Permutations and Combinations

Fundamental Principle of Counting

- The fundamental principle of counting states:
If an event can occur in m different ways, following which another event can occur in n different ways, then the total number of occurrence of the events in the given order is $m \times n$.

- Let us take a simple problem based on fundamental principle of counting.

Example 1: An ice-cream shop owner has a total of 15 flavours of ice-cream in his shop. In each cup, he puts 2 scoops of ice-cream. If a customer wants a cup with two different flavours of ice-cream, then in how many ways can the shopkeeper give the cup of ice-cream to the customer with two scoops?

Solution:

It is given that in an ice-cream shop, there are a total of 15 flavours of ice-cream.

Since there are a total of 15 flavours, the total number of ways in which the shopkeeper can put the first scoop of ice-cream in the cup is 15.

It is given that the customer wants an ice-cream cup with two different flavours. Hence, the total number of ways in which the shopkeeper can put the second scoop of ice-cream in the cup is 14, as he cannot use the flavour he used in the first scoop.

Thus, by the fundamental principle of counting, the required number of ways in which the shopkeeper can give an ice-cream cup to the customer with two different flavours is $15 \times 14 = 210$.

Example 2: Tushar daily goes to his college on buses that are run by the university. The total number of buses that run on his route daily is 20. In how many ways can Tushar go and come back on a bus provided that he comes back to his home in a different bus?

Solution:

It is given that the total number of buses that ply on Tushar's route is 20.

The number of ways in which Tushar can go to his college is 20.

Hence, the number of ways in which Tushar can come back in a different bus is 19.

Thus, by the fundamental principle of counting, the required number of ways in which Tushar can go and come back on a bus provided that he comes back to his home in a different bus is 20 × 19 = 380.

Example 3: Sonu goes to Pizza House. The owner gives him a choice among four types of vegetarian pizzas and choice among 5 types of cold drinks. In how many ways can Sonu buy one vegetarian pizza and one cold drink for himself?

Solution:

It is given that there are 4 types of vegetarian pizzas, out of which Sonu has to buy one pizza. Hence, he can choose one pizza in 4 different ways.

It is also given that there are 5 types of cold drinks, out of which Sonu has to buy one cold drink. Hence, he can choose one cold drink in 5 different ways.

Thus, by the fundamental principle of counting, the required number of ways in which Sonu can buy one vegetarian pizza and one cold drink for himself is 4 × 5 = 20.

Concept of Factorials

- **The notation *n*! , where *n* is a non-negative integer, is the product of all positive integers that are less than or equal to *n*.**
  ∴ $n! = n \times (n-1) \times (n-2) \times \ldots\ldots \times 5 \times 4 \times 3 \times 2 \times 1$

- For example:

  (i) 7! = 7 × 6 × 5 × 4 × 3 × 2 × 1 = 5040

  (ii) $\frac{12!}{8!} = \frac{12\times11\times10\times9\times8\times7\times6\times5\times4\times3\times2\times1}{8\times7\times6\times5\times4\times3\times2\times1} = 12\times11\times10\times9 = 11880$

- We can also write *n*! as follows:
  $n! = n \times (n-1)!$
  $= n \times (n-1) \times (n-2)!$[If $n \geq 2$]
  $= n \times (n-1) \times (n-2) \times (n-3)!$[If $n \geq 3$]
  And so on…

- For example, we can write 5! as 5 × 4!, or 5 × 4 × 3!, or 5 × 4 × 3 × 2!, or 5 × 4 × 3 × 2 × 1.

- In the concept of factorial, we use the following convention: 0! = 1

Solved Examples

Example 1: Simplify: $\frac{(p+1)!}{(p-3)!}$

Solution:

We have

$$\frac{(p+1)!}{(p-3)!}=\frac{(p+1)\times(p)\times(p-1)\times(p-2)\times(p-3)\times......\times3\times2\times1}{(p-3)\times......\times3\times2\times1}$$
$$=(p+1)\times(p)\times(p-1)\times(p-2)$$
$$=p\times(p-2)\times(p^2-1)$$
$$=p(p^3-2p^2-p+2)$$
$$=p^4-2p^3-p^2+2p$$

Example 2: Find the value of *y*, if $\frac{13!}{7!\,8!}=\frac{4!}{y}$.

Solution:

We have

$$\frac{13!}{7!\,8!}=\frac{4!}{y}$$
$$\Rightarrow\frac{13\times12\times11\times10\times9\times8\times7\times6\times5\times4\times3\times2\times1}{7!\times8!}=\frac{4!}{y}$$
$$\Rightarrow\frac{13\times12\times11\times10\times9\times8!}{7!\times8!}=\frac{4!}{y}$$
$$\Rightarrow\frac{13\times12\times11\times10\times9}{7!}=\frac{4!}{y}$$
$$\Rightarrow y=\frac{4!\times7!}{13\times12\times11\times10\times9}$$
$$\Rightarrow y=\frac{(4\times3\times2\times1)\times(7\times6\times5\times4\times3\times2\times1)}{13\times12\times11\times10\times9}$$
$$\Rightarrow y=\frac{112}{143}$$

Thus, the value of *y* is $\frac{112}{143}$.

Example 3: Find the value of the expression $\frac{a!(a+b-c)!}{b!}$, when *a* = 10, *b* = 7, *c* = 17.

Solution:

For *a* = 10, *b* = 7, *c* = 17, we have

$$\begin{aligned}\frac{a!(a+b-c)!}{b!} &= \frac{10!(10+7-17)!}{7!}\\ &= \frac{10!(17-17)!}{7!}\\ &= \frac{10!0!}{7!}\\ &= \frac{10!\times 1}{7!} \qquad [\because 0!=1]\\ &= \frac{10\times 9\times 8\times 7!}{7!}\\ &= 10\times 9\times 8\\ &= 720\end{aligned}$$

Example 4: Find the value of *n* from the expression $12\left[\frac{(n-3)!}{(n-1)!}\right]=1$.

Solution:

We have

$$12\left[\frac{(n-3)!}{(n-1)!}\right]=1$$

$$\Rightarrow 12\left[\frac{(n-3)!}{(n-1)(n-2)(n-3)!}\right]=1$$

$$\Rightarrow 12\left[\frac{1}{(n-1)(n-2)}\right]=1$$

$$\Rightarrow 12=(n-1)(n-2)$$

$$\Rightarrow n^2-3n+2-12=0$$

$$\Rightarrow n^2-3n-10=0$$

$$\Rightarrow n^2+2n-5n-10=0$$

$$\Rightarrow n(n+2)-5(n+2)=0$$

$$\Rightarrow n(n+2)-5(n+2)=0$$

$$\Rightarrow (n+2)(n-5)=0$$

$$\Rightarrow n=-2 \text{ or } n=5$$

However, for *n* = −2, $(n-3)$ and $(n-1)$ will be negative and we cannot calculate the factorial of negative integers. Hence, *n* cannot be −2.

Therefore, the value of *n* is 5.

Example 5 Find the number of 5 letter words, with or without meaning, which can be formed out of the letters of the word ‘TABLE’, where the repetition of letters is not allowed.

Solution:

There are as many words as there are ways of filling-in 5 vacant places □□□□□.

Since the repetition of letters is not allowed, the required number of words that can be made is given by

5! = 5 × 4 × 3 × 2 × 1 = 120

Concept of Permutations When All Objects Are Distinct

- **A permutation is an arrangement in a definite order of a number of objects taken some or all at a time.**

- The concept of permutations is used when we need to count the number of ways of arrangement of objects where the order of the given objects does matter.

- To find the number of permutations when the objects are distinct, the following theorem is used:
  The number of permutation of n different objects taken r at a time, where $0 < r \leq n$ and the objects do not repeat is $n(n - 1)(n - 2)(n - 3)\ldots\ldots(n - r + 1)$, which is denoted by ^nP_r.

- $$^nP_r = \frac{n!}{(n-r)!}, 0 \leq r \leq n$$

Example 1: In a class of 50 students, students who secure the first three positions in the annual test are rewarded with a trophy. In how many number of ways can the students win the trophy?

Solution:

It is given that the total number of students in the class is 50, out of which three students who secure the first three positions in the annual test are rewarded with a trophy.

To find the number of ways in which the students can win the trophy is the same as to find the number of permutations of 50 students taken 3 at a time.

∴Required number of ways = $^{50}P_3 = \frac{50!}{(50-3)!} = \frac{50!}{47!} = 50 \times 49 \times 48 = 117600$

Example 2: If $^8P_r : {}^{10}P_r = 7 : 15$, then find the value of r.

Solution:

It is given that $^8P_r : {}^{10}P_r = 7 : 15$. Hence,

$$\frac{\frac{8!}{(8-r)!}}{\frac{10!}{(10-r)!}} = \frac{7}{15}$$

$$\Rightarrow \frac{8!(10-r)!}{10!(8-r)!} = \frac{7}{15}$$

$$\Rightarrow \frac{8!(10-r)\times(9-r)\times(8-r)!}{10\times 9\times 8!(8-r)!} = \frac{7}{15}$$

$$\Rightarrow \frac{(10-r)\times(9-r)}{10\times 9} = \frac{7}{15}$$

$$\Rightarrow (10-r)(9-r) = \frac{7\times 10\times 9}{15} = 42$$

$$\Rightarrow 90-10r-9r+r^2 = 42$$

$$\Rightarrow r^2-19r+48=0$$

$$\Rightarrow r^2-16r-3r+48=0$$

$$\Rightarrow r(r-16)-3(r-16)=0$$

$$\Rightarrow (r-3)(r-16)=0$$

$$\Rightarrow r=3 \text{ or } r=16$$

However, since $0 \le r \le n$, r cannot be equal to 16.

Thus, the value of r is 3.

Example 3: Find the number of permutations of the letters in the word 'PLASTIC' taken four at a time.

Solution:

The given word is 'PLASTIC'.

Here, there are 7 different objects or letters.

Therefore, the required number of permutations = ${}^7P_4$ = $\frac{7!}{(7-4)!} = \frac{7!}{3!} = 7\times 6\times 5\times 4 = 840$

Example 4: Find the number of 5-dgit numbers that can be formed using the digits 2, 4, 5, 7, 8, and 9, which are divisible by 5, if no digit is repeated.

Solution:

The given digits are 2, 4, 5, 7, 8, and 9.

From the digits given above, we have to form different 5-digit numbers divisible by 5 provided no digit is repeated.

We know that a number is divisible by 5 if the digit at its units place is 0 or 5. However, among the given digits, there is no 0. Hence, the units place will be occupied by the digit 5.

Hence, the number of ways in which the units place is filled with digit 5 is 1.

Since the digits are not repeated, and units place is already occupied by a digit, the remaining places will be filled with the remaining 5 digits.

Hence, the number of ways in which the remaining places are filled is the permutation of 5 different digits taken 4 at a time.

Number of ways of filling remaining places $= {}^{5}P_{4} = \frac{5!}{(5-4)!} = \frac{5!}{1!}$

= 5 × 4 × 3 × 2 × 1 = 120

Thus, by the multiplication principle, number of 5-digit numbers that are divisible by 5 is 120 × 1 = 120.

Concept of Permutations When All Objects Are Not Distinct

- The number of permutations of n objects, where p objects are of the same kind and remaining objects are all different, is given by $\frac{n!}{p!}$.
- The number of permutations of n objects, where p_1 objects are of the first kind, p_2 are of the second kind, ……, p_k are of the k^{th} kind and the rest (if any) are of a different kind, is given by $\frac{n!}{p_1!p_2!\ldots\ldots p_k!}$.

Example 1: A carton contains 14 pieces of cloth of same lengths. Out of the 14 pieces, 5 are of blue colour, 2 are of red colour, 4 are of pink colour, and 3 are of black colour. Find the number of ways in which the 14 pieces can be arranged in a row.

Solution:

It is given that the carton contains 14 pieces of cloth of the same lengths.

It is also given that out of the 14 pieces, 5 are blue coloured, 2 are red coloured, 4 are pink coloured, and 3 are black coloured.

Therefore, number of ways in which the pieces can be arranged in a row = $\frac{14!}{5!\,2!\,4!\,3!}$

$$= \frac{14\times13\times12\times11\times10\times9\times8\times7\times6\times5!}{5!\times2\times1\times4\times3\times2\times1\times3\times2\times1}$$
$$= \frac{14\times13\times12\times11\times10\times9\times8\times7\times6}{2\times4\times3\times2\times3\times2}$$
$$= 2522520$$

Example 2: Find the number of arrangements of the letters in the word AMERICA that start with letter M.

Solution:

The given word i.e., AMERICA has 7 letters.

Let us fix M at the extreme left position. We now count the number of arrangements of the remaining 6 letters. We have 2 A's while the rest of the letters are distinct.

Therefore, the required number of words starting with letter M is

$$\frac{6!}{2!} = 6\times5\times4\times3 = 360$$

Concept of Combinations

- **A combination is an arrangement of objects in which the order of objects does not matter.**
- The number of combinations of n different objects taken r at a time is given by ${}^nC_r$, where ${}^nC_r = \frac{n!}{r!(n-r)!}$, $0 < r \le n$.
- From the above formula of ${}^nC_r$, we can derive the following results:

- $^{n}C_{n} = \frac{n!}{n!(n-n)!} = \frac{n!}{n!0!} = 1$
- $^{n}C_{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{n!} = 1$. This means that the formula of $^{n}C_{r}$ also holds for $r = 0$.
- $^{n}C_{n-r} = \frac{n!}{(n-r)!\left[n-(n-r)\right]!} = \frac{n!}{(n-r)!r!} = {}^{n}C_{r}$
- $^{n}C_{r} + {}^{n}C_{r-1} = {}^{n+1}C_{r}$
- $^{n}C_{a} = {}^{n}C_{b} \Rightarrow a = b \text{ or } a = n-b, \text{i.e, } n = a+b$

We have $^{n}C_{a} = {}^{n}C_{b}$

$\Rightarrow \frac{n!}{a!(n-a)!} = \frac{n!}{b!(n-b)!}$

$\Rightarrow \frac{1}{a!(n-a)!} = \frac{1}{b!(n-b)!}$

This is possible only if $a = b$ or $a = n-b$.

Solved Examples

Example 1: Find the value of $^{n}C_{5}$ if $^{n}C_{12} = {}^{n}C_{14}$.

Solution:

It is given that $^{n}C_{12} = {}^{n}C_{14}$.

We know that $^{n}C_{r} = \frac{n!}{r!(n-r)!}$. Thus,

$$\frac{n!}{12!(n-12)!}=\frac{n!}{14!(n-14)!}$$
$$\Rightarrow \frac{1}{12!(n-12)\times(n-13)\times(n-14)!}=\frac{1}{14\times13\times12!(n-14)!}$$
$$\Rightarrow \frac{1}{(n-12)\times(n-13)}=\frac{1}{14\times13}$$
$$\Rightarrow (n-12)\times(n-13)=14\times13=182$$
$$\Rightarrow n^2-12n-13n+156-182=0$$
$$\Rightarrow n^2-25n-26=0$$
$$\Rightarrow n^2+n-26n-26=0$$
$$\Rightarrow n(n+1)-26(n+1)=0$$
$$\Rightarrow (n-26)(n+1)=0$$
$$\Rightarrow n=26 \text{ or } n=-1$$

However, we know that n cannot be negative. Therefore, n = 26.

Now,

${}^{n}C_5 = {}^{26}C_5$

$$=\frac{26!}{5!(26-5)!}$$
$$=\frac{26!}{5!21!}$$
$$=\frac{26\times25\times24\times23\times22\times21!}{5\times4\times3\times2\times1\times21!}$$
$$=26\times5\times23\times22$$
$$=65780$$

Example 2: A carton contains 2 red, 6 pink, 7 green, and 5 black balls. Eight balls are drawn from the carton. Find the number of ways of selecting 8 balls such that the selection consists of 2 balls of each colour.

Solution:

It is given that the carton contains a total of 2 red, 6 pink, 7 green, and 5 black balls. 8 balls have to be selected in such a manner that the selection consists of 2 balls of each colour.

2 balls can be selected from the 2 red balls in $^2C_2$ ways.

2 balls can be selected from the 6 pink balls in $^6C_2$ ways.

2 balls can be selected from the 7 green balls in $^7C_2$ ways.

2 balls can be selected from the 5 black balls in $^5C_2$ ways.

Thus, by the multiplication principle, the required number of ways of selecting 8 balls

$$= {}^2C_2 \times {}^6C_2 \times {}^7C_2 \times {}^5C_2 = \frac{2!}{2!\,0!} \times \frac{6!}{2!\,4!} \times \frac{7!}{2!\,5!} \times \frac{5!}{3!\,2!}$$
$$= \frac{2!}{2!} \times \frac{6 \times 5 \times 4!}{2!\,4!} \times \frac{7 \times 6 \times 5!}{2!\,5!} \times \frac{5 \times 4 \times 3!}{3!\,2!}$$
$$= 1 \times 15 \times 21 \times 10 = 3150$$

Example 3: Sumit has a total 20 music CDs. On a particular occasion, he has to take 9 CDs to his brother's house. In how many ways can Sumit choose 9 CDs out of the 20 CDs?

Solution:

It is given that Sumit has 20 music CDs, out of which he has to choose 9 CDs.

In this case, the order does not matter. Hence, this is a case of combinations.

The required number of ways in which Sumit can choose 9 CDs is given by

$$^{20}C_9 = \frac{20!}{9!(20-9)!} = \frac{20!}{9!\,11!} = \frac{20 \times 19 \times 18 \times 17 \times 16 \times 15 \times 14 \times 13 \times 12}{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = 167960$$

Example 4: Anju won a coupon, using which she can purchase 1 purse, 3 belts and 2 pairs of artificial earings from a shop. She has to choose these items from 5 purses, 11 belts, and 20 pairs of artificial earings. Find the number of ways in which Anju can select the six items.

Solution:

In this case, the order does not matter. Hence, this is a case of combinations.

Hence,

Number of ways in which Anju can select 1 purse out of 5 purses is $^5C_1$.

Number of ways in which Anju can select 3 belts out of 11 belts is ${}^{11}C_3$.

Number of ways in which Anju can select 2 pairs of earings out of 20 pairs of earings is ${}^{20}C_2$.

Thus, the required number of ways in which Anju can choose the six items is

${}^{5}C_1 \times {}^{11}C_3 \times {}^{20}C_2$

$$= \frac{5!}{1!\,4!} \times \frac{11!}{3!\,8!} \times \frac{20!}{2!\,18!}$$
$$= 5 \times \frac{11 \times 10 \times 9}{3 \times 2 \times 1} \times \frac{20 \times 19}{2 \times 1}$$
$$= 5 \times 165 \times 190$$
$$= 156750$$