing on $\left(0, \frac{\pi}{2}\right)$.

Ans: The given function is $f(x) = \frac{\sin x}{x}$.

Differentiating the above function with respect to x we get,

$$f'(x) = \frac{x(\cos x) - \sin x(1)}{(x)^2}$$

$$f'(x) = \frac{x\cos x - \sin x}{x^2}$$

Let us assume that the above function is decreasing in the interval $\left(0, \frac{\pi}{2}\right)$.

We know that a function is decreasing if and only if $f'(x) < 0$ that is,

$$\frac{x\cos x - \sin x}{x^2} < 0$$
$$x\cos x < \sin x$$

As we know that the values of $\cos x$ is positive for all $x \in \left(0, \frac{\pi}{2}\right)$ we can divide both sides of the equation with $\cos x$ without changing the inequality sign that is,

$$x < \tan x$$

This equation is true for all values of $x \in \left(0, \frac{\pi}{2}\right)$.

Therefore, our assumption is true that is the given function $f(x) = \frac{\sin x}{x}$ is decreasing in the interval $\left(0, \frac{\pi}{2}\right)$.

Using differentials, find the approximate value of (Q. No. 60 to 64).

60. $\mathbf{(0.009)^{\frac{1}{3}}}$

Ans:

Using first order approximation,

$$f(x + \Delta x) = f(x) + f'(x)\Delta x$$

Here assume,

$$f(x) = x^{1/3}$$
$$\Rightarrow f'(x) = \frac{1}{3}x^{-2/3}$$

$$\therefore (0.009)^{1/3} = (0.008 + 0.001)^{1/3}$$
$$\Rightarrow (0.008)^{1/3} + \frac{1}{3}(0.008)^{-2/3} \times 0.001 \Rightarrow 0.208$$

61. $\mathbf{(255)^{\frac{1}{4}}}$

Ans: Let us consider,

$$x = 256$$
$$x + \Delta x = 255$$

Then,

$$\Delta x = -1$$

For $x = 256$,

$$y = (256)^{\frac{1}{4}} = 4$$

Let:

$$dx = \Delta x = -1$$

$$\text{Now, } y = (x)^{\frac{1}{4}}$$
$$\Rightarrow \frac{dy}{dx} = \frac{1}{4(x)^{\frac{3}{4}}}$$
$$\Rightarrow \left(\frac{dy}{dx}\right)_{x=256} = \frac{1}{256}$$
$$\therefore \Delta y = dy = \frac{dy}{dx}dx = \frac{1}{256} \times -1 = \frac{-1}{256}$$

$$\Rightarrow \Delta y = \frac{-1}{256} = -0.003906$$
$$\therefore (255)^{\frac{1}{4}} = y + \Delta y = 3.99609 \approx 3.9961$$

62. $\mathbf{\left(0.0037\right)^{\frac{1}{2}}}$

Ans: Let us consider $y = \sqrt{x}$

Let $x = 0.0036$ and $\Delta x = 0.0001$

Since $y = \sqrt{x}$

On differentiating both sides with respect to *x*, we get

$$\frac{dy}{dx} = \frac{d(\sqrt{x})}{dx}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

Now,

$$\Delta y = \frac{dy}{dx}\Delta x$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}(0.0001)$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{0.0036}}(0.0001)$$

$$\frac{dy}{dx} = \frac{1}{2\times 0.06}\times(0.0001)$$

$$\frac{dy}{dx} = \frac{0.0001}{0.12}$$

$$\frac{dy}{dx} = \frac{1}{1200} = 0.000833$$

Also,

$$\Delta y = f(x+\Delta x) - f(x)$$

So, $\Delta y = (x+\Delta x)^{\frac{1}{2}} - (x)^{\frac{1}{2}}$

On putting values,

$$0.000833 = (0.0036 + 0.0001)^{\frac{1}{2}} - (0.0036)^{\frac{1}{2}}$$

$$0.000833 = (0.0037)^{\frac{1}{2}} - \left(\frac{36}{10000}\right)^{\frac{1}{2}}$$

$$0.000833=(0.0037)^{\frac{1}{2}}-\left(\frac{6}{100}\right)^{2\times\frac{1}{2}}$$

$$0.000833=(0.0037)^{\frac{1}{2}}-(0.06)^{2\times\frac{1}{2}}$$

$$0.000833=(0.0037)^{\frac{1}{2}}-(0.06)$$

$$0.000833+0.06=(0.0037)^{\frac{1}{2}}$$

$$0.060833=(0.0037)^{\frac{1}{2}}$$

$$(0.0037)^{\frac{1}{2}}=0.060833$$

Thus, the Approximate Value of $(0.0037)^{\frac{1}{2}}=0.060833$.

63. $\sqrt{0.037}$

Ans: Let us consider $y=\sqrt{x}$.

Let $x=0.04$ and $\Delta x=-0.003$,

Then,

$$\Delta y=\sqrt{x+\Delta x}-\sqrt{x}$$

$$\Delta y=\sqrt{0.037}-\sqrt{0.04}$$
$$\Rightarrow \Delta y+0.2=\sqrt{0.037}$$

Now, *dy* is approximately equal to Δy and is given by

$$dy=\left(\frac{dy}{dx}\right)\Delta x$$
$$\Rightarrow \frac{1}{2\sqrt{x}}(-0.003)$$
$$\Rightarrow \frac{1}{2\sqrt{0.04}}(-0.003)$$
$$\Rightarrow -0.0075$$

Thus, the approximate value of $\sqrt{0.037}$ is $0.2-0.0075=0.1925$.

64. $\sqrt{25.3}$**.**

Ans: Let us consider $y=\sqrt{x}$

where $x=25$ and $\Delta x=0.3$

Since $y=\sqrt{x}$

$$\frac{dy}{dx}=\frac{d(\sqrt{x})}{dx}$$
$$\frac{dy}{dx}=\frac{1}{2\sqrt{x}}$$

Now,

$$\Delta y=\frac{dy}{dx}\Delta x$$

$$\Delta y=\frac{1}{2\sqrt{x}}(0.3)$$
$$\Delta y=\frac{1}{2\sqrt{25}}(0.3)$$
$$\Delta y=\frac{1}{2\times 5}\times 0.3$$
$$\Delta y=\frac{0.3}{10}=0.03$$

Also,

$$\Delta y=f(x+\Delta x)-f(x)$$

Putting values,

$$\Delta y=\sqrt{x+\Delta x}-\sqrt{x}$$
$$0.03=\sqrt{25+0.3}-\sqrt{25}$$

$0.03=\sqrt{25.3}-5$

$0.03+5=\sqrt{25.3}$

$\sqrt{25.3}=5.03$

Hence, approximate value of $\sqrt{25.3}$ is 5.03.

65. Find the approximate value of $f(5.001)$ where $f(x)=x^3-7x^2+15$.

Ans: First we need to break the number 5.001 as $x=5$ and $\Delta x=0.001$ and use the relation $f(x+\Delta x)=f(x)+\Delta x f'(x)$.

Let us consider,

$f(x)=x^3-7x^2+15$

On differentiating both sides with respect to *x*, we get

$\Rightarrow f'(x)=3x^2-14x$

Let $x=5$,

and $\Delta x=0.001$

Also,

$f(x+\Delta x)=f(x)+\Delta x f'(x)$

Therefore,

$f(x+\Delta x)=\left(x^3-7x^2+15\right)+\Delta x\left(3x^2-14x\right)$

$\Rightarrow f(5.001)=\left(5^3-7\times5^2+15\right)+\left(3\times5^2-14\times5\right)(0.001)$

(as $x=5, \Delta x=0.001$)

$\Rightarrow 125-175+15+(75-70)(0.001)$

$\Rightarrow 035+(5)(0.001)$

$\Rightarrow -35+0.005=-34.995$

66. Find the approximate value of $f(3.02)$ where $f(x)=3x^2+5x+3$.

Ans: It is given that the function is $f(x)=3x^2+5x+3$

where $x=3$ and $\Delta x=0.02$,

On differentiating the function with respect to *x*, we get

$f'(x)=6x+5$

Now,

$$\Delta y=f'(x)\Delta x$$
$$=(6x+5)0.02$$

Putting $x=3$,

$$=(6\times3+5)0.02$$
$$=23\times0.02$$
$$=0.46$$

Now,

$$f(x+\Delta x)=f(x)+\Delta y$$
$$f(3.02)\quad=f(3)+0.46$$
$$=\left(3\times3^2+5\times3+3\right)+0.46$$
$$=(27+15+3)+0.46$$
$$=45+0.46$$
$$=45.46$$

Hence, approximate value of $f(3.02)$ is 45.46.

Long Answer Type Questions 6 Marks

67. Show that of all rectangles inscribed in a given fixed circle, the square has the maximum area.

Ans: Let x and y be the length and breadth of a rectangle inscribed in a circle of radius r. If A be the area of rectangle then

$$A = x \cdot y$$

$$A = x \cdot \sqrt{4r^2 - x^2}$$

On differentiating the function with respect to *x*, we get

$$\frac{dA}{dx} = x \cdot \frac{1}{2\sqrt{4r^2 - x^2}} \times (-2x) + \sqrt{4r^2 - x^2}$$

$$\frac{dA}{dx} = -\frac{2x^2}{2\sqrt{4r^2 - x^2}} + \sqrt{4r^2 - x^2}$$

$$\frac{dA}{dx} = \frac{-x^2 + 4r^2 - x^2}{\sqrt{4r^2 - x^2}}$$

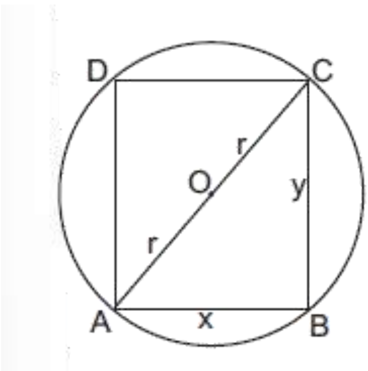
$$\frac{dA}{dx} = \frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}}$$

Since, ΔABC is a right angled triangle.

$$\Rightarrow 4r^2 = x^2 + y^2$$

$$\Rightarrow y^2 = 4r^2 - x^2$$

$$\Rightarrow y = \sqrt{4r^2 - x^2} \quad ...(i)$$



For maximum or minimum, $\frac{dA}{dx} = 0$.

$$\Rightarrow \frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}} = 0$$

$$\Rightarrow 2x^2 = 4r^2$$

$\Rightarrow x = \sqrt{2}r$

Again, on differentiating the function with respect to *x*, we get

$$\frac{d^2A}{dx^2} = \frac{\sqrt{4r^2 - x^2}.(-4x) - (4r^2 - x^2).\frac{1 \times -2x}{2\sqrt{4r^2 - x^2}}}{\left(\sqrt{4r^2 - x^2}\right)^2}$$

$$\frac{d^2A}{dx^2} = \frac{-4x(4r^2 - x^2) + x(4r^2 - 2x^2)}{(4r^2 - x^2)^{3/2}}$$

$$\frac{d^2A}{dx^2} = \frac{x\{-16r^2 + 4x^2 + 4r^2 - 2x^2\}}{(4r^2 - x^2)^{3/2}}$$

$$\frac{d^2A}{dx^2} = \frac{x(-12r^2 + 2x^2)}{(4r^2 - x^2)^{3/2}}$$

When $x = \sqrt{2}r$,

$$\left[\frac{d^2A}{dx^2}\right]_{x=\sqrt{2}r} = \frac{\sqrt{2}r\left(-12r^2 + 2\left(\sqrt{2}r\right)^2\right)}{\left(4r^2 - \left(\sqrt{2}r\right)^2\right)^{3/2}}$$

$$\left[\frac{d^2A}{dx^2}\right]_{x=\sqrt{2}r} = \frac{\sqrt{2}r \times -8r^2}{(2r^2)^{3/2}}$$

$$\left[\frac{d^2A}{dx^2}\right]_{x=\sqrt{2}r} = \frac{-8\sqrt{2}r^3}{2\sqrt{2}r^3} = -4 < 0$$

Hence, A is maximum when $x = \sqrt{2}r$.

On substituting $x = \sqrt{2}r$ in equation (i), we get

$y = \sqrt{4r^2 - 2r^2} = \sqrt{2}r$, i.e., $x = y = \sqrt{2}r$.

Therefore, area of rectangle is maximum when $x = y$, i.e., rectangle is square.

68. Find two positive numbers x and y such that their sum is 35 and the product x^2y^5 is maximum.

Ans: It is given that the sum of positive numbers is 35,

$x+y=35$
$\Rightarrow x=35-y \ ...(1)$

Let p be the product,

$p=x^2y^2=(35-y)^2\left(y^5\right)$
$p=\left(35^2+y^2-70y\right)y^5$
$p=1225y^5+y^7-70y^6 \ ...(2)$

For product to be maximum $\frac{dp}{dy}=0$

On differentiating the above equation (2) with respect to *y*, we get

$$\frac{dp}{dy}=1225\times 5y^4+7y^6-70\times 6y^5$$

$$\frac{dp}{dy}=7y^4\left(875+y^2-60y\right)=0$$

$$\frac{dp}{dy}=7y^4(y-35)(y-25)$$

$$\frac{dp}{dy}=0$$

$7y^4(y-35)(y-25)=0$

hence $y=0,25,35$

When $y=0\Rightarrow x=35$ and $x^2y^5=0$

When $y=25\Rightarrow x=10$ and $x^2y^5=976562500$

When $y=35\Rightarrow x=0$ and $x^2y^5=0$.

Hence maximum when two numbers are 25,10.

69. Show that of all the rectangles of given area, the square has the smallest perimeter.

Ans: Let us consider *x* and *y* be the length and breadth of rectangle whose are is A and perimeter is P.

We know,

$$\Rightarrow P = 2(x + y) \ ...(1)$$

$$A = x \times y$$

$$\therefore y = \frac{A}{x} \quad ...(2)$$

Substituting value of (2) in (1) we get,

$$\Rightarrow P = 2\left(x + \frac{A}{x}\right)$$

On differentiating the above equation with respect to *x*, we get

$$\Rightarrow \frac{dP}{dx} = 2\left(1 - \frac{A}{x^2}\right) \ldots(3)$$

For maximum or minimum values of perimeter P

$$\Rightarrow \frac{dP}{dx} = 0$$

$$\Rightarrow 2\left(1 - \frac{A}{x^2}\right) = 0$$

$$\Rightarrow 1 - \frac{A}{x^2} = 0$$

$$\Rightarrow x^2 = A$$

$$\Rightarrow x = \sqrt{A} \quad \text{[Dimension of rectangle is always positive]}$$

Now,

On again differentiating equation (3) with respect to *x*, we get

$$\frac{d^2P}{dx^2} = 2\left(0 - A \times \frac{(-1)}{x^3}\right) = \frac{2A}{x^3}$$

$$\therefore \left[\frac{d^2P}{dx^2}\right]_{x=\sqrt{A}} = \frac{2a}{(\sqrt{A})^3} > 0$$

i.e., for $x = \sqrt{A}$, P (perimeter of rectangle) is smallest.

$$y = \frac{A}{x} = \frac{A}{\sqrt{A}} = \sqrt{A}$$

Hence, for smallest perimeter, length and breadth of rectangle are equal.

$\left(x = y = \sqrt{A}\right)$ i.e., rectangle is square.

70. Show that the right circular cone of least curved surface area and given volume has an altitude equal to $\sqrt{2}$ times the radium of the base.

Ans: Let us consider $r, 1$ and h be the radius, slant height and altitude of the cone respectively, then

Volume of the cone $= V = \frac{1}{3}\pi r^2 h \quad ...(1)$

Height of the cone $= h = \frac{3V}{\pi r^2} ...(2)$

$S = \pi rl$, squaring both sides

$$S^2 = \pi^2 r^2 1^2 = \pi r^2\left(h^2 + r^2\right) \quad \left[\because \quad 1^2 = r^2 + h^2\right]$$

$$= \pi^2 r^2\left[\frac{9V^2}{\pi^2 r^4} + r^2\right] \quad \text{[from (2)]}$$

$S^2 = \frac{9V^2}{r^2} + \pi^2 r^4$, differentiating with respect to $'r'$

$$2S\frac{ds}{dr}=\frac{-18V^2}{r^3}+4\pi^2r^3 \quad \ldots(3)$$

For maximum or minimum,

$$\Rightarrow \frac{ds}{dr}=0$$

$$\Rightarrow \frac{-18V^2}{r^3}+4\pi^2r^3=0$$

$$\Rightarrow 4\pi^2r^3=\frac{18V^2}{r^3}$$

$$\Rightarrow 2\pi^2r^6=9V^2$$

$$\Rightarrow 9\times\frac{1}{9}\pi^2r^4h^2 \quad [\text{from}(1)]$$

$$\Rightarrow 2\pi^2r^6=\pi^2r^4h^2$$

$$\Rightarrow 2r^2=h^2$$

$$\Rightarrow h=\sqrt{2}r$$

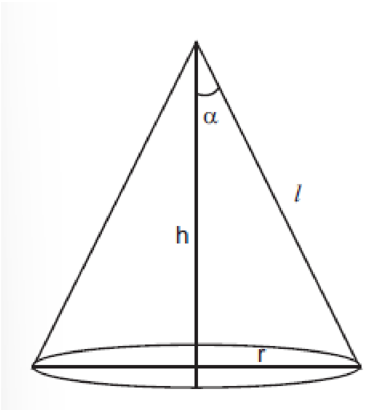
On again differentiating equation (3), we get

$$\frac{d^2S}{dr^2}=12\pi^2r^2+\frac{54V^2}{r^4}>0 \quad \text{for all values of } V \text{ and } r.$$

Hence, for the least surface area of a cone and given volume, altitude is equal to $\sqrt{2}$ times the radius of the base.

71. Show that the semi vertical angle of right circular cone of given surface area and maximum volume is $\sin^{-1}\left(\frac{1}{3}\right)$.

Ans: Let us consider $r, 1$ and h be the radius, slant height and altitude of the cone respectively, then



Surface area of cone is $S = \pi rl + \pi r^2 \; \ldots(1)$

or

$$l = \frac{S - \pi r^2}{\pi r}$$

The volume of the cone,

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2 \sqrt{l^2 - r^2}$$

$$V = \frac{\pi r^2}{3}\sqrt{\frac{\left(S - \pi r^2\right)^2}{\pi^2 r^2} - r^2}$$

$$V = \frac{\pi r^2}{3}\sqrt{\frac{\left(S - \pi r^2\right)^2 - \pi^2 r^4}{\pi^2 r^2}}$$

$$V = \frac{\pi r^2}{3} \times \frac{\sqrt{S^2 - \pi^2 r^4 - 2\pi S r^2 - \pi^2 r^4}}{\pi r}$$

$$V = \frac{r}{3}\sqrt{S\left(S - 2\pi r^2\right)}$$

On squaring both the sides, we get

$$V^2 = \frac{r^2}{9} S\left(S - 2\pi r^2\right)$$

$$V^2 = \frac{S}{9}\left(Sr^2 - 2\pi r^4\right)$$

$$\frac{d^2V^2}{dr^2} = \frac{S}{9}\left(2S - 24\pi r^2\right) \quad \cdots(ii)$$

Now $\frac{dV^2}{dr} = 0$

$\Rightarrow \frac{S}{9}\left(2Sr - 8\pi r^3\right) = 0$ or

$S - 4\pi r^2 = 0$

$\Rightarrow S = 4\pi r^2$

Putting $S = 4\pi r^2$ in (ii),

$$\frac{d^2V^2}{dr^2} = \frac{4\pi r^2}{9}\left[8\pi r^2 - 24\pi r^2\right] < 0$$

$\Rightarrow V$ is maximum when $S = 4\pi r^2$

Putting this value of S in (i)

$4\pi r^2 = \pi rl + \pi r^2$

or $3\pi r^2 = \pi rl$

or $\frac{r}{l} = \sin\alpha = \frac{1}{3}$

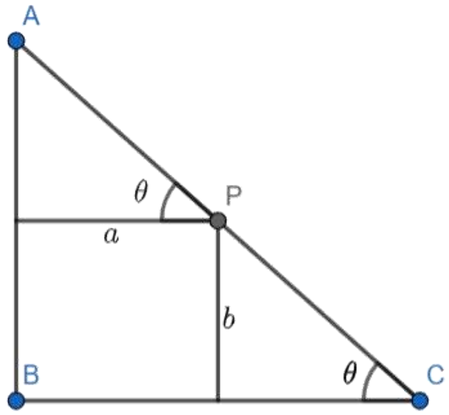
$\therefore \quad \alpha = \sin^{-1}\left(\frac{1}{3}\right)$

Thus V is maximum, when semi vertical angle is $\sin^{-1}\frac{1}{3}$.

72. A point on the hypotenuse of a triangle is at a distance a and b from the sides of the triangle. Show that the minimum length of the hypotenuse is $\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$.

Ans: Let us consider a ΔABC be right-angled at B and $AB = x$ and $BC = y$.

Let P be a point on the hypotenuse of the triangle such that P is at a distance of a and b from the sides AB and BC respectively.



Let $\angle C = \theta$

We have,

$$AC = \sqrt{x^2 + y^2}$$

Now,

$PC = b\,\text{cosec}\,\theta$ and $AP = a\sec\theta$.

$$\Rightarrow AC = AP + PC$$

$$\Rightarrow b\,\text{cosec}\theta + a\sec\theta \ldots(1)$$

$$\therefore \frac{d(AC)}{d\theta} = -b\,\text{cosec}\theta\cot\theta + a\sec\theta\tan\theta$$

$$\therefore \therefore \frac{d(AC)}{d\theta} = 0$$

$$\Rightarrow a\sec\theta\tan\theta = b\,\text{cosec}\theta\cot\theta$$

$$\Rightarrow \frac{a}{\cos\theta}.\frac{\sin\theta}{\cos\theta} = \frac{b}{\sin\theta}.\frac{\cos\theta}{\sin\theta}$$

$$\Rightarrow a\sin^3\theta = b\cos^3\theta$$

$$\Rightarrow a^{\frac{1}{3}}\sin\theta = b^{\frac{1}{3}}\cos\theta$$

$$\Rightarrow \tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$$

$$\therefore \sin\theta = \frac{b^{\frac{1}{3}}}{\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}} \text{ and } \cos\theta = \frac{a^{\frac{1}{3}}}{\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}} \quad ...(2)$$

It can be clearly shown that $\frac{d^2(AC)}{d\theta^2} < 0$ when . ..

Hence, by second derivative test, the length of the hypotenuse is the maximum when

$$\tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$$

Now, when $\tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$, we have:

$$AC = \frac{b\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}}{b^{\frac{1}{3}}} + \frac{a\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}}{a^{\frac{1}{3}}} \qquad [\text{using (1) and (2)}]$$

$$= \sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}\left(b^{\frac{2}{3}} + a^{\frac{2}{3}}\right)$$

$$= \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$$

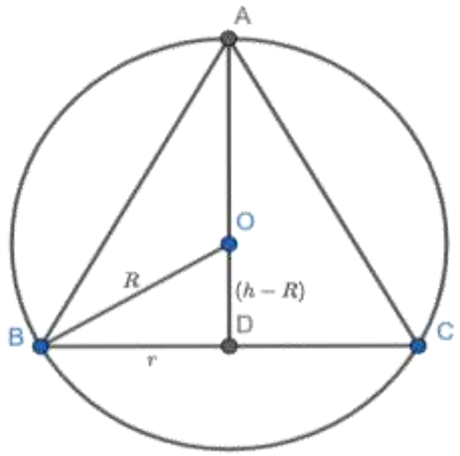
Hence, the maximum length of the hypotenuses is $\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$.

73. Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.

Ans: Let us consider the centre of the sphere be O and radius be R. Let the height and radius of the variable cone inside the sphere be h and r respectively.

So, in the diagram, $OA = OB = R, AD = h, BD = r$

$OD = AD - OA = h - R$.



Using Pythagoras Theorem in ΔOBD,

$OB^2 = OD^2 + BD^2$

$\Rightarrow R^2 = (h - R)^2 + r^2$

$\Rightarrow R^2 = h^2 + R^2 - 2hR + r^2$

$\Rightarrow r^2 = 2hR - h^2$

Volume of the cone $V = \frac{1}{3}\pi r^2 h$

$$V = \frac{1}{3}\pi\left(2hR - h^2\right)h$$

$$V = \frac{2\pi h^2 R}{3} - \frac{\pi h^3}{3}$$

For maximum volume, $\frac{dV}{dh} = 0$

$$\Rightarrow 0 = \frac{4\pi hR}{3} - \pi h^2$$

$$\Rightarrow h = \frac{4R}{3}$$

$$\therefore V = \frac{2\pi R}{3}\frac{16R^2}{9} - \frac{64\pi R^3}{81}$$

$$= \frac{(96-64)\pi R^3}{81}$$

$$= \frac{32\pi R^3}{81}$$

$$= \frac{8}{27} \times \frac{4}{3}\pi R^3$$

We know that the volume of the sphere is $V_s = \frac{4}{3}\pi R^3$

Therefore, $V = \frac{8}{27}V_s$

74. Find the interval in which the function f given by $\mathbf{f(x) = \sin x + \cos x}$, $\mathbf{0 \le x \le 2\pi}$ is strictly increasing or strictly decreasing.

Ans: The given function is $f(x) = \sin x + \cos x \quad 0 \le x \le 2\pi$

On differentiating with respect to *x*, we get

$$f'(x) = \cos x - \sin x$$

For the critical points of the function over the interval v $0 \le x \le 2\pi$ is given by

$$f'(x) = 0$$

$$\Rightarrow \cos x - \sin x = 0$$

$$\Rightarrow \cos x = \sin x$$

$$\Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}$$

Possible intervals are $\left(0, \frac{\pi}{4}\right), \left(\frac{\pi}{4}, \frac{5\pi}{4}\right), \left(\frac{5\pi}{4}, 2\pi\right)$

If $0 < x < \frac{\pi}{4}$,

$f'(x) = \cos x - \sin x > 0 \because \cos x > \sin x$ $

$\Rightarrow f'(x) > 0$

$\Rightarrow f(x)$ is strictly increasing.

If $\frac{\pi}{4} < x < \frac{5\pi}{4}$,

$f'(x) = \cos x - \sin x < 0 \quad \because \cos x < \sin x$

$\Rightarrow f(x)$ is strictly decreasing.

If $\frac{5\pi}{4} < x < 2\pi$,

$\Rightarrow f'(x) = \cos x - \sin x > 0 \because \cos x > \sin x$

$\Rightarrow f(x)$ is again strictly increasing.

Given function $f(x) = \sin x + \cos x,\ [0, 2\pi]$ is strictly increasing,

$\forall x \in \left(0, \frac{\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, 2\pi\right)$

while it is strictly decreasing $\forall x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$.

75. Find the intervals in which the function $f(x) = (x+1)^3(x-3)^3$ is strictly increasing or strictly decreasing.

Ans: The given function is $f(x) = (x+1)^3(x-3)^3$

On differentiating with respect to *x*, we get

$f'(x) = (x+1)^3 3(x-3)^2 + (x-3)^3 3(x+1)^2$

$f'(x) = 3(x+1)^2(x-3)^2(x+1+x-3)$

$f'(x) = 3(x+1)^2(x-3)^2(2x-2)$

$f'(x) = 6(x+1)^2(x-3)^2(x-1)$

To obtain the critical points, put $f'(x)=0$.

$$6(x+1)^2(x-3)^2(x-1)=0$$
$$\Rightarrow x=-1,1,3$$

The points $x=-1$, $x=1$ and $x=3$ divide the real line into four disjoints intervals.

The intervals are,

$(-\infty,-1),(-1,1),(1,3),(3,\infty)$

$f'(-2)=(-2-1)<0$

$\therefore$ Strictly decreasing in $(-\infty,-1)$

$f'(0)=(0-1)<0$

$\therefore$ Strictly decreasing in $(-1,1)$

$f'(2)=(2-1)>0$

$\therefore$ Strictly increasing in $(1,3)$

$f'(4)=(4-1)>0$

$\therefore$ Strictly increasing in $(3,\infty)$.

Hence, the function $f(x)=(x+1)^3(x-3)^3$ strictly increasing in $\left[0,\frac{\pi}{4}\right)\cup\left(\frac{5\pi}{4},2\pi\right]$ and strictly decreasing in $\left(\frac{\pi}{4},\frac{5\pi}{4}\right)$.

76. Find the local maximum and local minimum of $f(x)=\sin 2x-x$ $-\frac{\pi}{2}<x<\frac{\pi}{2}$.

Ans: The given function is $f(x)=\sin 2x-x,\quad -\frac{\pi}{2}\le x\le\frac{\pi}{2}$

On differentiating with respect to *x*, we get

$$f'(x) = 2\cos 2x - 1 \qquad \frac{-\pi}{2} \le x \le \frac{\pi}{2}$$
$$\Rightarrow f'(x) = 0 \qquad -\pi \le 2x \le \pi$$
$$\Rightarrow 2\cos 2x - 1 = 0$$
$$\Rightarrow \cos 2x = \frac{1}{2}$$
$$\therefore 2x = \frac{\pi}{3}, \frac{-\pi}{3}$$
$$\therefore \ x = \frac{\pi}{6}, \frac{-\pi}{6}$$

at $x = \frac{\pi}{6}, f(x)$ has local maximum,

$$f\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{3}\right) - \frac{\pi}{6}$$
$$f\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - \frac{\pi}{6}$$

at $x = -\frac{\pi}{6}, f(x)$ has local minima,

$$f\left(\frac{-\pi}{6}\right) = \sin\left(-\frac{\pi}{3}\right) + \frac{\pi}{6}$$
$$f\left(\frac{-\pi}{6}\right) = \frac{\pi}{6} - \frac{\sqrt{3}}{2}$$

77. Find the intervals in which the function $f(x) = 2x^3 - 15x^2 + 36x + 1$ is strictly increasing or decreasing. Also find the points on which the tangents are parallel to x-axis.

Ans: The given function is $f(x) = 2x^3 - 15x^2 + 36x + 1$

Differentiate the above equation with respect x, we get

$$f'(x) = \frac{d}{dx}\left(2x^3 - 15x^2 + 36x + 1\right)$$
$$\Rightarrow f'(x) = 6x^2 - 30x + 36$$

For the function $f(x)$ we have to find critical point, we must have

$$\Rightarrow f'(x) = 0$$
$$\Rightarrow 6x^2 - 30x + 36 = 0$$
$$\Rightarrow 6\left(x^2 - 5x + 6\right) = 0$$
$$\Rightarrow 3\left(x^2 - 3x - 2x + 6\right) = 0$$
$$\Rightarrow x^2 - 3x - 2x + 6 = 0$$
$$\Rightarrow (x-3)(x-2) = 0$$
$$\Rightarrow x = 3, 2$$

It is clear, $f'(x) > 0$ if $x < 2$ and $x > 3$ and $f'(x) < 0$ if $2 < x < 3$

Hence, $f(x)$ increases on $(-\infty, 2) \cup (3, \infty)$ and $f(x)$ is decreasing on interval $x \in (2,3)$.

Tangent is parallel to x-axis when $f'(x) = 0$ i.e., $6(x-2)(x-3) = 0$

When $6(x-2)(x-3) = 0$

then $x = 2, 3$.

When,

$$x = 2,$$
$$y = f(2)$$
$$y = 2(2)^3 - 15(2)^2 + 36(2) + 1 - 16 - 60 + 72 + 1$$
$$y = 29$$

When,

$$x = 3,$$
$$y = f(3)$$
$$y = 2(3)^3 - 15(3)^2 + 36(3) + 1 = 54 - 135 + 108 + 1$$

$y = 28$

$\therefore$ The points are $(2, 29), (3, 28)$.

78. A solid is formed by a cylinder of radius r and height h together with two hemisphere of radius r attached at each end. It the volume of the solid is constant but radius r is increasing at the rate of $\frac{1}{2\pi}$ metre/min. How fast must h (height) be changing when r and h are 10 metres.

Ans: It is given that a solid is formed by a cylinder of radius r and height h together with two hemispheres of radius r attached at each end.

Volume of solid $= V = \pi r^2 h + \frac{4}{3}\pi r^3$

Differentiate the above equation with respect t, we get

$$\frac{dV}{dt} = \frac{d}{dt}\left(\pi r^2 h + \frac{4}{3}\pi r^3\right)$$

$$\frac{dV}{dt} = \pi\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt} + \frac{4}{3}\times 3r^2\frac{dr}{dt}\right)$$

As V is constant.

$$\therefore \frac{dV}{dt} = 0$$

Given that the r and h are 10 metres and $\frac{dr}{dt} = \frac{1}{2\pi}$ meter/min.

$$\therefore 0 = \pi\left(2\times 10\times 10\times \frac{1}{2\pi} + 10^2\times \frac{dh}{dt} + 4\times 10^2\times \frac{1}{2\pi}\right)$$

$$\frac{100}{\pi} + 100\frac{dh}{dt} + \frac{200}{\pi} = 0$$

$$\frac{dh}{dt} = \frac{-300}{\pi\times 100}$$

$$\frac{dh}{dt} = -\frac{3}{\pi} \text{ meter/min}$$

Therefore, h decreases at the rate of $\frac{3}{\pi}$ meter/min.

79. Find the equation of the normal to the curve $x = a(\cos\theta + \theta\sin\theta); y = a(\sin\theta - \theta\cos\theta)$ at the point θ and show that its distance from the origin is a.

Ans: The given equation of the curves are

$x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$.

Firstly, let us find the slope of the tangent to the curve at a point θ.

$$m_T = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\cos\theta - a\cos\theta + a\theta\sin\theta}{-a\sin\theta + a\sin\theta + a\theta\cos\theta}$$

$$m_T = \frac{\sin\theta}{\cos\theta}$$

Since, normal is always perpendicular to the tangent, slope of the normal will be

$$m_N = -\frac{\cos\theta}{\sin\theta}$$

Normal passes through the point (x, y) given in the question. Using the equation of the line in slope-point form, we get

$$\frac{y - a\sin\theta + a\theta\cos\theta}{x - a\cos\theta - a\theta\sin\theta} = -\frac{\cos\theta}{\sin\theta}$$

$$\Rightarrow y\sin\theta - a\sin^2\theta + a\theta\sin\theta\cos\theta = a\cos^2\theta + a\theta\sin\theta\cos\theta - x\cos\theta$$

$$\Rightarrow x\cos\theta + y\sin\theta = a\left(\cos^2\theta + \sin^2\theta\right) = a$$

Distance of origin from this line will be

$$d = \frac{|0+0-a|}{\sqrt{\sin^2\theta + \cos^2\theta}} = |a|$$

Hence, the distance from origin to the normal at any point θ is a constant, say *a*.

80. For the curve $y = 4x^3 - 2x^5$, find all the points at which the tangent passes through the origin.

Ans: The given equation of the curve is $y = 4x^3 - 2x^5$.

Differentiating with respect to x, we get

$$\Rightarrow \frac{dy}{dx} = 12x^2 - 10x^4$$

Let $P(x_1, y_1)$ be the point on the curve at which tangent passes through the orgin.

Slope of the tangent at $P(x_1, y_1)$ is $\left(\frac{dy}{dx}\right)_{x_1, y_1} = 12{x_1}^2 - 10{x_1}^4$

Equation of the tangent at (x_1, y_1) is given by,

$$y - y_1 = \left(12x_1^2 - 10x_1^4\right)\left(x - x_1\right)\ldots(1) \quad \text{(using one-point form)}$$

Since, the tangent passes through the origin $(0,0)$.

So, equation (1) becomes

$$-y_1 = \left(12x_1^2 - 10x_1^4\right)\left(-x_1\right)$$
$$y_1 = 12x_1^3 - 10x_1^5 \quad ...(2)$$

Since, $P(x_1, y_1)$ lies on the curve

So, $y_1 = 4x_1^3 - 2x_1^5$

From equation (2) and (3), we get

$$12x_1^3 - 10x_1^5 = 4x_1^3 - 2x_1^5$$
$$\Rightarrow 8x_1^5 - 8x_1^3 = 0$$

$\Rightarrow x_1^5 - x_1^3 = 0$

$\Rightarrow x_1^3\left(x_1^2 - 1\right)$

$\Rightarrow x_1 = 0, \pm 1$

When $x_1 = 0,$

$y_1 = 4(0)^3 - 2(0)^5 = 0$

When $x_1 = 1,$

$y_1 = 4(1)^3 - 2(1)^5 = 2$

When $x_1 = -1,$

$y_1 = 4(-1)^3 - 2(-1)^5 = -2$

Hence, the required points are $(0,0),(1,2)$ and $(-1,-2)$.

81. Find the equation of the normal to the curve $x^2 = 4y$ which passes through the point $(1,2)$.

Ans: The given equation of the curve is $x^2 = 4y$

On differentiating both sides with respect to*x*, we get

$$\frac{dy}{dx} = \frac{x}{2}$$

If *m* be the slope of the normal to $x^2 = 4y$ then $m = \frac{-1}{\left(\frac{dy}{dx}\right)} = \frac{-2}{x}$

$\therefore$ Slope of tangent at $\left(x_1, y_1\right) = \frac{x_1}{2}$

$\therefore$ Slope of Normal $= -\frac{2}{x_1}$

Since normal passes through $(1,2)$

$\therefore$ Slope of normal line $= \frac{y_1 - 2}{x_1 - 1} = \frac{\frac{x_1^2}{4} - 2}{x_1 - 1}$

Or $-\frac{2}{x_1} = \frac{x_1^2 - 8}{4(x_1 - 1)}$

$x_1^3 - 8x_1 = -8x_1 + 8$

or

$x_1 = 2$

$x_1 = 2, \text{ or } y_1 = 1$

$\therefore$ Point of contact is $(2,1)$

Equation of normal is

$y - 1 = -\frac{2}{2}(x - 2) \text{ or } x + y = 3$

82. Find the equation of the tangents at the points where the curve $2y = 3x^2 - 2x - 8$ cuts the x-axis and show that they make supplementary angles with the x-axis.

Ans: The given equation of the curve is $2y = 3x^2 - 2x - 8$.

Let the tangent meet the x-axis at point $(x,0)$

On differentiating both sides with respect to*x*, we get

$2y = 3x^2 - 2x - 8$

$\Rightarrow 2\frac{dy}{dx} = 6x - 2$

$\Rightarrow \frac{dy}{dx} = 3x - 1$

The tangent passes through point $(x,0)$

$$\therefore 0 = \frac{3}{2}x^2 - x - 4 = 0$$
$$\Rightarrow 3x^2 - 2x - 8 = 0$$
$$\Rightarrow 3x^2 - 6x + 4x - 8 = 0$$
$$\Rightarrow (3x + 4)(x - 2) = 0$$
$$\Rightarrow x = 2, x = \frac{-4}{3}$$

Case I: when $x = 2$,

Slope of tangent,

$$\left(\frac{dy}{dx}\right)_{(2,0)} = 3.2 - 1$$
$$\left(\frac{dy}{dx}\right)_{(2,0)} = 6 - 1 = 5$$

$$\therefore \left(x_1, y_1\right) = (2,0)$$

Equation of tangent,

$$y - y_1 = m\left(x - x_1\right)$$
$$\Rightarrow y - 0 = 5(x - 2)$$
$$\Rightarrow 5x - y - 10 = 0$$

Case II: When $x = -\frac{4}{3}$,

Slope of tangent,

$$\left(\frac{dy}{dx}\right)_{(2,0)} = 3.\frac{-4}{3} - 1$$
$$\left(\frac{dy}{dx}\right)_{(2,0)} = -5$$

$$\therefore \left(x_1, y_1\right) = \left(\frac{-4}{3}, 0\right)$$

Equation of tangent,

$$y-y_1=m(x-x_1)$$

$$\Rightarrow y-0=-5\left(x-\left(\frac{-4}{3}\right)\right)$$

$$\Rightarrow y=-5x-\frac{20}{3}$$

$$\Rightarrow 15x+3y+20=0$$

83. Find the equations of the tangent and normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point (x_0,y_0).

Ans: The given equation of the hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Differentiating $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with respect to *x*, we get

$$\frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\Rightarrow \frac{2y}{b^2}\frac{dy}{dx}=\frac{2x}{a^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{b^2x}{a^2y}$$

Therefore, the slope of the tangent at (x_0,y_0) is,

$$\left(\frac{dy}{dx}\right)_{(x_0,y_0)}=\frac{b^2x_0}{a^2y_0}$$

Then, the equation of the tangent at (x_0,y_0)

$$y-y_0=\frac{b^2x_0}{a^2y_0}(x-x_0)$$

$$\Rightarrow a^2yy_0 - a^2y_0^2 = b^2xx_0 - b^2x_0^2$$

$$\Rightarrow b^2xx_0 - a^2yy_0 - b^2x_0^2 + a^2y_0^2 = 0$$

$$\Rightarrow \frac{xx_0}{a^2} - \frac{yy_0}{b^2} - \left(\frac{x_0^2}{a^2} - \frac{y_0^2}{b^2}\right) = 0 \quad [\text{On dividing both sides by } a^2b^2]$$

$$\Rightarrow \frac{xx_0}{a^2} - \frac{yy_0}{b^2} - 1 = 0 \quad \left[(x_0, y_0) \text{ lies on the hyperbola } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\right]$$

$$\Rightarrow \frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1$$

Now, the slope of the normal at (x_0, y_0) is given by,

$$\frac{-1}{\text{Slope of the tangent at } (x_0, y_0)} = \frac{-a^2y_0}{b^2x_0}$$

Hence, the equation of the normal at (x_0, y_0) is given by,

$$\frac{-1}{\text{Slope of the tangent at } (x_0, y_0)} = \frac{-a^2y_0}{b^2x_0}$$

Hence, the equation of the normal at (x_0, y_0) is given by,

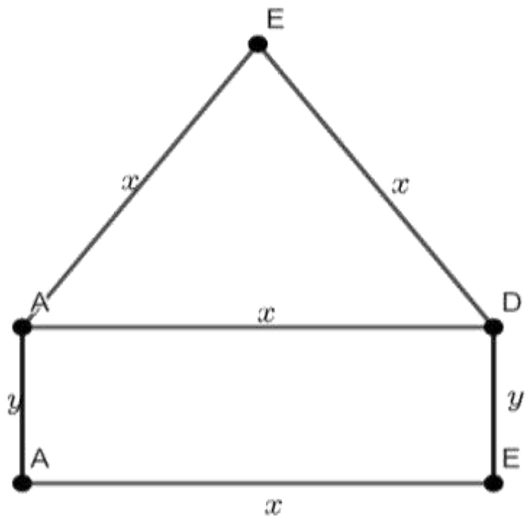
$$y - y_0 = \frac{-a^2y_0}{b^2x_0}(x - x_0)$$

$$\Rightarrow \frac{y - y_0}{a^2y_0} = \frac{-(x - x_0)}{b^2x_0}$$

$$\Rightarrow \frac{y - y_0}{a^2y_0} + \frac{(x - x_0)}{b^2x_0} = 0$$

84. A window is in the form of a rectangle surmounted by an equilateral triangle. Given that the perimeter is 16 metres. Find the width of the window in order that the maximum amount of light may be admitted.

Ans: Consider the side $AB = x$ and $BC = y$,



Perimeter of given window is $16\,m$.

$\Rightarrow 3x + 2y = 16$

$\Rightarrow y = \dfrac{16 - 3x}{2} (A)$

Let the area of a given figure is A.

$\therefore$ A = Area of an equilateral triangle + Area of a rectangle.

$$\Rightarrow A = \frac{\sqrt{3}}{4}x^2 + xy$$

(area of equilateral triangle is given by $= \dfrac{\sqrt{3}}{4}x^2$ and area of rectangle is xy)

$$\Rightarrow A = \frac{\sqrt{3}}{4}x^2 + x\left(\frac{16 - 3x}{2}\right) ...(1)$$

Now, differentiating the above equation (1) with respect to *x*, we get

$$\Rightarrow \frac{dA}{dx} = \frac{d}{dx}\left(\frac{\sqrt{3}}{4}x^2 + \left(\frac{16x - 3x^2}{2}\right)\right)$$

$$\Rightarrow \frac{dA}{dx} = \frac{\sqrt{3}}{4}(2x) + \left(\frac{16 - 6x}{2}\right)$$

$$\Rightarrow \frac{dA}{dx} = \frac{\sqrt{3}}{2}(x) + 8 - 3x \ldots(2)$$

For maximum or minimum we have,

$$\frac{dA}{dx} = 0$$

On substituting the value of $\frac{dA}{dx}$ from (2) we get,

$$\Rightarrow \frac{\sqrt{3}}{2}(x) + 8 - 3x = 0$$

Taking the terms of x together and simplifying further we get,

$$\Rightarrow x\left(3 - \frac{\sqrt{3}}{2}\right) = 8$$

$$\Rightarrow x\left(\frac{6 - \sqrt{3}}{2}\right) = 8$$

$$\Rightarrow x(6 - \sqrt{3}) = 16$$

$$\Rightarrow x = \frac{16}{(6 - \sqrt{3})}$$

Again on differentiating equation (2) with respect to *x*, we get

$$\frac{d^2 A}{dx^2} = \frac{\sqrt{3}}{2} - 3$$

Now as $\frac{d^2 A}{dx^2} = \frac{\sqrt{3}}{2} - 3 < 0$, Hence the area is maximum when

$$x = \frac{16}{6 - \sqrt{3}}$$

Now we substitute the value of x in equation (A), we get,

$$\Rightarrow y = \frac{16 - 3\left(\frac{16}{6-\sqrt{3}}\right)}{2}$$

On further simplification we get,

$$\Rightarrow y = \frac{16 - \frac{48}{6-\sqrt{3}}}{2}$$

$$\Rightarrow y = \frac{16(6-\sqrt{3})-48}{2(6-\sqrt{3})}$$

$$\Rightarrow y = \frac{96-16\sqrt{3}-48}{2(6-\sqrt{3})}$$

$$\Rightarrow y = \frac{48-16\sqrt{3}}{2(6-\sqrt{3})}$$

$$\Rightarrow y = \frac{24-8\sqrt{3}}{(6-\sqrt{3})}$$

Therefore, the breadth of a window $x = \frac{16}{(6-\sqrt{3})} \approx 3.75$ and length of a window is $y = \frac{24-8\sqrt{3}}{(6-\sqrt{3})} \approx 2.375$.

85. A jet of an enemy is flying along the curve $y = x^2 + 2$. A soldier is placed at the point $(3, 2)$. What is the nearest distance between the soldier and the jet?

Ans: Let us consider a point $P(x, y)$ be the position of the jet and the soldier is placed at $A(3, 2)$.

or $AP = \sqrt{(x-3)^2 + (y-2)^2} \quad \ldots(i)$

The given equation of the curve is $y = x^2 + 2$.

As,

$y = x^2 + 2$ or $y - 2 = x^2$

$\therefore \quad AP^2 = (x-3)^2 + x^4 = z$ (say)

Differentiating the above equation with respect to *x*, we get

$$\frac{dz}{dx} = 2(x-3) + 4x^3 \text{ ...(ii)}$$

$$\frac{dz}{dx} = 0$$

$\therefore \quad 2x - 6 + 4x^3 = 0$

Put $\quad x = 1$

$2 - 6 + 4 = 0$

$\therefore x - 1$ is a factor

And,

Again differentiating the above equation (ii) with respect to *x*, we get

$$\frac{d^2z}{dx^2} = 12x^2 + 2$$

$$\frac{dz}{dx} = 0 \text{ or } x = 1$$

and $\frac{d^2z}{dx^2}(\text{ at } x = 1) > 0$

$\therefore z$ is minimum when $x = 1, y = 1 + 2 = 3$

$\therefore$ minimum distance $= \sqrt{(1-3)^2 + (3-2)^2} = \sqrt{5}$

86. Find a point on the parabola $y^2 = 4x$ which is nearest to the point $(2,-8)$

Ans: Let us consider the point on the curve be (x_1, y_1).

The equation of the curve is $y^2 = 4x$

Point (x_1, y_1) on curve,

$$y_1^2 = 4x_1$$
$$D = \sqrt{(x_1 - 2)^2 + (y_1 + 8)^2}$$

D is always positive,

$$D^2 = (x_1 - 2)^2 + + (y_1 + 8)^2$$
$$D^2 = \left(\frac{y_1^2}{4} - 2\right)^2 + (y_1 + 8)^2$$

On differentiating the above equation with respect to y_1, we get

$$\frac{dD^2}{dy_1} = 2\left(\frac{y_1^2}{4} - 2\right)\frac{2y_1}{4} + 2(y_1 + 8)$$
$$\frac{dD^2}{dy_1} = \frac{y_1(y_1^2 - 8)}{4} + 2y_1 + 16$$
$$\frac{dD^2}{dy_1} = y_1^3 - 8y_1 + 8y_1 + 64$$
$$\frac{dD^2}{dy_1} = y_1^3 + 64$$
$$\frac{dD^2}{dy_1} = 0$$
$$y_1^3 = -64$$
$$(y_1)^2 = -4$$
$$\Rightarrow (-4)^2 = 4x_1$$
$$\Rightarrow x_1 = 4$$

Point is $(4, -4)$

87. A square piece of tin of side 18cm is to be made into a box without top by cutting a square from each cover and folding up the flaps to form the box. What should be the side of the square to be cut off so that the volume of the box is the maximum.

Ans: Let us consider the length of the each side of the square which is cut from each corner of the tin sheet be $x\,\text{cm}$.

When the flaps are folded, a cuboidal box is formed whose length, breadth and height are $18-2x, 18-2x$ and x respectively.

Then, its volume V is given by,

$$V=(18-2x)(18-2x)x$$

On differentiating both sides with respect to *x*, we get

$$\Rightarrow \frac{dV}{dx}=324-144x+12x^2$$

On further differentiation of both sides with respect to *x*,

$$\frac{d^2V}{dx^2}=-144+24x$$

The critical numbers of V are given by,

$$\frac{dV}{dx}=0$$

$$\Rightarrow 324-144x+12x^2=0$$

$$\Rightarrow x^2-12x+27=0$$

$$\Rightarrow x=3,9$$

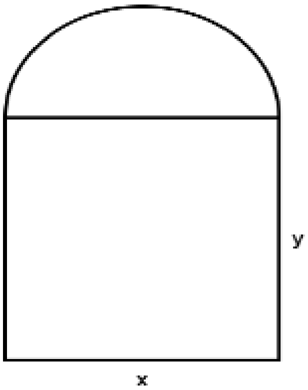
But, $x=9$ is not possible. Therefore, $x=3$.

Clearly, $\left(\frac{d^2V}{dx^2}\right)_{x=3}=-144+72=-72<0$

So, V is maximum when $x=3$ i.e., the length of each side of the square to be cut is 3cm.

88. A window in the form of a rectangle is surmounted by a semi-circular pening. The total perimeter of the window is 30 metres. Find the imensions of the rectangular part of the window to admit maximum light through the whole opening.

Ans:



Let x and y be the length and width of rectangle part of window respectively.

Let A be the opening area of the window which admits light.

Obviously, for admitting the maximum light through the opening, A must be maximum.

First taking the perimeter,

$$x + y + y + \pi\left(\frac{x}{2}\right) = 30$$

$$y = 15 - \left(\frac{2+\pi}{4}\right)x$$

Now

$A =$ area of rectangle + Area of the semicircle

Therefore,

$$\Rightarrow A = xy + \frac{1}{2}\pi.\frac{x^2}{4}$$

$$\Rightarrow A = xy + \frac{1}{2}\frac{\pi x^2}{4}$$

$$\Rightarrow A = 15x - \left(\frac{\pi + 2}{4} - \frac{\pi}{8}\right)x^2$$

$$\Rightarrow A = 15x - \frac{\pi + 4}{8}x^2$$

$$\Rightarrow \frac{dA}{dx} = 15 - \left(\frac{\pi + 4}{8}\right)2x$$

For maximum or minimum value of A,

$$\frac{dA}{dx} = 0$$

$$\Rightarrow 15 - \left(\frac{\pi + 4}{8}\right)2x = 0$$

$$\Rightarrow x = \frac{60}{\pi + 4}$$

Now,

$$\frac{d^2A}{dx^2} = \frac{\pi + 4}{2} \times 2 = \frac{\pi + 4}{4}$$

Hence for $x = \frac{60}{\pi + 4}$, A is maximum

And thus $y = 15 - \frac{60}{\pi + 4} \times \frac{\pi + 2}{4}$

$$= 15 - \frac{15(\pi + 2)}{\pi + 4}$$

$$= \frac{15\pi + 60 - 15\pi - 30}{\pi + 4} = \frac{30}{\pi + 4}$$

Therefore, for maximum A, i.e. for admitting the maximum light

Length of rectangle

$x = \frac{60}{\pi + 4}$ and

Breadth of rectangle $y=\frac{30}{\pi+4}$.

89. An open box with square base is to be made out of a given iron sheet of area 27 sq. meter, show that the maximum value of the box is 13.5 cubic metres.

Ans: Let us consider ' x ' be the side of the square base and ' y ' the height of the box.

Area of the square base $=x^2$

and area of four walls $=4xy$

According to the question,

$x^2+4xy=27$…...(1)

Let V = volume of the box =area of base $\times$ height

$$V=x^2y$$

$$V=\frac{x^2\left(27-x^2\right)}{4x}$$

$$V=\frac{1}{4}\left(27x-x^3\right)...(2)$$

On differentiating equation (2) with respect to *x*, we get

$$\frac{dV}{dx}=\frac{1}{4}\left(27-3x^2\right)...(3)$$

and

$$\frac{d^2V}{dx^2}=-\frac{3x}{2}...(4)$$

Now,

$$\frac{dV}{dx}=0$$

$$\frac{1}{4}\left(24-3x^2\right)=0$$

$$\Rightarrow x^2=\frac{27}{3}=9$$

$$\Rightarrow x=\pm 3$$

$\Rightarrow x=3$ [since *x* cannot be negative]

V can only max. when $x=3$ m.

When $x=3$,

$\frac{d^2V}{dx^2}=\frac{-3\times 3}{2}=-\frac{9}{2}$, which is negative.

Therefore, $x=3$ gives maximum value of V.

Hence,

Maximum V = $\frac{1}{4}\left[27\times 3-27\right]$ [putting $x=3$ in (2)]

$$V=\frac{27}{4}(3-1)$$

$$V=\frac{27}{2}$$

$$V=13.5\text{ cm}^3$$

90. A wire of length 28cm is to be cut into two pieces. One of the two pieces is to be made into a square and other in to a circle. What should be the length of two pieces so that the combined area of the square and the circle is minimum?

Ans: Let us consider the length of one part of the wire be $x$cm, then the other part will be $28-x$.

Let the part of the length x be covered into a circle of radius.

$2\pi r = x$

$\Rightarrow r = \frac{x}{2\pi}$

Area of circle $= \pi r^2$

Area of circle $= \pi\left(\frac{x}{2\pi}\right)^2 = \frac{x^2}{4\pi}$

Now second part of length $28 - x$ is covered into a square.

Side of a square $= \frac{28 - x}{4}$

Area of square $= \left[\frac{28 - x}{4}\right]^2$

Thus, total area covered = $A = \frac{x^2}{4\pi} + \left[\frac{28 - x}{4}\right]^2$

Differentiating the above equation with respect to *x*, we get

$\frac{dA}{dx} = \frac{2x}{4\pi} + \frac{2}{16}(28 - x)(-1)$

$\frac{dA}{dx} = \frac{x}{2\pi} - \frac{28 - x}{8}$

Lets, take $\frac{dA}{dx} = 0$

Thus,

$\frac{x}{2\pi} - \frac{28 - x}{8} = 0 \text{ ....(1)}$

$4x = 28\pi - \pi x$

$4x + \pi x = 28\pi$

$x[4 + \pi] = 28\pi$

$$x = \frac{28\pi}{4+\pi}$$

Other part $= 28 - x = 28 - \frac{28\pi}{4+\pi}$

$$= \frac{112 + 28\pi - 28\pi}{4+\pi}$$

$$= \frac{112}{4+\pi}$$

Now again differentiating, we get

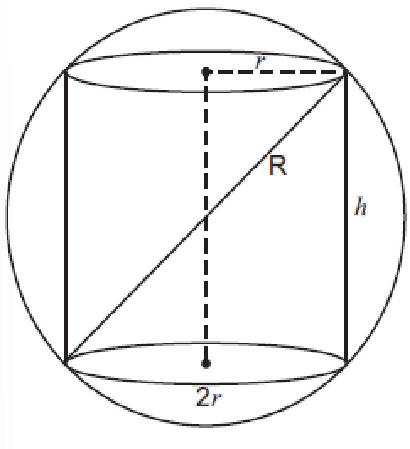
$$\frac{d^2 A}{dx^2} = \frac{1}{2\pi} + \frac{1}{8} = \text{positive}$$

A is minimum.

When $x = \frac{28\pi}{4+\pi}$ and $28 - x = \frac{112}{4+\pi}$.

91. Show that the height of the cylinder of maximum volume which can be inscribed in a sphere of radius $\mathbf{R}$ is $\frac{\mathbf{2R}}{\sqrt{\mathbf{3}}}$. Also find the maximum volume.

Ans: Let us consider r, h be the radius and height of the cylinder inscribed in the sphere of radius R.



By using Pythagoras theorem,

$$4r^2 + h^2 = 4R^2$$

$$\Rightarrow r^2 = \frac{4R^2 - h^2}{4} \ ...(1)$$

Volume of cylinder $= V = \pi r^2 h$

$$\Rightarrow V = \pi \cdot h\left(\frac{4R^2 - h^2}{4}\right) = \pi R^2 h - \frac{\pi}{4}h^3$$

$$\Rightarrow \frac{dV}{dh} = \pi R^2 - \frac{3\pi}{4}h^2 \ ...(2)$$

For finding maximum volume,

$$\frac{dV}{dh} = 0$$

$$\Rightarrow \pi R^2 = \frac{3\pi}{4}h^2$$

$$\Rightarrow h = \frac{2}{\sqrt{3}}R$$

Differentiating equation (2) with respect to *x*, we get

$$\frac{d^2V}{dh^2} = -\frac{6\pi}{4}h$$

$$\frac{d^2V}{dh^2}_{\left(h=\frac{2}{\sqrt{3}}R\right)} = -\frac{3\pi}{2}\left(\frac{2}{\sqrt{3}}R\right)$$

$$\frac{d^2V}{dh^2}_{\left(h=\frac{2}{\sqrt{3}}R\right)} = -\sqrt{3}R\pi < 0$$

Hence volume is maximum when $h = \frac{2}{\sqrt{3}}R$.

Maximum volume $= V_{h=\frac{2R}{\sqrt{3}}} = \pi h\left(\frac{4R^2 - h^2}{4}\right)$

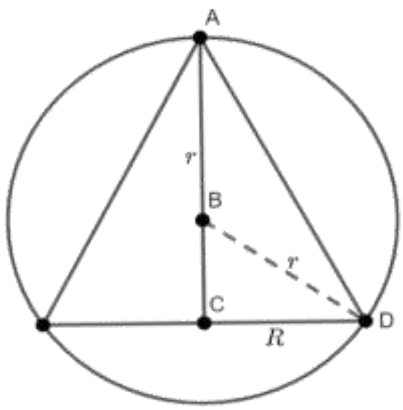
$$V_{max} = \pi \times \frac{2R}{\sqrt{3}} \left(\frac{4R^2 - \frac{4R^2}{3}}{4} \right)$$

$$V_{max} = \frac{2\pi R}{\sqrt{3}} \cdot \frac{2R^2}{3} = \frac{4\pi R^3}{3\sqrt{3}} \text{ cubic units.}$$

92. Show that the altitude of the right circular cone of maximum volume that can be inscribed is a sphere of radius r is $\frac{4r}{3}$.

Ans: It is given that the radius of a fixed sphere is *r*.

Let *R* and *h* be the radius and height of the cone respectively.



The volume (V) of the cone is given by,

$$V = \frac{1}{3}\pi R^2 h$$

Now, from the right triangle BCD, we have:

$$BC = \sqrt{r^2 - R^2}$$

$$h = r + \sqrt{r^2 - R^2}$$

$$\therefore V = \frac{1}{3}\pi R^2 \left(r + \sqrt{r^2 - R^2} \right)$$

$$V=\frac{1}{3}\pi R^2 r+\frac{1}{3}\pi R^2\sqrt{r^2-R^2}$$

Differentiating the above equation with respect to *x*, we get

$$\therefore \frac{dV}{dR}=\frac{2}{3}\pi Rr+\frac{2\pi}{3}\pi R\sqrt{r^2-R^2}+\frac{R^2}{3}\cdot\frac{(-2R)}{2\sqrt{r^2-R^2}}$$

$$=\frac{2}{3}\pi Rr+\frac{2\pi}{3}\pi R\sqrt{r^2-R^2}-\frac{R^3}{3\sqrt{r^2-R^2}}$$

$$=\frac{2}{3}\pi Rr+\frac{2\pi R\left(r^2-R^2\right)-\pi R^3}{3\sqrt{r^2-R^2}}$$

$$=\frac{2}{3}\pi Rr+\frac{2\pi Rr^2-3\pi R^3}{3\sqrt{r^2-R^2}}$$

Now,

$$\frac{dV}{dR^2}=0$$

$$\Rightarrow \frac{2\pi rR}{3}=\frac{3\pi R^3-2\pi Rr^2}{3\sqrt{r^2-R^2}}$$

$$\Rightarrow 2r\sqrt{r^2-R^2}=3R^2-2r^2$$

$$\Rightarrow 4r^2\left(r^2-R^2\right)=\left(3R^2-2r^2\right)^2$$

$$\Rightarrow 4r^4-4r^2R^2=9R^4+4r^4-12R^2r^2$$

$$\Rightarrow 9R^4-8r^2R^2=0$$

$$\Rightarrow 9R^2=8r^2$$

$$\Rightarrow R^2=\frac{8r^2}{9}$$

Now,

$$\frac{d^2V}{dR^2}=\frac{2\pi r}{3}+\frac{3\sqrt{r^2-R^2}\left(2\pi r^2-9\pi R^2\right)-\left(2\pi Rr^2-3\pi R^3\right)(-6R)\frac{1}{2\sqrt{r^2-R^2}}}{9\left(r^2-R^2\right)}$$

$$\frac{d^2V}{dR^2}=\frac{2\pi r}{3}+\frac{3\sqrt{r^2-R^2}\left(2\pi r^2-9\pi R^2\right)+\left(2\pi Rr^2-3\pi R^3\right)(3R)\frac{1}{2\sqrt{r^2-R^2}}}{9\left(r^2-R^2\right)}$$

Now, when $R^2=\frac{8r^2}{9}$, it can be shown that $\frac{d^2V}{dR^2}<0$

The volume is the maximum when $R^2=\frac{8r^2}{9}$

When $R^2=\frac{8r^2}{9}$, height of the cone,

$$=r+\sqrt{r^2-\frac{8r^2}{9}}=r+\sqrt{\frac{r^2}{9}}=r+\frac{r}{3}=\frac{4r}{3}$$

From above we can conclude that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$.

93. Prove that the surface area of solid cuboid of a square base and given volume is minimum, when it is a cube.

Ans: Let us consider x be the side of square base of cuboid and other side be y

Then volume of cuboid with square base,

$$V=x\cdot x\cdot y=x^2y$$

As volume of cuboid is given so volume is taken constant throughout the question,

Therefore,

$$y = \frac{V}{x^2} \ldots \text{ (i)}$$

In order to show that surface area is minimum when the given cuboid is cube, we have to show $S'' > 0$ and $x = y$

Let S be the surface area of cuboid, then

$$S = x^2 + xy + xy + xy + xy + x^2$$

$$S = 2x^2 + 4xy \ldots \text{(ii)}$$

$$\Rightarrow S = 2x^2 + 4x \cdot \frac{V}{x^2}$$

$$\Rightarrow S = 2x^2 + \frac{4V}{x} \ldots \text{(iii)}$$

On differentiating equation (iii) with respect to *x*, we get

$$\Rightarrow \frac{dS}{dx} = 4x - \frac{4V}{x^2} \quad \cdots \text{(iv)}$$

For maximum/minimum value of S, we have

$$\frac{dS}{dx} = 0$$

$$\Rightarrow 4x - \frac{4V}{x^2} = 0$$

$$\Rightarrow 4V = 4x^3$$

$$\Rightarrow V = x^3 \ldots \text{(v)}$$

Putting $V = x^3$ in equation (i), we get

$$y = \frac{x^3}{x^2} = x$$

Here,

$$y = x$$

$\Rightarrow$ cuboid is a cube.

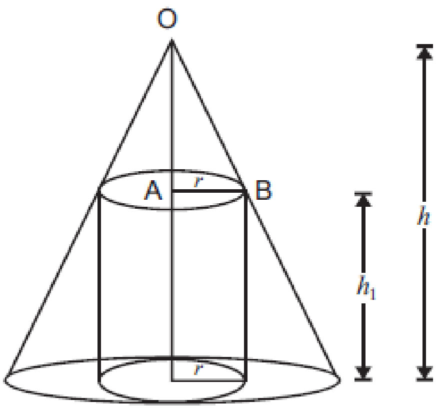
Differentiating (iv) with respect to *x*, we get

$$\frac{d^2S}{dx^2} = \left(4 + \frac{8V}{x^3}\right) > 0$$

Hence, surface area is minimum when given cuboid is a cube.

94. Show that the volume of the greatest cylinder which can be inscribed in a right circular cone of height h and semi-vertical angle α is $\frac{4}{27}\pi h^3 \tan^2\alpha$.

Ans: Let a cylinder of base radius r and height h_1 is included in a cone of height h and semi-vertical angle α.



Then,

$AB = r, OA = (h - h_1)$

In right angled triangle OAB,

$$\frac{AB}{OA} = \tan\alpha$$

$$\Rightarrow \frac{r}{h - h_1} = \tan\alpha$$

or $r = (h - h_1)\tan\alpha$

$$\therefore V = \pi\left[(h - h_1)\tan\alpha\right]^2 \cdot h_1$$

(Volume of cylinder $= \pi r^2 h$)

$$V = \pi \tan^2 \alpha \cdot h_1 \left(h - h_1\right)^2 \ldots(1)$$

Differentiating equation (1) with respect to h_1, we get

$$\frac{dV}{dh_1} = \pi \tan^2 \alpha \left[h_1 \cdot 2\left(h - h_1\right)(-1) + \left(h - h_1\right)^2 \times 1 \right]$$

$$\frac{dV}{dh_1} = \pi \tan^2 \alpha \left(h - h_1\right)\left[-2h_1 + h - h_1\right]$$

$$\frac{dV}{dh_1} = \pi \tan^2 \alpha \left(h - h_1\right)\left(h - 3h_1\right)$$

For maximum volume $V, \dfrac{dV}{dh_1} = 0$

$\Rightarrow h - h_1 = 0 \quad \text{or} \quad h - 3h_1 = 0$

$\Rightarrow \quad h = h_1 \quad \text{or} \quad h_1 = \dfrac{1}{3}h$

$\Rightarrow \quad h_1 = \dfrac{1}{3}h$

$\left(h = h_1\right.$ is not possible)

Again differentiating with respect to h_1, we get

$$\frac{d^2V}{dh_1^2} = \pi \tan^2 \alpha \left[\left(h - h_1\right)(-3) + \left(h - 3h_1\right)(-1) \right]$$

At $h_1 = \dfrac{1}{3}h$,

$$\frac{d^2V}{dh_1^2} = \pi \tan^2 \alpha \left[\left(h - \frac{1}{3}h\right)(-3) + 0 \right]$$

$$\frac{d^2V}{dh_1^2} = -2\pi h \tan^2 \alpha < 0$$

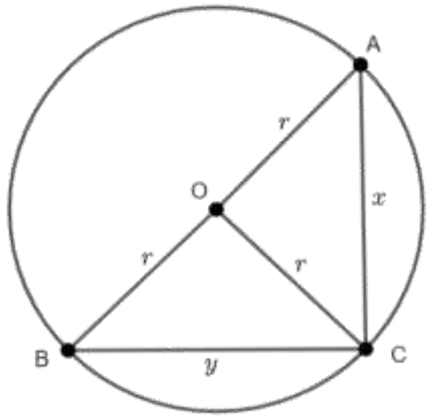
$\therefore$ Volume is maximum for $h_1 = \dfrac{1}{3}h$

$$V_{max} = \pi \tan^2 \alpha \cdot \left(\frac{1}{3}h\right)\left(h - \frac{1}{3}h\right)^2$$
$$V_{max} = \frac{4}{27}\pi h^3 \tan^2 \alpha$$

95. Show that the right triangle of maximum area that can be inscribed in a circle is an isosceles triangle.

Ans: It is given that a right angle triangle is inscribed inside the circle.

Let us consider, $'r'$ is the radius of the circle and x and y be the base and height of the right angle triangle.



The hypotenuse of the $\Delta ABC = AB^2 = AC^2 + BC^2$

$AB = 2r,\ AC = y$ and $BC = x$

Hence,

$$4r^2 = x^2 + y^2$$
$$y^2 = 4r^2 - x^2$$
$$y = \sqrt{4r^2 - x^2} \ldots(1)$$

Now, Area of the ΔABC is,

$$A = \frac{1}{2} \times \text{base} \times \text{height}$$

$$A=\frac{1}{2}\times x\times y$$

Now, substituting (1) in the area of the triangle,

$$A=\frac{1}{2}\times\left(\sqrt{4r^2-x^2}\right)\quad \text{(Squaring both sides)}$$

$$Z=A^2=\frac{1}{4}\times x^2\times\left(4r^2-x^2\right)\ldots(2)$$

For finding the maximum/minimum of given function, we can find it by differentiating it with *x* and then equating it to zero. This is because if the function $f\left(x\right)$ has a maximum/minimum at a point *c* then $f'\left(c\right)=0$.

Differentiating the equation (2) with respect to *x*, we get

$$\frac{dZ}{dx}=\frac{d}{dx}\left[\frac{1}{4}x^2\left(4r^2-x^2\right)\right]$$

$$\frac{dZ}{dx}=\frac{1}{4}\left[\left(4r^2-x^2\right)\frac{d}{dx}\left(x^2\right)+x^2\frac{d}{dx}\left(4r^2-x^2\right)\right]$$

$$\frac{dZ}{dx}=\frac{1}{4}\left[\left(4r^2-x^2\right)\times(2x)+x^2(0-2x)\right]$$

Since $\frac{d}{dx}\left(x^n\right)=nx^{n-1}$.

If *u* and *v* are two functions of *x*, then

$$\left.\frac{d}{dx}(u.v)=v\frac{du}{dx}+u\frac{dv}{dx}\right]$$

$$\frac{dZ}{dx}=\frac{1}{4}\left[8r^2x-2x^3-2x^3\right]$$

$$\frac{dZ}{dx}=\frac{1}{4}\left[8r^2x-4x^3\right]$$

$$\frac{dZ}{dx}=\frac{4x}{4}\left[2r^2-x^2\right]$$

$$\frac{dZ}{dx}=2r^2x-x^3\cdots(3)$$

To find the critical point, we need to equate equation (3) to zero.

$$\Rightarrow \frac{dZ}{dx} = 0$$

$$\Rightarrow 2r^2x - x^3 = 0$$

$$\Rightarrow 2r^2x = x^3$$

$$\Rightarrow x^2 = 2r^2$$

$$\Rightarrow x = \pm\sqrt{2r^2}$$

$$\Rightarrow x = r\sqrt{2}$$

(as the base of the triangle cannot be negative).

Now, to check if this critical point will determine the maximum area of the triangle, we need to check with second differential which needs to be negative.

Consider differentiating the equation (3) with *x*,

$$\frac{d^2Z}{dx^2} = \frac{d}{dx}\left[2r^2x - x^3\right]$$

$$\frac{d^2Z}{dx^2} = \frac{d}{dx}\left(2r^2x\right) - \frac{d}{dx}\left(x^3\right)$$

$$\frac{d^2Z}{dx^2} = 2r^2 - 3x^2 \cdots (4)$$

[Since $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$]

Now, consider the value of

$$\left(\frac{d^2Z}{dx^2}\right)_{x=r\sqrt{2}} = 2r^2 - 3(r\sqrt{2})^2$$

$$\left(\frac{d^2Z}{dx^2}\right)_{x=r\sqrt{2}} = 2r^2 - 6r^2 = -4r^2$$

As $\left(\frac{d^2Z}{dx^2}\right)_{x=r\sqrt{2}} = -4r^2 < 0$

so the function A is maximum at $x = r\sqrt{2}$.

Now substituting $x = r\sqrt{2}$ In equation (1):

$$y = \sqrt{4r^2 - (r\sqrt{2})^2}$$
$$y = \sqrt{4r^2 - 2r^2} = \sqrt{2r^2} = r\sqrt{2}$$

As $x = y = r\sqrt{2}$, the base and height of the triangle are equal, which means that two sides of a right angled triangle are equal,

Hence the given triangle, which is Inscribed In a circle, Is an Isosceles triangle with sides AC and BC equal.

96. A given quantity of metal is to be cast half cylinder with a rectangular box and semi-circular ends. Show that the total surface area is minimum when the ratio of the length of cylinder to the diameter of its semi-circular ends is $\pi : (\pi + 2)$.

Ans: Let r be the radius and h be the height of half cylinder.

Volume $= \frac{1}{2}\pi r^2 h = V$ (constant) …(1)

Total surface area of half cylinder is,

$$S = 2\left(\frac{1}{2}\pi r^2\right) + \pi rh + 2rh \ \ldots(2)$$

From equation (1) and (2), we get

$$S = \left(\pi r^2\right) + \pi r\left(\frac{2V}{\pi r^2}\right) + 2r\left(\frac{2V}{\pi r^2}\right)$$
$$S = \left(\pi r^2\right) + \left(\frac{1}{r}\right)\left[\frac{4V}{\pi} + 2V\right]$$

On differentiating the above equation with respect to *r*, we get

$$\frac{dS}{dr} = (2\pi r) + \left(\frac{-1}{r^2}\right)\left[\frac{4V}{\pi} + 2V\right] \quad \ldots(3)$$

For maxima/minima,

$$\Rightarrow \frac{dS}{dr} = 0$$

$$\Rightarrow (2\pi r) + \left(\frac{-1}{r^2}\right)\left[\frac{4V}{\pi} + 2V\right] = 0$$

$$\Rightarrow (2\pi r) = \left(\frac{1}{r^2}\right)\left[\frac{4V + 2V\pi}{\pi}\right]$$

$$\Rightarrow \pi r^3 = V\left[\frac{2+\pi}{\pi}\right]$$

$$\Rightarrow V = \frac{(\pi^2 r^3)}{(\pi + 2)} \ldots(4)$$

From equation (1) and (4), we get

$$\Rightarrow \frac{1}{2}\pi r^2 h = \frac{\pi^2 r^3}{\pi + 2}$$

$$\Rightarrow \frac{h}{2r} = \frac{\pi}{\pi + 2}$$

Height : diameter = $\pi : \pi + 2$

Differentiating equation (3) with respect to *r*, we get

$$\frac{d^2S}{dr^2} = (2\pi) + \left(\frac{2}{r^3}\right)\left[\frac{4V}{\pi} + 2V\right]$$

= positive (as all quantities are positive)

So, S is minimum when,

Height : diameter = $\pi : \pi + 2$.