8 Trigonometric Equations

KEY FACTS

1. A **trigonometric equation** is an equation involving the trigonometric function or functions of unknown angles, *e.g.* $\cos x = 0$, $\sin^2 x = \frac{1}{2}$, $\cos\left(\theta + \frac{\pi}{4}\right) = \frac{1}{2}$, etc.

2. The **solution** of a trigonometric equation is a value of the unknown angle that satisfies the equation.
A trigonometric equation may have an unlimited number of solutions.
For example, if $\sin x = 0$, then $x = 0, \pi, 2\pi, 3\pi, ...$

- **A solution lying between 0° and 360°** is called the **principal solution.**
- Since the trigonometric functions are periodic, a **solution generalized by means of periodicity** is known as the **general solution.**
  Every equation will have a principal solution as well as a general solution.

3. Solutions of equations of type $\sin\theta = 0$, $\cos\theta = 0$ and $\tan\theta = 0$

(*a*) $\sin\theta = 0$

$\because$ $\sin\theta = \sin 0 = \sin\pi = \sin 2\pi = \sin(-2\pi) = ... = 0$
Therefore $\sin\theta = 0$ is satisfied by the following values of θ.
$\therefore$ $\theta = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \pm 4\pi, ...$
$\Rightarrow$ The general solution of **$\sin\theta = 0$ is $\theta = n\pi$, $n \in I$**

(*b*) $\cos\theta = 0$

$\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$, ... satisfy the equation $\cos\theta = 0$ as

$$\cos\theta = \cos\left(\pm\frac{\pi}{2}\right) = \cos\left(\pm\frac{3\pi}{2}\right) = ... = 0$$

$\Rightarrow$ The general solution of **$\cos\theta = 0$ is $\theta = (2n+1)\frac{\pi}{2}, n \in I$**

(*c*) $\tan\theta = 0$

This is satisfied by $\theta = 0, \pm\pi, \pm 2\pi, ...$
$\Rightarrow$ The general solution of **$\tan\theta = 0$ is $\theta = n\pi$, $n \in I$**

4. Solution of equations of type $\sin\theta = \sin\alpha$

$\sin\theta = \sin\alpha \Rightarrow \sin\theta - \sin\alpha = 0$

$$\Rightarrow 2\cos\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2} = 0$$

$\Rightarrow$ Either $\cos\dfrac{\theta+\alpha}{2}=0$ or $\sin\dfrac{\theta-\alpha}{2}=0$

$\Rightarrow \dfrac{\theta+\alpha}{2}=(2m+1)\dfrac{\pi}{2}, m\in I$ | $\dfrac{\theta-\alpha}{2}=m\pi, m\in I$

$\Rightarrow \theta+\alpha=(2m+1)\pi$ | $\theta-\alpha=2m\pi, m\in I$

$\Rightarrow \theta=(2m+1)\pi-\alpha$ ....(*i*) | $\Rightarrow \theta=2m\pi+\alpha, m\in I$

Thus θ = ($-\alpha$ + odd multiple of π) or (α + even multiple of π)

$\therefore$ $\boldsymbol{\theta = n\pi + (-1)^n \alpha}$ **where** $\boldsymbol{n \in I}$.

5. Solution of equations of type cos θ = cos α

$\cos\theta=\cos\alpha \Rightarrow \cos\theta-\cos\alpha=0$

$\Rightarrow 2\sin\dfrac{\theta+\alpha}{2}\sin\dfrac{\alpha-\theta}{2}=0 \Rightarrow -\sin\dfrac{\theta+\alpha}{2}\sin\dfrac{\theta-\alpha}{2}=0 \Rightarrow -\sin\dfrac{\theta+\alpha}{2}=0$ or $\sin\dfrac{\theta-\alpha}{2}=0$

$\Rightarrow \sin\dfrac{\theta+\alpha}{2}=0$ or $\sin\dfrac{\theta-\alpha}{2}=0$

$\Rightarrow \dfrac{\theta+\alpha}{2}=n\pi$ | $\dfrac{\theta-\alpha}{2}=n\pi$

$\Rightarrow \theta+\alpha=2n\pi$ | $\theta-\alpha=2n\pi$

$\Rightarrow \theta=2n\pi-\alpha$ | $\theta=2n\pi+\alpha$

$\therefore$ $\boldsymbol{\theta = 2n\pi \pm \alpha, n\in I}$

6. Solution of equations of type tan θ = tan α

$\tan\theta=\tan\alpha$

$\Rightarrow \dfrac{\sin\theta}{\cos\theta}=\dfrac{\sin\alpha}{\cos\alpha} \Rightarrow \sin\theta\cos\alpha-\cos\theta\sin\alpha=0$

$\Rightarrow \sin(\theta-\alpha)=0 \Rightarrow \theta-\alpha=n\pi \Rightarrow$ $\boldsymbol{\theta = n\pi + \alpha, n\in I}$.

Note: 1. In the above results, α is numerically the least angle which should be expressed in radians as far as possible.

2. $\operatorname{cosec}\theta=\operatorname{cosec}\alpha$ follows from $\sin\theta=\sin\alpha$

$\sec\theta=\sec\alpha$ follows from $\cos\theta=\cos\alpha$

$\cot\theta=\cot\alpha$ follows from $\tan\theta=\tan\alpha$.

7. Solution of the equations sin²θ = sin²α, cos²θ = cos²α, tan²θ = tan²α

(*a*) $\sin^2\theta=\sin^2\alpha$

$\Rightarrow \dfrac{1-\cos2\theta}{2}=\dfrac{1-\cos2\alpha}{2}$

$\Rightarrow \cos2\theta=\cos2\alpha$

$\Rightarrow 2\theta=2n\pi\pm2\alpha$

$\Rightarrow \boldsymbol{\theta=n\pi\pm\alpha, n\in I}$

(*b*) $\cos^2\theta=\cos^2\alpha$

$\Rightarrow \dfrac{1+\cos2\theta}{2}=\dfrac{1+\cos2\alpha}{2}$

$\Rightarrow \cos2\theta=\cos2\alpha$

$\Rightarrow 2\theta=2n\pi\pm2\alpha$

$\Rightarrow \boldsymbol{\theta=n\pi\pm\alpha, n\in I}$

(*c*) $\tan^2\theta=\tan^2\alpha$

$\Rightarrow \dfrac{1}{\tan^2\theta}=\dfrac{1}{\tan^2\alpha}$

$\Rightarrow \dfrac{1-\tan^2\theta}{1+\tan^2\theta}=\dfrac{1-\tan^2\alpha}{1+\tan^2\alpha}$

$\Rightarrow \cos2\theta=\cos2\alpha$

$\Rightarrow 2\theta=2n\pi\pm2\alpha$

$\Rightarrow \boldsymbol{\theta=n\pi\pm\alpha, n\in I}$

SOLVED EXAMPLES

Ex. 1. Solve for *x*, $\sin x=-\dfrac{\sqrt3}{2}$, $(0<x<2\pi)$

Sol. $\sin x=-\dfrac{\sqrt3}{2}=-\sin60°=\sin(180°+60°)=\sin(360°-60°)$

$\Rightarrow$ $\boldsymbol{x = 240°, 300°}$.

Ex. 2. Solve $4\cos^2\theta = 3$ $(0° \le \theta \le 360°)$

Sol. $4\cos^2\theta = 3 \Rightarrow \cos^2\theta = \frac{3}{4} \Rightarrow \cos\theta = \pm\frac{\sqrt{3}}{2}$

$\cos\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = 30°, 330°$ (∵ cos θ is +ve and so θ lies in 1st and 4th quad.)

$\cos\theta = -\frac{\sqrt{3}}{2} \Rightarrow \theta = 150°, 210°$ (∵ cos θ is –ve and so θ lies in 2nd and 3rd quad.)

$\Rightarrow$ **$\theta = 30°, 150°, 210°, 330°$.**

Note: Recall that if the value of θ is α when θ lies in Ist quadrant then it is 180° – α, 180° + α and 360° – α. If θ lies in 2nd, 3rd and 4th quadrant respectively.

Ex. 3. Solve $\sin^2 x + \sin x - 2 = 0$. $(0° < \theta < 360°)$

Sol. $\sin^2 x + \sin x - 2 = 0 \Rightarrow (\sin x + 2)(\sin x - 1) = 0$

or

$\Rightarrow \sin x + 2 = 0$	$\sin x - 1 = 0$
$\Rightarrow \sin x = -2$	$\sin x = 1 \Rightarrow x = 90°$
This value of x being numerically > 1 is inadmissible as sin x cannot be numerically greater than 1.	

∴ **$x = 90°$.**

Ex. 4. Solve $\cos^2\theta - \sin\theta - \frac{1}{4} = 0$ $(0° < \theta < 360°)$

Sol. $\cos^2\theta - \sin\theta - \frac{1}{4} = 0 \Rightarrow 1 - \sin^2\theta - \sin\theta - \frac{1}{4} = 0$

$\Rightarrow 4\sin^2\theta + 4\sin\theta - 3 = 0 \Rightarrow (2\sin\theta + 3)(2\sin\theta - 1) = 0$

$\Rightarrow 2\sin\theta + 3 = 0$ or $2\sin\theta - 1 = 0$

$\Rightarrow \sin\theta = -\frac{3}{2}$ or $\sin\theta = \frac{1}{2} \Rightarrow \theta = 30°, 150°$.

Since $|\sin\theta| = \frac{3}{2}$ is > 1, the value $\sin\theta = -\frac{3}{2}$ is inadmissible.

∴ **$\theta = 30°, 150°$.**

Ex. 5. Solve $2\sin\theta\cos\theta = \cos\theta$ $(0° < \theta < 360°)$

Sol. The given equation may be written as:

$2\sin\theta\cos\theta - \cos\theta = 0 \Rightarrow \cos\theta(2\sin\theta - 1) = 0$

$\Rightarrow \cos\theta = 0$ or $2\sin\theta - 1 = 0$

$\Rightarrow \theta = 90°, 270°$ | $\sin\theta = \frac{1}{2} \Rightarrow \theta = 30°, 150°$

∴ **$\theta = 30°, 90°, 150°$ $270°$.**

Ex. 6. Solve $\tan^2\alpha + \sec\alpha - 1 = 0$, $0 \le \alpha \le 2\pi$

Sol. $\tan^2\alpha + \sec\alpha - 1 = 0 \Rightarrow \sec^2\alpha - 1 + \sec\alpha - 1 = 0$

$\Rightarrow \sec^2\alpha + \sec\alpha - 2 = 0 \Rightarrow (\sec\alpha + 2)(\sec\alpha - 1) = 0$

$\Rightarrow$ sec α + 2 = 0 or sec α – 1 = 0

$\Rightarrow \sec\alpha = -2$ | $\sec\alpha = 1$

$\Rightarrow \cos\alpha = -\frac{1}{2}$ | $\cos\alpha = 1$

$\Rightarrow \alpha = \frac{2\pi}{3}, \frac{4\pi}{3}$ | $\alpha = 0.$

$\therefore$ $\boldsymbol{\alpha = 0, \frac{2\pi}{3}, \frac{4\pi}{3}}$.

Ex. 7. Find the general values of θ if

(*i*) $2\sin\theta - 1 = 0$ (*ii*) $\cos\theta = -\frac{1}{2}$ (*iii*) $4\sin^2\theta = 1$

(*iv*) $\tan 2x - \sqrt{3} = 0$ (*v*) $2\cot^2\theta = \text{cosec}^2\theta$

Sol. (*i*) $2\sin\theta - 1 = 0 \Rightarrow \sin\theta = \frac{1}{2} = \sin\frac{\pi}{6} \Rightarrow \boldsymbol{\theta = n\pi + (-1)^n \frac{\pi}{6}, n \in I.}$

(*ii*) $\cos\theta = -\frac{1}{2} = \cos(180° - 60°) = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3} \therefore \boldsymbol{\theta = 2n\pi \pm \frac{2\pi}{3}, n \in I.}$

(*iii*) $4\sin^2\theta = 1 \Rightarrow \frac{4(1-\cos 2\theta)}{2} = 1 \Rightarrow 2(1 - \cos 2\theta) = 1 \Rightarrow 1 - \cos 2\theta = \frac{1}{2} \Rightarrow \cos 2\theta = \frac{1}{2} = \cos\frac{\pi}{3}$

$\therefore 2\theta = 2n\pi \pm \frac{\pi}{3} \Rightarrow \boldsymbol{\theta = n\pi \pm \frac{\pi}{6}}.$

(*iv*) $\tan 2x - \sqrt{3} = 0 \Rightarrow \tan 2x = \sqrt{3} = \tan\frac{\pi}{3} \Rightarrow 2x = n\pi + \frac{\pi}{3} \Rightarrow \boldsymbol{x = \frac{n\pi}{2} + \frac{\pi}{6}, n \in I}$

(*v*) $2\cot^2\theta = \text{cosec}^2\theta \Rightarrow 2\cot^2\theta = 1 + \cot^2\theta \Rightarrow \cot^2\theta = 1$

$\Rightarrow \cot\theta = \pm 1 = \cot\left(\pm\frac{\pi}{4}\right) \Rightarrow \boldsymbol{\theta = n\pi \pm \frac{\pi}{4}, n \in I.}$

Ex. 8. If $3\cos x \neq 2\sin x$, then find the general solution of $\sin^2 x - \cos 2x = 2 - \sin 2x$. (***EAMCET 2009***)

Sol. $\sin^2 x - \cos 2x = 2 - \sin 2x$

$\Rightarrow 1 - \cos^2 x - (2\cos^2 x - 1) = 2 - 2\sin x\cos x$

$\Rightarrow -3\cos^2 x + 2\sin x\cos x = 0$

$\Rightarrow \cos x(2\sin x - 3\cos x) = 0 \Rightarrow \cos x = 0$ ($\because 2\sin x \neq 3\cos x$)

$\Rightarrow x = 2n\pi \pm \frac{\pi}{2} \Rightarrow \boldsymbol{x = (4n \pm 1)\frac{\pi}{2}}.$

Ex. 9. Find the general solution of $|\sin x| = \cos x, n \in I.$

Sol. $|\sin x| = \cos x \Rightarrow \sin^2 x = \cos^2 x \Rightarrow 1 - \cos^2 x = \cos^2 x \Rightarrow 2\cos^2 x = 1$

$\Rightarrow \cos x = +\frac{1}{\sqrt{2}}$ [$\because \cos x = |\sin x|$, it cannot be negative]

$\Rightarrow \cos x = \cos\frac{\pi}{4} \Rightarrow \boldsymbol{x = 2n\pi \pm \frac{\pi}{4}}.$

Ex. 10. Find the general solution of tan 2 θ tan θ = 1. (***Gujarat CET 2007***)

Sol. Given, $\tan 2\theta \tan\theta = 1$

$\Rightarrow \frac{2\tan\theta}{1-\tan^2\theta} . \tan\theta = 1 \Rightarrow \frac{2\tan^2\theta}{1-\tan^2\theta} = 1 \Rightarrow 2\tan^2\theta = 1 - \tan^2\theta$

$\Rightarrow 3\tan^2\theta = 1 \Rightarrow \tan^2\theta = \frac{1}{3} \Rightarrow \tan^2\theta = \left(\frac{1}{\sqrt{3}}\right)^2 = \tan^2\frac{\pi}{6} \Rightarrow \theta = n\pi \pm \boldsymbol{\frac{\pi}{6}}$

Ex. 11. Find the number of solutions of the equation tan x + sec x = 2 cos x, $x \in [0, \pi]$. (***WBJEE 2012***)

Sol. $\tan x + \sec x = 2\cos x \Rightarrow \dfrac{\sin x}{\cos x} + \dfrac{1}{\cos x} = 2\cos x$

$\Rightarrow 1 + \sin x = 2\cos^2 x \Rightarrow 1 + \sin x = 2(1 - \sin^2 x) = 2 - 2\sin^2 x$

$\Rightarrow 2\sin^2 x + \sin x - 1 = 0 \Rightarrow (\sin x + 2)(2\sin x - 1) = 0$

$\Rightarrow (\sin x + 2) = 0$ or $2\sin x = 1 \Rightarrow \sin x = -2$ or $\sin x = \dfrac{1}{2}$

Since $\sin x = -2$ is inadmissible, therefore, $\sin x = \dfrac{1}{2}$

$\Rightarrow x = 30°, 150°$, *i.e.* $x = \boldsymbol{\dfrac{\pi}{6}, \dfrac{5\pi}{6}}$.

$\therefore$ The number of solutions $x \in [0, \pi]$ are **2.**

Ex. 12. Find the general solution of sin 9θ = sin θ.

Sol. $\sin 9\theta = \sin\theta$

$\Rightarrow 9\theta = 2n\pi + \theta$ or $9\theta = (2n+1)\pi - \theta, n \in I$ $\quad (\because \sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n \alpha$, where $n \in I)$

$\Rightarrow \theta = \dfrac{2n\pi}{8}$ or $\theta = \dfrac{(2n+1)\pi}{10} \Rightarrow \boldsymbol{\theta = \dfrac{n\pi}{4}}$ **or** $\boldsymbol{\dfrac{(2n+1)\pi}{10}}$.

Ex. 13. Solve tan 3x = cot 5x (0 < x < 2π).

Sol. $\tan 3x = \tan\left(\dfrac{\pi}{2} - 5x\right)$

$\Rightarrow 3x = n\pi + \left(\dfrac{\pi}{2} - 5x\right)$ $\quad (\because \tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha)$

$\Rightarrow 8x = (2n+1)\dfrac{\pi}{2} \Rightarrow x = (2n+1)\boldsymbol{\dfrac{\pi}{16}}$

$\therefore$ Putting $n = 0, 1, 2, ... 15$, we see that the values of x between 0 and 2π are

$\boldsymbol{x = \dfrac{\pi}{16}, \dfrac{3\pi}{16}, \dfrac{5\pi}{16}, ..., \dfrac{31\pi}{16}}$.

Ex. 14. Solve $\boldsymbol{\cos 3x + \cos 2x = \sin\dfrac{3}{2}x + \sin\dfrac{1}{2}x, \quad 0 < x \le \pi}$. (***IIT***)

Sol. $\cos 3x + \cos 2x = \sin\dfrac{3}{2}x + \sin\dfrac{1}{2}x \Rightarrow 2\cos\dfrac{5}{2}x\cos\dfrac{x}{2} = 2\sin x\cos\dfrac{x}{2}$

$\Rightarrow \cos\dfrac{x}{2}\left[\cos\dfrac{5x}{2} - \sin x\right] = 0 \Rightarrow \cos\dfrac{x}{2} = 0$ or $\cos\dfrac{5x}{2} - \sin x = 0$

Now, $\cos\dfrac{x}{2} = 0 \Rightarrow \dfrac{x}{2} = \dfrac{\pi}{2} \Rightarrow x = \pi$

and $\cos\dfrac{5x}{2} - \sin x = 0 \Rightarrow \cos\dfrac{5x}{2} = \sin x \Rightarrow \cos\dfrac{5x}{2} = \cos\left(\dfrac{\pi}{2} - x\right)$ or $\sin\left(2\pi + \dfrac{\pi}{2} - x\right)$

$\Rightarrow \dfrac{5x}{2} = \dfrac{\pi}{2} - x$ or $\dfrac{5x}{2} = 2\pi + \dfrac{\pi}{2} - x \Rightarrow \dfrac{7}{2}x = \dfrac{\pi}{2}$ or $\dfrac{7x}{2} = \dfrac{5\pi}{2} \Rightarrow x = \dfrac{\pi}{7}$ or $\dfrac{5\pi}{7}$

$\therefore \boldsymbol{x = \dfrac{\pi}{7}, \dfrac{5\pi}{7}}$ **or π.**

Ex. 15. Solve $\boldsymbol{\sqrt{3}\sin\theta - \cos\theta = \sqrt{2}}$.

Sol. Dividing both sides of the equation by $\sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2$, we get $\dfrac{\sqrt{3}}{2}\sin\theta - \dfrac{1}{2}\cos\theta = \dfrac{1}{\sqrt{2}}$

$\Rightarrow \cos 30° \sin \theta - \sin 30° \cos \theta = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \sin(\theta - 30°) = \sin\left(\theta - \frac{\pi}{6}\right) = \frac{1}{\sqrt{2}}$

$\Rightarrow \sin\left(\theta - \frac{\pi}{6}\right) = \sin\frac{\pi}{4} \quad \Rightarrow \quad \theta - \frac{\pi}{6} = n\pi + (-1)^n \frac{\pi}{4}, n \in I \quad \Rightarrow \quad \boldsymbol{\theta = n\pi + \frac{\pi}{6} + (-1)^n \frac{\pi}{4}, n \in I}$.

Ex. 16. Solve $\cos x - \sqrt{3} \sin x = 1, 0° \le x \le 360°$.

Sol. Dividing both the sides of the equation $\cos x - \sqrt{3} \sin x = 1$ by $\sqrt{(1)^2 + (-\sqrt{3})^2} = 2$, we get

$\frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x = \frac{1}{2} \Rightarrow \cos 60° \cos x - \sin 60° \sin x = \frac{1}{2} \quad \Rightarrow \quad \cos(x + 60°) = \cos 60°$

$\Rightarrow \cos(x + 60°) = \cos 60° = \cos(360° - 60°) = \cos(360° + 60°)$

$\Rightarrow x + 60° = 60°$ or $300°$ or $420° \Rightarrow$ **$x = 0°, 240°, 360°$.**

Ex. 17. Show that the equation $k \sin x + \cos 2x = 2k - 7$ has a solution only if $2 \le k \le 6$. (***Kerala PET 2011***)

Sol. $k \sin x + \cos 2x = 2k - 7 \quad \Rightarrow \quad k \sin x + 1 - 2\sin^2 x = 2k - 7$

$\Rightarrow 2\sin^2 x - k \sin x - 1 + 2k - 7 = 0 \quad \Rightarrow \quad 2\sin^2 x - k \sin x + 2k - 8 = 0$, which is a quadratic in $\sin x$.

$\therefore \sin x = \frac{k \pm \sqrt{k^2 - 4(2)(2k-8)}}{4} = \frac{k \pm \sqrt{k^2 - 16k + 64}}{4} = \frac{k \pm (k-8)}{4}$

$= \frac{k + k - 8}{4}$ or $\frac{k - k + 8}{4} = \frac{2k - 8}{4}$ or $2 = \frac{k-4}{2}$ or 2

$\because -1 \le \sin x \le 1, \therefore \sin x = 2$ is inadmissible.

$\therefore -1 \le \frac{k-4}{2} \le 1 \quad \Rightarrow \quad -2 \le k - 4 \le 2 \quad \Rightarrow \quad \boldsymbol{2 \le k \le 6}$.

Ex. 18. Find the total number of solutions of the equation $\sin^4 x + \cos^4 x = \sin x \cos x$ in $[0, 2\pi]$.

Sol. $\sin^4 x + \cos^4 x = \sin x \cos x$

$\Rightarrow (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x = \sin x \cos x \quad \Rightarrow \quad 1 - \frac{(2\sin x \cos x)^2}{2} = \frac{2\sin x \cos x}{2}$

$\Rightarrow 1 - \frac{\sin^2 2x}{2} = \frac{\sin 2x}{2} \quad \Rightarrow \sin^2 2x + \sin 2x - 2 = 0$

$\Rightarrow (\sin 2x + 2)(\sin 2x - 1) = 0 \Rightarrow \sin 2x = 1 \Rightarrow \sin 2x = \sin\frac{\pi}{2} = \sin\left(2\pi + \frac{\pi}{2}\right)$

($\because \sin 2x \ne -2$ is indivisible)

$\Rightarrow 2x = \frac{\pi}{2}$ or $\left(2\pi + \frac{\pi}{2}\right) \Rightarrow 2x = \frac{\pi}{2}$ or $\frac{5\pi}{2} \Rightarrow \boldsymbol{x = \frac{\pi}{4}}$ **or** $\boldsymbol{\frac{5\pi}{4}}$.

Ex. 19. If $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$, then what is the number of distinct real roots of this equation in the interval $-\pi/2 < x < \pi/2$? (***AMU 2011***)

Sol. Let $A = \begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix}$ $\therefore$ Applying $R_1 \to R_1 + R_2 + R_3$, we have

$$A = \begin{vmatrix} \sin x + 2\cos x & 2\cos x + \sin x & 2\cos x + \sin x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = (\sin x + 2\cos x)\begin{vmatrix} 1 & 1 & 1 \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we have

$$= (\sin x + 2\cos x)\begin{vmatrix} 1 & 0 & 0 \\ \cos x & \sin x - \cos x & 0 \\ \cos x & 0 & \sin x - \cos x \end{vmatrix}$$

$= (\sin x + 2\cos x) \times (\sin x - \cos x)^2$ (Expanding along R_1)

$\therefore$ The given equation reduces to

$(\sin x + 2\cos x)(\sin x - \cos x)^2 = 0 \Rightarrow (\tan x + 2)(\tan x - 1)^2 = 0.$

$\Rightarrow (\tan x + 2) = 0$ or $(\tan x - 1) = 0 \quad \Rightarrow \tan x = -2$ or $\tan x = 1$

$\Rightarrow$ **$x = \tan^{-1}(-2)$ or $\frac{\pi}{4}$** $\quad (\because -\infty < \tan x < \infty)$.

$\therefore$ **There are two possible values of x.**

Ex. 20. Find the smallest positive value of x satisfying the equation $\log_{\cos x} \sin x + \log_{\sin x} \cos x = 2$. (*EAMCET*)

Sol. Let $\log_{\cos x} \sin x = p$. Then, $\log_{\sin x} \cos x = \frac{1}{p}$.

$\therefore \log_{\cos x} \sin x + \log_{\sin x} \cos x = 2$

$\Rightarrow p + \frac{1}{p} = 2 \Rightarrow p^2 - 2p + 1 = 0 \Rightarrow (p-1)^2 = 0 \Rightarrow p = 1$

$\Rightarrow \log_{\cos x} \sin x = 1 \Rightarrow \sin x = \cos x$

The smallest positive value of x for which $\sin x = \cos x$ is **$x = \frac{\pi}{4}$.**

PRACTICE SHEET

1. The value of θ $(0 < \theta < 2\pi)$ satisfying $\text{cosec}\,\theta + 2 = 0$ are
(*a*) 210°, 330° (*b*) 210°, 240°
(*c*) 210°, 330° (*d*) 240°, 300° **(*EAMCET*)**

2. The value of x in $\left(0, \frac{\pi}{2}\right)$ satisfying the equation $\sin x \cos x = \frac{1}{4}$ is
(*a*) $\frac{\pi}{6}$ (*b*) $\frac{\pi}{8}$ (*c*) $\frac{\pi}{12}$ (*d*) $\frac{\pi}{4}$
(*Kerala CEE 2010*)

3. If the equation $\tan\theta + \tan 2\theta + \tan\theta \,.\, \tan 2\theta = 1$, $\theta =$
(*a*) $\frac{n\pi}{6} + \frac{\pi}{6}$ (*b*) $\frac{n\pi}{2} + 6$ (*c*) $\frac{n\pi}{3} + \frac{\pi}{12}$ (*d*) $\frac{n\pi}{2} + \frac{\pi}{12}$
(*Odisha JEE 2011*)

4. The general value of θ obtained from the equation $\cos 2\theta = \sin\alpha$ is
(*a*) $\theta = 2n\pi \pm \left(\frac{\pi}{2} - \alpha\right)$ (*b*) $\theta = \frac{n\pi + (-1)^n \alpha}{2}$
(*c*) $\theta = n\pi \pm \left(\frac{\pi}{4} - \frac{\alpha}{2}\right)$ (*d*) $2\theta = \frac{\pi}{2} - \alpha$
(*MPPET, AMU 2002*)

5. The general value of θ for which $\frac{\tan\theta - 1}{\tan\theta + 1} = \sqrt{3}$ is
(*a*) $\frac{n\pi}{3} + \frac{\pi}{12}$ (*b*) $\frac{n\pi}{3} + \frac{7\pi}{36}$ (*c*) $n\pi + \frac{\pi}{12}$ (*d*) $n\pi + \frac{7\pi}{12}$
(*MPPET 2012*)

6. The solution of the equation $4\cos^2 x + 6\sin^2 x = 5$ are
(*a*) $x = n\pi \pm \frac{\pi}{4}$ (*b*) $x = n\pi \pm \frac{\pi}{3}$
(*c*) $x = n\pi \pm \frac{\pi}{2}$ (*d*) $x = n\pi \pm \frac{2\pi}{3}$
(*MPPET 2009*)

7. The root of the equation $1 - \cos\theta = \sin\theta . \sin\frac{\theta}{2}$ is
(*a*) $k\pi, k \in I$ (*b*) $2k\pi, k \in I$
(*c*) $k.\frac{\pi}{2}, k \in I$ (*d*) None of these **(*DCE 2008*)**

8. The most general value of θ satisfying the equation $\sin\theta = \sin\alpha$ and $\cos\theta = \cos\alpha$ is
(*a*) $2n\pi + \alpha$ (*b*) $2n\pi - \alpha$ (*c*) $n\pi + \alpha$ (*d*) $n\pi - \alpha$
(*UPSEE 2008*)

9. $x \in R$: $\cos 2x + 2\cos^2 x = 2$ is equal to
(*a*) $2n\pi + \frac{\pi}{3}, n \in Z$ (*b*) $n\pi \pm \frac{\pi}{6}, n \in Z$
(*c*) $n\pi + \frac{\pi}{3}, n \in Z$ (*d*) $2n\pi - \frac{\pi}{3}, n \in Z$
(*EAMCET 2008*)

10. If $2\sec 2\alpha = \tan\beta + \cot\beta$, then one of the values of $\alpha + \beta$ is
(*a*) $\frac{\pi}{4}$ (*b*) $\frac{\pi}{2}$ (*c*) π (*d*) $n\pi - \frac{\pi}{4}, n \in I$
(*VITEEE 2009*)

11. If $\cos x \neq -\frac{1}{2}$, then the solutions of $\cos x + \cos 2x + \cos 3x = 0$ are

(*a*) $2n\pi \pm \frac{\pi}{4}, n \in Z$ (*b*) $2n\pi \pm \frac{\pi}{3}, n \in Z$

(*c*) $2n\pi \pm \frac{\pi}{6}, n \in Z$ (*d*) $2n\pi \pm \frac{\pi}{2}, n \in Z$

(*J&K CET 2008*)

12. The most general solution of the equation $\sec^2 x = \sqrt{2}\left(1 - \tan^2 x\right)$ are given by

(*a*) $n\pi \pm \frac{\pi}{4}$ (*b*) $2n\pi + \frac{\pi}{4}$ (*c*) $n\pi \pm \frac{\pi}{8}$ (*d*) None of these

(*BITSAT 2013*)

13. If $\sin 2x = 4 \cos x$, then x is equal to

(*a*) $\frac{n\pi}{2} \pm \frac{\pi}{4}, n \in Z$ (*b*) No value

(*c*) $n\pi + (-1)^n \frac{\pi}{4}, n \in Z$ (*d*) $2n\pi \pm \frac{\pi}{2}, n \in Z$

(*Karnataka CET 2012*)

14. If $\sin 6\theta + \sin 4\theta + \sin 2\theta = 0$, then the general value of θ is

(*a*) $\frac{n\pi}{4}, n\pi \pm \frac{\pi}{3}$ (*b*) $\frac{n\pi}{4}, n\pi \pm \frac{\pi}{6}$

(*c*) $\frac{n\pi}{4}, 2n\pi \pm \frac{\pi}{3}$ (*d*) $\frac{n\pi}{4}, 2n\pi \pm \frac{\pi}{6}$

(*WBJEE 2010*)

15. If $2 \sin^2 \theta + \sqrt{3} \cos \theta + 1 = 0$, then the value of θ is

(*a*) $\frac{\pi}{6}$ (*b*) $\frac{2\pi}{3}$ (*c*) $\frac{5\pi}{6}$ (*d*) π

(*Odisha JEE 2008*)

16. If $1 + \sin\theta + \sin^2\theta + ... \infty = 4 + 2\sqrt{3}, 0 < \theta < \pi, \theta \neq \frac{\pi}{2}$, then

(*a*) $\theta = \frac{\pi}{3}$ (*b*) $\theta = \frac{\pi}{6}$

(*c*) $\theta = \frac{\pi}{3}$ or $\theta = \frac{\pi}{6}$ (*d*) $\theta = \frac{\pi}{3}$ or $\theta = \frac{2\pi}{3}$

(*Manipal 2008*)

17. The equation $3 \sin^2 x + 10 \cos x - 6 = 0$ is satisfied, if

(*a*) $x = n\pi \pm \cos^{-1}\left(\frac{1}{3}\right)$ (*b*) $x = 2n\pi \pm \cos^{-1}\left(\frac{1}{3}\right)$

(*c*) $x = n\pi \pm \cos^{-1}\left(\frac{1}{6}\right)$ (*d*) $x = 2n\pi \pm \cos^{-1}\left(\frac{1}{6}\right)$

(*WBJEE 2007*)

18. If $\tan\theta + \sec\theta = \sqrt{3}$, then the principal value of $\theta + \frac{\pi}{6}$ is equal to

(*a*) $\frac{\pi}{4}$ (*b*) $\frac{\pi}{3}$ (*c*) $\frac{2\pi}{3}$ (*d*) $\frac{3\pi}{4}$

(*EAMCET 2000*)

19. The solution of equation $\cos^2 \theta + \sin \theta + 1 = 0$ lies in the interval:

(*a*) $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ (*b*) $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$ (*c*) $\left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)$ (*d*) $\left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$

(*MPPET 2006*)

20. If $12 \cot^2\theta - 31 \operatorname{cosec} \theta + 32 = 0$, then the value of $\sin \theta$ is

(*a*) $\frac{3}{5}$ or 1 (*b*) $\frac{2}{3}$ or $-\frac{2}{3}$

(*c*) $\frac{4}{5}$ or $\frac{3}{4}$ (*d*) $\pm\frac{1}{2}$

(*Karnataka CET 2005*)

21. Which one of the following equations has no solution?

(*a*) $\cos \theta + \sin \theta = \sqrt{2}$ (*b*) $\operatorname{cosec} \theta \sec \theta = 1$

(*c*) $\sqrt{3} \sin\theta - \cos\theta = 2$ (*d*) $\operatorname{cosec}\theta - \sec\theta = \operatorname{cosec}\theta \sec\theta$

(*KCET 2006*)

22. The number of real roots of the equation $\sin^4 \theta - 2 \sin^2\theta - 1 = 0$ in the interval $(0, 2\pi)$ is

(*a*) 0 (*b*) 1 (*c*) 2 (*d*) 4

(*IIT 1983*)

23. The general solution of $\sin x - \cos x = \sqrt{2}$ for any integer *n is*

(*a*) $n\pi$ (*b*) $(2n + 1)\pi$ (*c*) $2n\pi$ (*d*) $2n\pi + \frac{3\pi}{4}$

(*BITSAT 2006*)

24. The number of values of x in the interval $[0, 5\pi]$ satisfying the equation $3\sin^2 x - 7\sin x + 2 = 0$ is

(*a*) 0 (*b*) 5 (*c*) 6 (*d*) 10 **(*IIT*)**

25. One of the solutions of the equation $4 \sin^4 x + \cos^4 x = 1$ is

(*a*) $n\pi$ (*b*) $\frac{2n\pi}{3}$ (*c*) $(n-1)\frac{\pi}{4}$ (*d*) $(2n+1)\frac{\pi}{2}$

(*Odisha JEE 2008*)

26. The solution set of $\sin\left(x + \frac{\pi}{4}\right) = \sin 2x$ equals

(*a*) $\frac{n\pi + \pi/4}{1 - (-1)^n\ 2}$ (*b*) $\frac{n\pi - \pi/4}{1 - (-1)^n\ 2}$

(*c*) $\frac{n\pi + \pi/4}{1 + (-1)^n\ 2}$ (*d*) $\frac{n\pi - \pi/4}{1 + (-1)^n\ 2}$ **(*IIT*)**

27. If $\tan\left(\frac{\alpha\pi}{4}\right) = \cot\left(\frac{\beta\pi}{4}\right)$, then

(*a*) $\alpha + \beta = 0$ (*b*) $\alpha + \beta = 2n, n \in I$

(*c*) $\alpha + \beta = 2n + 1$ (*d*) $\alpha + \beta = 2(2n + 1), n \in I$

28. The solution of the equation $(\sin x + \cos x)^{1 + \sin 2x} = 2$, $-\pi \leq x \leq \pi$ is

(*a*) $\pi/2$ (*b*) π (*c*) $\pi/4$ (*d*) $\frac{3\pi}{4}$

(*Kerala CEE 2009*)

29. The general solution of sin 3x + sin x – 3 sin 2x = cos 3x + cos x – 3 cos 2x is

(*a*) $\frac{n\pi}{2}+\frac{\pi}{8}, n\in I$ (*b*) $\frac{n\pi}{2}-\frac{\pi}{8}, n\in I$

(*c*) $n\pi+\frac{\pi}{8}, n\in I$ (*d*) $n\pi-\frac{\pi}{8}, n\in I$

(*AMU 2013*)

30. Solve:

cot (θ + π/4) + cot (θ – π/4) = 2 tan θ . cot (θ – π/4) . cot (θ + π/4) for the general value of θ.

(*a*) θ = nπ (*b*) θ = $n\pi + (-1)^n \pi/4$

(*c*) θ = $n\pi + (-1)^n (-\pi/4)$ (*d*) θ = 2nπ ± π/4

ANSWERS

1. (*a*)	**2.** (*c*)	**3.** (*c*)	**4.** (*c*)	**5.** (*b*)	**6.** (*a*)	**7.** (*b*)	**8.** (*c*)	**9.** (*b*)	**10.** (*a*)
11. (*a*)	**12.** (*c*)	**13.** (*d*)	**14.** (*a*)	**15.** (*c*)	**16.** (*d*)	**17.** (*b*)	**18.** (*b*)	**19.** (*d*)	**20.** (*c*)
21. (*b*)	**22.** (*a*)	**23.** (*d*)	**24.** (*c*)	**25.** (*a*)	**26.** (*b*)	**27.** (*d*)	**28.** (*c*)	**29.** (*a*)	**30.** (*a*)

HINTS AND SOLUTIONS

1. Given, cosec θ + 2 = 0

$\Rightarrow$ cosec θ = –2 $\Rightarrow$ $\sin\theta = -\frac{1}{2} = -\sin 30°$

$\Rightarrow$ sin θ = – sin 30° = sin (180° + 30°) or sin (360° – 30°)

$\Rightarrow$ sin θ = sin 210° or sin 330°

$\Rightarrow$ **θ = 210°, 330°.**

2. $\sin x\cos x = \frac{1}{4} \Rightarrow 2\sin x\cos x = \frac{1}{2}$

$\Rightarrow \sin 2x = \frac{1}{2} \Rightarrow \sin 2x = \sin\frac{\pi}{6}$

$\Rightarrow 2x = \frac{\pi}{6}, x\in\left(0,\frac{\pi}{2}\right) \Rightarrow x = \boldsymbol{\frac{\pi}{12}}.$

3. tan θ + tan 2θ = 1 – tan θ . tan 2 θ

$\Rightarrow \frac{\tan\theta+\tan 2\theta}{1-\tan\theta\tan 2\theta}=1$

$\Rightarrow \tan(\theta+2\theta)=1 \quad \left(\because \tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}\right)$

$\Rightarrow \tan 3\theta = \tan\frac{\pi}{4} \Rightarrow 3\theta = n\pi+\frac{\pi}{4} \Rightarrow \theta = \boldsymbol{\frac{n\pi}{3}+\frac{\pi}{12}}.$

4. $\cos 2\theta = \sin\alpha \Rightarrow \cos 2\theta = \cos\left(\frac{\pi}{2}-\alpha\right)$

$\Rightarrow 2\theta = 2n\pi \pm \left(\frac{\pi}{2}-\alpha\right)$ ($\because$ cos θ = cos α $\Rightarrow$ θ = 2nπ ± α)

$\Rightarrow \theta = \boldsymbol{n\pi \pm \left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}.$

5. $\frac{\tan 3\theta - 1}{\tan 3\theta+1}=\sqrt{3} \Rightarrow \frac{\tan 3\theta - \tan\pi/4}{1+\tan\frac{\pi}{4}.\tan 3\theta}=\sqrt{3}=\tan\frac{\pi}{3}$

$\Rightarrow \tan(3\theta-\pi/4)=\tan\pi/3 \Rightarrow 3\theta-\pi/4 = n\pi+\frac{\pi}{3}$

$\Rightarrow 3\theta = n\pi+\frac{\pi}{3}+\frac{\pi}{4} \Rightarrow 3\theta = n\pi+\frac{7\pi}{12} \Rightarrow \theta = \boldsymbol{\frac{n\pi}{3}+\frac{7\pi}{36}}.$

6. $4\cos^2 x + 6\sin^2 x = 5$

$\Rightarrow 4(\cos^2 x+\sin^2 x)+2\sin^2 x = 5$

$\Rightarrow 2\sin^2 x = 5-4 = 1 \Rightarrow \sin^2 x = \frac{1}{2}$

$\Rightarrow \sin x = \pm\frac{1}{\sqrt{2}} \Rightarrow \boldsymbol{x = n\pi \pm \frac{\pi}{4}}.$

7. 1 – cos θ = sin θ . sin θ/2

$\Rightarrow 2\sin^2\theta/2 = 2\sin\theta/2 . \cos\theta/2 . \sin\theta/2$

$\Rightarrow 2\sin^2\theta/2\,(1-\cos\theta/2)=0$

$\Rightarrow 2\sin^2\theta/2 = 0$ or $1-\cos\theta/2 = 0 \Rightarrow \sin\theta/2 = 0$

or $2\sin^2\theta/4 = 0$ (Using, $1-\cos 2\theta = 2\sin^2\theta$)

$\Rightarrow \frac{\theta}{2}=k\pi$ or $\frac{\theta}{4}=k\pi$, where $\boldsymbol{k\in I}$.

$\Rightarrow$ θ = 2kπ or θ = 4kπ, $k\in I$.

$\Rightarrow$ **θ = 2*k*π, *k*∈*I*.**

8. Given, sin θ = sin α ...(*i*)

cos θ = cos α ...(*ii*)

$\therefore$ Dividing eqn (*i*) by eqn (*ii*), we have

tan θ = tan α

$\Rightarrow$ **θ = *n*π + α.**

9. $\cos 2x + 2\cos^2 x = 2 \Rightarrow 2\cos^2 x - 1 + 2\cos^2 x = 2$

$\Rightarrow 4\cos^2 x = 3 \Rightarrow \cos x = \pm\frac{\sqrt{3}}{2}$

$\Rightarrow \boldsymbol{x = n\pi \pm \frac{\pi}{6}, n\in Z}.$

10. 2 sec 2α = tan β + cot β

$\Rightarrow 2\sec 2\alpha = \frac{1+\tan^2\beta}{\tan\beta}=\frac{\sec^2\beta}{\tan\beta}$

$= \frac{2}{2\cos\beta\sin\beta} = 2\,\text{cosec}\,2\beta$

$\Rightarrow \sec 2\alpha = \sec\left(\frac{\pi}{2}-2\beta\right) \Rightarrow 2\alpha = 2n\pi \pm (\pi/2 - 2\beta)$

For (α + β), taking positive sign, we have

2 (α + β) = 2nπ + π/2

$\Rightarrow$ α + β = nπ + π/4, $n\in I$

$\therefore$ For n = 0

$\alpha+\beta = \boldsymbol{\frac{\pi}{4}}.$

11. $\cos x + \cos 2x + \cos 3x = 0$

$\Rightarrow \cos x + \cos 3x + \cos 2x = 0$

$\Rightarrow 2\cos\left(\frac{x+3x}{2}\right)\cos\left(\frac{3x-x}{2}\right) + \cos 2x = 0$

$\left(\because \cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{D-C}{2}\right)\right).$

$\Rightarrow 2\cos 2x \cos x + \cos 2x = 0$

$\Rightarrow \cos 2x\,(2\cos x + 1) = 0$

$\Rightarrow \cos 2x = 0$ or $2\cos x = -1/2$

(which is not possible, as given)

$\Rightarrow \cos 2x = 0 = \cos\frac{\pi}{2} \Rightarrow 2x = \pi/2 \Rightarrow x = \pi/4$

$\therefore$ The general solution of the given equation are

$\mathbf{2n\pi \pm \frac{\pi}{4},\ n \in I.}$

12. $\sec^2 x = \sqrt{2}\,(1-\tan^2\alpha) \Rightarrow \tan^2\alpha + 1 = \sqrt{2}\,(1-\tan^2\alpha)$

$\Rightarrow \tan^2\alpha\,(1+\sqrt{2}) = \sqrt{2} - 1$

$\Rightarrow \tan^2\alpha = \frac{\sqrt{2}-1}{\sqrt{2}+1} = \frac{(\sqrt{2}-1)^2}{(\sqrt{2}+1)(\sqrt{2}-1)} = (\sqrt{2}-1)^2 = \tan^2\frac{\pi}{8}$

$\therefore \tan\alpha = \tan\left(\pm\frac{\pi}{8}\right)$

$\therefore \mathbf{\alpha = n\pi \pm \frac{\pi}{8}.}$

13. $\sin 2x = 4\cos x$

$\Rightarrow 2\sin x\cos x = 4\cos x \Rightarrow 2\cos x(\sin x - 2) = 0$

$\Rightarrow \sin x - 2 = 0$ | or $\cos x = 0$

$\Rightarrow \sin x = 2$ | or $\cos x = \cos\pi/2$

which is not possible | $\Rightarrow \mathbf{x = 2n\pi \pm \pi/2,\ n \in Z.}$

14. $\sin 6\theta + \sin 4\theta + \sin 2\theta = 0$

$\Rightarrow (\sin 6\theta + \sin 2\theta) + \sin 4\theta = 0$

$\Rightarrow 2\sin 4\theta\cos 2\theta + \sin 4\theta = 0$

$\Rightarrow \sin 4\theta\,(2\cos 2\theta + 1) = 0$

$\Rightarrow$ Either $\sin 4\theta = 0$ or $(2\cos 2\theta + 1) = 0$

$\Rightarrow \sin 4\theta = 0$ | $\cos 2\theta = -\frac{1}{2} = \cos 2\pi/3$

$\Rightarrow 4\theta = n\pi$ | $\Rightarrow 2\theta = 2n\pi \pm 2\pi/3$

$\Rightarrow \theta = n\pi/4$ | $\Rightarrow \theta = n\pi \pm \pi/3.$

$\therefore \mathbf{\theta = \frac{n\pi}{4}}$ **or** $\mathbf{n\pi \pm \pi/3,\ n \in I.}$

15. $2\sin^2\theta + \sqrt{3}\cos\theta + 1 = 0$

$\Rightarrow 2(1-\cos^2\theta) + \sqrt{3}\cos\theta + 1 = 0$

$\Rightarrow 2\cos^2\theta - \sqrt{3}\cos\theta - 3 = 0$

$\Rightarrow \cos\theta = \frac{\sqrt{3} \pm \sqrt{3 - 4\times 2\times(-3)}}{2\times 2}$

$\Rightarrow \cos\theta = \frac{\sqrt{3} \pm 3\sqrt{3}}{4} = \frac{4\sqrt{3}}{4}$ or $-\frac{2\sqrt{3}}{4} = \sqrt{3}$ or $-\frac{\sqrt{3}}{2}$

Since $\cos\theta = \sqrt{3}$, is inadmissible, therefore

$\cos\theta = -\frac{\sqrt{3}}{2} = \cos\frac{5\pi}{6} \Rightarrow \mathbf{\theta = \frac{5\pi}{6}.}$

16. $1 + \sin\theta + \sin^2\theta + ... + \infty = 4 + 2\sqrt{3}$

$\Rightarrow \frac{1}{1-\sin\theta} = 4 + 2\sqrt{3}$

($\because$ Sum of an infinite G.P $= \frac{a}{1-r}$, where $r < 1$. This is an infinite G.P with common ratio **sin θ** and $0 < \sin\alpha < 1$)

$\Rightarrow 1 - \sin\theta = \frac{1}{4+2\sqrt{3}}$

$\Rightarrow 1 - \sin\theta = \frac{1}{4+2\sqrt{3}} \times \frac{4-2\sqrt{3}}{4-2\sqrt{3}}$

$\Rightarrow 1 - \sin\theta = \frac{4-2\sqrt{3}}{16-12} = \frac{4-2\sqrt{3}}{4} = 1 - \frac{\sqrt{3}}{2}$

$\Rightarrow \sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \mathbf{\theta = \frac{\pi}{3}}$ **or** $\mathbf{\frac{2\pi}{3}.}$

17. $3\sin^2 x + 10\cos x - 6 = 0$

$\Rightarrow 3(1-\cos^2 x) + 10\cos x - 6 = 0$

$\Rightarrow -3\cos^2 x + 10\cos x - 3 = 0$

$\Rightarrow (\cos x - 3)(1 - 3\cos x) = 0 \Rightarrow \cos x = 3$ or $\cos x = \frac{1}{3}$

Since $\cos x = 3$ is inadmissible, therefore, $\cos x = \frac{1}{3}$

$\Rightarrow x = \cos^{-1}\left(\frac{1}{3}\right)$

$\therefore$ The general solution is $\mathbf{x = 2n\pi \pm \cos^{-1}\left(\frac{1}{3}\right).}$

18. $\tan\theta + \sec\theta = \sqrt{3}$

$\Rightarrow \frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta} = \sqrt{3} \Rightarrow \sin\theta + 1 = \sqrt{3}\cos\theta$

$\Rightarrow \frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta = \frac{1}{2}$

$\Rightarrow \cos\frac{\pi}{6}\cos\theta - \sin\frac{\pi}{6}\sin\theta = \frac{1}{2}$

$\Rightarrow \cos(\theta + \pi/6) = \cos\pi/3$

$\Rightarrow$ Principal value of $\mathbf{\theta + \pi/6 = \pi/3.}$

19. $\cos^2\theta + \sin\theta + 1 = 0 \Rightarrow (1 - \sin^2\theta) + \sin\theta + 1 = 0$

$\Rightarrow \sin^2\theta - \sin\theta - 2 = 0 \Rightarrow (\sin\theta + 1)(\sin\theta - 2) = 0$

$\Rightarrow (\sin\theta + 1) = 0$ or $(\sin\theta - 2) = 0$

$\Rightarrow \sin\theta = -1$ ($\because \sin\theta = 2$ is inadmissible)

$\Rightarrow \sin\theta = \sin\frac{3\pi}{2} \Rightarrow \theta = \frac{3\pi}{2} \in \left(\mathbf{\frac{5\pi}{4}, \frac{7\pi}{4}.}\right)$

20. $12\cot^2\theta - 31\,\text{cosec}\,\theta + 32 = 0$

$\Rightarrow 12\frac{\cos^2\theta}{\sin^2\theta} - \frac{31}{\sin\theta} + 32 = 0$

$\Rightarrow 12\cos^2\theta - 31\sin\theta + 32\sin^2\theta = 0$

$\Rightarrow 12(1-\sin^2\theta) - 31\sin\theta + 32\sin^2\theta = 0$

$\Rightarrow 20\sin^2\theta - 31\sin\theta + 12 = 0$

$\therefore \sin\theta = \frac{31 \pm \sqrt{31^2 - 4\times 20\times 12}}{2\times 20}$

$= \frac{31 \pm \sqrt{961 - 960}}{40} = \frac{31 \pm 1}{40}$

$= \frac{32}{40}$ or $\frac{30}{40} = \frac{4}{5}$ or $\frac{3}{4}$

$\therefore$ $\sin\theta = \mathbf{\frac{4}{5}}$ **or** $\mathbf{\frac{3}{4}}$**.**

21. (*a*) $\cos\theta + \sin\theta = \sqrt{2} \Rightarrow \frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta = 1$

$\Rightarrow$ $\cos\pi/4 \cos\theta + \sin\pi/4 \sin\theta = 1$

$\Rightarrow$ $\cos(\theta - \pi/4) = \cos 0 \Rightarrow \theta - \pi/4 = 2n\pi$

$\Rightarrow$ $\theta = 2n\pi + \pi/4 \Rightarrow$ Solutions exist

(*b*) $\operatorname{cosec}\theta \,.\, \sec\theta = 1 \Rightarrow \frac{1}{\sin\theta}\cdot\frac{1}{\cos\theta} = 1$

$\Rightarrow$ $\sin\theta \,.\, \cos\theta = 1 \Rightarrow 2\sin\theta\cos\theta = 2$

$\Rightarrow$ $\sin 2\theta = 2$ which is not possible as $\mathbf{-1 \le \sin x \le 1}$.

Hence no solutions exist.

(*c*) $\sqrt{3}\sin\theta - \cos\theta = \sqrt{2}$

Dividing throughout by $\sqrt{(\sqrt{3})^2 + (-1)^2} = 2$, we get

$\frac{\sqrt{3}}{2}\sin\theta - \frac{1}{2}\cos\theta = \frac{1}{\sqrt{2}}$

$\Rightarrow$ $\cos\frac{\pi}{6}\sin\theta - \sin\frac{\pi}{6}\cos\theta = \frac{1}{\sqrt{2}}$

$\Rightarrow$ $\sin(\theta - \pi/6) = \sin\pi/4$

$\Rightarrow$ $\theta - \pi/6 = n\pi + (-1)^n \pi/4, n \in I$

$\Rightarrow$ $\theta = n\pi + (-1)^n \pi/4 + \pi/6, n \in I$

$\Rightarrow$ Solution exist.

(*d*) $\operatorname{cosec}\theta - \sec\theta = \operatorname{cosec}\theta \,.\, \sec\theta$

$\Rightarrow$ $\frac{1}{\sin\theta} - \frac{1}{\cos\theta} = \frac{1}{\sin\theta}\cdot\frac{1}{\cos\theta} \Rightarrow \cos\theta - \sin\theta = 1$

Dividing throughout by $\sqrt{1^2 + (-1)^2} = \sqrt{2}$, we get

$\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta = \frac{1}{\sqrt{2}}$

$\Rightarrow$ $\cos\theta\cos\pi/4 - \sin\theta\sin\pi/4 = \frac{1}{\sqrt{2}}$

$\Rightarrow$ $\cos(\theta + \pi/4) = \cos(\pi/4) \Rightarrow \theta + \pi/4 = 2n\pi \pm \pi/4, n \in I$

$\Rightarrow$ $\theta = 2n\pi$ or $\theta = 2n\pi - \pi/2, n \in I$.

$\Rightarrow$ Solutions exist.

22. $\sin^4\theta - 2\sin^2\theta - 1 = 0 \Rightarrow (\sin^2\theta)^2 - 2\sin^2\theta - 1 = 0$

$\Rightarrow$ $\sin^2\theta = \frac{2 \pm \sqrt{4 - 4\times 1\times(-1)}}{2\times 1}$

$= \frac{2 \pm \sqrt{8}}{2} = \frac{2 \pm 2\sqrt{2}}{2} = 1 - \sqrt{2}$ or $1 + \sqrt{2}$

$\sin^2\theta = 1 - \sqrt{2}$ is inadmissible as it is not real.

$\because$ $-1 \le \sin\theta \le 1 \Rightarrow 0 \le \sin^2\theta \le 1$

$\Rightarrow \sin^2\theta = 1 + \sqrt{2}$ is not possible.

Hence the given equation has no real root.

23. $\sin x - \cos x = \sqrt{2} \Rightarrow \frac{1}{\sqrt{2}}\sin x - \frac{1}{\sqrt{2}}\cos x = 1$

$\Rightarrow$ $\sin\pi/4 \sin x - \cos\pi/4 \cos x = 1$

$\Rightarrow$ $-(\cos\pi/4 \cos x - \sin\pi/4 \sin x) = 1$

$\Rightarrow$ $-\cos(x + \pi/4) = 1 \Rightarrow \cos(x + \pi/4) = -1 = \cos\pi$

$\Rightarrow$ $x + \pi/4 = 2n\pi + \pi \Rightarrow x = \mathbf{2n\pi + \frac{3\pi}{4}}$**.**

24. $3\sin^2 x - 7\sin x + 2 = 0 \Rightarrow (3\sin x - 1)(\sin x - 2) = 0$

$\Rightarrow$ $(3\sin x - 1) = 0$ or $(\sin x - 2) = 0$

$\because$ $\sin x = 2$ is inadmissible, therefore, $\sin x = \frac{1}{3}$

Since, $\sin x = \sin\alpha$ where $\sin\alpha = \frac{1}{3}$, so α lies in the 1st quadrant

$\Rightarrow$ $x = n\pi + (-1)^n\alpha, n \in I$, where $0 < \alpha < \pi/2$

Since x lies in the interval $[0, 5\pi]$, so we have one value of x corresponding to each of the values 0, 1, 2, 3, 4, 5 or n.

$\therefore$ The number of values of x in the interval $[0, 5\pi]$ is **6**.

25. $4\sin^4 x + \cos^4 x = 1 \Rightarrow 4\sin^4 x + (1 - \sin^2 x)^2 = 1$

$\Rightarrow$ $4\sin^4 x + 1 + \sin^4 x - 2\sin^2 x = 1$

$\Rightarrow$ $5\sin^4 x - 2\sin^2 x = 0 \Rightarrow \sin^2 x(5\sin^2 x - 2) = 0$

$\Rightarrow$ $\sin^2 x = 0$ or $5\sin^2 x - 2 = 0$

$\Rightarrow$ $\sin x = 0$ or $\sin x = \sqrt{\frac{2}{5}}$

$\Rightarrow$ $x = n\pi$ or $x = \sin^{-1}\left(\sqrt{\frac{2}{5}}\right)$

$= n\pi + (-1)^n \sin^{-1}\left(\sqrt{\frac{2}{5}}\right), n \in I.$

$\therefore$ $x = n\pi$ is one of the solutions of the given equation.

26. $\sin(x + \pi/4) = \sin 2x \Rightarrow x + \pi/4 = n\pi + (-1)^n 2x, n \in I$

$\Rightarrow$ $x - (-1)^n 2x = n\pi - \pi/4, n \in I$

$\Rightarrow$ $x\{1 - (-1)^n .2\} = n\pi - \pi/4 \Rightarrow x = \mathbf{\frac{n\pi - \pi/4}{1 - (-1)^n . 2}}, n \in I.$

27. $\tan\left(\frac{\alpha\pi}{4}\right) = \cot\left(\frac{\beta\pi}{4}\right) \Rightarrow \tan\left(\frac{\alpha\pi}{4}\right) = \tan\left(\frac{\pi}{2} - \frac{\beta\pi}{4}\right)$

$\Rightarrow$ $\frac{\alpha\pi}{4} = n\pi + \frac{\pi}{2} - \frac{\beta\pi}{4}, n \in I$

$\Rightarrow$ $(\alpha + \beta)\pi/4 = n\pi + \pi/2, n \in I$

$\Rightarrow$ $(\alpha + \beta)\pi/4 = (2n + 1)\pi/2, n \in I$

$\Rightarrow$ $\alpha + \beta = \mathbf{2(2n + 1), n \in I.}$

28. $[\sin x + \cos x]^{1 + 2\sin x\cos x} = 2$

$\Rightarrow$ $[\sin x + \cos x]^{(\sin x + \cos x)^2} = 2$

$\Rightarrow$ $[\sin x + \cos x]^{(\sin x + \cos x)^2} = (\sqrt{2})^{(\sqrt{2})^2}$

Comparing both the sides, we have

$\sin x + \cos x = \sqrt{2}$

$\Rightarrow$ $\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = 1$

$\Rightarrow$ $\sin\pi/4 \sin x + \cos\pi/4 \cos x = 1$

$\Rightarrow$ $\sin(x + \pi/4) = 1 = \sin\pi/2$

$\Rightarrow$ $x + \pi/4 = n\pi + (-1)^n \pi/2 \Rightarrow x = n\pi + (-1)^n \pi/2 - \pi/4$

For $-\pi \le x \le \pi$, put $n = 0$.

Then $x = \mathbf{\pi/4}$ **.**

29. $(\sin 3x + \sin x) - 3 \sin 2x = (\cos 3x + \cos x) - 3 \cos 2x$

$\Rightarrow$ $2 \sin 2x \cos x - 3 \sin 2x = 2 \cos 2x \cdot \cos x - 3 \cos 2x$

$\Rightarrow$ $\sin 2x (2 \cos x - 3) = \cos 2x (2 \cos x - 3)$

$\Rightarrow$ $(2 \cos x - 3)(\sin 2x - \cos 2x) = 0$

$\Rightarrow$ $(2 \cos x - 3) = 0$ or $(\sin 2x - \cos 2x) = 0$

$\Rightarrow$ $\cos x = \frac{3}{2}$ or $\sin 2x = \cos 2x$

$\because$ $-1 \le \cos x \le 1$, $\cos x \ne \frac{3}{2}$.

$\therefore$ $\sin 2x = \cos 2x \Rightarrow \tan 2x = 1$

$\Rightarrow$ $\tan 2x = \tan (\pi/4) \Rightarrow 2x = n\pi + \pi/4$

$\Rightarrow$ $x = \boldsymbol{\frac{n\pi}{2} + \frac{\pi}{8}, n \in I.}$

30. $\cot (\theta - \pi/4) + \cot (\theta + \pi/4)$

$= 2 \tan \theta \cdot \cot (\theta - \pi/4) \cdot \cot (\theta + \pi/4)$

Dividing both sides by $\cot (\theta - \pi/4) \cdot \cot (\theta + \pi/4)$, we have

$$\frac{1}{\cot (\theta + \pi/4)} + \frac{1}{\cot (\theta - \pi/4)} = 2 \tan \theta$$

$\Rightarrow$ $\tan (\theta + \pi/4) + \tan (\theta - \pi/4) = 2 \tan \theta$

$\Rightarrow$ $\frac{\tan \theta + 1}{1 - \tan \theta} + \frac{\tan \theta - 1}{1 + \tan \theta} = 2 \tan \theta$

$\Rightarrow$ $(1 + \tan \theta)^2 - (1 - \tan \theta)^2 = 2 \tan \theta (1 - \tan^2 \theta)$

$\Rightarrow$ $1 + 2 \tan \theta + \tan^2 \theta - 1 + 2 \tan \theta - \tan^2 \theta$

$= 2 \tan \theta - 2 \tan^3 \theta$

$\Rightarrow$ $4 \tan \theta = 2 \tan \theta - 2 \tan^3 \theta$

$\Rightarrow$ $2 \tan \theta + 2 \tan^3 \theta = 0$

$\Rightarrow$ $2 \tan \theta (1 + \tan^2 \theta) = 0$

$\Rightarrow$ $2 \tan \theta = 0$ $(\because 1 + \tan^2 \theta \ne 0)$

$\Rightarrow$ $\tan \theta = 0 \Rightarrow \theta = \boldsymbol{n\pi}, n \in I.$

SELF ASSESSMENT SHEET

1. The general solution of $\tan 5\theta = \cot 2\theta$ is

(*a*) $\theta = \frac{n\pi}{7} + \frac{\pi}{14}$ (*b*) $\theta = \frac{n\pi}{7} + \frac{\pi}{5}$

(*c*) $\theta = \frac{n\pi}{7} + \frac{\pi}{3}$ (*d*) $\theta = \frac{n\pi}{7} + \frac{\pi}{2}$

(*AIEEE 2002*)

2. The most general solution of $\tan \theta = -1$ and $\cos \theta = \frac{1}{\sqrt{2}}$ is

(*a*) $n\pi + (-1)^n \frac{\pi}{4}$ (*b*) $2n\pi + \frac{3\pi}{4}$

(*c*) $n\pi + (-1)^n \frac{5\pi}{4}$ (*d*) $2n\pi + \frac{7\pi}{4}$

(*EAMCET 2011*)

3. If $\sqrt{3} \cos \theta - \sin \theta = 1$, then θ is

(*a*) π (*b*) $\pi/2$ (*c*) $\pi/3$ (*d*) $\pi/6$

4. If $\sin \theta = \sqrt{3} \cos \theta$, $-\pi < \theta < 0$, then θ is equal to

(*a*) $\frac{-5\pi}{6}$ (*b*) $\frac{-2\pi}{3}$

(*c*) $\frac{2\pi}{3}$ (*d*) $\frac{5\pi}{6}$

5. If $\tan m\theta + \cot n\theta = 0$, then the general value of θ is

(*a*) $\frac{r\pi}{m + n}$ (*b*) $\frac{r\pi}{m - n}$

(*c*) $\frac{(2r + 1)\pi}{(m + n)}$ (*d*) $\frac{(2r + 1)\pi}{2(m - n)}$ **(*Roorkee*)**

6. The number of values of x in $(0, 2\pi)$ satisfying the equation $\sin 3\theta = \sin \theta$ are

(*a*) 8 (*b*) 9 (*c*) 5 (*d*) 7

(*Karnataka CET 2007*)

7. The number of solutions of the equation $\sin x \cos 3x = \sin 3x \cos 5x$ in $[0, \pi/2]$ is

(*a*) 3 (*b*) 4 (*c*) 5 (*d*) 6

(*J&K CET 2009*)

8. The most general solutions of the equation $\sec x - 1 = (\sqrt{2} - 1) \tan x$ are given by

(*a*) $n\pi + \pi/8$ (*b*) $2n\pi$, $2n\pi + \pi/4$

(*c*) $2n\pi$ (*d*) None of these

(*WBJEE 2007*)

9. If $0 \le x \le \pi$ and $81^{\sin^2 x} + 81^{\cos^2 x} = 30$, then x is equal to

(*a*) $\pi/6$ (*b*) $\pi/2$ (*c*) $\pi/4$ (*d*) $3\pi/4$

10. If $3 \tan (\theta - 15°) = \tan (\theta + 15°)$, $0 < \theta < \pi$, then θ is equal to

(*a*) π (*b*) $\pi/2$ (*c*) $\pi/4$ (*d*) 2π

(*EAMCET*)

ANSWERS

1. (*a*) **2.** (*d*) **3.** (*d*) **4.** (*b*) **5.** (*d*) **6.** (*c*) **7.** (*c*) **8.** (*b*) **9.** (*a*) **10.** (*c*)

HINTS AND SOLUTIONS

1. $\tan 5\theta = \cot 2\theta \Rightarrow \tan 5\theta = \tan (\pi/2 - 2\theta)$

$\Rightarrow$ $5\theta = n\pi + (\pi/2 - 2\theta), n \in I \Rightarrow 7\theta = n\pi + \pi/2, n \in I$

$\Rightarrow$ $\theta = \boldsymbol{\frac{n\pi}{7} + \frac{\pi}{14}}, n \in I.$

2. $\tan \theta = -1$

$\Rightarrow$ $\tan \theta = -\tan \pi/4 = \tan (\pi - \pi/4) = \tan (2\pi - \pi/4)$

$\Rightarrow$ $\tan \theta = \tan \frac{3\pi}{4}$ or $\tan \frac{7\pi}{4} \Rightarrow \theta = \frac{3\pi}{4}, \frac{7\pi}{4}$ ...(*i*)

$\cos \theta = \frac{1}{\sqrt{2}}$

$\Rightarrow$ $\cos \theta = \cos \frac{\pi}{4} = \cos (2\pi - \pi/4)$

$\Rightarrow$ $\cos\theta = \cos\dfrac{\pi}{4} = \cos\dfrac{7\pi}{4}$ $\Rightarrow$ $\theta = \dfrac{\pi}{4}, \dfrac{7\pi}{4}$ ...(*ii*)

$\therefore$ From (*i*) and (*ii*) $\theta = \dfrac{7\pi}{4}$ is the only value of θ in $[0, 2\pi[$ which satisfies both the equations.

$\therefore$ The general value of θ satisfying both the equations is

$\boldsymbol{\theta = 2n\pi + \dfrac{7\pi}{4}, n \in I.}$

3. $\sqrt{3}\cos\theta - \sin\theta = 1$

Dividing throughout by $\sqrt{\left(\sqrt{3}\right)^2 + (-1)^2} = 2$, we have

$\dfrac{\sqrt{3}}{2}\cos\theta - \dfrac{1}{2}\sin\theta = \dfrac{1}{2}$

$\Rightarrow$ $\sin\pi/3\cos\theta - \cos\pi/3\sin\theta = \dfrac{1}{2}$

$\Rightarrow$ $\sin(\pi/3 - \theta) = \sin\pi/6$ $\Rightarrow$ $\pi/3 - \theta = \pi/6 \Rightarrow \theta = \boldsymbol{\pi/6.}$

4. $\sin\theta = \sqrt{3}\cos\theta$ $\Rightarrow$ $\tan\theta = \sqrt{3}$

$\Rightarrow$ $\tan\theta = \tan\pi/3$ $\Rightarrow$ $\theta = n\pi + \pi/3, n \in I$

Putting $n = -1$, we get $\theta = -\pi + \dfrac{\pi}{3} = -\boldsymbol{\dfrac{2\pi}{3}},$ which lies between $-\pi$ and 0.

5. $\tan m\theta + \cot n\theta = 0$ $\Rightarrow$ $\tan m\theta = -\cot n\theta$

$\Rightarrow$ $\tan m\theta = \tan(\pi/2 + n\theta)$ $(\because \tan(\pi/2 + x) = -\cot x)$

$\Rightarrow$ $m\theta = r\pi + \pi/2 + n\theta, r \in I$

$\Rightarrow$ $(m - n)\theta = (2r + 1)\pi/2, r \in I$

$\Rightarrow$ $\theta = \boldsymbol{\dfrac{(2r+1)\pi}{2(m-n)}}, r \in I.$

6. $\sin 3\theta - \sin\theta = 0$

$\Rightarrow$ $2\cos\left(\dfrac{3\theta + \theta}{2}\right)\sin\left(\dfrac{3\theta - \theta}{2}\right) = 0$

$\Rightarrow$ $\cos 2\theta \,.\, \cos\theta = 0$ $\Rightarrow$ $\cos 2\theta = 0$ or $\cos\theta = 0$

$\Rightarrow$ $2\theta = \pi/2, 3\pi/2, 5\pi/2, 7\pi/2$ or $\theta = 0, \pi, 2\pi$

$\Rightarrow$ $\theta = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$ or $\theta = \pi$

Hence the total number of solutions is **5.**

7. $\sin x\cos 3x = \sin 3x\cos 5x$

$\Rightarrow$ $2\sin x\cos 3x - 2\sin 3x\cos 5x = 0$

$\Rightarrow$ $[\sin(3x + x) - \sin(3x - x)]$

$- [\sin(3x + 5x) - \sin(5x - 3x)] = 0$

$\Rightarrow$ $\sin 4x - \sin 2x - \sin 8x + \sin 2x = 0$

$\Rightarrow$ $\sin 4x - \sin 8x = 0$

$\Rightarrow$ $2\cos\left(\dfrac{4x + 8x}{2}\right)\sin\left(\dfrac{8x - 4x}{2}\right) = 0$

$\Rightarrow$ $2\cos 6x\sin 2x = 0$ $\Rightarrow$ $\cos 6x = 0$ or $\sin 2x = 0$

$\Rightarrow$ $6x = (2n + 1)\pi/2$ or $2x = n\pi$

$\Rightarrow$ $x = (2n + 1)\pi/2$ or $x = \dfrac{n\pi}{2}$

$\Rightarrow$ $x = 0, \dfrac{\pi}{2}, \dfrac{\pi}{12}, \dfrac{3\pi}{12}, \dfrac{5\pi}{12}$ in $\left[0, \dfrac{\pi}{2}\right].$

8. $\sec x - 1 = \left(\sqrt{2} - 1\right)\tan x$

$\Rightarrow$ $\dfrac{1}{\cos x} - 1 = \left(\sqrt{2} - 1\right)\dfrac{\sin x}{\cos x}$

$\Rightarrow$ $\dfrac{1 - \cos x}{\cos x} = \left(\sqrt{2} - 1\right)\dfrac{\sin x}{\cos x}$

$\Rightarrow$ $1 - \cos x = \left(\sqrt{2} - 1\right)\sin x$

$\Rightarrow$ $2\sin^2 x/2 - \left(\sqrt{2} - 1\right).\, 2\sin x/2\cos x/2 = 0$

$\Rightarrow$ $\sin x/2\,[\sin x/2 - \left(\sqrt{2} - 1\right)\cos x/2] = 0$

$\Rightarrow$ $\sin x/2 = 0$ or $\sin x/2 - \left(\sqrt{2} - 1\right)\cos x/2 = 0$

$\Rightarrow$ $x/2 = n\pi$ or $\tan x/2 = \sqrt{2} - 1 = \tan\dfrac{45°}{2} = \tan\pi/8$

$\Rightarrow$ $x = 2n\pi$ or $\dfrac{x}{2} = n\pi + \pi/8$

$\Rightarrow$ $x = \boldsymbol{2n\pi}$ or $\boldsymbol{2n\pi + \pi/4.}$

9. $81^{\sin^2 x} + 81^{\cos^2 x} = 30$

$\Rightarrow$ $81^{\sin^2 x} + 81^{1 - \sin^2 x} = 30$

$\Rightarrow$ $81^{\sin^2 x} + \dfrac{81}{81^{\sin^2 x}} = 30$

$\Rightarrow$ $y + \dfrac{81}{y} = 30$ (putting $y = 81^{\sin^2 x}$)

$\Rightarrow$ $y^2 - 30y + 81 = 0$ $\Rightarrow$ $(y - 27)(y - 3) = 0$

$\Rightarrow$ $81^{\sin^2 x} = 27$ or $81^{\sin^2 x} = 3$

$\Rightarrow$ $3^{4\sin^2 x} = 3^3$ or $3^{4\sin^2 x} = 3^1$

$\Rightarrow$ $4\sin^2 x = 3$ or $4\sin^2 x = 1$

$\Rightarrow$ $\sin^2 x = \dfrac{3}{4}$ or $\sin^2 x = \dfrac{1}{4}$

$\Rightarrow$ $\sin x = \dfrac{\sqrt{3}}{2}$ or $\sin x = \dfrac{1}{4}$

$\Rightarrow$ $x = \pi/3, 2\pi/3$ or $x = \pi/6, 5\pi/6.$

10. $3\tan(\theta - 15°) = \tan(\theta + 15°)$

$\Rightarrow$ $\dfrac{3\sin(\theta - 15°)}{\cos(\theta - 15°)} = \dfrac{\sin(\theta + 15°)}{\cos(\theta + 15°)}$

$\Rightarrow$ $3\cos(\theta + 15°)\sin(\theta - 15°) = \sin(\theta + 15°)\cos(\theta - 15°)$

$\Rightarrow$ $3[\sin(2\theta) - \sin 30°] = \sin 2\theta + \sin 30°$

$\Rightarrow$ $3\left(\sin 2\theta - \dfrac{1}{2}\right) = \sin 2\theta + \dfrac{1}{2}$

$\Rightarrow$ $3\sin 2\theta = \sin 2\theta + \dfrac{1}{2} + \dfrac{3}{2}$

$\Rightarrow$ $2\sin 2\theta = 2$

$\Rightarrow$ $\sin 2\theta = 1$ $\Rightarrow$ $2\theta = \pi/2 \Rightarrow \theta = \boldsymbol{\pi/4.}$