# Chapter 2

# Kinematics

## DISTANCE TRAVERSED AND SPEED

### Distance traversed (Path length)

1. The total length of actual path traversed by the body between initial and final positions is called distance.
2. It has no direction and is always positive.
3. Distance covered by particle never decreases.
4. Its SI unit is metre (m) and dimensional formula is $[M^0L^1T^0]$.

## EQUATIONS OF MOTION

**General equations of motion :**

$$v = \frac{dx}{dt} \Rightarrow dx = vdt \Rightarrow \int dx = \int vdt \text{ = area enclosed by velocity-time graph}$$

$$a = \frac{dv}{dt} \Rightarrow dv = adt \Rightarrow \int dv = \int adt \text{ = area enclosed by acceleration-time graph}$$

$$a = \frac{vdv}{dx} \Rightarrow vdv = adx \Rightarrow \int vdv = \int adx \text{ = area enclosed by acceleration-position graph}$$

**Equations of motion of a particle moving with uniform acceleration in straight line :**

1. $v = u + at$
2. $S = ut + \frac{1}{2}at^2 = \left(\frac{v+u}{2}\right)t = vt - \frac{1}{2}at^2$
3. $v^2 = u^2 + 2aS$
4. $S_{n^{th}} = u + \frac{1}{2}a(2n-1)$
5. $x = x_0 + ut + \frac{1}{2}at^2$

Here,

$u$ = velocity of particle at $t = 0$

$S$ = Displacement of particle between 0 to $t$

$= x - x_0$ ($x_0$ = position of particle at $t = 0$, $x$ = position of particle at time $t$)

$a$ = uniform acceleration

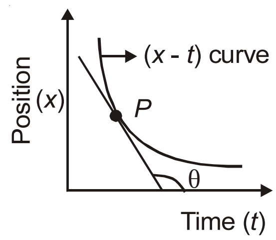
$v$ = velocity of particle at time $t$

$S_{n^{th}}$ = Displacement of the particle in $n^{th}$ second

## GRAPHS

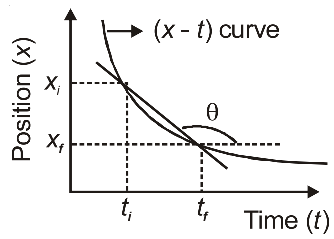
The important properties of various graphs are given below :

1. Slope of the tangent at a point on the position-time graph gives the instantaneous velocity at that point.



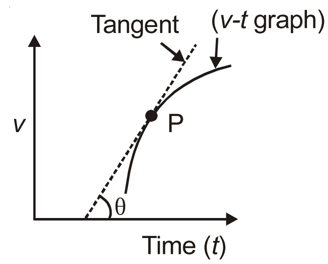

$\tan\theta = \dfrac{dx}{dt} = v$ (**Instantaneous Velocity** at point $P$)

2. Slope of a chord joining two points on the Position-time graph gives the average velocity during the time interval between those points.



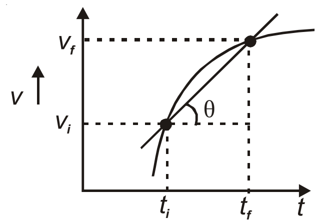

$\tan\theta = \dfrac{x_f - x_i}{t_f - t_i} = v_{av}$

3. Slope of the tangent at a point on the velocity-time graph gives the instantaneous acceleration at that point.



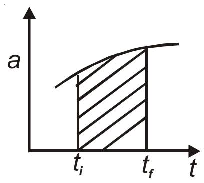

$\tan\theta = \dfrac{dv}{dt} = a$ (Instantaneous acceleration at $P$)

4. Slope of the chord joining two points on the velocity-time graph gives the average acceleration during the time interval between those points.



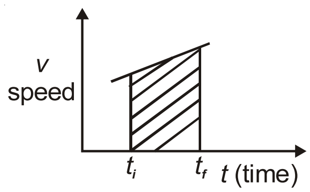

$\tan\theta = \dfrac{v_f - v_i}{t_f - t_i} = a_{av}$ = Average acceleration in time interval $t_f - t_i$

5. The area under the acceleration-time graph between $t_i$ and $t_f$ gives the change in velocity $(v_f - v_i)$ between the two instants.



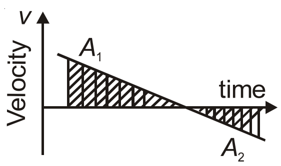

Shaded area = $v_f - v_i$ = change in velocity during interval $t_i$ to $t_f$

6. The area under speed-time graph between $t_i$ and $t_f$ gives distance covered by particle in the interval $t_f - t_i$.




Shaded area = distance covered in time $(t_f - t_i)$

7. The area under the velocity-time graph between $t_i$ and $t_f$ gives the displacement $(x_f - x_i)$ between the two instants.

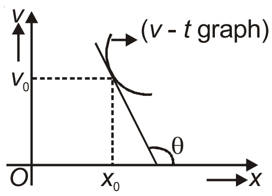



Shaded area $(A_1 - A_2)$ = Displacement in time $(t_f - t_i)$

Also, $A_1 + A_2$ = Distance covered in time $(t_f - t_i)$

8. In velocity-position graph, the acceleration of particle of any position $x_0$ is given as.

$$a = v_0 \tan\theta = v_0 \left|\frac{dv}{dx}\right|_{x=x_0}$$

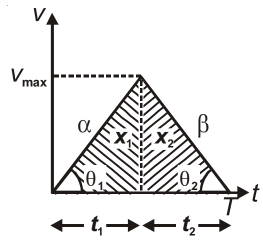



9. The position-time graph cannot be symmetric about the time-axis because at an instant a particle cannot have two displacements.
10. The distance-time graph is always an increasing curve for a moving body.
11. The displacement-time graph does not show the trajectory of the particle.

## Applications

1. If a particle is moving with uniform acceleration on a straight line and have velocity $v_A$ at $A$ and $v_B$ at $B$, then velocity of particle midway on line $AB$ is $v = \sqrt{\dfrac{v_A^2 + v_B^2}{2}}$.
2. If a body starts from rest with acceleration $\alpha$ and then retards to rest with retardation $\beta$, such that total time of journey is $T$, then

(a) Maximum velocity during the trip $v_{max.} = \left(\frac{\alpha\beta}{\alpha+\beta}\right).T$




(b) Length of the journey $L = \frac{1}{2}\left(\frac{\alpha\beta}{\alpha+\beta}\right)T^2$

(c) Average velocity of the trip = $\frac{v_{max.}}{2} = \frac{\alpha\beta T}{2(\alpha+\beta)}$

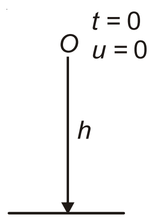
(d) $\frac{x_1}{x_2} = \frac{\beta}{\alpha} = \frac{t_1}{t_2}$ .

## MOTION UNDER GRAVITY

If height of object is very small as compared to radius of earth, motion of object will be uniformly accelerated. Equation of motion can be applied with proper sign convention.

**Following are the important cases of interest.**

1. Object is released from a height *h.*




Time taken to reach ground

$-h = 0 - \frac{1}{2}gT^2$ (taking up as positive)

$\Rightarrow \ T = \sqrt{\frac{2h}{g}}$

Velocity of ball when it reaches ground

$v = 0 - gT = -g\sqrt{\frac{2h'}{g}} = -\sqrt{2gh}$

'–' sign indicate that velocity will be in downward direction.

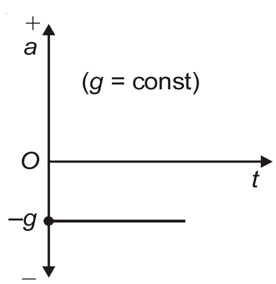
2. A particle is projected from ground with velocity *u* in vertically upward direction then

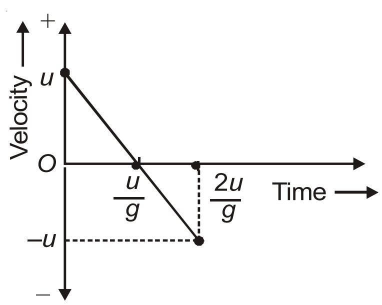
(a) Time of ascent = Time of descent = $\frac{\text{Time of flight}}{2} = \frac{T}{2} = \frac{u}{g}$

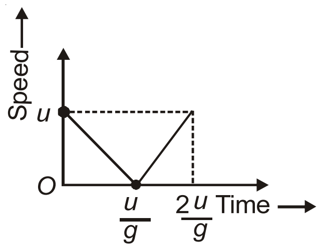
(b) Maximum height attained = $\frac{u^2}{2g}$

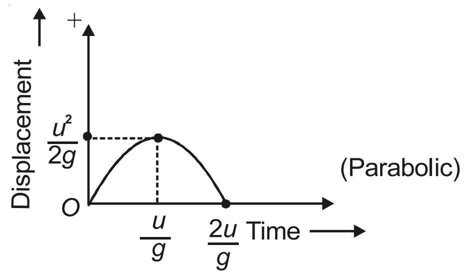
(c) Speed of particle when it hits the ground = *u*

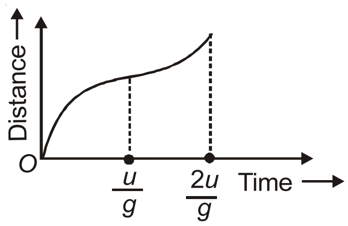
(d) Graphs






(e) Displacement of particle in complete journey = zero $\Rightarrow$ average velocity $v_{av} = 0$

(f) Distance covered by particle in complete journey = $\dfrac{u^2}{g}$

$\Rightarrow$ Average speed in complete journey = $\dfrac{u}{2}$

3. A body is thrown upward such that it takes $t$ seconds to reach its highest point.
   (a) Distance travelled in $(t)^{th}$ second = distance travelled in $(t + 1)^{th}$ second.
   (b) Distance travelled in $(t - 1)^{th}$ second = distance travelled in $(t + 2)^{th}$ second.
   (c) Distance travelled in $(t - r)^{th}$ second = distance travelled in $(t + r + 1)^{th}$ second.
4. A body is projected upward from certain height $h$ with initial speed $u$.
   (a) Its speed when it acquires the same level is $u$.
   (b) Its speed at the ground level is

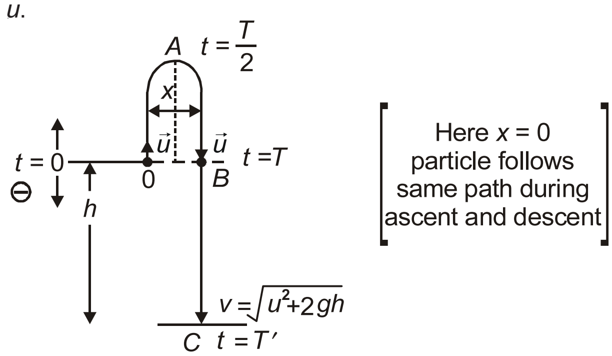
   $$v = \sqrt{u^2 + 2gh}$$

   (c) The time required to attain same level is

   $$T = \frac{2u}{g}$$

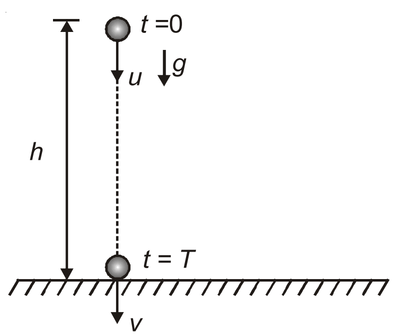
   (d) Total time of flight ($T'$) is obtained by solving

   $$\Rightarrow -h = uT' - \frac{1}{2}gT'^2 \text{ or } T' = \frac{u + \sqrt{u^2 + 2gh}}{g}$$

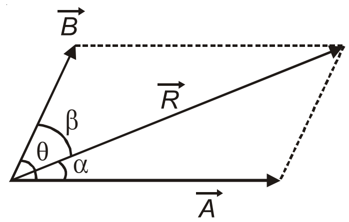



5. A body is projected from a certain height $h$ with initial speed $u$ downward.
   (a) Its speed at ground level is $v = \sqrt{u^2 + 2gh}$
   (b) Time of flight ($T$)

   $$T = \frac{-u + \sqrt{u^2 + 2gh}}{g}$$



**Parallelogram Law of Vector Addition :** If two vectors having common origin are represented both in magnitude and direction as the two adjacent sides of a parallelogram, then the diagonal which originates from the common origin represents the resultant of these two vectors. The results are listed below




(a) $\vec{R} = \vec{A} + \vec{B}$

(b) $|\vec{R}| = (A^2 + B^2 + 2AB\cos\theta)^{1/2}$

(c) $\tan\alpha = \dfrac{B\sin\theta}{A + B\cos\theta}$, $\tan\beta = \dfrac{A\sin\theta}{B + A\cos\theta}$

(d) If $|\vec{A}| = |\vec{B}| = x\,(\text{say})$, then $R = x\sqrt{2(1+\cos\theta)} = 2x\cos\dfrac{\theta}{2}$ and $\alpha = \beta = \dfrac{\theta}{2}$ *i.e.,* resultant bisect angle between $\vec{A}$ and $\vec{B}$.

(e) If $|\vec{A}|\ \ |\vec{B}|$ then $\alpha < \beta$

(f) $R_{max} = A + B$, when $\theta = 0$ and $R_{min} = |A - B|$ when $\theta = 180°$.

(g) $R^2 = A^2 + B^2$, if $\theta = 90°$ *i.e.*, $\vec{A}$ and $\vec{B}$ are perpendicular.

(h) If $|\vec{A}| = |\vec{B}| = |\vec{R}|$, then $\theta = 120°$.

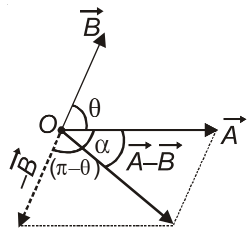
(i) If $\vec{R}$ is perpendicular to $\vec{A}$, then $\cos\theta = -\dfrac{A}{B}$ and $A^2 + R^2 = B^2$.

(j) For $n$ coplanar vectors of same magnitude acting at a point such that angle between consecutive vectors are equal $\left(\dfrac{360}{n}\right)$, the resultant is zero.

## VECTOR SUBTRACTION

Subtraction of vector $\vec{B}$ from vector $\vec{A}$ is simply addition of vector $-\vec{B}$ with $\vec{A}$ *i.e.,* $\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$

Using parallelogram law,




**Result :** $R = |\vec{A} - \vec{B}| = \sqrt{A^2 + B^2 - 2AB\cos\theta}$, $\tan\alpha = \dfrac{B\sin\ \pi-\theta}{A + B\cos(\pi-\theta)} = \dfrac{B\sin\theta}{A - B\cos\theta}$

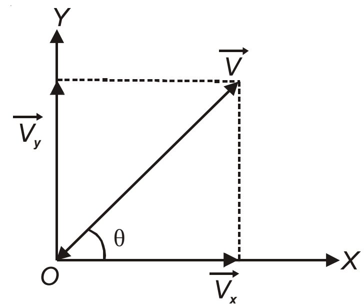
> ***Note :*** *If* $|\vec{A}| = |\vec{B}| = x\,(\text{say})$, *then* $R = x\sqrt{2(1-\cos\theta)} = 2x\sin\dfrac{\theta}{2}$.

## RESOLUTION OF VECTORS

Any vector $\vec{V}$ can be represented as a sum of two vectors $\vec{P}$ and $\vec{Q}$ which are in same plane as $\vec{V} = \lambda\vec{P} + \mu\vec{Q}$, where $\lambda$ and $\mu$ are two real numbers. We say that $\vec{V}$ has been resolved in two component vector $\lambda\vec{P}$ and $\mu\vec{Q}$ along $\vec{P}$ and $\vec{Q}$ respectively.

**Rectangular components in two dimensions :**

$\vec{V} = \vec{V}_x + \vec{V}_y, \vec{V} = V_x\hat{i} + V_y\hat{j},\ V = \sqrt{V_x^2 + V_y^2}$




$\vec{V}_x$ and $\vec{V}_y$ are rectangular component of vector in 2-dimension.

$V_x = V \cos\theta$

$V_y = V \sin\theta = V\cos(90 - \theta)$

$V_z$ = zero.

$\vec{V} = V\cos\theta\,\hat{i} + V\sin\theta\,\hat{j}$

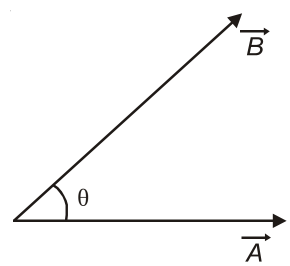
> ***Note :*** *Unit vector along* $\vec{V}$ is $\cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

## SCALAR AND VECTOR PRODUCTS

**Scalar (dot) Product of Two Vectors :** The scalar product of two vectors $\vec{A}$ and $\vec{B}$ is defined as

$\vec{A}\,.\,\vec{B} = AB\cos\theta$

$\cos\theta = \dfrac{\vec{A}\,.\,\vec{B}}{AB}$




If $\vec{A}$ and $\vec{B}$ are perpendicular, then $\vec{A}.\vec{B} = 0$

If $\theta < 90°$, then $\vec{A}.\vec{B}$ 0 and if $\theta > 90°$ then $\vec{A}.\vec{B}$ 0.

Projection of vector $\vec{A}$ on $\vec{B}$ is $(\vec{A}.\vec{B})\dfrac{\vec{B}}{B^2}$.

$A^2 = \vec{A}.\vec{A}$

$\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1.$

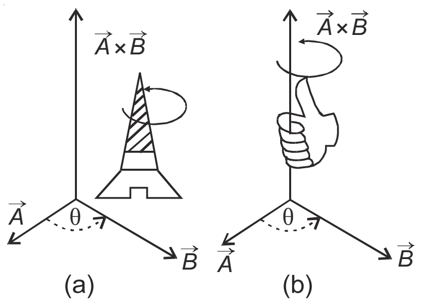
Scalar product is commutative *i.e.*, $\vec{A}.\vec{B} = \vec{B}.\vec{A}$.

Scalar product is distributive *i.e.*, $\vec{A}\ (\vec{B}\ \ \vec{C}) = \vec{A}\ \vec{B} + \vec{A}\ \vec{C}$

**Vector Product of two Vectors :**

Mathematically, if $\theta$ is the angle between vectors $A$ and $B$, then

$A \times B = AB\sin\theta\,\hat{n}$ ...(i)




The direction of vector $A \times B$ is the same as that of unit vector $\hat{n}$. It is decided by any of the following two rules :

**(a)** **Right handed screw rule :** Rotate a right handed screw from vector $A$ to $B$ through the smaller angle between them; then the direction of motion of screw gives the direction of vector $A \times B$ (Fig. a).

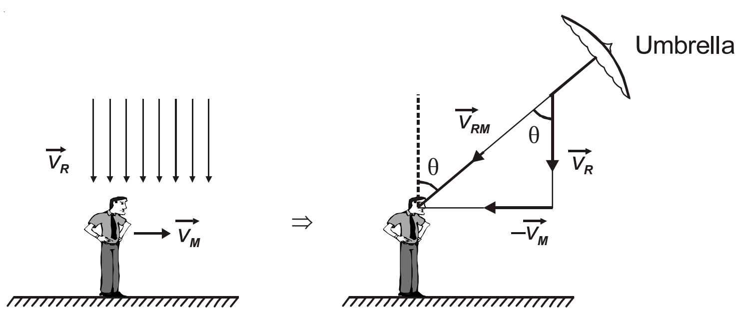
**(b)** **Right hand thumb rule :** Bend the finger of the right hand in such a way that they point in the direction of rotation from vector $A$ to $B$ through the smaller angle between them; then the thumb points in the direction of vector $A \times B$ (Fig. b).

## RELATIVE MOTION IN TWO DIMENSIONS

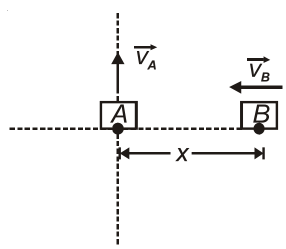
**Relative velocity :**

Velocity of object *A* w.r.t. object *B* is $\vec{v}_{AB} = \vec{v}_A - \vec{v}_B, \vec{v}_{BA} = \vec{v}_B - \vec{v}_A$

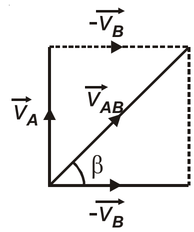
**1.** **Direction of Umbrella :** A person moving on straight road has to hold his umbrella opposite to direction of relative velocity of rain. The angle θ is given by $\tan\theta = \dfrac{v_M}{v_R}$ with vertical in forward direction.




**2.** **Closest approach :** Two objects *A* and *B* having velocities $\vec{v}_A$ and $\vec{v}_B$ at separation *x* are shown in figure



The relative velocity of $A$ with respect to $B$ is given by

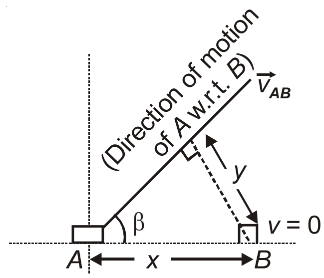



$$\vec{v}_{AB} = \vec{v}_A - \vec{v}_B$$

$$\tan\beta = \frac{v_A}{v_B}$$

The above situation is similar to figure given below.

$y$ is the distance of closest approach.

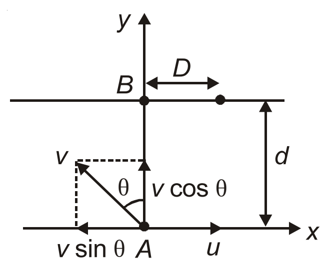



Now, $\sin\beta = \frac{y}{x}$

$\Rightarrow$ $y = x \sin\beta$

$$y = \frac{x\tan\beta}{\sqrt{1+\tan^2\beta}} = \frac{xV_A}{\sqrt{V_B^2 + V_A^2}}$$

**3. Crossing a river :**




$v$ = velocity of the man in still water.

θ = angle at which man swims w.r.t. normal to bank such that

$v_x = -v\sin\theta$, $v_y = v\cos\theta$

Time taken to cross the river is given by

$$t = \frac{d}{v_y} = \frac{d}{v\cos\theta}$$

Velocity along the river

$$v'_x = u - v\sin\theta$$

Distance drifted along the river $D = t\,v'_x$

$$D = \frac{d}{v\cos\theta}(u - v\sin\theta)$$

***Case I : (Shortest time)***

The Minimum time to cross the river is given by

$$_{min} = \frac{d}{v} \qquad \text{(when } \cos\theta = 1,\ \theta = 0°,\ u \quad v\text{)}$$

Distance drifted is given by

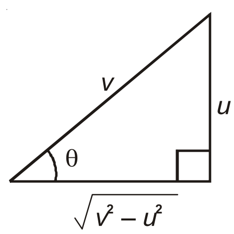
$$D = \frac{d}{v} \times u$$

***Case II : (Shortest path)***

To cross the river straight

drift $D = 0$ $\Rightarrow$ $u - v\sin\theta = 0$

$\sin\theta = \frac{u}{v}$ $\Rightarrow$ provided $v > u$




Time to cross the river straight across is given by

$$t = \frac{d}{v\cos\theta} = \frac{d}{\sqrt{v^2 - u^2}}$$

**PROJECTILE MOTION**

An object moving in space under the influence of gravity is called projectile. Two important cases of interest are discussed below :

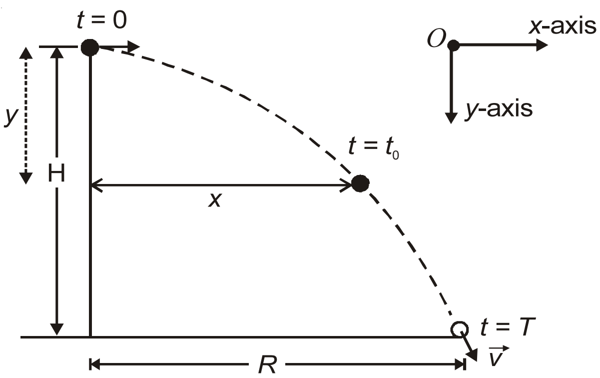
**1. Horizontal projection :**

A body of mass $m$ is projected horizontally with a speed $u$ from a height $h$ at the moment $t = 0$. The path followed by it is a parabola.

It hits the ground at the moment $t = T$, with a velocity $\vec{v}$ such that

$$T = \sqrt{\frac{2H}{g}}$$

$$|\vec{v}| = \sqrt{u^2 + 2gH} = |u\hat{i} + gT\hat{j}|$$




The position at any instant $t_0$ is given by

$$x = ut_0$$

$$y = \frac{1}{2}gt_0^2$$

$$y = \frac{gx^2}{2u^2} \text{ (trajectory of particle)}$$

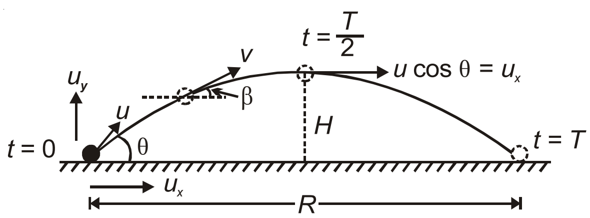
The velocity at any instant $t_0$ is given by

$$\overrightarrow{v_0} = u\hat{i} + gt_0\hat{j}$$

The range $R$ will be given by $\boxed{R = u\sqrt{\frac{2H}{g}}}$

**2. Oblique projection :** A body of mass $m$ is projected from ground with speed $u$ at an angle $\theta$ above horizontal at the moment $t = 0$.

It hits the ground at a horizontal distance $R$ at the moment $t = T$.




1. Time of flight $\quad T = \frac{2u_y}{g} = \frac{2u\sin\theta}{g}$

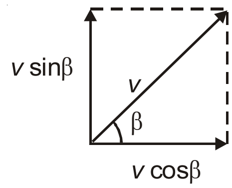
2. Maximum height $\quad H = \frac{u_y^2}{2g} = \frac{u^2\sin^2\theta}{2g}$

3. Horizontal range $\quad R = u_x \times T = \frac{2u_xu_y}{g} = \frac{u^2\sin 2\theta}{g}$

4. Equation of trajectory: $\quad y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$

or $y = x\tan\theta\left(1 - \frac{x}{R}\right)$

5. Instantaneous velocity $v = \sqrt{u^2 + (gt)^2 - 2u(gt)\sin\theta}$



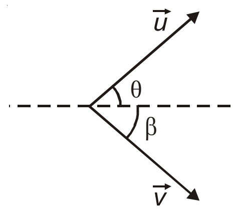

and direction of motion is such that, $\tan\beta = \dfrac{u\sin\theta - gt}{u\cos\theta}$

(a) $v = \dfrac{u\cos\theta}{\cos\beta}$ [∵ Horizontal component is same everywhere]

(b) $v\sin\beta = u\sin\theta - gt$

(c) When $\vec{v}$ (velocity at any instant '$t$') is perpendicular to $\vec{u}$ (initial velocity)

$\Rightarrow$ $\beta = -(90° - \theta)$




(i) $v = \dfrac{u\cos\theta}{\cos(90 - \theta)} = u\cot\theta$

(ii) $t = \dfrac{u}{g\sin\theta}$

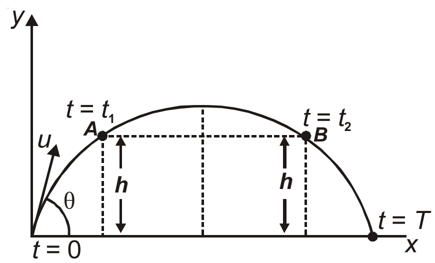
**Applications :**

1. The height attained by the particle is largest when $\theta = 90°$. In this situation, time of flight is maximum and range is minimum (zero).
2. When $R$ is range, $T$ is time of flight and $H$ is maximum height, then

   (a) $\tan\theta = \dfrac{gT^2}{2R}$

   (b) $\tan\theta = \dfrac{4H}{R}$

3. When horizontal range is maximum, $H = \dfrac{R_{max}}{4}$
4. The horizontal range is same for complimentary angles like $(\theta, 90° - \theta)$ or $(45° + \theta, 45° - \theta)$. It is maximum for $\theta = 45°$.
5. If $A$ and $B$ are two points at same level such that the object passes $A$ at $t = t_1$ and $B$ at $T = t_2$, then



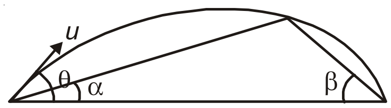

   (i) $T = \dfrac{2u\sin\theta}{g} = t_1 + t_2$

   (ii) $h = \dfrac{1}{2}gt_1t_2$

   (iii) Average velocity in the interval $AB$ is

   $v_{av} = u\cos\theta$ [∵ vertical displacement is zero]

6. If a projectile is projected from one vertex of a triangle such that it grazes second vertex and finally fall down on 3rd vertex of the triangle on the same horizontal level, then $\boxed{\tan\theta = \tan\alpha + \tan\beta}$.




7. A projectile has same range for two angle of projection. If time of flight in two cases are $T_1$ and $T_2$, maximum height is $H_1$ and $H_2$ and the horizontal range is $R$. Then

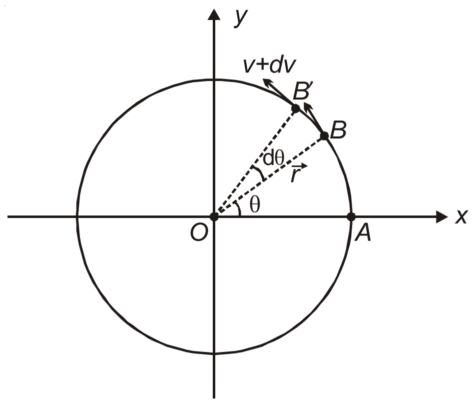
   (i) Range of projectile is $R = \frac{1}{2} g T_1 T_2$

   (ii) Velocity of projection of projectile is $u = \frac{1}{2} g \left[T_1^2 + T_2^2\right]^{1/2}$

   (iii) $R = 4\sqrt{H_1 H_2}$

## CIRCULAR MOTION

An object of mass $m$ is moving on a circular track of radius $r$. At $t$ = 0, it was at $A$. At any moment of time '$t$', it has moved to $B$, such that $AOB = \theta$. Let its speed at this instant be $v$ and direction is along the tangent. In a small time $dt$, it moves to $B'$ such that $B'OB = d\theta$.




The angular displacement vector is $\overrightarrow{d\theta} = d\theta\hat{k}$

The angular velocity vector is $\vec{\ } = \frac{d\theta}{dt}\hat{k}$.

At $B'$, the speed of the object has become $v + dv$.

The tangential acceleration is $a_t = \frac{dv}{dt} = r\alpha$

The radial (centripetal acceleration) is $a_c = \frac{v^2}{r} = \ ^2 r$

The angular acceleration is $\alpha = \frac{d}{dt}$

Relations among various quantities.

1. $\vec{v} = \vec{\ } \times \vec{r}$

2. $\vec{a} = \frac{d\vec{v}}{dt} = \vec{\ } \times \frac{d\vec{r}}{dt} + \frac{d\vec{\ }}{dt} \times \vec{r} = \overrightarrow{a_c} + \overrightarrow{a_t}$

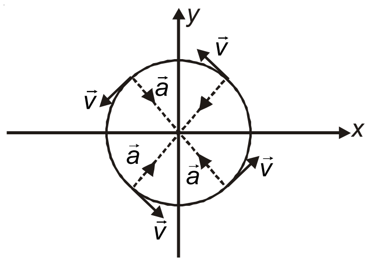
3. $\overrightarrow{a_c} = \vec{\ } \times \vec{v}$

4. $\overrightarrow{a_t} = \vec{\alpha} \times \vec{r}$

## Uniform Circular Motion :

1. In uniform circular motion, the speed ($v$) of particle is remain constant (   = constant)

2. $a_T = 0$ and $\alpha = \frac{d}{dt} = 0$



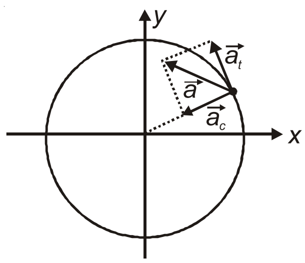

3. Only centripetal acceleration (also called normal acceleration) exists in uniform circular motion

   $a_c = r\ \ ^2 = \frac{v^2}{r}$

4. In uniform circular motion $\vec{v}$   $\vec{a}$

## Nonuniform Circular Motion :

1. In nonuniform circular motion the speed ($v$) and angular velocity (  ) change w.r.t. time.
2. Net acceleration of particle in non-uniform circular motion.




$$a = \sqrt{a_c^2 + a_t^2} = \sqrt{\left(\frac{v^2}{r}\right)^2 + r\alpha\ \ ^2}$$

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