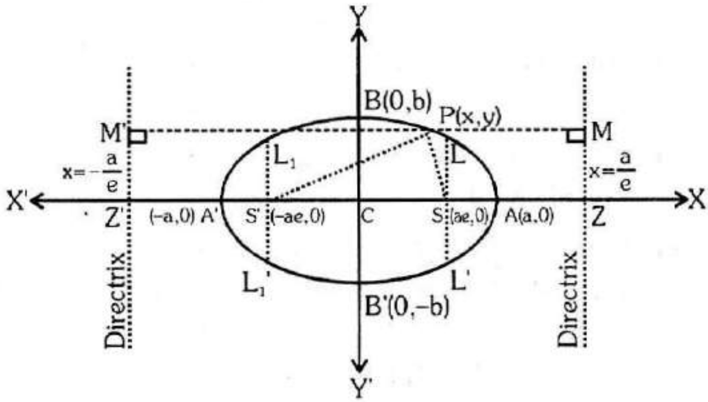
ELLIPSE

1. STANDARD EQUATION & DEFINITION :

Standard equation of an ellipse referred to its principal axis along the co-ordinate axis is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. where $a > b$ & $b^2 = a^2(1-e^2)$

$\Rightarrow a^2 - b^2 = a^2 e^2$.

where e = eccentricity ($0 < e < 1$).



FOCI : $S \equiv (ae, 0)$ & $S' \equiv (-ae, 0)$.

(a) Equation of directrices :

$$x=\frac{a}{e} \ \& \ x=-\frac{a}{e}.$$

(b) Vertices :

$A' \equiv (-a, 0)$ & $A \equiv (a, 0)$.

(c) Major axis : The line segment A' A in which the foci S' & S lie is of length 2a & is called the **major axis** ($a > b$) of the ellipse. Point of intersection of major axis with directrix is called **the foot of the directrix (Z)** $\left(\pm\frac{a}{e}, 0\right)$.

(d) Minor Axis : The y-axis intersects the ellipse in the points $B' \equiv (0,-b)$ & $B \equiv (0, b)$. The line segment B'B of length 2b (b < a) is called the **Minor Axis** of the ellipse.

(e) Principal Axis : The major & minor axis together are called **Principal Axis** of the ellipse.

(f) Centre : The point which bisects every chord of the conic drawn through it is called the **centre** of the conic. $C \equiv (0,0)$ the origin is the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(g) Diameter : A chord of the conic which passes through the centre is called a **diameter** of the conic.

(h) Focal Chord : A chord which passes through a focus is called a **focal chord.**

(i) Double Ordinate : A chord perpendicular to the major axis is called a **double ordinate** with respect to major axis as diameter.

(j) Latus Rectum : The focal chord perpendicular to the major axis is called the **latus rectum**.

(i) Length of latus rectum

$$(LL') = \frac{2b^2}{a} = \frac{(\text{minor axis})^2}{\text{major axis}} = 2a(1-e^2)$$

(ii) Equation of latus rectum : $x = \pm ae$.

(iii) Ends of the latus rectum are $L\left(ae, \frac{b^2}{a}\right)$, $L'\left(ae, -\frac{b^2}{a}\right)$,

$$L_1\left(-ae, \frac{b^2}{a}\right) \text{ and } L_1'\left(-ae, -\frac{b^2}{a}\right).$$

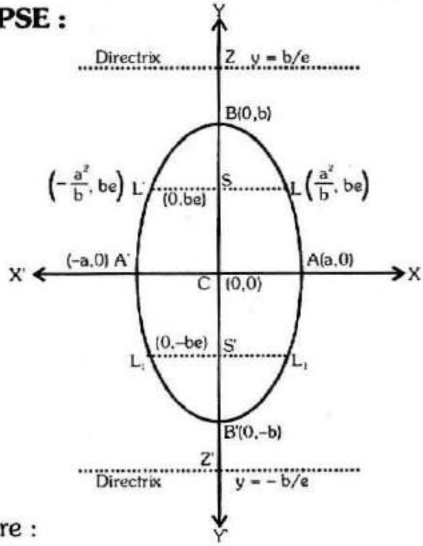
(k) Focal radii : $SP = a - ex$ and $S'P = a + ex$

$\Rightarrow SP + S'P = 2a =$ Major axis.

(l) Eccentricity : $e = \sqrt{1 - \frac{b^2}{a^2}}$

2. ANOTHER FORM OF ELLIPSE :

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, (a <b)



(a) AA' = Minor axis = 2a

(b) BB' = Major axis = 2b

(c) $a^2 = b^2(1-e^2)$

(d) Latus rectum

$$LL' = L_1L_1' = \frac{2a^2}{b}.$$

equation $y = \pm be$

(e) Ends of the latus rectum are :

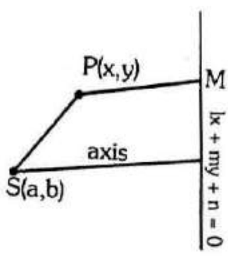
$$L\left(\frac{a^2}{b}, be\right), L'\left(-\frac{a^2}{b}, be\right), L_1\left(\frac{a^2}{b}, -be\right), L_1'\left(-\frac{a^2}{b}, -be\right)$$

(f) Equation of directrix $y = \pm\frac{b}{e}$.

(g) Eccentricity : $e = \sqrt{1-\frac{a^2}{b^2}}$

3. GENERAL EQUATION OF AN ELLIPSE :

Let (a,b) be the focus S, and $lx + my + n = 0$ is the equation of directrix. Let P(x,y) be any point on the ellipse. Then by definition.



$\Rightarrow$ SP = e PM (e is the eccentricity)

$$\Rightarrow (x-a)^2 + (y-b)^2 = e^2\frac{(lx+my+n)^2}{(l^2+m^2)}$$

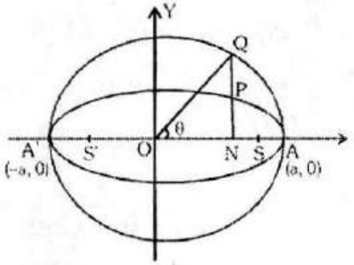
$$\Rightarrow (l^2+m^2)\{(x-a)^2+(y-b)^2\} = e^2\{lx+my+n\}^2$$

4. POSITION OF A POINT W.R.T. AN ELLIPSE :

The point $P(x_1, y_1)$ lies outside, inside or on the ellipse according as ; $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1 > < \text{or} = 0$.

5. AUXILLIARY CIRCLE/ECCENTRIC ANGLE :

A circle described on major axis as diameter is called the **auxiliary circle.** Let Q be a point on the auxiliary circle $x^2 + y^2 = a^2$ such that QP produced is perpendicular to the x-axis then P & Q are called as the CORRESPONDING POINTS on the ellipse & the auxiliary circle respectively.'θ' is called the **ECCENTRIC ANGLE** of the point P on the ellipse $(0 \le \theta < 2\pi)$.



Note that $\frac{l(PN)}{l(QN)} = \frac{b}{a} = \frac{\text{Semi minor axis}}{\text{Semi major axis}}$

Hence "If from each point of a circle perpendiculars are drawn upon a fixed diameter then the locus of the points dividing these perpendiculars in a given ratio is an ellipse of which the given circle is the auxiliary circle".

6. PARAMATRIC REPRESENTATION :

The equations $x = a\cos\theta$ & $y = b\sin\theta$ together represent the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

where θ is a parameter (eccentric angle).

Note that if $P(\theta) \equiv (a\cos\theta, b\sin\theta)$ is on the ellipse then ;
$Q(\theta) \equiv (a\cos\theta, a\sin\theta)$ is on the auxiliary circle.

7. LINE AND AN ELLIPSE :

The line $y = mx + c$ meets the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in two real points, coincident or imaginary according as c^2 is $<=$ or $>$ $a^2m^2+b^2$.

Hence $y = mx + c$ is tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if $c^2 = a^2m^2+b^2$.

The equation to the chord of the ellipse joining two points with eccentric angles α & β is given by $\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2}$.

8. TANGENT TO THE ELLIPSE $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$:

(a) Point form :

Equation of tangent to the given ellipse at its point (x_1, y_1) is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$$

(b) Slope form :

Equation of tangent to the given ellipse whose slope is 'm', is

$$y = mx \pm \sqrt{a^2m^2+b^2}$$

Point of contact are $\left(\frac{\pm a^2m}{\sqrt{a^2m^2+b^2}},\frac{\mp b^2}{\sqrt{a^2m^2+b^2}}\right)$

(c) Parametric form :

Equation of tangent to the given ellipse at its point $(a\cos\theta, b\sin\theta)$, is $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$

9. NORMAL TO THE ELLIPSE $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$:

(a) Point form : Equation of the normal to the given ellipse at (x_1, y_1) is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2=a^2e^2$.

(b) Slope form : Equation of a normal to the given ellipse whose slope is 'm' is $y = mx \mp \frac{(a^2 - b^2)m}{\sqrt{a^2 + b^2m^2}}$.

(c) Parametric form : Equation of the normal to the given ellipse at the point $(a\cos\theta, b\sin\theta)$ is $ax.\sec\theta - by.\text{cosec}\,\theta = (a^2 - b^2)$.

10. CHORD OF CONTACT :

If PA and PB be the tangents from point $P(x_1, y_1)$ to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

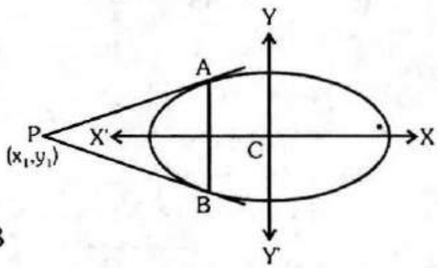
then the equation of the chord of contact AB is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ or $T = 0$ at (x_1, y_1)

11. PAIR OR TANGENTS :

If $P(x_1, y_1)$ be any point lies outside the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and a pair of tangents PA, PB can be drawn to it from P.



Then the equation of pair of tangents of PA and PB is $SS_1 = T^2$

where $S_1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$, $T = \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$

i.e. $$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right) = \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1\right)^2$$

12. DIRECTOR CIRCLE :

Locus of the point of intersection of the tangents which meet at right angles is called the **Director Circle.** The equation to this locus is $x^2 + y^2 = a^2 + b^2$ i.e. a circle whose centre is the centre of the ellipse & whose radius is the length of the line joining the ends of the major & minor axis.

13. EQUATION OF CHORD WITH MID POINT (x_1, y_1) :

The equation of the chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,

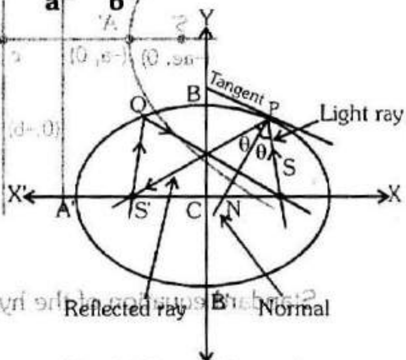
whose mid-point be (x_1, y_1) is $T = S_1$

where $T = \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$, $S_1 = \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$

i.e. $\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)=\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)$

14. IMPORTANT HIGHLIGHTS for $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$:

(I) The tangent & normal at a point P on the ellipse bisect the external & internal angles between the focal distances of P. This refers to the well known reflection property of the ellipse which states that rays from one focus are reflected through other focus & vice-versa.



(II) Point of intersection of the tangents at the point α & β is $\left(a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$

(III) If $A(\alpha)$, $B(\beta)$, $C(\gamma)$ & $D(\delta)$ are conormal points then sum of their eccentric angles is odd multiple of π. i.e. $\alpha+\beta+\gamma+\delta = (2n+1)\pi$.

(IV) If $A(\alpha)$, $B(\beta)$, $C(\gamma)$ & $D(\delta)$ are four concyclic points then sum of their eccentric angles is even multiple of π.

i.e. $\alpha+\beta+\gamma+\delta = 2n\pi$

(V) The product of the length's of the perpendicular segments from the foci on any tangent to the ellipse is b^2 and the feet of these perpendiculars lie on its auxiliary circle.