7 Application of Derivatives

BASIC CONCEPTS

1. **Rate of Change:** If a quantity y varies with another quantity x, satisfying some rule $y = f(x)$, then $\left.\dfrac{dy}{dx}\right|_{x = x_0}$ (or $f'(x_0)$) represents the rate of change of y with respect to x at $x = x_0$.

2. **Differentials:** Let $y = f(x)$ be any function of x which is differentiable in (a, b). The derivative of this function at some point x of (a, b) is given by the relation

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x)$$

$$\Rightarrow \qquad \frac{dy}{dx} = f'(x)$$

$\Rightarrow \qquad dy = f'(x)\, dx$, where dy is called the differential of the function.

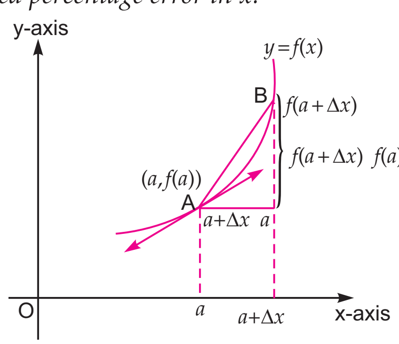
Note: Formula $dy = f'(x)\, dx$ or $\Delta y = f'(x)\, \Delta x$ is very useful in measuring the errors in the dependent variable for given error in independent variable.

(*i*) **Absolute Error:** *The error Δx in x is called the absolute error.*

(*ii*) **Relative Error:** *If Δx is error in x then ratio $\dfrac{\Delta x}{x}$ is called the relative error.*

(*iii*) **Percentage Error:** *If $\dfrac{\Delta x}{x}$ is relative error, then $\dfrac{\Delta x}{x} \times 100$ is called percentage error in x.*

3. **Tangents and Normals:**



(*i*) Slope of chord $AB = \dfrac{f(a + \Delta x) - f(a)}{a + \Delta x - a} = \dfrac{f(a + \Delta x) - f(a)}{\Delta x}$

Obviously, if $\Delta x \to 0$ comes very close to A and then chord AB becomes tangent at A *i.e.*, $x = a$.

i.e., slope of tangent at $(a, f(a)) = \displaystyle\lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}$

$$= [f'(x)]_{(a, f(a))} = \left(\frac{dy}{dx}\right)_{(a,\, f(a))}$$

Hence, equation of tangent to the curve $y = f(x)$ at the point (x_1, y_1) is given by

$$(y - y_1) = \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} (x - x_1)$$

[$\because$ Equation of line is $y - y_1 = m(x - x_1)$, where m is slope] (*ii*) If $\frac{dy}{dx} = \infty$ at the point $P(x_1, y_1)$, then the tangent at P is parallel to y-axis and its equation is given by

$$\infty = \frac{y - y_1}{x - x_1} \quad \Rightarrow \quad x - x_1 = 0 \quad \Rightarrow \quad x = x_1$$

[**Note :** In this case, $\frac{dx}{dy}$ at $P(x_1, y_1) = 0$]

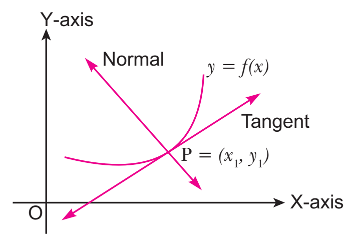
(*iii*) If $\frac{dy}{dx} = 0$, at the point $P(x_1, y_1)$, then the tangent at P is parallel to x-axis and equation is given as

$$0 = \frac{y - y_1}{x - x_1} \quad \Rightarrow \quad y - y_1 = 0 \quad \Rightarrow \quad y = y_1$$

(*iv*) Obviously, normal to the curve $y = f(x)$ at $P(x_1, y_1)$ is perpendicular to the tangent at $P(x_1, y_1)$.

$\therefore$ Slope of normal $= \frac{-1}{\text{slope of tangent}}$

$$= \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}$$



Hence, equation of normal to the curve $y = f(x)$ at $P(x_1, y_1)$ is

$$(y - y_1) = \frac{-1}{\left[\frac{dy}{dx}\right]_{(x_1, y_1)}}(x - x_1)$$

(*v*) If $\frac{dy}{dx}$ at the point $P(x_1, y_1)$ is zero, then the equation of normal is $x = x_1$.

(*vi*) If $\left(\frac{dy}{dx}\right)$ at the point (x_1, y_1) does not exist, then the equation of normal is $y = y_1$.

(*vii*) The angle θ between two given curves $y_1 = f_1(x)$ and $y_2 = f_2(x)$ at a point (x_1, y_1) is given by $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$ where (x_1, y_1) is the point of intersection and m_1, m_2 are slopes of their tangents at point (x_1, y_1).

Note: The gradient of a curve at a point is defined as the slope of tangent to the curve at that point.

4. **Nature of Function:** To know the behaviour of a function in an interval, we study the properties of increasing or decreasing functions, maximum and minimum of the functions.

5. **Monotonic Function:** A function is said to be monotonic in an interval, if it is either increasing or decreasing in the given interval.

6. **Increasing Function:** A function $f(x)$ is said to be an increasing function in (a, b) if

$$x_1 < x_2 \quad \Rightarrow \quad f(x_1) \le f(x_2) \quad \forall\, x_1, x_2 \in (a, b)$$

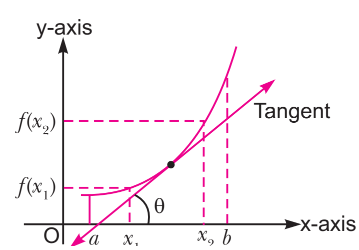
In this way, we can say

$$f(x) \text{ is increasing in } (a, b) \text{ if } \forall\, x \in (a, b), f'(x) > 0$$

Obviously, the angle θ made by tangent with +ve direction of x-axis in interval (a, b) is acute.

$\Rightarrow$ $\tan\theta$ is +ve $\Rightarrow$ slope is +ve

$\Rightarrow$ $\frac{dy}{dx} = f'(x) > 0$



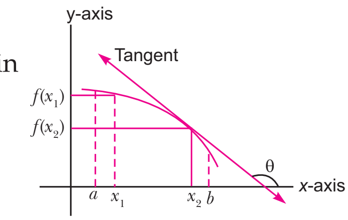
7. **Decreasing Function:** A function $f(x)$ is said to be decreasing in the interval (a, b) if

$$x_1 < x_2 \quad \Rightarrow \quad f(x_1) \ge f(x_2) \quad \forall\, x_1, x_2 \in (a, b)$$

In other way,

$f(x)$ is decreasing in interval (a, b) if $\forall\ x \in (a, b), f'(x) < 0$

Obviously, the angle θ made by tangent with +ve direction of x-axis in interval (a, b) is obtuse.



$\Rightarrow \qquad \tan\theta$ is –ve $\qquad \Rightarrow \qquad$ slope is –ve

$$\Rightarrow \qquad \frac{dy}{dx} = f'(x) < 0$$

[**Note:** A function $f(x)$ is said to be:

Strictly increasing if $x_1 < x_2 \;\Rightarrow\; f(x_1) < f(x_2)\ \forall\ x_1, x_2 \in (a, b)$

Strictly decreasing if $x_1 < x_2 \;\Rightarrow\; f(x_1) > f(x_2)\ \forall\ x_1, x_2 \in (a, b)$

8. Maximum and Minimum Value of a Function (or Absolute Maximum or Minimum Value)

A function f is said to attain maximum value at a point $a \in D_f$, if $f(a) \geq f(x)\ \ \forall\ x \in D_f$ then $f(a)$ is called absolute maximum value of f.

A function f attains minimum value at $x = b \in D_f$, if $f(b) \leq f(x)\ \ \forall\ x \in D_f$ then $f(b)$ is called absolute minimum value of f.

Note that a function 'f' may have maximum (or minimum) values in some parts (intervals) of the domain. Such values may occur at more than one point. These are therefore, called local (or relative) maxima (or minima).

9. Local Maxima and Local Minima (or Relative Extrema)

Local Maxima: *A function $f(x)$ is said to attain a local maxima at $x = a$, if there exists a neighbourhood $(a - \delta, a + \delta)$ of 'a' such that $f(x) < f(a)\ \forall\ x \in (a - \delta, a + \delta), x \neq a$, then $f(a)$ is the local maximum value of $f(x)$ at $x = a$.*

Local Minima: *A function $f(x)$ is said to attain a local minima at $x = a$, if there exists a neighbourhood $(a - \delta, a + \delta)$ of 'a' such that $f(x) > f(a)\ \forall\ x \in (a - \delta, a + \delta), x \neq a$, then $f(a)$ is called the local minimum value at $x = a$.*

Caution:

(*i*) *A function defined in an interval can reach maximum or minimum values only for those values of x which lie within the given interval.*

(*ii*) *One should not think that the maximum and minimum of a function are its respective largest and smallest values over a given interval.*

10. Test for Identifying Relative (Local) Maxima or Minima

(*i*) First Derivative Test

Step I: Find $f'(x)$

Step II: The equation $f'(x) = 0$ is solved to get critical points $x = c_1, c_2, \ldots\ldots\ldots, c_n$.

Step III: The sign of $f'(x)$ is studied in the neighbourhood of each critical points.

Let one critical point be $x = c$.

If the sign of $f'(x)$ changes from +ve to –ve as x increases through c (from left to right of c).	If the sign of $f'(x)$ changes from –ve to +ve as x increases through c (from left to right of c).
↓	↓
$x = c$ is relative maxima and $f(c)$ is relative maximum value.	$x = c$ is relative minima and $f(c)$ is relative minimum value.

(*ii*) Second order derivative test

Step I: Find $f'(x) = 0$

Step II: The equation $f'(x) = 0$ is solved to get critical points $x = c_1, c_2, \ldots\ldots\ldots, c_n$.

Step III: $f''(x)$ is obtained and the sign of $f''(x)$ is studied for all critical points $x = c_1, c_2, \ldots\ldots\ldots, c_n$.

Let $x = c$ be one critical point.

- If $f''(c) > 0$ → $x = c$ is relative minima and $f(c)$ is relative minimum value.
- If $f''(c) < 0$ → $x = c$ is relative maxima and $f(c)$ is relative maximum value.
- If $f''(c) = 0$ → Second derivative test fail and first derivative test is used.

11. **Critical point:** A point $x = c$ is called critical point of the function $f(x)$, if $f(c)$ exists and either $f'(c) = 0$ or $f'(c) = \infty$ (does not exist).

12. **Point of Inflexion:** If $f(x)$ is a function and $x = c$ is critical point, then $x = c$ is called point of inflexion if

(*i*) $f'(c) = 0$ (*ii*) $f''(c) = 0$ (*iii*) $f'''(c) \neq 0$

SOME USEFUL RESULTS

- Area of a square $= x^2$ and perimeter $= 4x$, where x is side of the square.
- Area of a rectangle $= x \cdot y$, as x and y are length and breadth of rectangle and perimeter $= 2(x + y)$.
- Area of a trapezium $= \frac{1}{2}$ (sum of parallel sides) × perpendicular distance between them.
- Area of a circle $= \pi r^2$, Circumference of a circle $= 2\pi r$, where r is the radius.
- Volume of sphere $= \frac{4}{3}\pi r^3$; Surface area $= 4\pi r^2$, where r is the radius.
- Total surface area of a right circular cylinder $= 2\pi rh + 2\pi r^2$; Curved surface area $= 2\pi rh$.
  Volume $= \pi r^2 h$, where r is the radius and h is the height of the cylinder.
- Volume of a right circular cone $= \frac{1}{3}\pi r^2 h$; Curved surface area $= \pi rl$, Total surface area $= \pi r^2 + \pi rl$,
  where r is the radius, h is the height and l is the slant height of the cone.
- Volume of a parallelopiped $= xyz$ and surface area $= 2(xy + yz + zx)$, where x, y and z are the dimensions of parallelopiped.
- Volume of a cube $= x^3$ and surface area $= 6x^2$, where x is the side of the cube.
- Area of an equilateral triangle $= \frac{\sqrt{3}}{4}(\text{side})^2$.

Selected NCERT Questions

1. **A stone is dropped into a quiet lake and waves move in circles at the speed of 5 cm/s. At the instant when the radius of the circular wave is 8 cm, how fast is the enclosed area increasing?**

Sol. Let x be the radius and A be the area of the given circle at any time t.

Then $\frac{dx}{dt} = 5$ cm/s and $x = 8$ cm

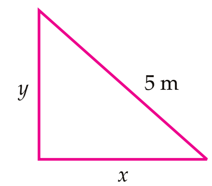
Now, $A = \pi x^2 \Rightarrow \frac{dA}{dt} = 2\pi x \frac{dx}{dt} \Rightarrow \frac{dA}{dt} = 2\pi \times 8 \times 5 = 80\pi \text{ cm}^2/\text{s}$

Thus, rate of change of area is 80π cm^2/s.

2. A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2 cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall? **[*CBSE (AI) 2012*]**

Sol. Let x, y be the distance of the bottom and top of the ladder respectively from the edge of the wall.

Here, $\dfrac{dx}{dt} = 2 \text{ cm/s}$



From figure, $x^2 + y^2 = 25$

When $x = 4$ m,

$$(4)^2 + y^2 = 25 \quad \Rightarrow \quad y^2 = 25 - 16 = 9 \quad \Rightarrow \quad y = 3 \text{ m}$$

Now, $x^2 + y^2 = 25$

Differentiating with respect to t, we have

$$\Rightarrow \quad 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \quad \Rightarrow \quad x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

$$\Rightarrow \quad 4 \times 2 + 3 \times \frac{dy}{dt} = 0 \quad \Rightarrow \quad \frac{dy}{dt} = -\frac{8}{3}$$

Hence, the rate of decrease of its height $= \dfrac{8}{3} \text{cm / s}$

3. Sand is pouring from a pipe at the rate of 12 cm^3/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm? **[*CBSE Delhi 2011*]**

Sol. Let r be the radius and h be the height of the cone so that $V = \dfrac{1}{3}\pi r^2 h$

We have, $$\frac{dV}{dt} = 12\text{cm}^3/\text{s} \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{1}{3}\pi r^2 h\right) = 12 \qquad \ldots(i)$$

As $$h = \frac{1}{6}r \quad \Rightarrow \quad r = 6h$$

Putting in (i), we get

$$\frac{d}{dt}\left(\frac{1}{3}\pi (6h)^2 \times h\right) = 12 \quad \Rightarrow \quad \frac{d}{dt}(12\pi h^3) = 12$$

$$\Rightarrow \quad 12\pi \times 3h^2 \frac{dh}{dt} = 12 \quad \Rightarrow \quad \frac{dh}{dt} = \frac{1}{3\pi h^2}$$

when $h = 4$ cm, $\dfrac{dh}{dt} = \dfrac{1}{3\pi(4)^2} = \dfrac{1}{48\pi}\text{cm/s}$

4. The total cost $C(x)$ in rupees associated with the production of x units of an item is given by

$$C(x) = 0.007\,x^3 - 0.003\,x^2 + 15x + 4000$$

Find the marginal cost when 17 units are produced.

Sol. Given, $C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000$

$$\therefore \quad \text{Marginal cost} = \frac{d}{dx}C(x) = 0.021x^2 - 0.006x + 15$$

When $x = 17$

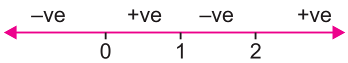
Marginal cost $= 0.021 \times (17)^2 - 0.006 \times 17 + 15 =$ ₹ 20.967

5. Find the values of x for which $y = [x(x-2)]^2$ is an increasing function. **[*CBSE (AI) 2014*]**

Sol. Given, $y = [x\,(x-2)]^2$

$$\therefore \quad \frac{dy}{dx} = 2[x(x-2)] \times (2x-2) = 4x(x-1)(x-2)$$

For increasing function, $\frac{dy}{dx} > 0$



Sign rule

$$4x(x-1)(x-2) > 0 \qquad \Rightarrow \qquad x(x-1)(x-2) > 0$$

From sign rule,

For $\quad \frac{dy}{dx} > 0$ value of $x = 0 < x < 1$ and $x > 2$

Therefore, y is increasing $\forall\ x \in (0, 1) \cup (2, \infty)$.

6. Prove that $y = \dfrac{4\sin\theta}{(2+\cos\theta)} - \theta$ is an increasing function of θ in $\left[0, \dfrac{\pi}{2}\right]$.

[*CBSE (AI) 2011; (North) 2016*]

Sol. Given, $\quad y = \dfrac{4\sin\theta}{2+\cos\theta} - \theta$

$$\frac{dy}{dx} = \frac{(2+\cos\theta).4\cos\theta - 4\sin\theta.(0-\sin\theta)}{(2+\cos\theta)^2} - 1$$

$$= \frac{8\cos\theta + 4\cos^2\theta + 4\sin^2\theta - (2+\cos\theta)^2}{(2+\cos\theta)^2} = \frac{8\cos\theta + 4 - 4 - \cos^2\theta - 4\cos\theta}{(2+\cos\theta)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{4\cos\theta - \cos^2\theta}{(2+\cos\theta)^2} = \frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{+ve \times (+ve)}{+ve} \qquad \left[\begin{array}{l} \because\ \theta \in [0, \pi/2] \Rightarrow \cos\theta > 0 \\ 4 - \cos\theta \text{ is } +\text{ve as } -1 \le \cos\theta \le 1 \end{array}\right]$$

$$\Rightarrow \quad \frac{dy}{dx} > 0$$

i.e., $\quad y = \dfrac{4\sin\theta}{2+\cos\theta} - \theta$ is increasing function in $\left[0, \dfrac{\pi}{2}\right]$.

7. Prove that the function f given by $f(x) = \log \sin x$ is strictly increasing on $\left(0, \dfrac{\pi}{2}\right)$ and strictly decreasing on $\left(\dfrac{\pi}{2}, \pi\right)$.

Sol. Here, $\quad f(x) = \log \sin x \Rightarrow f'(x) = \dfrac{1}{\sin x}(\cos x) = \cot x$

when $x \in \left(0, \dfrac{\pi}{2}\right)$ then $f'(x) > 0 \Rightarrow f(x)$ is strictly increasing on $\left(0, \dfrac{\pi}{2}\right)$.

when $x \in \left(\dfrac{\pi}{2}, \pi\right)$ then $f'(x) < 0 \Rightarrow f(x)$ is strictly decreasing on $\left(\dfrac{\pi}{2}, \pi\right)$.

8. Find the equation of tangent to the curve $y = \dfrac{x-7}{(x-2)(x-3)}$ at the point, where it cuts the x-axis. **[*CBSE Delhi 2010; Guwahati 2015*]**

Sol. We have $\qquad y = \dfrac{x-7}{(x-2)(x-3)} \qquad \ldots(i)$

Let (i) cuts the x-axis at $(x, 0)$ *i.e.*, $y = 0$

then $\qquad \dfrac{x-7}{(x-2)(x-3)} = 0 \quad \Rightarrow \quad x = 7$

$\therefore$ The required point is $(7, 0)$.

Differentiating equation (*i*) with respect to x, we get

$$\frac{dy}{dx}=\frac{(x-2)(x-3)1-(x-7)[(x-2)+(x-3)]}{[(x-2)(x-3)]^2}$$

$$=\frac{x^2-5x+6-2x^2+19x-35}{(x^2-5x+6)^2}=\frac{-x^2+14x-29}{(x^2+6-5x)^2}$$

$$\left.\frac{dy}{dx}\right]_{(7,0)}=\frac{-49+98-29}{(49-35+6)^2}=\frac{20}{400}=\frac{1}{20}$$

$\therefore$ Equation of tangent is $y-y_1=\frac{1}{20}(x-x_1)$

$\Rightarrow$ $y-0=\frac{1}{20}(x-7)$ or $x-20y-7=0$

9. Find the equations of the tangent and normal to the curve $x=a\sin^3\theta$ and $y=a\cos^3\theta$ at $\theta=\frac{\pi}{4}$.

[*CBSE Delhi 2014*]

Sol. Given, $x=a\sin^3\theta$ and $y=a\cos^3\theta$

$\Rightarrow$ $\frac{dx}{d\theta}=3a\sin^2\theta.\cos\theta$ and $\frac{dy}{d\theta}=-3a\cos^2\theta\sin\theta$

$$\Rightarrow\quad \frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{-3a\cos^2\theta.\sin\theta}{3a\sin^2\theta.\cos\theta}=-\cot\theta$$

$\Rightarrow$ Slope of tangent to the given curve at $\theta=\frac{\pi}{4}$ is $\left[\frac{dy}{dx}\right]_{\theta=\frac{\pi}{4}}=-\cot\frac{\pi}{4}=-1$.

Since for $\theta=\frac{\pi}{4}, x=a\sin^3\frac{\pi}{4}$ and $y=a\cos^3\frac{\pi}{4}$

$\Rightarrow$ $x=a\left(\frac{1}{\sqrt{2}}\right)^3$ and $y=a\left(\frac{1}{\sqrt{2}}\right)^3$ $\Rightarrow$ $x=\frac{a}{2\sqrt{2}}$ and $y=\frac{a}{2\sqrt{2}}$

i.e., co-ordinates of the point of contact $=\left(\frac{a}{2\sqrt{2}},\frac{a}{2\sqrt{2}}\right)$

$\therefore$ Equation of tangent is

$$\left(y-\frac{a}{2\sqrt{2}}\right)=(-1).\left(x-\frac{a}{2\sqrt{2}}\right)\quad\Rightarrow\quad y-\frac{a}{2\sqrt{2}}=-x+\frac{a}{2\sqrt{2}}$$

$\Rightarrow$ $x+y=\frac{a}{\sqrt{2}}$

Also slope of normal $\left(\text{at } \theta=\frac{\pi}{4}\right)=-\frac{1}{\text{slope of tangent}}=-\frac{1}{-1}=1$

$\therefore$ Equation of normal is $\left(y-\frac{a}{2\sqrt{2}}\right)=(1).\left(x-\frac{a}{2\sqrt{2}}\right)$

$\Rightarrow$ $y-\frac{a}{2\sqrt{2}}=x-\frac{a}{2\sqrt{2}}$ $\Rightarrow$ $y-x=0$

10. Find the slope of the normal to the curve $x=1-a\sin\theta$, $y=b\cos^2\theta$ at $\theta=\frac{\pi}{2}$.

Sol. Here $x=1-a\sin\theta$ and $y=b\cos^2\theta$

Differentiating both sides w.r.t. θ, we have

$$\frac{dx}{d\theta}=-a\cos\theta,\ \frac{dy}{d\theta}=-2b\sin\theta\cos\theta$$

Now, $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{-2b\sin\theta\cos\theta}{-a\cos\theta} = \dfrac{2b\sin\theta}{a}$

Slope of normal at $\theta = \dfrac{\pi}{2}$ is $\dfrac{-1}{\left[\dfrac{dy}{dx}\right]_{\theta=\frac{\pi}{2}}} = \dfrac{-1}{\left[\dfrac{2b}{a}\sin\theta\right]_{\theta=\frac{\pi}{2}}}$

$$= \frac{-a}{2b\sin\frac{\pi}{2}} = \frac{-a}{2b}$$

Thus, slope of normal at $\theta = \dfrac{\pi}{2}$ is $-\dfrac{a}{2b}$.

11. Find the point on the curve $y = x^3 - 11x + 5$ at which the equation of tangent is $y = x - 11$.
[*CBSE Delhi 2012*]

Sol. Let the required point of contact be (x_1, y_1).

Given curve is $\quad y = x^3 - 11x + 5 \qquad ...(i)$

$\therefore \quad \dfrac{dy}{dx} = 3x^2 - 11 \quad \Rightarrow \quad \left[\dfrac{dy}{dx}\right]_{(x_1, y_1)} = 3x_1^2 - 11$

i.e., slope of tangent at (x_1, y_1) to give curve (i) = $3x_1^2 - 11$

From question

$3x_1^2 - 11$ = slope of line $y = x - 11$, which is also tangent.

i.e., $\quad 3x_1^2 - 11 = 1 \quad \Rightarrow \quad x_1^2 = 4 \quad \Rightarrow \quad x_1 = \pm 2$

i.e., since (x_1, y_1) lie on curve (i)

$\therefore \quad y_1 = x_1^3 - 11x_1 + 5$

When $\quad x_1 = 2, y_1 = 2^3 - 11 \times 2 + 5 = -9$

$\quad x_1 = -2, y_1 = (-2)^3 - 11 \times (-2) + 5 = 19$

But $(-2, 19)$ does not satisfy the line $y = x - 11$.

Therefore $(2, -9)$ is the required point of curve at which tangent is $y = x - 11$.

12. Find the equation of the tangent line of the curve $y = x^2 - 2x + 7$ which is *(a)* parallel to the line $2x - y + 9 = 0$, *(b)* perpendicular to the line $5y - 15x = 13$.

Sol. Here, $\quad y = x^2 - 2x + 7$

$$\frac{dy}{dx} = 2x - 2 = 2(x - 1)$$

(*a*) Slope of the line $2x - y + 9 = 0$ is

$$-\frac{\text{Coefficient of } x}{\text{Coefficient of } y} = \frac{-2}{-1} = 2$$

It is given that tangent is parallel to the line.

$\therefore \quad 2(x - 1) = 2 \quad \Rightarrow \quad x - 1 = 1 \quad \Rightarrow \quad x = 2$

when $x = 2$ then $y = (2)^2 - 2 \times 2 + 7 = 4 - 4 + 7 = 7$

$\therefore$ Equation of tangent at $(2, 7)$ is

$$y - 7 = 2(x - 2) \quad \Rightarrow \quad y - 7 = 2x - 4 \quad \Rightarrow \quad 2x - y + 3 = 0$$

(*b*) Slope of line $5y - 15x = 13$ is

$$-\frac{\text{Coefficient of } x}{\text{Coefficient of } y} = -\frac{(-15)}{5} = 3$$

It is given that tangent is perpendicular to the line $5y - 15x = 13$

$$\therefore \quad 2(x-1) \times 3 = -1 \quad \Rightarrow \quad 6x - 6 = -1 \quad \Rightarrow \quad x = \frac{5}{6}$$

when $x = \frac{5}{6}$ then $y = \left(\frac{5}{6}\right)^2 - 2 \times \frac{5}{6} + 7 = \frac{25}{36} - \frac{5}{3} + 7$

$$= \frac{25 - 60 + 252}{36} = \frac{217}{36}$$

$\therefore$ Equation of tangent at $\left(\frac{5}{6}, \frac{217}{36}\right)$ is

$$y - \frac{217}{36} = -\frac{1}{3}\left(x - \frac{5}{6}\right) \quad \Rightarrow \quad \frac{36y - 217}{36} = \frac{-6x + 5}{18}$$

$$\Rightarrow \quad 36y - 217 = -12x + 10 \quad \Rightarrow \quad 12x + 36y - 227 = 0$$

13. Prove that the curves $x = y^2$ and $xy = k$ cut at right angles if $8k^2 = 1$. **[CBSE (AI) 2008]**

Sol. As we know, two curves intersect at right angles if the tangents to the curves at the point of intersection are perpendicular to each other, *i.e.*, product of slope of these two curves is –1.

We have $\quad x = y^2$

Differentiating with respect to x, we have

$$1 = 2y\frac{dy}{dx} \quad \Rightarrow \frac{dy}{dx} = \frac{1}{2y} = m_1, \quad \text{(let)} \qquad \ldots(i)$$

$$xy = k$$

Also, differentiating with respect to x, we have

$$x\frac{dy}{dx} + y.1 = 0 \;\Rightarrow\; \frac{dy}{dx} = -\frac{y}{x} = m_2 \quad \text{(let)} \qquad \ldots(ii)$$

On solving the equations of the two curves

$$xy = k \quad \text{and} \quad x = y^2$$

We get $\quad x = k^{2/3} \quad$ and $\quad y = k^{1/3}$

Putting these values in (*i*) and (*ii*), we have

$$m_1 = \frac{1}{2k^{1/3}} \quad \text{and} \quad m_2 = \frac{-k^{1/3}}{k^{2/3}} = -k^{-1/3}$$

For the curves to intersect at right angles

$$m_1 \times m_2 = -1 \quad \Rightarrow \quad \frac{1}{2k^{1/3}} \times (-k^{-1/3}) = -1 \quad \Rightarrow \quad \frac{1}{2k^{2/3}} = 1$$

$$\left(\frac{1}{2}\right)^3 = (k^{2/3})^3 \quad \Rightarrow \quad \frac{1}{8} = k^2 \quad \Rightarrow \quad 8k^2 = 1$$

Hence, the result is proved.

14. Find the equation of the tangent and normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_0, y_0).

Sol. Here, equation of the given curve is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Differentiating both sides w.r.t. x, we have

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{2x}{a^2} \times \frac{b^2}{-2y} = \frac{b^2 x}{a^2 y}$$

$$\Rightarrow \quad \frac{dy}{dx} \text{ at } (x_0, y_0) = \frac{b^2 x_0}{a^2 y_0}$$

$\therefore$ Equation of tangent at (x_0, y_0) is

$$y - y_0 = \frac{b^2 x_0}{a^2 y_0}(x - x_0) \quad \Rightarrow \quad \frac{yy_0}{b^2} - \frac{y_0^2}{b^2} = \frac{xx_0}{a^2} - \frac{x_0^2}{a^2}$$

$$\Rightarrow \quad \frac{xx_0}{a^2} - \frac{yy_0}{b^2} = \frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} \quad \Rightarrow \quad \frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1 \qquad \left[\because \frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = 1\right]$$

Equation of normal at (x_0, y_0) is

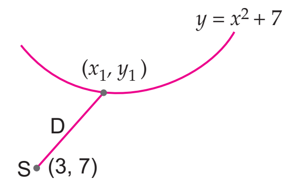
$$y - y_0 = -\frac{1}{b^2 x_0 / a^2 y_0}(x - x_0)$$

$$\Rightarrow \quad y - y_0 = -\frac{a^2 y_0}{b^2 x_0}(x - x_0) \quad \Rightarrow \quad y - y_0 = \frac{-a^2 y_0}{b^2 x_0}(x - x_0)$$

$$\Rightarrow \quad \frac{y - y_0}{a^2 y_0} = \frac{-(x - x_0)}{b^2 x_0} \quad \Rightarrow \quad \frac{y - y_0}{a^2 y_0} + \frac{x - x_0}{b^2 x_0} = 0$$

15. An Apache helicopter of enemy is flying along the curve given by $y = x^2 + 7$. A soldier, placed at (3, 7), wants to shoot down the helicopter when it is nearest to him. Find the nearest distance. **[*HOTS*]**

Sol. Let (x_1, y_1) be the position of helicopter on the curve $y = x^2 + 7$, when the distance D from soldier placed at (3, 7) is minimum.



Now, $D = \sqrt{(x_1 - 3)^2 + (y_1 - 7)^2} \quad \Rightarrow \quad D^2 = (x_1 - 3)^2 + (y_1 - 7)^2$

$\because$ (x_1, y_1) lie on curve $y = x^2 + 7$

$\therefore$ $y_1 = x_1^2 + 7$

$\Rightarrow$ $D^2 = (x_1 - 3)^2 + (x_1^2 + 7 - 7)^2 \quad \Rightarrow \quad D^2 = x_1^2 - 6x_1 + 9 + x_1^4$

Differentiate D^2 w.r.t. x_1, we get

$$\frac{d(D^2)}{dx_1} = 2x_1 - 6 + 4x_1^3 = 4x_1^3 + 2x_1 - 6$$

Now, for maximum or minimum distance $\frac{d(D^2)}{dx_1} = 0$

$\Rightarrow \quad 4x_1^3 + 2x_1 - 6 = 0 \quad \Rightarrow \quad 4x_1^2(x_1 - 1) + 4x_1(x_1 - 1) + 6(x_1 - 1) = 0$

$\Rightarrow \quad (x_1 - 1)(4x_1^2 + 4x_1 + 6) = 0 \quad \Rightarrow \quad x_1 - 1 = 0$ or $4x_1^2 + 4x_1 + 6 = 0$

$\Rightarrow \quad x_1 = 1 \qquad$ [$4x_1^2 + 4x_1 + 6$ have no real roots]

Again $\frac{d^2(D^2)}{dx_1^2} = 12x_1^2 + 2 \quad \Rightarrow \quad \left.\frac{d^2(D^2)}{dx_1^2}\right]_{x_1 = 3} = +\text{ve}$

Hence, for $x_1 = 1$, D^2 is minimum, *i.e.*, D is minimum.

Also, for $x_1 = 1$, $y_1 = 1^2 + 7 = 8$

$\therefore$ Minimum distance, $D = \sqrt{(1 - 3)^2 + (8 - 7)^2} = \sqrt{5}$ unit.

Thus, when helicopter is at (1, 8) then it is at nearest distance $\sqrt{5}$ unit from soldier.

16. Find the local maximum or minimum if any of the function $f(x) = \sin x - \cos x$, $0 < x < 2\pi$. Also, find the local extrema values.

Sol. $f'(x) = \cos x + \sin x = 0 \quad \Rightarrow \quad \cos x + \sin x = 0$

$\Rightarrow \quad \tan x = -1 \quad \Rightarrow \quad x = \frac{3\pi}{4}, \frac{7\pi}{4}$

At $x = \frac{3\pi}{4}$, $f'(x) = \cos x + \sin x$,

when $x < \frac{3\pi}{4}$, $f'(x) = +ve$

when $x > \frac{3\pi}{4}$, $f'(x) = -ve \quad \Rightarrow \quad x = \frac{3\pi}{4}$ is a point of local maxima.

At $x = \frac{7\pi}{4}$, $f'(x) = \cos x + \sin x$

when $x < \frac{7\pi}{4}$, $f'(x) = -ve$

when $x > \frac{7\pi}{4}$, $f'(x) = +ve \quad \Rightarrow \quad x = \frac{7\pi}{4}$ is a point of local minima.

$f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{4} - \cos\frac{3\pi}{4} = \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right) = \sqrt{2}$ is the local maximum value.

$f\left(\frac{7\pi}{4}\right) = \sin\frac{7\pi}{4} - \cos\frac{7\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}$ is the local minimum value.

17. Find the absolute maximum value and the absolute minimum value of the following function in the given interval.

$$f(x) = 4x - \frac{1}{2}x^2,\ x \in \left[-2, \frac{9}{2}\right]$$

Sol. Here, $f(x) = 4x - \frac{1}{2}x^2,\ x \in \left[-2, \frac{9}{2}\right] \Rightarrow f'(x) = 4 - x$

For maximum or minimum $f'(x) = 0 \quad \Rightarrow \quad 4 - x = 0 \Rightarrow x = 4$

$$f(-2) = 4(-2) - \frac{1}{2}(-2)^2 = -8 - 2 = -10$$

$$f(4) = 4(4) - \frac{1}{2}(4)^2 = 16 - 8 = 8$$

$$f\left(\frac{9}{2}\right) = 4\left(\frac{9}{2}\right) - \frac{1}{2}\left(\frac{9}{2}\right)^2 = 18 - \frac{81}{8} = \frac{63}{8}$$

Thus, absolute maximum value is 8 at $x = 4$ and absolute minimum value is -10 at $x = -2$.

18. A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off square from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum?

Sol. Let x cm be the length of square, then volume of the box,

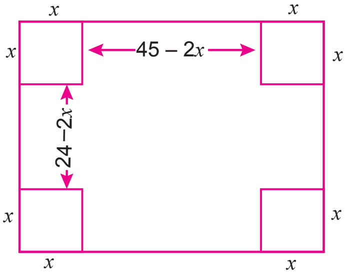
$$V = (45 - 2x)(24 - 2x).\, x$$

$$\frac{dV}{dx} = (-2)(24 - 2x)x + (45 - 2x)(-2)x + (45 - 2x)(24 - 2x).1$$

$$\frac{dV}{dx} = -48x + 4x^2 - 90x + 4x^2 + 1080 - 138x + 4x^2$$

$$= 12x^2 - 276x + 1080$$

$$= 12(x^2 - 23x + 90) = 12(x - 18)(x - 5)$$



For maximum or minimum, $\frac{dV}{dx} = 0$

$\Rightarrow \quad 12(x - 18)(x - 5) = 0 \quad \Rightarrow \quad x = 5, 18$

$x = 18$ is not possible as breadth is 24 cm.

Now, $\dfrac{d^2V}{dx^2} = 12\,(1)\,(x-5) + 12\,(x-18).\,1$

$$\left(\frac{d^2V}{dx^2}\right)_{x=5} = 0 + 12\,(5-18) = -156 < 0$$

$\Rightarrow$ $x = 5$ is the point of maximum.

Thus, 5 cm is the side of the square to be cut off from rectangle for box of maximum volume.

19. Prove that the volume of the largest cone that can be inscribed in a sphere of radius a is $\dfrac{8}{27}$ of the volume of the sphere. *[CBSE Delhi 2016; (AI) 2014; (F) 2013]*

OR

Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius R is $\dfrac{4R}{3}$. *[CBSE (F) 2012]*

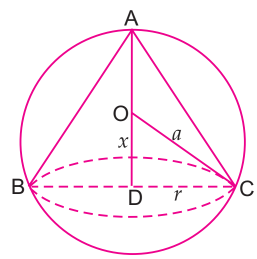
[Hint: replace a by R and you can get the result]

Sol. Consider a sphere of radius a with centre at O such that $OD = x$ and $DC = r$.

Let h be the height of the cone.

Then $h = AD = AO + OD = a + x$ ...(*i*)

($OA = OC =$ radius)



In the right angle ΔODC,

$a^2 = r^2 + x^2$ (By Pythagoras theorem) ...(*ii*)

Let V be the volume the cone, then $V = \dfrac{1}{3}\pi r^2 h$

$\Rightarrow$ $V(x) = \dfrac{1}{3}\pi(a^2 - x^2)(a + x)$ [From (*i*) and (*ii*)]

$\Rightarrow$ $V'(x) = \dfrac{1}{3}\pi\left[(a^2 - x^2)\dfrac{d}{dx}(a + x) + (a + x)\dfrac{d}{dx}(a^2 - x^2)\right]$

$$= \frac{1}{3}\pi[(a^2 - x^2)(1) + (a + x)(-2x)] = \frac{1}{3}\pi[(a + x)(a - x - 2x)] = \frac{1}{3}\pi(a + x)(a - 3x)$$

Also, $V''(x) = \dfrac{1}{3}\pi\left[(a + x)\dfrac{d}{dx}(a - 3x) + (a - 3x)\dfrac{d}{dx}(a + x)\right]$

$\Rightarrow$ $V''(x) = \dfrac{1}{3}\pi[(a + x)(-3) + (a - 3x)(1)]$

For maximum or minimum value, we have $V'(x) = 0$

$$\frac{1}{3}\pi(a + x)(a - 3x) = 0 \quad \Rightarrow \quad x = -a \text{ or } x = \frac{a}{3}$$

Neglecting $x = -a$ [$\because x > 0$]

$$V''\left(\frac{a}{3}\right) = \frac{1}{3}\pi\left[\left(a + \frac{a}{3}\right)(-3) + \left(a - 3\left(\frac{a}{3}\right)\right)\right] = \frac{-4\pi a}{3} < 0$$

$\therefore$ Volume is maximum when $x = \dfrac{a}{3}$.

Putting $x = \dfrac{a}{3}$ in equation (*i*) and (*ii*), we get $h = a + \dfrac{a}{3} = \dfrac{4a}{3}$ and $r^2 = a^2 - \dfrac{a^2}{9} = \dfrac{8a^2}{9}$

Now, volume of cone $= \dfrac{1}{3}\pi r^2 h = \dfrac{1}{3}\pi\left(\dfrac{8a^2}{9}\right)\left(\dfrac{4a}{3}\right) = \dfrac{8}{27}\left(\dfrac{4}{3}\pi a^3\right)$

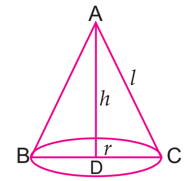
Thus, volume of the cone $= \dfrac{8}{27}$ (volume of the sphere).

20. Show that the right-circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ times the radius of the base. **[*CBSE (AI) 2011*] [*HOTS*]**

Sol. Let ABC be right-circular cone having radius 'r' and height 'h'. If V and S are its volume and surface area (curved) respectively, then

$$S = \pi r l$$

$$S = \pi r\sqrt{h^2 + r^2} \qquad \ldots(i)$$



Also, $V = \dfrac{1}{3}\pi r^2 h \quad \Rightarrow \quad h = \dfrac{3V}{\pi r^2}$

Putting the value of h in (i), we get

$$S = \pi r\sqrt{\frac{9V^2}{\pi^2 r^4} + r^2}$$

$\Rightarrow \quad S^2 = \pi^2 r^2\left(\dfrac{9V^2 + \pi^2 r^6}{\pi^2 r^4}\right)$ [Maxima or Minima is same for S or S^2]

$\Rightarrow \quad S^2 = \dfrac{9V^2}{r^2} + \pi^2 r^4$

Differentiating with respect to 'r', we get

$\Rightarrow \quad (S^2)' = \dfrac{-18V^2}{r^3} + 4\pi^2 r^3 \qquad \ldots(ii)$

Now, for max. or min. $(S^2)' = 0 \Rightarrow -18\dfrac{V^2}{r^3} + 4\pi^2 r^3 = 0 \Rightarrow 4\pi^2 r^6 = 18V^2$

Putting value of V

$\Rightarrow \quad 4\pi^2 r^6 = 18 \times \dfrac{1}{9}\pi^2 r^4 h^2 \Rightarrow 2r^2 = h^2 \Rightarrow r = \dfrac{h}{\sqrt{2}}$

Differentiating (ii) with respect to 'r', again

$$(S^2)'' = \frac{54V^2}{r^4} + 12\pi^2 r^2$$

$\Rightarrow \quad (S^2)''\Big]_{r=\frac{h}{\sqrt{2}}} > 0$ (For any value of r)

Hence, S^2 *i.e.*, S is minimum for $r = \dfrac{h}{\sqrt{2}}$ or $h = \sqrt{2}\,r$

i.e., for least curved surface, altitude is equal to $\sqrt{2}$ times the radius of the base.

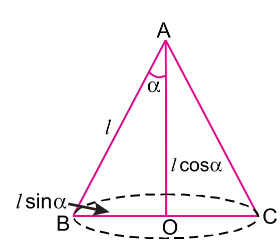
21. Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.

Sol. Let α be the semi-vertical angle of a cone and slant height is l.

then $AO = l\cos\alpha$ and $BO = l\sin\alpha$

$\Rightarrow \quad V = \dfrac{1}{3}\pi(l\sin\alpha)^2(l\cos\alpha) = \dfrac{1}{3}\pi l^3\sin^2\alpha\cos\alpha$

$\Rightarrow \quad \dfrac{dV}{d\alpha} = \dfrac{\pi}{3}l^3[-\sin^3\alpha + 2\sin\alpha\cos^2\alpha]$



For maximum or minimum volume V, we have

$\dfrac{dV}{d\alpha} = 0 \Rightarrow \dfrac{\pi}{3}l^3\sin\alpha(-\sin^2\alpha + 2\cos^2\alpha) = 0 \Rightarrow 2\cos^2\alpha - \sin^2\alpha = 0$

$\Rightarrow 2\cos^2\alpha = \sin^2\alpha \quad \Rightarrow \tan\alpha = \sqrt{2} \quad \Rightarrow \cos\alpha = \dfrac{1}{\sqrt{3}}$

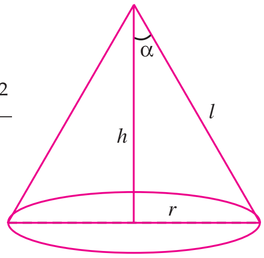
Now, $\dfrac{d^2V}{d\alpha^2} = \dfrac{1}{3}\pi l^3 [2\cos^3\alpha - 7\sin^2\alpha\cos\alpha] = \dfrac{1}{3}\pi l^3 \cos^3\alpha\,(2 - 7\tan^2\alpha)$

$$= \frac{1}{3}\pi l^3 \left(\frac{1}{\sqrt{3}}\right)^3 (2 - 7\times 2) < 0$$

Thus, V is maximum when $\tan\alpha = \sqrt{2}$, $\alpha = \tan^{-1}\sqrt{2}$.

22. Show that the semi-vertical angle of the right circular cone of given total surface area and maximum volume is $\sin^{-1}\dfrac{1}{3}$. **[*CBSE (AI) 2008*]**

Sol. Let r, h, l and α be the radius, height, slant height and semi-vertical angle of cone respectively. If V be the volume of cone then



$$V = \frac{1}{3}\pi r^2 h \quad \Rightarrow \quad V = \frac{1}{3}\pi r^2\sqrt{l^2 - r^2}$$

$\Rightarrow \quad V = \dfrac{1}{3}\pi r^2 \sqrt{\left(\dfrac{S - \pi r^2}{\pi r}\right)^2 - r^2}$, where Area $S = \pi r l + \pi r^2 \Rightarrow l = \dfrac{S - \pi r^2}{\pi r}$

$\Rightarrow \quad V = \dfrac{\pi r^2}{3} \times \dfrac{\sqrt{S^2 - 2S\pi r^2 + \pi^2 r^4 - \pi^2 r^4}}{\pi r} = \dfrac{r}{3}\times\sqrt{S^2 - 2S\pi r^2}$

$\Rightarrow \quad V^2 = \dfrac{r^2}{9}(S^2 - 2S\pi r^2) = \dfrac{(S^2 r^2 - 2S\pi r^4)}{9}$

Differentiating with respect to r, we get

$$\frac{d(V^2)}{dr} = \frac{1}{9}(2S^2 r - 8S\pi r^3)$$

For maxima or minima

$\dfrac{d(V^2)}{dr} = 0 \quad \Rightarrow \quad \dfrac{1}{9}(2S^2 r - 8S\pi r^3) = 0$

$\Rightarrow \quad 2S^2 r - 8S\pi r^3 = 0 \quad \Rightarrow \quad 2Sr\,(S - 4\pi r^2) = 0$

$\Rightarrow \quad S - 4\pi r^2 = 0 \quad \Rightarrow \quad 4\pi r^2 = S \quad \Rightarrow \quad r = \sqrt{\dfrac{S}{4\pi}}$

Again $\dfrac{d^2(V^2)}{dr^2} = \dfrac{1}{9}(2S^2 - 24S\pi r^2)$

$\therefore \quad \left.\dfrac{d^2(V^2)}{dr^2}\right]_{r=\sqrt{\frac{S}{4\pi}}} = \dfrac{1}{9}\left(2S^2 - 24S\pi.\dfrac{S}{4\pi}\right) = \dfrac{1}{9}(2S^2 - 6S^2) < 0$

Hence, for $r = \sqrt{\dfrac{S}{4\pi}}$, V^2 is maximum *i.e.*, V is maximum.

$\Rightarrow$ For $S = 4\pi r^2$, V is maximum.

Now, since $S = \pi r l + \pi r^2 \quad \Rightarrow \quad 4\pi r^2 = \pi r l + \pi r^2$ [For maximum volume $S = 4\pi r^2$]

$\Rightarrow \quad 3\pi r^2 = \pi r l \quad \Rightarrow \quad \dfrac{\pi r^2}{\pi r l} = \dfrac{1}{3}$

$\Rightarrow \quad \dfrac{r}{l} = \dfrac{1}{3} \quad \Rightarrow \quad \sin\alpha = \dfrac{1}{3} \quad \left[\because \sin\alpha = \dfrac{r}{l}\right]$

$\Rightarrow \quad \alpha = \sin^{-1}\left(\dfrac{1}{3}\right)$

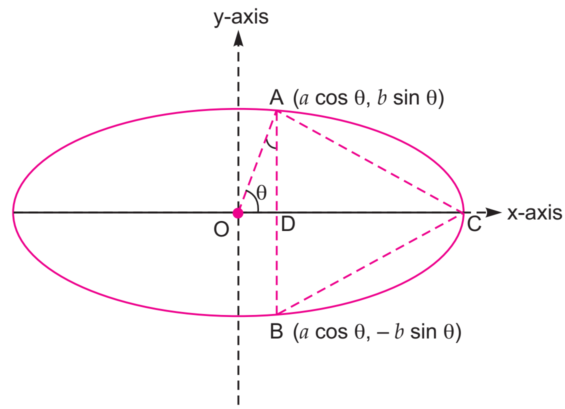
Hence, for maximum volume α (semi-vertical angle) $= \sin^{-1}\left(\dfrac{1}{3}\right)$

23. Find the maximum area of the isosceles triangle inscribed in the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ with its vertex at one end of major axis. *[CBSE Bhubaneshwar 2015, (AI) 2008]*

Sol. Let ΔABC be an isosceles triangle inscribed in the ellipse.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Such that 'C' lies on end of major axis and $AC = BC$.



Let coordinates of A and B be $(a\cos\theta, b\sin\theta)$ and $(a\cos\theta, -b\sin\theta)$ respectively.

If 'A' be the area of inscribed triangle then

$$A=\frac{1}{2}\times AB\times CD=\frac{1}{2}\times 2b\sin\theta\times(a-a\cos\theta)$$

$$= ab\sin\theta\,(1-\cos\theta)$$

Differentiating with respect to θ, we get

$$\frac{dA}{d\theta}=ab[\sin\theta.\sin\theta+(1-\cos\theta).\cos\theta]=ab(\sin^2\theta+\cos\theta-\cos^2\theta)$$

For maxima and minima $\dfrac{dA}{d\theta}=0$

$\Rightarrow \quad ab(\sin^2\theta+\cos\theta-\cos^2\theta)=0$

$\Rightarrow \quad \cos\theta-\cos 2\theta=0$

$\Rightarrow \quad \cos 2\theta=\cos\theta$

$\Rightarrow \quad 2\theta=2n\pi\pm\theta \qquad [\because \cos\theta=\cos\alpha;\ \theta=2n\pi\pm\alpha]$

$\Rightarrow \quad \theta=n\pi+\dfrac{\theta}{2}$ or $n\pi-\dfrac{\theta}{2}$, where $n=0,\pm 1,\pm 2,\ldots$

$\Rightarrow \quad \theta=\dfrac{2\pi}{3}\in(0,\pi)$

$\therefore \quad \dfrac{d^2A}{d\theta^2}=ab(2\sin\theta.\cos\theta-\sin\theta+2\cos\theta.\sin\theta)=ab(2\sin 2\theta-\sin\theta)$

$$\left[\frac{d^2A}{d\theta^2}\right]_{\theta=\frac{2\pi}{3}}<0$$

Hence, for $\theta=\dfrac{2\pi}{3}$, A is maximum.

Hence, maximum area of triangle $A=ab\sin\dfrac{2\pi}{3}.\left(1-\cos\dfrac{2\pi}{3}\right)=ab\dfrac{\sqrt{3}}{2}\left(1+\dfrac{1}{2}\right)=\dfrac{3\sqrt{3}}{4}ab$ sq units.

24. A window is in the form of rectangle surmounted by a semi-circular opening. Total perimeter of the window is 10 m. What will be the dimensions of the whole opening to admit maximum light and air through the whole opening? *[CBSE 2006; (AI) 2017; (F) 2011, 2014]*

Sol. Let $ABCED$ be required window having length and width y. If A is the area of window. Then

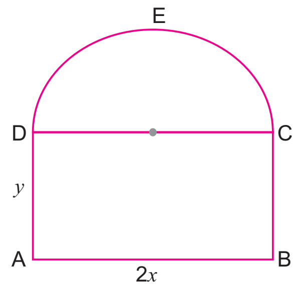
$$A=2xy+\frac{1}{2}\pi r^2$$

$\left[\text{Given perimeter} = 10 \Rightarrow 2x+y+y+\dfrac{1}{2}2\pi x=10 \quad \Rightarrow 2y=10-2x-\pi x\right]$

$$= x(10 - 2x - \pi x) + \frac{1}{2}\pi x^2 = 10x - 2x^2 - \pi x^2 + \frac{1}{2}\pi x^2$$

$$= 10x - 2x^2 - \frac{1}{2}\pi x^2 = 10x - \left(2 + \frac{1}{2}\pi\right)x^2$$

Obviously, window will admit maximum light and air if its area A is maximum.



Now, $\frac{dA}{dx} = 10 - 2x\left(2 + \frac{1}{2}\pi\right)$

For maxima or minima of A, $\frac{dA}{dx} = 0$.

$\Rightarrow \quad 10 - 2x\left(2 + \frac{1}{2}\pi\right) = 0 \quad \Rightarrow \quad 10 - x(4 + \pi) = 0$

$\Rightarrow \quad x = \frac{10}{4 + \pi}$ and $\frac{d^2A}{dx^2} = -(4 + \pi) < 0$

$\Rightarrow$ For maximum value of $A, x = \frac{10}{4+\pi}$ and thus $y = \frac{10}{4+\pi}$

Therefore, for maximum area, *i.e.*, for admitting maximum light and air,

Length $= 2x = \frac{20}{4+\pi}$ and width $= \frac{10}{4+\pi}$ of rectangular part of window.

Multiple Choice Questions [1 mark]

Choose and write the correct option in the following questions.

1. **The interval in which the function f given by $f(x) = x^2 e^{-x}$ is strictly increasing, is** *[CBSE 2020 (65/2/1)]*

(*a*) $(-\infty, \infty)$ (*b*) $(-\infty, 0)$ (*c*) $(2, \infty)$ (*d*) $(0, 2)$

2. **$y = x(x - 3)^2$ decreases for the values of x given by**

(*a*) $1 < x < 3$ (*b*) $x < 0$ (*c*) $x > 0$ (*d*) $0 < x < \frac{3}{2}$

3. **The abscissa of the point on the curve $3y = 6x - 5x^3$, the normal at which passes through origin is**

(*a*) 1 (*b*) $\frac{1}{3}$ (*c*) 2 (*d*) $\frac{1}{2}$

4. **The curve $y = x^{1/5}$ has at (0, 0)**

(*a*) a vertical tangent (parallel to y-axis) (*b*) a horizontal tangent (parallel to x-axis)
(*c*) an oblique tangent (*d*) no tangent

5. **The equation of normal to the curve $3x^2 - y^2 = 8$ which is parallel to the line $x + 3y = 8$ is**

(*a*) $3x - y = 8$ (*b*) $3x + y + 8 = 0$ (*c*) $x + 3y \pm 8 = 0$ (*d*) $x + 3y = 0$

6. **The tangent to the curve $y = e^{2x}$ at the point (0, 1) meets x-axis at**

(*a*) (0, 1) (*b*) $\left(-\frac{1}{2}, 0\right)$ (*c*) (2, 0) (d) (0, 2)

7. **$f(x) = x^x$ has a stationary point at**

(*a*) $x = e$ (*b*) $x = \frac{1}{e}$ (*c*) $x = 1$ (*d*) $x = \sqrt{e}$

8. **The maximum value of $\left(\frac{1}{x}\right)^x$ is**

(*a*) e (*b*) e^e (*c*) $e^{1/e}$ (*d*) $\left(\frac{1}{e}\right)^{1/e}$

9. **The two curves $x^3 - 3xy^2 + 2 = 0$ and $3x^2y - y^3 = 2$** *[NCERT Exemplar]*

(*a*) touch each other (*b*) cut at right angle (*c*) cut at an angle $\frac{\pi}{3}$ (*d*) cut at an angle $\frac{\pi}{4}$

10. **The tangent to the curve given by $x = e^t.\cos t$, $y = e^t.\sin t$ at $t = \frac{\pi}{4}$ makes with x-axis an angle**

(*a*) 0 (*b*) $\frac{\pi}{4}$ (*c*) $\frac{\pi}{3}$ (*d*) $\frac{\pi}{2}$

11. **The slope of normal to the curve $y = 2x^2 + 3\sin x$ at $x = 0$ is**

(*a*) 3 (*b*) $\frac{1}{3}$ (*c*) –3 (*d*) $-\frac{1}{3}$

12. **The equation of the normal to the curve $y = \sin x$ at (0, 0) is**

(*a*) $x = 0$ (*b*) $y = 0$ (*c*) $x + y = 0$ (*d*) $x - y = 0$

13. **The point on the curve $y^2 = x$, where the tangent makes an angle of $\frac{\pi}{4}$ with x-axis is** *[NCERT Exemplar]*

(*a*) $\left(\frac{1}{2}, \frac{1}{4}\right)$ (*b*) $\left(\frac{1}{4}, \frac{1}{2}\right)$ (*c*) (4, 2) (*d*) (1, 1)

14. **The point on the curve $x^2 = 2y$ which is nearest to the point (0, 5) is**

(*a*) $(2\sqrt{2}, 4)$ (*b*) $(2\sqrt{2}, 0)$ (*c*) (0, 0) (*d*) (2, 2)

15. **The maximum value of $[x(x - 1) + 1]^{1/3}$, $0 \le x \le 1$ is**

(*a*) $\left(\frac{1}{3}\right)^{1/3}$ (*b*) $\frac{1}{2}$ (*c*) 1 (*d*) 0

16. **The line $y = x + 1$ is a tangent to the curve $y^2 = 4x$ at the point**

(*a*) (1, 2) (*b*) (2, 1) (*c*) (1, – 2) (*d*) (–1, 2)

17. **A ladder, 5 meter long, standing on a horizontal floor, leans against a vertical wall. If the top of the ladder slides downwards at the rate of 10 cm/sec, then the rate at which the angle between the floor and the ladder is decreasing when lower end of ladder is 2 metres from the wall is** *[NCERT Exemplar]*

(*a*) $\frac{1}{10}$ radian/sec (*b*) $\frac{1}{20}$ radian/sec (*c*) 20 radian/sec (*d*) 10 radian/sec

18. **The rate of change of the area of a circle with respect to its radius r at $r = 6$ cm is**

(*a*) 10π (*b*) 12π (*c*) 8π (*d*) 11π

19. **The total revenue in rupees received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$. The marginal revenue, when $x = 15$ is**

(*a*) 116 (*b*) 96 (*c*) 90 (*d*) 126

20. **If x is real, the minimum value of $x^2 - 8x + 17$ is** *[NCERT Exemplar]*

(*a*) – 1 (*b*) 0 (*c*) 1 (*d*) 2

Answers

1. (*d*)	**2.** (*a*)	**3.** (*a*)	**4.** (*a*)	**5.** (*c*)	**6.** (*b*)
7. (*b*)	**8.** (*c*)	**9.** (*b*)	**10.** (*d*)	**11.** (*d*)	**12.** (*c*)
13. (*b*)	**14.** (*a*)	**15.** (*c*)	**16.** (*a*)	**17.** (*b*)	**18.** (*b*)
19. (*d*)	**20.** (*c*)				

Solutions of Selected Multiple Choice

1. We have, $f(x) = x^2 e^{-x}$

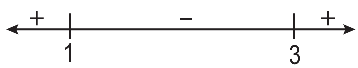
$\Rightarrow \quad f'(x) = -x^2 e^{-x} + 2x e^{-x} = x e^{-x}(2 - x)$

for $f(x)$ to be strictly increasing, $f'(x) > 0$

$$\Rightarrow \quad x e^{-x}(2 - x) > 0$$
$$\Rightarrow \quad x(2 - x) > 0$$
$$\Rightarrow \quad x(x - 2) < 0$$
$$\Rightarrow \quad 0 < x < 2$$
$$\therefore \quad x \in (0, 2)$$

2. We have, $\quad y = x(x - 3)^2$

$$\frac{dy}{dx} = x.2(x - 3).1 + (x - 3)^2.1$$
$$= 2x^2 - 6x + x^2 + 9 - 6x = 3x^2 - 12x + 9$$
$$= 3(x^2 - 3x - x + 3) = 3(x - 3)(x - 1)$$



So, $y = x(x - 3)^2$ decreases for $(1, 3)$.

[Since, $y' < 0$ for all $x \in (1, 3)$, hence y is decreasing on $(1, 3)$]

3. Let (x_1, y_1) be the point on the given curve $3y = 6x - 5x^3$ at which the normal passes through the origin. Then we have $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 2 - 5x_1^2$. Again the equation of the normal at (x_1, y_1) passing through the origin gives $2 - 5x_1^2 = \frac{-x_1}{y_1} = \frac{-3}{6 - 5x_1^2}$. Since $x_1 = 1$ satisfies the equation, therefore, correct answer is (*a*).

4. We have, $\quad y = x^{1/5}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{5}x^{\frac{1}{5}-1} = \frac{1}{5}x^{-4/5}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{(0, 0)} = \frac{1}{5} \times (0)^{-4/5} = \infty$$

So, the curve $y = x^{1/5}$ has a vertical tangent at $(0, 0)$, which is parallel to y-axis.

7. We have, $\quad f(x) = x^x$

Let $\quad y = x^x$

and $\quad \log y = x \log x$

$$\therefore \quad \frac{1}{y}.\frac{dy}{dx} = x.\frac{1}{x} + \log x.1 \qquad \text{[Differentiate both sides]}$$

$$\Rightarrow \quad \frac{dy}{dx} = (1 + \log x).x^x$$

$$\because \quad \frac{dy}{dx} = 0 \quad \Rightarrow \quad (1 + \log x).\, x^x = 0$$

$$\Rightarrow \quad \log x = -1 \quad \Rightarrow \quad \log x = \log e^{-1}$$

$$\Rightarrow \quad x = e^{-1} \quad \Rightarrow \quad x = \frac{1}{e}$$

Hence, $f(x)$ has a stationary point at $x = \frac{1}{e}$.

8. Let $\quad y = \left(\frac{1}{x}\right)^x$

$$\Rightarrow \quad \log y = x.\log\frac{1}{x}$$

$$\Rightarrow \qquad \frac{1}{y}\cdot\frac{dy}{dx} = x\cdot\frac{1}{\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right) + \log\frac{1}{x}\cdot 1 = -1 + \log\frac{1}{x} \qquad \text{[Differentiate both sides]}$$

$$\Rightarrow \qquad \frac{dy}{dx} = \left(\log\frac{1}{x} - 1\right)\cdot\left(\frac{1}{x}\right)^x$$

Now, $\frac{dy}{dx} = 0 \quad \Rightarrow \quad \log\frac{1}{x} = 1 = \log e$

$$\Rightarrow \qquad \frac{1}{x} = e$$

$$\therefore \qquad x = \frac{1}{e}$$

Hence, the maximum value of $f\left(\frac{1}{e}\right) = (e)^{1/e}$.

9. From first equation of the curve, we have $3x^2 - 3y^2 - 6xy\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{x^2 - y^2}{2xy} = (m_1)$ say and second equation of the curve gives

$$6xy + 3x^2\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 0 \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{-2xy}{x^2 - y^2} = (m_2)\text{say}$$

Since $m_1 \,.\, m_2 = -1$. Therefore, correct answer is (*b*).

12. $\frac{dy}{dx} = \cos x$. Therefore, slope of normal $= \left(\frac{-1}{\cos x}\right)_{x=0} = -1$.

Hence, the equation of normal is $y - 0 = -1\,(x - 0)$ or $x + y = 0$.

Therefore, correct answer is (*c*).

Fill in the Blanks [1 mark]

1. If the radius of the circle is increasing at the rate of 0.5 cm/s, then the rate of increase of its circumference is ______________ . [CBSE 2020 (65/2/1)]
2. The values of a for which the function $f(x) = \sin x - ax + b$ increases on R are ______________ .
3. The rate of change of volume of a sphere with respect to its surface area, when radius is 2 cm, is ______________ .
4. The equation of the tangent to the curve $y = \sec x$ at the point (0, 1) is ______________ . [CBSE 2020 (65/3/1)]
5. The least value of the function $f(x) = ax + \frac{b}{x}$, $(a > 0, b > 0, x > 0)$ is ______________ .

Answers

1. π cm/sec. **2.** $(-\infty, -1)$ **3.** $1\ \text{cm}^3/\text{cm}^2$ **4.** $y = 1$ **5.** $2\sqrt{ab}$

Solutions of Selected Fill in the Blanks

1. Given, $\frac{dr}{dt} = 0.5$ cm/s

$$\therefore \qquad \frac{dC}{dt} = \frac{d2\pi r}{dt} = 2\pi\frac{dr}{dt} = 2\pi \times 0.5\ \text{cm/s}$$

$$= \pi\ \text{cm/sec.}$$

4. We have, slope of the tangent to the curve $y = \sec x$ is given by

$$\frac{dy}{dx} = \sec x \,.\, \tan x$$

$\therefore$ Slope of tangent at (0, 1) = $\sec 0 \times \tan 0 = 0$

$\therefore$ Equation of tangent at the point (0, 1) be

$$\frac{y-1}{x-0} = 0 \quad \Rightarrow \quad y - 1 = 0 \quad \Rightarrow \quad y = 1$$

Very Short Answer Questions [1 mark]

1. Find the slope of the tangent to the curve $y = 2\sin^2(3x)$ at $x = \frac{\pi}{6}$. *[CBSE 2020 (65/5/1)]*

Sol. We have,

$$y = 2\sin^2(3x)$$

$$\Rightarrow \quad \frac{dy}{dx} = 2 \times 2\sin(3x) \times \cos(3x) \times 3$$

$$\Rightarrow \quad \frac{dy}{dx} = 6\sin(6x)$$

$\therefore$ Slope of tangent at $x = \frac{\pi}{6}$ is $6\sin\left(6 \times \frac{\pi}{6}\right)$

$$= 6\sin\pi = 6 \times 0 = 0$$

2. Find the rate of change of the area of a circle with respect to its radius 'r' when $r = 4$ cm. *[NCERT Exemplar]*

Sol. If A is area and r is the radius of a circle, then

$$A = \pi r^2 \qquad \Rightarrow \qquad \frac{dA}{dr} = 2\pi r$$

$$\therefore \quad \left[\frac{dA}{dr}\right]_{r=4} = 8\pi \text{cm}^2/\text{cm}$$

3. An edge of a variable cube is increasing at the rate of 5 cm per second. How fast is the volume increasing when the side is 15 cm?

Sol. Let x be the edge of the cube and V be the volume of the cube at any time t.

Given, $\frac{dx}{dt} = 5$ cm/s, $x = 15$ cm

Since we know the volume of cube = (side)^3 *i.e.*, $V = x^3$.

$$\Rightarrow \quad \frac{dV}{dt} = 3x^2 \cdot \frac{dx}{dt} \qquad \Rightarrow \quad \frac{dV}{dt} = 3 \cdot (15)^2 \times 5 = 3375 \text{ cm}^3/\text{sec}$$

4. Find the slope of the tangent to the curve $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$ at $t = 2$.

Sol. Slope of the tangent $= \dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{4t-2}{2t+3}$

$$\Rightarrow \quad \left.\frac{dy}{dx}\right|_{\text{at } t=2} = \left(\frac{4t-2}{2t+3}\right)_{\text{at } t=2} = \frac{4(2)-2}{2(2)+3} = \frac{6}{7}$$

5. If $y = \log_e x$, then find Δy when $x = 3$ and $\Delta x = 0.03$.

Sol. We have, $y = \log_e x$

$$\therefore \quad \Delta y = \frac{dy}{dx} \cdot \Delta x = \frac{1}{x} \cdot \Delta x = \frac{0.03}{3} = 0.01$$

Short Answer Questions-I [2 marks]

1. The money to be spent for the welfare of the employees of a firm is proportional to the rate of change of its total revenue (marginal revenue). If the total revenue (in rupees) received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$, find the marginal revenue when $x = 5$. [*CBSE (AI) 2013*]

Sol. Given: $R(x) = 3x^2 + 36x + 5$

$\Rightarrow \quad R'(x) = 6x + 36$

$\therefore$ Marginal revenue (when $x = 5$) $= R'(x)]_{x=5}$

$= 6 \times 5 + 36 = ₹\ 66.$

2. The amount of pollution content added in air in a city due to x-diesel vehicles is given by $P(x) = 0.005x^3 + 0.02x^2 + 30x$. Find the marginal increase in pollution content when 3 diesel vehicles are added. [*CBSE Delhi 2013*]

Sol. We have to find $[P'(x)]_{x=3}$

Now, $\quad P(x) = 0.005x^3 + 0.02x^2 + 30x$

$\therefore \quad P'(x) = 0.015x^2 + 0.04x + 30$

$\Rightarrow \quad [P'(x)]_{x=3} = 0.015 \times 9 + 0.04 \times 3 + 30$

$= 0.135 + 0.12 + 30 = 30.255$

3. If x and y are the sides of two squares such that $y = x - x^2$, then find the rate of change of the area of second square with respect to the area of first square. [*NCERT Exemplar*]

Sol. Since, x and y are the sides of two squares such that $y = x - x^2$.

$\therefore$ Area of the first square $(A_1) = x^2$

and area of the second square $(A_2) = y^2 = (x - x^2)^2$

$$\therefore \quad \frac{dA_2}{dt} = \frac{d}{dt}(x - x^2)^2 = 2(x - x^2)\left(\frac{dx}{dt} - 2x.\frac{dx}{dt}\right)$$

$$= \frac{dx}{dt}(1 - 2x)2(x - x^2)$$

and $$\frac{dA_1}{dt} = \frac{d}{dx}x^2 = 2x.\frac{dx}{dt}$$

$$\therefore \quad \frac{dA_2}{dA_1} = \frac{dA_2/dt}{dA_1/dt} = \frac{\frac{dx}{dt}.(1 - 2x)(2x - 2x^2)}{2x.\frac{dx}{dt}}$$

$$= \frac{(1 - 2x)\,2x(1 - x)}{2x}$$

$$= (1 - 2x)(1 - x) = 1 - x - 2x + 2x^2 = 2x^2 - 3x + 1$$

4. Using differentials, find the approximate value of $\sqrt{49.5}$. [*CBSE Delhi 2012*]

Sol. Let $\quad f(x) = \sqrt{x}$, where $x = 49$ and $\delta x = 0.5$

$\therefore \quad f(x + \delta x) = \sqrt{x + \delta x} = \sqrt{49.5}$

Now by definition, approximately we can write

$$f'(x) = \frac{f(x + \delta x) - f(x)}{\delta x} \qquad ...(i)$$

Here $\quad f(x) = \sqrt{x} = \sqrt{49} = 7$ and $\delta x = 0.5$

$$\Rightarrow \quad f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{49}} = \frac{1}{14}$$

Putting these values in (*i*), we get

$$\frac{1}{14} = \frac{\sqrt{49.5} - 7}{0.5}$$

$$\sqrt{49.5} = \frac{0.5}{14} + 7 = \frac{0.5 + 98}{14} = \frac{98.5}{14} = 7.036$$

5. Show that the function f given by $f(x) = \tan^{-1}(\sin x + \cos x)$ is decreasing for all $\in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

[*CBSE (F) 2017*]

Sol. We have

$$f(x) = \tan^{-1}(\sin x + \cos x)$$

$$\Rightarrow \quad f'(x) = \frac{1}{1 + (\sin x + \cos x)^2} \times (\cos x - \sin x)$$

$$f'(x) = \frac{\cos x - \sin x}{1 + (\sin x + \cos x)^2}$$

$$\because \quad 1 + (\sin x + \cos x)^2 > 0 \ \forall x \in R$$

Also, $\forall x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \sin x > \cos x \quad \Rightarrow \quad \cos x - \sin x < 0$

$\therefore \quad f'(x) = \frac{-ve}{+ve} = -ve \quad$ *i.e.,* $\quad f'(x) < 0$

$\Rightarrow \quad f(x)$ is decreasing in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

6. The volume of a cube is increasing at the rate of 9 cm^3/s. How fast is its surface area increasing when the length of an edge is 10 cm? **[*CBSE (AI) 2017*]**

Sol. Let V and S be the volume and surface area of a cube of side x cm respectively.

Given $\frac{dV}{dt} = 9 \text{ cm}^3/\text{sec}$

We require $\left.\frac{dS}{dt}\right]_{x = 10 \text{ cm}}$

Now $V = x^3$

$$\Rightarrow \quad \frac{dV}{dt} = 3x^2 . \frac{dx}{dt} \quad \Rightarrow \quad 9 = 3x^2 . \frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{9}{3x^2} = \frac{3}{x^2}$$

Again, $\because \ S = 6x^2$ [By formula for surface area of a cube]

$$\Rightarrow \quad \frac{dS}{dt} = 12.x.\frac{dx}{dt}$$

$$= 12x.\frac{3}{x^2} = \frac{36}{x}$$

$$\Rightarrow \quad \left.\frac{dS}{dt}\right]_{x = 10 \text{ cm}} = \frac{36}{10} = 3.6 \text{ cm}^2/\text{sec.}$$

7. Find the approximate change in the value of $\frac{1}{x^2}$, when x changes from $x = 2$ to $x = 2.002$.

[*CBSE Sample Paper 2018*]

Sol. Let $y = \frac{1}{x^2}$. Let δx be small change in x and δy the corresponding change in y.

Given $\delta x = 2.002 - 2 = 0.002$, where $x = 2$.

Now $y = \frac{1}{x^2} \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{2}{x^3} \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{2}{2^3} = -\frac{2}{8}$

We know that, approximately,

$$\delta y = \frac{dy}{dx}.\delta x \qquad \therefore \quad \delta y = -\frac{2}{8} \times 0.002 = -0.0005$$

8. Find whether the function $f(x) = \cos(2x + \frac{\pi}{4})$; is increasing or decreasing in the interval $\left(\frac{3\pi}{8}, \frac{7\pi}{8}\right)$. **[*CBSE 2019(65/5/3)*]**

Sol. We have $f(x) = \cos\left(2x + \frac{\pi}{4}\right)$

We know that function $f(x)$ is increasing in (a, b) if $f'(x) > 0 \ \forall\ x \in (a, b)$ & is decreasing if $f'(x) < 0 \ \forall\ x \in (a, b)$.

Now, $f'(x) = -2\sin\left(2x + \frac{\pi}{4}\right)$

$\therefore \quad f'(x) > 0 \quad \Rightarrow \quad -2\sin\left(2x + \frac{\pi}{4}\right) > 0$

$\Rightarrow \quad \sin\left(2x + \frac{\pi}{4}\right) < 0$

$\Rightarrow \quad \pi < 2x + \frac{\pi}{4} < 2\pi \qquad [\because \sin x < 0\ \forall\ x \in (\pi, 2\pi)]$

$\Rightarrow \quad \pi - \frac{\pi}{4} < 2x < 2\pi - \frac{\pi}{4} \quad \Rightarrow \quad \frac{3\pi}{4} < 2x < \frac{7\pi}{4}$

$\Rightarrow \quad \frac{3\pi}{8} < x < \frac{7\pi}{8}$

Thus $f'(x) > 0\ \forall\ x \in \left(\frac{3\pi}{8}, \frac{7\pi}{8}\right)$

$\Rightarrow f(x)$ is increasing function on $\left(\frac{3\pi}{8}, \frac{7\pi}{8}\right)$.

Short Answer Questions-II [3 marks]

1. Prove that the curves $xy = 4$ and $x^2 + y^2 = 8$ touch each other. **[*NCERT Exemplar*]**

Sol. Given equation of curves are

$$xy = 4 \qquad \text{... }(i)$$

and $$x^2 + y^2 = 8 \qquad \text{... }(ii)$$

$\Rightarrow \quad x.\frac{dy}{dx} + y = 0$

and $\quad 2x + 2y\frac{dy}{dx} = 0$

$\Rightarrow \quad \frac{dy}{dx} = \frac{-y}{x} \quad$ and $\quad \frac{dy}{dx} = \frac{-2x}{2y}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{-y}{x} = m_1 \qquad$ (say)

and $\quad \frac{dy}{dx} = \frac{-x}{y} = m_2 \qquad$ (say)

$\therefore$ Both the curves should have same slope.

$\therefore \qquad \dfrac{-y}{x} = \dfrac{-x}{y} \quad \Rightarrow \quad -y^2 = -x^2$ and $x^2 = y^2$ ... (*iii*)

Using the value of x^2 in equation (*ii*), we get

$$y^2 + y^2 = 8 \quad \Rightarrow \quad y^2 = 4 \Rightarrow y = \pm 2$$

For $\quad y = 2, x = \dfrac{4}{2} = 2$ and for $y = -2, x = \dfrac{4}{-2} = -2$

Thus, the required points of intersection are (2, 2) and (–2, –2)

For (2, 2), $\qquad m_1 = \dfrac{-y}{x} = \dfrac{-2}{2} = -1$

and $\qquad m_2 = \dfrac{-x}{y} = \dfrac{-2}{2} = -1$

$\because \qquad m_1 = m_2$

For (–2, – 2), $\qquad m_1 = \dfrac{-y}{x} = \dfrac{-(-2)}{-2} = -1$

$\because \qquad m_2 = \dfrac{-x}{y} = \dfrac{-(-2)}{-2} = -1$

For both the intersection points, we see that slope of both curve are same.

2. Find the intervals in which the function $f(x) = \dfrac{3}{2}x^4 - 4x^3 - 45x^2 + 51$ is
(*a*) strictly increasing (*b*) strictly decreasing. **[*CBSE (F) 2014*]**

Sol. Here, $\quad f(x) = \dfrac{3}{2}x^4 - 4x^3 - 45x^2 + 51$

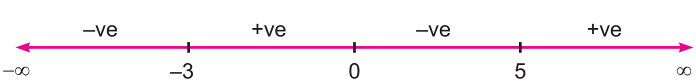
$\Rightarrow \quad f'(x) = 6x^3 - 12x^2 - 90x \quad \Rightarrow \quad f'(x) = 6x(x^2 - 2x - 15) = 6x(x+3)(x-5)$

Now for critical point $f'(x) = 0$

$$6x(x+3)(x-5) = 0 \quad \Rightarrow \quad x = 0, -3, 5$$

i.e., –3, 0, 5 are critical points which divides domain R of given function into four disjoint sub intervals $(-\infty, -3)$, $(-3, 0)$, $(0, 5)$, $(5, \infty)$.

For $(-\infty, -3)$



$f'(x) = +\text{ve} \times (-\text{ve}) \times (-\text{ve}) \times (-\text{ve}) = -\text{ve}$

i.e., $f(x)$ is decreasing in $(-\infty, -3)$.

For $(-3, 0)$

$f'(x) = +\text{ve} \times (-\text{ve}) \times (+\text{ve}) \times (-\text{ve}) = +\text{ve}$

i.e., $f(x)$ is increasing in $(-3, 0)$.

For $(0, 5)$

$f'(x) = +\text{ve} \times (+\text{ve}) \times (+\text{ve}) \times (-\text{ve}) = -\text{ve}$

i.e., $f(x)$ is decreasing in $(0, 5)$.

For $(5, \infty)$

$f'(x) = +\text{ve} \times (+\text{ve}) \times (+\text{ve}) \times (+\text{ve}) = +\text{ve}$

i.e., $f(x)$ is increasing in $(5, \infty)$.

Hence, $f(x)$ is (*a*) strictly increasing in $(-3, 0) \cup (5, \infty)$

(*b*) strictly decreasing in $(-\infty, -3) \cup (0, 5)$.

3. Find the intervals in which $f(x) = \sin 3x - \cos 3x, 0 < x < \pi$, is strictly increasing or strictly decreasing. **[*CBSE Delhi 2016*]**

Sol. Given function is

$$f(x) = \sin 3x - \cos 3x$$

$$f'(x) = 3\cos 3x + 3\sin 3x$$

For critical points of function $f(x)$

$$f'(x) = 0$$

$\Rightarrow \quad 3\cos 3x + 3\sin 3x = 0 \quad \Rightarrow \quad \cos 3x + \sin 3x = 0$

$\Rightarrow \quad \sin 3x = -\cos 3x \quad \Rightarrow \quad \dfrac{\sin 3x}{\cos 3x} = -1$

$\Rightarrow \quad \tan 3x = -\tan\dfrac{\pi}{4} \quad \Rightarrow \quad \tan 3x = \tan\left(\pi - \dfrac{\pi}{4}\right)$

$\Rightarrow \quad \tan 3x = \tan\dfrac{3\pi}{4}$

$\Rightarrow \quad 3x = n\pi + \dfrac{3\pi}{4}$, where $n = 0, \pm 1, \pm 2, \ldots.$

Putting $n = 0, \pm 1, \pm 2, \ldots.$, we get

$x = \dfrac{\pi}{4}, \dfrac{7\pi}{12}, \dfrac{11\pi}{12} \in (0, \pi)$

Hence, required possible intervals are $\left(0, \dfrac{\pi}{4}\right), \left(\dfrac{\pi}{4}, \dfrac{7\pi}{12}\right) \left(\dfrac{7\pi}{12}, \dfrac{11\pi}{12}\right) \left(\dfrac{11\pi}{12}, \pi\right)$

For $\left(0, \dfrac{\pi}{4}\right), f'(x) = +\text{ve}$

For $\left(\dfrac{\pi}{4}, \dfrac{7\pi}{12}\right), f'(x) = -\text{ve}$

For $\left(\dfrac{7\pi}{12}, \dfrac{11\pi}{12}\right), f'(x) = +\text{ve}$

For $\left(\dfrac{11\pi}{12}, \pi\right), f'(x) = -\text{ve}$

Hence, given function $f(x)$ is strictly increasing in $\left(0, \dfrac{\pi}{4}\right) \cup \left(\dfrac{7\pi}{12}, \dfrac{11\pi}{12}\right)$ and strictly decreasing in $\left(\dfrac{\pi}{4}, \dfrac{7\pi}{12}\right) \cup \left(\dfrac{11\pi}{12}, \pi\right)$.

4. Find the equation of the normal at the point (am^2, am^3) for the curve $ay^2 = x^3$. **[*CBSE (F) 2012; (South) 2016*]**

Sol. Given, curve $\quad ay^2 = x^3$

On differentiating, we get

$$2ay\frac{dy}{dx} = 3x^2 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{3x^2}{2ay}$$

$$\Rightarrow \quad \left.\frac{dy}{dx}\right|_{\text{at }(am^2, am^3)} = \frac{3 \times a^2m^4}{2a \times am^3} = \frac{3m}{2}$$

$$\therefore \quad \text{Slope of normal} = -\frac{1}{\text{slope of tangent}} = -\frac{1}{\frac{3m}{2}} = -\frac{2}{3m}$$

Equation of normal at the point (am^2, am^3) is given by

$$\frac{y - am^3}{x - am^2} = -\frac{2}{3m} \quad \Rightarrow \quad 3my - 3am^4 = -2x + 2am^2$$

$\Rightarrow \quad 2x + 3my - am^2(2 + 3m^2) = 0$

Hence, equation of normal is $2x + 3my - am^2(2 + 3m^2) = 0$

5. Show that $y = \log(1+x) - \frac{2x}{2+x}, x > -1$ is an increasing function of x throughout its domain. **[*CBSE (F) 2012*]**

Sol. Here, $f(x) = \log(1+x) - \frac{2x}{2+x}$ [where $y = f(x)$]

$$\Rightarrow \quad f'(x) = \frac{1}{1+x} - 2\left[\frac{(2+x).1 - x}{(2+x)^2}\right]$$

$$= \frac{1}{1+x} - \frac{2(2+x-x)}{(2+x)^2} = \frac{1}{1+x} - \frac{4}{(2+x)^2}$$

$$= \frac{4+x^2+4x-4-4x}{(x+1)(x+2)^2} = \frac{x^2}{(x+1)(x+2)^2}$$

For $f(x)$ being increasing function

$f'(x) > 0$

$$\Rightarrow \quad \frac{x^2}{(x+1)(x+2)^2} > 0 \qquad \Rightarrow \qquad \frac{1}{x+1}.\frac{x^2}{(x+2)^2} > 0$$

$$\Rightarrow \quad \frac{1}{x+1} > 0 \qquad \left[\frac{x^2}{(x+2)^2} > 0\right]$$

$\Rightarrow \quad x + 1 > 0 \quad$ or $\quad x > -1$

i.e., $f(x) = y = \log(1+x) - \frac{2x}{2+x}$ is increasing function in its domain $x > -1$ *i.e.*, $(-1, \infty)$.

6. Show that $f(x) = 2x + \cot^{-1}x + \log\left(\sqrt{1+x^2} - x\right)$ is increasing in R. **[*NCERT Exemplar*]**

Sol. We have, $f(x) = 2x + \cot^{-1}x + \log\left(\sqrt{1+x^2} - x\right)$

$$f'(x) = 2 + \left(\frac{-1}{1+x^2}\right) + \frac{1}{\left(\sqrt{1+x^2} - x\right)}\left(\frac{1}{2\sqrt{1+x^2}}.2x - 1\right)$$

$$= 2 - \frac{1}{1+x^2} + \frac{1}{\left(\sqrt{1+x^2} - x\right)}.\frac{\left(x - \sqrt{1+x^2}\right)}{\sqrt{1+x^2}} = 2 - \frac{1}{1+x^2} - \frac{1}{\sqrt{1+x^2}}$$

$$= \frac{2+2x^2-1-\sqrt{1+x^2}}{1+x^2} = \frac{1+2x^2-\sqrt{1+x^2}}{1+x^2}$$

For increasing function, $f'(x) \geq 0$

$$\frac{1+2x^2-\sqrt{1+x^2}}{1+x^2} \geq 0 \quad \Rightarrow \quad 1 + 2x^2 \geq \sqrt{1+x^2}$$

$\Rightarrow \quad (1+2x^2)^2 \geq 1 + x^2 \qquad \Rightarrow \quad 1 + 4x^4 + 4x^2 \geq 1 + x^2$

$\Rightarrow \quad 4x^4 + 3x^2 \geq 0 \qquad \Rightarrow \quad x^2(4x^2 + 3) \geq 0$

It is true for any real value of x.

Hence, $f(x)$ is increasing in R.

7. Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to the y-coordinate of the point. **[*CBSE Delhi 2010; (F) 2011*]**

Sol. Let $P(x_1, y_1)$ be the required point on the curve

$$y = x^3 \qquad \ldots(i)$$

$$\Rightarrow \quad \frac{dy}{dx} = 3x^2 \qquad \Rightarrow \qquad \left[\frac{dy}{dx}\right]_{(x_1,y_1)} = 3x_1^2$$

$\Rightarrow$ Slope of tangent at $(x_1, y_1) = 3x_1^2$

According to the question,

$$3x_1^2 = y_1 \qquad \ldots(ii)$$

Also (x_1, y_1) lies on (i)

$$\Rightarrow \qquad y_1 = x_1^3 \qquad \ldots(iii)$$

From (ii) and (iii), we get

$$3x_1^2 = x_1^3$$

$$\Rightarrow \quad x_1^3 - 3x_1^2 = 0 \qquad \Rightarrow \quad x_1^2(x_1 - 3) = 0$$

$$\Rightarrow \quad x_1 = 0 \quad \text{or} \quad x_1 = 3$$

$$\Rightarrow \quad y_1 = 0 \quad \text{or} \quad y_1 = 27$$

Hence, required points are (0, 0) and (3, 27).

8. Find the intervals in which the function $f(x) = -3\log(1 + x) + 4\log(2 + x) - \dfrac{4}{2+x}$ is strictly increasing or strictly decreasing. **[*CBSE Sample Paper 2018*]**

Sol. Given $f(x) = -3\log(1 + x) + 4\log(2 + x) - \dfrac{4}{2+x}$

$$\Rightarrow \quad f'(x) = \frac{-3}{1+x} + \frac{4}{2+x} + \frac{4}{(2+x)^2} = \frac{-3(2+x)^2 + 4(1+x)(2+x) + 4(1+x)}{(1+x)(2+x)^2}$$

$$= \frac{-3(4+4x+x^2) + 4(2+x+2x+x^2) + 4 + 4x}{(1+x)(2+x)^2}$$

$$= \frac{-12 - 12x - 3x^2 + 8 + 12x + 4x^2 + 4 + 4x}{(1+x)(2+x)^2}$$

$$f'(x) = \frac{x(x+4)}{(1+x)(2+x)^2}$$

Now, $f'(x) = 0$

$$\Rightarrow \quad \frac{x(x+4)}{(1+x)(2+x)^2} = 0 \qquad \Rightarrow \quad x(x+4) = 0$$

$$\Rightarrow \quad x = 0 \qquad [\because x \neq -4 \text{ as } f(x) \text{ is defined on } (-1, \infty)]$$

Hence, required intervals are $(-1, 0)$ and $(0, \infty)$.

For $(-1, 0)$

$$f'(x) = \frac{(-\text{ve}) \times (+\text{ve})}{(+\text{ve}) \times (+\text{ve})} = -\text{ve} \qquad \Rightarrow \quad f(x) \text{ is strictly decreasing in } (-1, 0)$$

For $(0, \infty)$

$$f'(x) = \frac{(+\text{ve}) \times (+\text{ve})}{(+\text{ve}) \times (+\text{ve})} = +\text{ve} \qquad \Rightarrow \quad f(x) \text{ is strictly increasing in } (0, \infty)$$

i.e., $f(x)$ is strictly decreasing on $(-1, 0)$ and strictly increasing on $(0, \infty)$.

9. Find the condition that curves $2x = y^2$ and $2xy = k$ intersect orthogonally. **[*NCERT Exemplar*]**

Sol. Given, equation of curves are $\quad 2x = y^2 \qquad \ldots(i)$

and $\quad 2xy = k \qquad \ldots(ii)$

$$\Rightarrow \qquad y = \frac{k}{2x}$$

From equation (i) $\quad 2x = \left(\dfrac{k}{2x}\right)^2$

$$\Rightarrow \qquad 8x^3 = k^2$$

$$\Rightarrow \quad x^3 = \frac{1}{8}k^2 \quad \Rightarrow \quad x = \frac{1}{2}k^{2/3}$$

$$\Rightarrow \quad y = \frac{k}{2x} = \frac{1}{2.\frac{1}{2}k^{2/3}}k \quad \Rightarrow \quad y = k^{(1/3)}$$

Thus, we get point of intersection of curves which is from equations (i) and (ii). $\left(\left(\frac{1}{2}\right)k^{(2/3)}, k^{(1/3)}\right)$

$$2 = 2y\frac{dy}{dx} \text{ and } 2\left[x.\frac{dy}{dx} + y.1\right] = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{y} \text{ and } \left(\frac{dy}{dx}\right) = \frac{-2y}{2x} = -\frac{y}{x}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)\left(\frac{1}{2}k^{2/3}, k^{1/3}\right) = \frac{1}{k^{1/3}} \qquad \text{[say } m_1\text{]}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)\left(\frac{1}{2}k^{2/3}, k^{1/3}\right) = \frac{-k^{1/3}}{\frac{1}{2}k^{2/3}} = -2k^{-1/3} \qquad \text{[say } m_2\text{]}$$

Since, the curves intersect orthogonally.

i.e., $\quad m_1.m_2 = -1$

$$\Rightarrow \quad \frac{1}{k^{1/3}}.(-2k^{-1/3}) = -1 \quad \Rightarrow \quad -2k^{-2/3} = -1$$

$$\Rightarrow \quad \frac{2}{k^{2/3}} = 1 \quad \Rightarrow \quad k^{2/3} = 2$$

$$\therefore \quad k^2 = 8$$

which is the required condition.

Long Answer Questions [5 marks]

1. Find the minimum value of ($ax + by$), where $xy = c^2$. ***[CBSE Delhi 2015, 2020 (65/5/1)]***

Sol. Let $z = ax + by$ ...(i)

Given $xy = c^2 \quad \Rightarrow \quad y = \frac{c^2}{x}$

Putting $y = \frac{c^2}{x}$ in equation (i), we have

$$z = ax + \frac{bc^2}{x}$$

For z to be maximum or, minimum

$$\frac{dz}{dx} = a - \frac{bc^2}{x^2} = 0 \quad \Rightarrow \quad a = \frac{bc^2}{x^2}$$

$$\Rightarrow \quad x^2 = \frac{bc^2}{a} \quad \Rightarrow \quad x = \pm c\sqrt{\frac{b}{a}}$$

Now, $\quad \frac{d^2z}{dx^2} = \frac{2bc^2}{x^3}$

$$\therefore \quad \text{at } x = c\sqrt{\frac{b}{a}}, \frac{d^2z}{dx^2} = \frac{2bc^2}{\left(c\sqrt{\frac{b}{a}}\right)^3} > 0$$

$\therefore$ z will be minimum at $x = c\sqrt{\frac{b}{a}}$

$\therefore$ $y = \frac{c^2}{x} = \frac{c^2}{c\sqrt{\frac{b}{a}}} = c\sqrt{\frac{a}{b}}$

$\therefore$ Minimum value of $z = ax + by$

$$= a \times c\sqrt{\frac{b}{a}} + b \times c\sqrt{\frac{a}{b}}$$

$$= c\sqrt{ab} + c\sqrt{ab}$$

$$= 2c\sqrt{ab}$$

2. Of all the closed right circular cylindrical cans of volume 128π cm^3, find the dimensions of the can which has minimum surface area. **[*CBSE Delhi 2014*]**

Sol. Let r, h be radius and height of closed right circular cylinder having volume 128π cm^3.

If S be the surface area then

$$S = 2\pi rh + 2\pi r^2 \quad \Rightarrow \quad S = 2\pi(rh + r^2)$$

$$\left[\begin{array}{l} \because \quad V = \pi r^2 h \\ \Rightarrow 128\,\pi = \pi r^2 h \\ \therefore \quad h = \frac{128}{r^2} \end{array}\right]$$

$$S = 2\pi\left(r.\frac{128}{r^2} + r^2\right)$$

$$S = 2\pi\left(\frac{128}{r} + r^2\right) \quad \Rightarrow \quad \frac{dS}{dr} = 2\pi\left(-\frac{128}{r^2} + 2r\right)$$

For extreme value of S

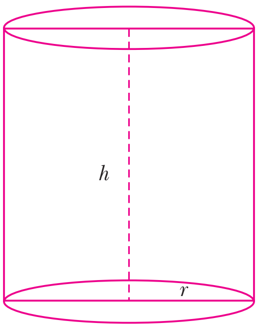
$$\frac{dS}{dr} = 0 \quad \Rightarrow \quad 2\pi\left(-\frac{128}{r^2} + 2r\right) = 0$$

$$\Rightarrow \quad -\frac{128}{r^2} + 2r = 0$$

$$\Rightarrow \quad 2r = \frac{128}{r^2} \quad \Rightarrow \quad r^3 = \frac{128}{2}$$

$$\Rightarrow \quad r^3 = 64 \quad \Rightarrow \quad r = 4$$

Again $\frac{d^2S}{dr^2} = 2\pi\left(\frac{128 \times 2}{r^3} + 2\right) \quad \Rightarrow \quad \left.\frac{d^2S}{dr^2}\right]_{r=4} = +\text{ ve}$



Hence, for $r = 4$ cm, S (surface area) is minimum.

Therefore, dimensions for minimum surface area of cylindrical can are

radius $r = 4$ cm and $h = \frac{128}{r^2} = \frac{128}{16} = 8$ cm.

3. Prove that the surface area of a solid cuboid, of square base and given volume, is minimum when it is a cube. **[*CBSE (AI) 2017; (F) 2009; CBSE 2005*]**

Sol. Let x be the side of square base of cuboid and other side be y.

Then volume of cuboid with square base, $V = x\,.\,x\,.\,y = x^2y$

As volume of cuboid is given so volume is taken constant throughout the question, therefore,

$$y = \frac{V}{x^2} \qquad ...(i)$$

In order to show that surface area is minimum when the given cuboid is cube, we have to show $S'' > 0$ and $x = y$.

Let S be the surface area of cuboid, then

$$S = x^2 + xy + xy + xy + xy + x^2 = 2x^2 + 4xy \qquad ...(ii)$$

$$= 2x^2 + 4x.\frac{V}{x^2} \quad \Rightarrow \quad S = 2x^2 + \frac{4V}{x} \quad ...(iii)$$

$$\Rightarrow \quad \frac{dS}{dx} = 4x - \frac{4V}{x^2} \quad ...(iv)$$

For maximum/minimum value of S, we have $\frac{dS}{dx} = 0$

$$\Rightarrow \quad 4x - \frac{4V}{x^2} = 0 \Rightarrow 4V = 4x^3 \quad \Rightarrow \quad V = x^3 \quad ...(v)$$

Putting $V = x^3$ in (i), we have

$$y = \frac{x^3}{x^2} = x$$

Here, $y = x \Rightarrow$ cuboid is a cube.

Differentiating (iv) w.r.t x, we get

$$\frac{d^2S}{dx^2} = \left(4 + \frac{8V}{x^3}\right) > 0$$

Hence, surface area is minimum when given cuboid is a cube.

4. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume. **[*CBSE 2019 (65/2/1)*]**

Sol. Let x be radius and $(y + R)$ be the height of cylinder given radius of sphere be R.

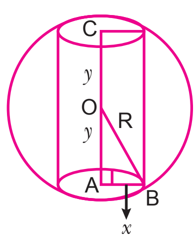
In ΔOAB, we have,

$$OB^2 = OA^2 + AB^2$$

$$\Rightarrow \quad R^2 = y^2 + x^2 \quad \Rightarrow \quad x^2 + y^2 = R^2 \quad \Rightarrow \quad x^2 = R^2 - y^2 \quad ...(i)$$

Now, volume of cylinder = $\pi x^2 \times 2y$

$$\Rightarrow \quad V = \pi(R^2 - y^2) \times 2y$$



For volume to be maximum or minimum

$$\frac{dV}{dy} = 0 \quad \Rightarrow \quad 2\pi\{(R^2 - y^2) \times 1 + y \times (-2y)\}$$

$$\Rightarrow \quad R^2 - y^2 - 2y^2 = 0 \quad \Rightarrow \quad R^2 - 3y^2 = 0 \quad \Rightarrow \quad R^2 = 3y^2$$

$$\Rightarrow \quad y^2 = \frac{R^2}{3} \quad \Rightarrow \quad y = \frac{R}{\sqrt{3}}$$

$$\therefore \quad \frac{d^2V}{dy^2}_{\left(\text{at } y = \frac{R}{\sqrt{3}}\right)} = 2\pi(-6y) = -12\pi y = \frac{-12R\pi}{\sqrt{3}} < 0$$

$\therefore$ Volume will be maximum when $y = \frac{R}{\sqrt{3}}$

$\therefore$ Height of cylinder = $2y = \frac{2R}{\sqrt{3}}$

and maximum volume $= \pi\left(R^2 - y^2\right) \times 2y$

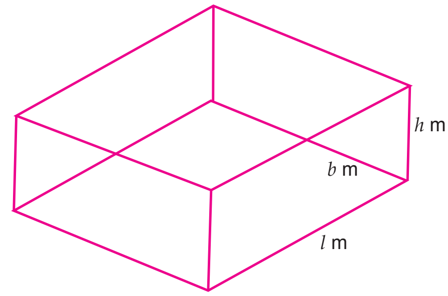
$$= \pi\left(R^2 - \frac{R^2}{3}\right) \times \frac{2R}{\sqrt{3}} = \pi \times \frac{2R^2}{3} \times \frac{2R}{\sqrt{3}} = \frac{4\pi R^3}{3\sqrt{3}}$$

5. A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m^3. If building of tank costs ₹70 per square metre for the base and ₹45 per square metre for the sides, what is the cost of least expensive tank? **[*CBSE 2019 (65/1/1)*]**

Sol. Let l, b and h metre be the length, breadth and height of the tank respectively.

Given $h = 2$m

and volume of tank $= l \times b \times h$



$$\Rightarrow \quad 8 = l \times b \times 2 \Rightarrow lb = 4 \Rightarrow b = \frac{4}{l}$$

$$\left[\begin{array}{l} \because V = lbh \\ 8 = lb2 \\ \therefore b = \dfrac{8}{2l} = \dfrac{4}{l} \end{array}\right]$$

Now, area of the base, $\quad lb = 4 \text{ m}^2$

and, area of four walls, $\quad A = 2(l+b) \times h$

$$= 2\left(l + \frac{4}{l}\right) \times 2 \Rightarrow A = 4\left(l + \frac{4}{l}\right)$$

For minimum cost

$$\frac{dA}{dl} = 0 \qquad \Rightarrow \qquad 4\left(1 - \frac{4}{l^2}\right) = 0$$

$$\Rightarrow \qquad \frac{4}{l^2} = 1 \qquad \Rightarrow \qquad l^2 = 4 \qquad \Rightarrow \qquad l = 2\text{m}$$

$$\therefore \qquad l = 2\text{m} \ (\because \text{ length cannot be negative}) \text{ and } b = \frac{4}{l} = \frac{4}{2} = 2\text{m}$$

Now, $\quad \dfrac{d^2A}{dl^2} = \dfrac{32}{l^2} = \dfrac{32}{4} = 8 > 0$

$\therefore$ Area will be minimum when $l = 2$m, $b = 2$m, $h = 2$m

$\therefore$ Cost of building of tank $= 70 \times (l \times b) = 70 \times 2 \times 2 =$ ₹ 280

and cost of building the walls $= 45 \times 2h\,(l + b) = 90 \times 2\,(2 + 2) =$ ₹ 720

$\therefore$ Total cost for building the tank $= 280 + 720 =$ ₹ 1,000

6. If the sum of hypotenuse and a side of a right angled triangle is given, show that the area of the triangle is maximum when the angle between them is $\frac{\pi}{3}$.

[*CBSE Delhi 2017; (AI) 2009, 2014; (Central) 2016*]

Sol. Let h and x be the length of hypotenuse and one side of a right triangle and y is length of the third side.

If A be the area of triangle, then

$$A = \frac{1}{2}xy = \frac{1}{2}x\sqrt{h^2 - x^2}$$

$$\left[\begin{array}{l} \text{also given} \\ h + x = k \text{ (constant)} \\ \therefore \ h = k - x \end{array}\right]$$

$$A = \frac{1}{2}x\sqrt{(k-x)^2 - x^2} = \frac{1}{2}x\sqrt{k^2 - 2kx + x^2 - x^2}$$

$$\Rightarrow \qquad A^2 = \frac{x^2}{4}(k^2 - 2kx) \quad \Rightarrow \quad A^2 = \frac{1}{4}(k^2x^2 - 2kx^3)$$

Differentiating with respect to x we get

$$\frac{d(A^2)}{dx} = \frac{1}{4}(2k^2x - 6kx^2) \qquad \ldots(i)$$

For maxima or minima of A^2

$$\frac{d(A^2)}{dx} = 0 \quad \Rightarrow \quad \frac{1}{4}(2k^2x - 6kx^2) = 0$$

$$\Rightarrow \qquad 2k^2x - 6kx^2 = 0 \quad \Rightarrow \quad 2kx\,(k - 3x) = 0$$

$$\Rightarrow \qquad k - 3x = 0\,; \qquad 2kx \neq 0$$

$$\Rightarrow \qquad x = \frac{k}{3}$$

Differentiating (i) again with respect to x, we get

$$\frac{d^2(A^2)}{dx^2} = \frac{1}{4}(2k^2 - 12kx)$$

$$\left.\frac{d^2(A^2)}{dx^2}\right]_{x=k/3} = \frac{1}{4}\left(2k^2 - 12k.\frac{k}{3}\right) = -\frac{k^2}{2} < 0$$

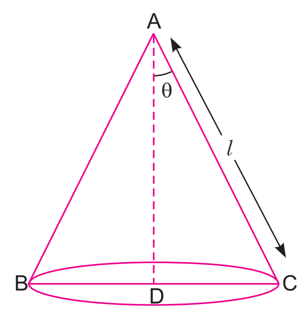
Hence, A^2 is maximum when $x = \frac{k}{3}$ and $h = k - \frac{k}{3} = \frac{2k}{3}$.

i.e., A is maximum when $x = \frac{k}{3}, h = \frac{2k}{3}$

$$\therefore \quad \cos\theta = \frac{x}{h} = \frac{k}{3} \times \frac{3}{2k} = \frac{1}{2} \quad \Rightarrow \quad \cos\theta = \frac{1}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{3}$$

7. Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $\cos^{-1}\frac{1}{\sqrt{3}}$. **[*CBSE Delhi 2014; (North) 2016*]**

Sol. Let ABC be cone having slant height l and semi-vertical angle θ.



If V be the volume of cone then

$$V = \frac{1}{3}.\pi \times DC^2 \times AD = \frac{\pi}{3} \times l^2 \sin^2\theta \times l\cos\theta$$

$$\Rightarrow \quad V = \frac{\pi l^3}{3}\sin^2\theta\cos\theta$$

$$\Rightarrow \quad \frac{dV}{d\theta} = \frac{\pi l^3}{3}[-\sin^3\theta + 2\sin\theta.\cos^2\theta]$$

For maximum value of V.

$$\frac{dV}{d\theta} = 0 \quad \Rightarrow \quad \frac{\pi l^3}{3}[-\sin^3\theta + 2\sin\theta.\cos^2\theta]. = 0$$

$$\Rightarrow \quad -\sin^3\theta + 2\sin\theta.\cos^2\theta = 0 \quad \Rightarrow \quad -\sin\theta(\sin^2\theta - 2\cos^2\theta) = 0$$

$$\Rightarrow \quad \sin\theta = 0 \quad \text{or} \quad 1 - \cos^2\theta - 2\cos^2\theta = 0$$

$$\Rightarrow \quad \theta = 0 \quad \text{or} \quad 1 - 3\cos^2\theta = 0$$

$$\Rightarrow \quad \theta = 0 \quad \text{or} \quad \cos\theta = \frac{1}{\sqrt{3}}$$

Now $$\frac{d^2V}{d\theta^2} = \frac{\pi l^3}{3}\{-3\sin^2\theta.\cos\theta - 4\sin^2\theta.\cos\theta + 2\cos^3\theta\}$$

$$\Rightarrow \quad \frac{d^2V}{d\theta^2} = \frac{\pi l^3}{3}\{-7\sin^2\theta\cos\theta + 2\cos^3\theta\} \quad \Rightarrow \quad \left.\frac{d^2V}{d\theta^2}\right]_{\theta=0} = +\text{ve}$$

and $$\left.\frac{d^2V}{d\theta^2}\right]_{\cos\theta=\frac{1}{\sqrt{3}}} = -\text{ve} \quad \left[\text{Putting } \cos\theta = \frac{1}{\sqrt{3}} \text{ and } \sin\theta = \sqrt{1-\left(\frac{1}{\sqrt{3}}\right)^2} = \frac{\sqrt{2}}{\sqrt{3}}\right]$$

Hence, for $\cos\theta = \frac{1}{\sqrt{3}}$ or $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$, V is maximum.

8. Show that the height of a closed right circular cylinder of given surface and maximum volume, is equal to the diameter of its base. **[*CBSE Delhi 2012*]**

Sol. Let r and h be radius and height of given cylinder of surface area S.

If V be the volume of cylinder then

$$V = \pi r^2 h$$

$$V = \frac{\pi r^2.(S - 2\pi r^2)}{2\pi r} \quad \left[\because \; S = 2\pi r^2 + 2\pi rh \Rightarrow \frac{S - 2\pi r^2}{2\pi r} = h\right]$$

$$V = \frac{Sr - 2\pi r^3}{2} \quad \Rightarrow \quad \frac{dV}{dr} = \frac{1}{2}(S - 6\pi r^2)$$

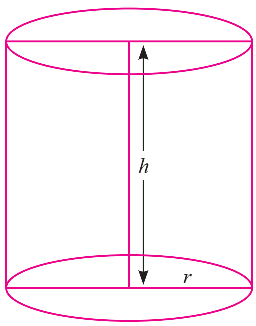
For maximum or minimum value of V

$$\frac{dV}{dr} = 0 \quad \Rightarrow \quad \frac{1}{2}(S - 6\pi r^2) = 0 \quad \Rightarrow \quad S - 6\pi r^2 = 0$$

$$\Rightarrow \quad r^2 = \frac{S}{6\pi} \quad \Rightarrow \quad r = \sqrt{\frac{S}{6\pi}}$$

Now $\dfrac{d^2V}{dr^2} = -\dfrac{1}{2} \times 12\pi r \quad \Rightarrow \quad \dfrac{d^2V}{dr^2} = -6\pi r \quad \Rightarrow \quad \left[\dfrac{d^2V}{dr^2}\right]_{r=\sqrt{\frac{S}{6\pi}}} < 0$

Hence, for $r = \sqrt{\dfrac{S}{6\pi}}$, volume V is maximum.



$$\Rightarrow \quad h = \frac{S - 2\pi.\frac{S}{6\pi}}{2\pi\sqrt{\frac{S}{6\pi}}} \quad \Rightarrow \quad h = \frac{3S - S}{3\times 2\pi} \times \sqrt{\frac{6\pi}{S}}$$

$$\Rightarrow \quad h = \frac{2S}{6\pi}.\frac{\sqrt{6\pi}}{\sqrt{S}} = 2\sqrt{\frac{S}{6\pi}}$$

$$\Rightarrow \quad h = 2r \text{ (diameter)} \qquad \left[\because r = \sqrt{\frac{S}{6\pi}}\right]$$

Therefore, for maximum volume, height of cylinder is equal to diameter of its base.

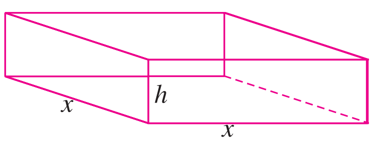
9. An open box with a square base is to be made out of a given quantity of cardboard of area c^2 square units. Show that the maximum volume of the box is $\dfrac{c^3}{6\sqrt{3}}$ cubic units.

[*NCERT Exemplar; CBSE (AI) 2012*]

Sol. Let the length, breadth and height of open box with square be x, x and h unit respectively.

If V be the volume of box then $V = x.x.\,h \quad \Rightarrow \quad V = x^2h$ ....(i)

Also $\quad c^2 = x^2 + 4xh \quad \Rightarrow \quad h = \dfrac{c^2 - x^2}{4x}$



Putting it in (i), we get

$$V = \frac{x^2(c^2 - x^2)}{4x} \quad \Rightarrow \quad V = \frac{c^2x}{4} - \frac{x^3}{4}$$

Differentiating with respect to x, we get

$$\frac{dV}{dx} = \frac{c^2}{4} - \frac{3x^2}{4}$$

Now for maxima or minima $\dfrac{dV}{dx} = 0$

$$\Rightarrow \quad \frac{c^2}{4} - \frac{3x^2}{4} = 0 \quad \Rightarrow \quad \frac{3x^2}{4} = \frac{c^2}{4}$$

$$\Rightarrow \quad x^2 = \frac{c^2}{3} \quad \Rightarrow \quad x = \frac{c}{\sqrt{3}}$$

Now, $\quad \dfrac{d^2V}{dx^2} = -\dfrac{6x}{4} = -\dfrac{3x}{2} \quad \Rightarrow \quad \left[\dfrac{d^2V}{dx^2}\right]_{x=c/\sqrt{3}} = -\dfrac{3c}{2\sqrt{3}} = -\text{ve}$

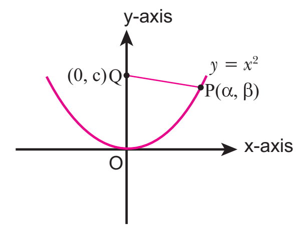
Hence, for $x = \dfrac{c}{\sqrt{3}}$ volume of box is maximum.

$$\therefore \quad h = \frac{c^2 - x^2}{4x} = \frac{c^2 - \frac{c^2}{3}}{4\frac{c}{\sqrt{3}}} = \frac{2c^2}{3} \times \frac{\sqrt{3}}{4c} = \frac{c}{2\sqrt{3}}$$

Therefore maximum volume = $x^2 \, . \, h \; = \dfrac{c^2}{3} . \dfrac{c}{2\sqrt{3}} = \dfrac{c^3}{6\sqrt{3}}$ cubic units

10. Find the shortest distance of the point (0, *c*) from the parabola $y = x^2$, where $1 \le c \le 5$. [*CBSE 2013*]

Sol. Let $P\,(\alpha, \beta)$ be required point on parabola $y = x^2$ such that the distance of P to given point $Q(0, c)$ is shortest.



Let $PQ = D$

$\therefore \qquad D = \sqrt{(\alpha - 0)^2 + (\beta - c)^2} \qquad \Rightarrow \qquad D^2 = \alpha^2 + (\beta - c)^2$

$\Rightarrow \qquad D^2 = \alpha^2 + (\alpha^2 - c)^2 \qquad [\because (\alpha, \beta) \text{ lie on } y = x^2 \Rightarrow \beta = \alpha^2] \quad ...(i)$

Now, $\quad \dfrac{d(D^2)}{d\alpha} = 2\alpha + 2(\alpha^2 - c).2\alpha = 2\alpha(1 + 2\alpha^2 - 2c) = 2\alpha + 4\alpha^3 - 4\alpha c$

For extremum value of D or D^2

$$\frac{d(D^2)}{d\alpha} = 0 \qquad \Rightarrow \qquad 2\alpha(1 + 2\alpha^2 - 2c) = 0$$

$$\Rightarrow \qquad \alpha = 0, \text{ or } 1 + 2\alpha^2 - 2c = 0 \qquad \Rightarrow \qquad \alpha = 0 \;\text{ or } \alpha = \pm\sqrt{\frac{2c-1}{2}}$$

Again $\quad \dfrac{d^2(D^2)}{d\alpha^2} = 2 + 12\alpha^2 - 4c \qquad \Rightarrow \qquad \left.\dfrac{d^2(D^2)}{d\alpha^2}\right]_{\alpha = 0} = 2 - 4c = -\text{ve} \qquad [\because 1 \le c \le 5]$

$$\left[\frac{d^2(D^2)}{d\alpha^2}\right]_{\alpha = \pm\sqrt{\frac{2c-1}{2}}} = 2 + 12\left(\frac{2c-1}{2}\right) - 4c = 2 + 12c - 6 - 4c = 8c - 4 > 0 \qquad [\because 1 \le c \le 5]$$

i.e., for $\alpha = \pm\sqrt{\dfrac{2c-1}{2}}$ $\quad D^2$ *i.e.*, D is minimum (shortest)

Now, the shortest distance D is

$$D = \sqrt{\alpha^2 + (\alpha^2 - c)^2} = \sqrt{\alpha^4 + \alpha^2 + c^2 - 2\alpha^2 c} \quad [\text{From } (i)]$$

$$= \sqrt{\left(\frac{2c-1}{2}\right)^2 + \frac{2c-1}{2} + c^2 - 2c\left(\frac{2c-1}{2}\right)} \qquad \left[\because \alpha^2 = \frac{2c-1}{2}\right]$$

$$= \sqrt{\frac{(2c-1)^2 + 4c^2 + 4c - 2 - 4c(2c-1)}{4}}$$

$$= \frac{1}{2}\sqrt{4c^2 + 1 - 4c + 4c^2 + 4c - 2 - 8c^2 + 4c} = \frac{1}{2}\sqrt{4c-1}$$

Hence, required shortest distance is $\dfrac{1}{2}\sqrt{4c-1}$.

11. Show that the volume of the greatest cylinder that can be inscribed in a cone of height '*h*' and semi-vertical angle 'α' is $\dfrac{4}{27}\pi h^3 \tan^2\alpha$. [*CBSE (AI) 2010, (East) 2016*]

Sol. Let a cylinder of base radius r and height h_1 is included in a cone of height h and semi-vertical angle α.

Then $AB = r$, $OA = (h - h_1)$. In right angle triangle OAB,

$$\frac{AB}{OA} = \tan\alpha \qquad \Rightarrow \qquad \frac{r}{h - h_1} = \tan\alpha \qquad \Rightarrow \qquad r = (h - h_1)\tan\alpha$$

$\therefore \qquad V = \pi\,[(h - h_1)\tan\alpha]^2 \,.\, h_1 \qquad (\because \text{ Volume of cylinder} = \pi r^2 h)$

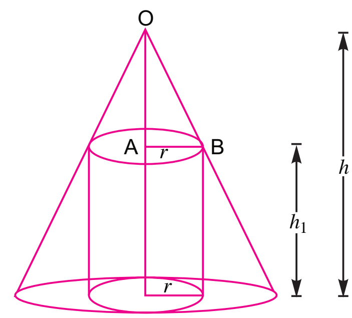
$\qquad\qquad = \pi\tan^2\alpha \,.\, h_1\,(h - h_1)^2 \qquad ...(i)$

Differentiating with respect to h_1, we get

$$\frac{dV}{dh_1} = \pi\tan^2\alpha[h_1.2(h - h_1)(-1) + (h - h_1)^2 \times 1]$$

$= \pi \tan^2 \alpha \, (h - h_1) \, [-2h_1 + h - h_1]$

$= \pi \tan^2 \alpha \, (h - h_1) \, (h - 3h_1)$



For maximum volume $V, \dfrac{dV}{dh_1} = 0$

$\Rightarrow \quad h - h_1 = 0 \;$ or $\; h - 3h_1 = 0 \;\Rightarrow\; h = h_1 \;$ or $\; h_1 = \dfrac{1}{3}h$

$\Rightarrow \quad h_1 = \dfrac{1}{3}h \qquad (\because\ h = h_1$ is not possible$)$

Again differentiating with respect to h_1, we get

$$\frac{d^2V}{dh_1^2} = \pi \tan^2\alpha[(h - h_1)(-3) + (h - 3h_1)(-1)]$$

At $h_1 = \dfrac{1}{3}h, \quad \dfrac{d^2V}{dh_1^2} = \pi \tan^2\alpha\left[\left(h - \dfrac{1}{3}h\right)(-3) + 0\right] = -2\pi h \tan^2\alpha < 0$

$\therefore$ Volume is maximum for $h_1 = \dfrac{1}{3}h$

$$V_{\max} = \pi \tan^2\alpha.\left(\frac{1}{3}h\right)\left(h - \frac{1}{3}h\right)^2 \qquad \text{[Using } (i)\text{]}$$

$$= \frac{4}{27}\pi h^3 \tan^2\alpha$$

12. The sum of the perimeter of a circle and a square is *k*, where *k* is some constant. Prove that the sum of their areas is least when the side of the square is double the radius of the circle.

[CBSE (F) 2010, 2014]

Sol. Let side of square be a units and radius of circle be r units.

It is given that $4a + 2\pi r = k$, where k is a constant

$\Rightarrow \quad r = \dfrac{k - 4a}{2\pi}$

Sum of areas, $A = a^2 + \pi r^2$

$$= a^2 + \pi\left[\frac{k - 4a}{2\pi}\right]^2 = a^2 + \frac{1}{4\pi}(k - 4a)^2$$

Differentiating with respect to a, we get

$$\frac{dA}{da} = 2a + \frac{1}{4\pi}.2(k - 4a).(-4) = 2a - \frac{2(k - 4a)}{\pi} \qquad \ldots(i)$$

For minimum area, $\dfrac{dA}{da} = 0$

$\Rightarrow \quad 2a - \dfrac{2(k - 4a)}{\pi} = 0 \;\Rightarrow\; 2a = \dfrac{2(k - 4a)}{\pi}$

$\Rightarrow \quad 2a = \dfrac{2(2\pi r)}{\pi} \qquad$ [As $k = 4a + 2\pi r$ given]

$\Rightarrow \quad a = 2r$

Now, again differentiating equation (i) with respect to a, we get

$$\frac{d^2A}{da^2} = 2 - \frac{2}{\pi}(-4) = 2 + \frac{8}{\pi} \qquad \text{at } a = 2\pi, \quad \frac{d^2A}{da^2} = 2 + \frac{8}{\pi} > 0$$

$\therefore$ For $ax = 2r$, sum of areas is least.

Hence, sum of areas is least when side of the square is double the radius of the circle.

13. Find the area of the greatest rectangle that can be inscribed in an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. **[CBSE (AI) 2013]**

Sol. Let $ABCD$ be rectangle having area A inscribed in an ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad ...(i)$$

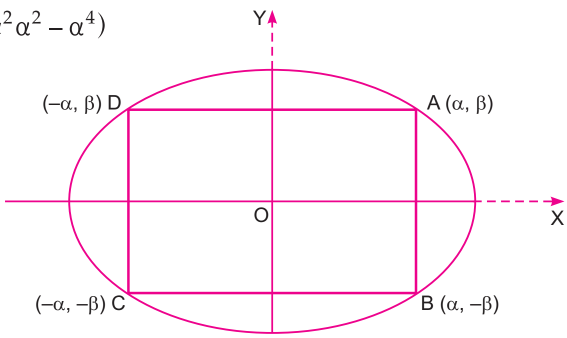
Let the coordinate of A be (α, β).

$\therefore$ Coordinate of $B \equiv (\alpha, -\beta)$, $C \equiv (-\alpha, -\beta)$, $D \equiv (-\alpha, \beta)$

Now A = Length × Breadth = $2\alpha \times 2\beta = 4\alpha\beta$

$$\Rightarrow \quad = 4\alpha.\sqrt{b^2\left(1-\frac{\alpha^2}{a^2}\right)} \qquad \left[\begin{array}{l}\because (\alpha,\beta) \text{ lies on ellipse } (i)\\ \therefore \frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}=1 \; i.e., \; \beta=\sqrt{b^2\left(1-\frac{\alpha^2}{a^2}\right)}\end{array}\right]$$

$$\Rightarrow \quad A^2 = 16\alpha^2\left\{b^2\left(1-\frac{\alpha^2}{a^2}\right)\right\} \Rightarrow A^2 = \frac{16b^2}{a^2}(a^2\alpha^2-\alpha^4)$$



$$\Rightarrow \quad \frac{d(A^2)}{d\alpha} = \frac{16b^2}{a^2}(2a^2\alpha-4\alpha^3)$$

For maximum or minimum value

$$\frac{d(A^2)}{d\alpha}=0$$

$$\Rightarrow \quad 2a^2\alpha-4\alpha^3=0 \quad \Rightarrow \quad 2\alpha(a^2-2\alpha^2)=0$$

$$\Rightarrow \quad \alpha=0, \alpha=\frac{a}{\sqrt{2}}$$

Again $$\frac{d^2(A^2)}{d\alpha^2}=\frac{16b^2}{a^2}(2a^2-12\alpha^2) \Rightarrow \left.\frac{d^2(A^2)}{d\alpha^2}\right]_{\alpha=\frac{a}{\sqrt{2}}} = \frac{16b^2}{a^2}\left(2a^2-12\times\frac{a^2}{2}\right)<0$$

$\Rightarrow$ For $\alpha=\frac{a}{\sqrt{2}}$, A^2 *i.e.*, A is maximum.

i.e., for greatest area A, $\alpha=\frac{a}{\sqrt{2}}$ and $\beta=\frac{b}{\sqrt{2}}$ (Using (i))

$\therefore$ Greatest area $=4\alpha.\beta = 4\frac{a}{\sqrt{2}}\times\frac{b}{\sqrt{2}}=2ab$

14. Tangent to the circle $x^2+y^2=4$ at any point on it in the first quadrant makes intercepts OA and OB on x and y axes respectively, O being the centre of the circle. Find the minimum value of $(OA+OB)$. **[CBSE Ajmer 2015]**

Sol. Let AB be the tangent in the first quadrant to the circle $x^2+y^2=4$ which make intercepts OA and OB on x and y axis respectively. Let $S = OA + OB$.

$$S = OA + OB \qquad ...(i)$$

Let θ be the angle made by OP with positive direction of x-axis.

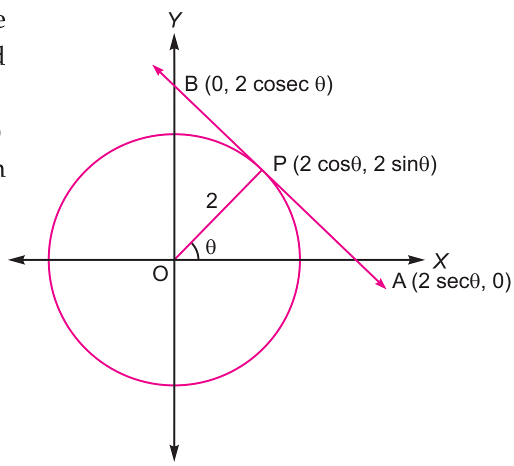
$\therefore$ Coordinates of $P = (2\cos\theta, 2\sin\theta)$

Coordinates of $A = (2\sec\theta, 0)$

Coordinates of $B = (0, 2\operatorname{cosec}\theta)$

$(i) \Rightarrow S = 2\sec\theta + 2\operatorname{cosec}\theta$

$$\Rightarrow \quad \frac{dS}{d\theta}=2\{\sec\theta\tan\theta-\operatorname{cosec}\theta\cot\theta\}$$



For extremum value of V

$\Rightarrow \quad \dfrac{dS}{d\theta} = 0 \quad \Rightarrow \quad 2\{\sec\theta\tan\theta - \text{cosec}\,\theta\cot\theta\} = 0$

$\Rightarrow \quad \sec\theta\tan\theta - \text{cosec}\,\theta\cot\theta = 0$

$\Rightarrow \quad \dfrac{1}{\cos\theta}\dfrac{\sin\theta}{\cos\theta} = \dfrac{1}{\sin\theta}\dfrac{\cos\theta}{\sin\theta} \quad \Rightarrow \quad \dfrac{\sin\theta}{\cos^2\theta} = \dfrac{\cos\theta}{\sin^2\theta}$

$\Rightarrow \quad \sin^3\theta = \cos^3\theta \qquad \Rightarrow \quad \sin\theta = \cos\theta$

$\Rightarrow \quad \theta = \frac{\pi}{4}\left[\because \theta \text{ lies in first quadrant } \Rightarrow 0 \le \theta \le \frac{\pi}{4}\right]$

Now, $\quad \dfrac{d^2S}{d\theta^2} = 2\{(\sec^3\theta + \tan^2\theta\sec\theta) + (\text{cosec}^3\theta + \text{cosec}\,\theta\cot^2\theta)\}$

$\Rightarrow \quad \left.\dfrac{d^2S}{d\theta^2}\right]_{\theta=\frac{\pi}{4}} = +\text{ve} \qquad \Rightarrow \quad S$ is minimum when $\theta = \dfrac{\pi}{4}$

$\therefore$ Minimum value of $S = OA + OB$ is $2\sec\dfrac{\pi}{4} + 2\,\text{cosec}\dfrac{\pi}{4} = 2\sqrt{2} + 2\sqrt{2} = 4\sqrt{2}$ units.

15. Find the absolute maximum and absolute minimum values of the function f given by $f(x) = \sin^2 x - \cos x,\ x \in [0, \pi]$. **[*CBSE Panchkula 2015*]**

Sol. Here, $f(x) = \sin^2 x - \cos x$

$f'(x) = 2\sin x.\cos x + \sin x \Rightarrow \quad f'(x) = \sin x(2\cos x + 1)$

For critical point: $f'(x) = 0$

$\Rightarrow \quad \sin x(2\cos x + 1) = 0 \qquad \Rightarrow \quad \sin x = 0 \text{ or } \cos x = -\dfrac{1}{2}$

$\Rightarrow \quad x = 0 \text{ or } \cos x = \cos\dfrac{2\pi}{3} \quad \Rightarrow \quad x = 0 \text{ or } x = 2n\pi \pm \dfrac{2\pi}{3}$, where $n = 0, \pm 1, \pm 2 \ldots$

$\Rightarrow \quad x = 0$ or $x = \dfrac{2\pi}{2}$ other values does not belong to $[0, \pi]$.

For absolute maximum or minimum values:

$f(0) = \sin^2 0 - \cos 0 = 0 - 1 = -1$

$f\left(\dfrac{2\pi}{3}\right) = \sin^2\dfrac{2\pi}{3} - \cos\dfrac{2\pi}{3} = \left(\dfrac{\sqrt{3}}{2}\right)^2 - \left(-\dfrac{1}{2}\right) = \dfrac{3}{4} + \dfrac{1}{2} = \dfrac{5}{4}$

$f(\pi) = \sin^2\pi - \cos\pi = 0 - (-1) = 1$

Hence, absolute maximum value $= \dfrac{5}{4}$ and absolute minimum value $= -1$.

16. If the function $f(x) = 2x^3 - 9mx^2 + 12m^2x + 1$, where $m > 0$ attains its maximum and minimum at p and q respectively such that $p^2 = q$, then find the value of m. **[*CBSE Patna 2015*]**

Sol. Given, $f(x) = 2x^3 - 9mx^2 + 12m^2x + 1$

$\Rightarrow \quad f'(x) = 6x^2 - 18mx + 12m^2$

For extremum value of $f(x)$, $f'(x) = 0$

$\Rightarrow \quad 6x^2 - 18mx + 12m^2 = 0 \qquad \Rightarrow \quad x^2 - 3mx + 2m^2 = 0$

$\Rightarrow \quad x^2 - 2mx - mx + 2m^2 = 0 \qquad \Rightarrow \quad x(x - 2m) - m(x - 2m) = 0$

$\Rightarrow \quad (x - m)(x - 2m) = 0 \qquad \Rightarrow \quad x = m \text{ or } x = 2m$

Now, $\quad f''(x) = 12x - 18m$

$\Rightarrow \quad f''(x)$ at $[x = m] = f''(m) = 12m - 18m = -6m < 0$

And, $\quad f''(x)$ at $[x = 2m] = f''(2m) = 24m - 18m = 6m > 0$

Hence, $f(x)$ attains maximum and minimum value at m and $2m$ respectively.

$\Rightarrow \quad m = p$ and $2m = q$

But, $\quad p^2 = q$ [Given]

$\therefore \quad m^2 = 2m \quad \Rightarrow \quad m^2 - 2m = 0$

$\Rightarrow \quad m(m-2) \quad \Rightarrow \quad m = 0$ or $m = 2$

$\Rightarrow \quad m = 2$ as $m > 0$

17. The sum of the surface areas of a cuboid with sides x, $2x$ and $\frac{x}{3}$ and a sphere is given to be constant. Prove that the sum of their volumes is minimum, if x is equal to three times the radius of sphere. Also find the minimum value of the sum of their volumes. [*CBSE (F) 2016*]

Sol. Let r be the radius of sphere and S, V be the sum of surface area and volume of cuboid and sphere.

Now $\quad V = \left(x \,.\, 2x \,.\, \frac{x}{3}\right) + \frac{4}{3}\pi r^3$

$\Rightarrow \quad V = \frac{2}{3}x^3 + \frac{4}{3}\pi r^3 \quad \Rightarrow \quad V = \frac{2}{3}(x^3 + 2\pi r^3)$

$\because \quad S = 2\left[x.2x + x.\frac{x}{3} + \frac{x}{3}.2x\right] + 4\pi r^2$

$\Rightarrow \quad S = \frac{18x^2}{3} + 4\pi r^2 = 6x^2 + 4\pi r^2$

$\Rightarrow \quad x^2 = \frac{S - 4\pi r^2}{6} \quad \Rightarrow \quad x^3 = \left(\frac{S - 4\pi r^2}{6}\right)^{3/2}$

$\Rightarrow \quad V = \frac{2}{3}\left\{\left(\frac{S - 4\pi r^2}{6}\right)^{\frac{3}{2}} + 2\pi r^3\right\}$

$\Rightarrow \quad \frac{dV}{dr} = \frac{2}{3}\left\{\frac{3}{2}\left(\frac{S - 4\pi r^2}{6}\right)^{\frac{1}{2}} . \frac{1}{6} . (-8\pi r) + 6\pi r^2\right\}$

For maximum or minimum value

$\frac{dV}{dr} = 0$

$\Rightarrow \quad \frac{2}{3}\left\{-2\pi r\left(\frac{S - 4\pi r^2}{6}\right)^{\frac{1}{2}} + 6\pi r^2\right\} = 0 \quad \Rightarrow \quad \left(\frac{S - 4\pi r^2}{6}\right)^{\frac{1}{2}} = \frac{6\pi r^2}{2\pi r}$

$\Rightarrow \quad \left(\frac{S - 4\pi r^2}{6}\right)^{\frac{1}{2}} = 3r \quad \Rightarrow \quad r = \frac{1}{3} . \left(\frac{S - 4\pi r^2}{6}\right)^{\frac{1}{2}}$

Obviously, $\left.\frac{d^2V}{dr^2}\right]_{r = \frac{1}{3}\left(\frac{S - 4\pi r^2}{6}\right)^{\frac{1}{2}}} = +\text{ve}$

$\therefore \quad V$ is minimum when $r = \frac{1}{3}\left(\frac{S - 4\pi r^2}{6}\right)^{\frac{1}{2}}$

$\Rightarrow \quad 3r = \left(\frac{S - 4\pi r^2}{6}\right)^{\frac{1}{2}} \quad \Rightarrow \quad 9r^2 = \left(\frac{S - 4\pi r^2}{6}\right) \quad \Rightarrow \quad 54r^2 = S - 4\pi r^2$

$\Rightarrow \quad 54r^2 = 6x^2 + 4\pi r^2 - 4\pi r^2 \qquad [\because \; S = 6x^2 + 4\pi r^2]$

$\Rightarrow \quad x^2 = 9r^2 \quad \Rightarrow \quad x = 3r$

i.e., x is equal to three times the radius of sphere.

Now, minimum value of V (sum of volume) $= \frac{2}{3}\left\{x^3 + 2\pi\left(\frac{x}{3}\right)^3\right\}$

$= \frac{2}{3}\left\{x^3 + \frac{2\pi}{27}x^3\right\} = \frac{2}{81}x^3(27 + 2\pi)$ cubic unit.

18. Find the maximum and minimum values of $f(x) = \sec x + \log \cos^2 x$, $0 < x < 2\pi$. [*CBSE (South) 2016*]

Sol. We have $f(x) = \sec x + \log \cos^2 x$

$$f'(x) = \sec x . \tan x + \frac{1}{\cos^2 x} . 2\cos x(-\sin x) = \sec x . \tan x - 2\tan x = \tan x(\sec x - 2)$$

For critical point

$f'(x) = 0$

$\Rightarrow \quad \tan x\,(\sec x - 2) = 0 \qquad \Rightarrow \quad \tan x = 0 \text{ or } \sec x - 2 = 0$

$\Rightarrow \quad x = n\pi \text{ or } \sec x = 2 \qquad \Rightarrow \quad x = n\pi \text{ or } \cos x = \frac{1}{2}$

$\Rightarrow \quad x = n\pi \text{ or } \cos x = \cos\frac{\pi}{3} \qquad \Rightarrow \quad x = n\pi \text{ or } x = 2n\pi \pm \frac{\pi}{3},\ n = 0, \pm 1, \pm 2.......$

Thus possible value of x in interval $0 < x < 2\pi$ are $x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$

Now, $f\left(\frac{\pi}{3}\right) = \sec\frac{\pi}{3} + \log\cos^2\frac{\pi}{3} = 2 + \log\left(\frac{1}{2}\right)^2$

$= 2 + 2(\log 1 - \log 2) = 2 - 2\log 2 = 2(1 - \log 2) \qquad [\because \log 1 = 0]$

$f(\pi) = \sec\pi + \log\cos^2\pi = -1 + \log(-1)^2 = -1$

$f\left(\frac{5\pi}{3}\right) = \sec\frac{5\pi}{3} + 2\log\cos\frac{5\pi}{3} = \sec\left(2\pi - \frac{\pi}{3}\right) + 2\log\cos\left(2\pi - \frac{\pi}{3}\right)$

$= \sec\frac{\pi}{3} + 2\log\cos\frac{\pi}{3} = 2 + 2\log\frac{1}{2}$

$= 2 + 2(\log 1 - \log 2) = 2 - 2\log 2 = 2(1 - \log 2)$

Hence, maximum value of $f(x) = 2(1 - \log 2)$

minimum value of $f(x) = -1$

19. Prove that the least perimeter of an isosceles triangle in which a circle of radius r can be inscribed is $6\sqrt{3}r$. **[*CBSE (Central) 2016*]**

Sol. Let ΔABC be isosceles triangle having $AB = AC$ in which a circle with centre O and radius r is inscribed touching sides AB, BC and AC at E, D and F respectively.

Let $AE = AF = x$, $BE = BD = y$

Obviously, $CF = CD = y$

Let P be the perimeter of ΔABC.

$ar(\Delta ABC) = ar(\Delta BOC) + ar(\Delta AOC) + ar(\Delta AOB)$

$\Rightarrow \frac{1}{2}AD.BC = \frac{1}{2}.BC.OD + \frac{1}{2}.AC.OF + \frac{1}{2}.AB.OE$

$\Rightarrow 2y.(r + \sqrt{r^2 + x^2}) = 2y.r + (x + y).r + (x + y).r$

$\Rightarrow 2y.(r + \sqrt{r^2 + x^2}) = 2yr + 2(x + y).r$

$\Rightarrow yr + y\sqrt{r^2 + x^2} = yr + xr + yr$

$\Rightarrow y\sqrt{r^2 + x^2} = xr + yr$

$\Rightarrow y^2(r^2 + x^2) = x^2r^2 + y^2r^2 + 2xyr^2$

$\Rightarrow y^2r^2 + x^2y^2 = x^2r^2 + y^2r^2 + 2xyr^2$

$\Rightarrow x^2y^2 = x^2r^2 + 2xyr^2$

$\Rightarrow xy^2 = xr^2 + 2yr^2$

$\Rightarrow x = \frac{2yr^2}{(y^2 - r^2)} \qquad ...(i)$

$\therefore \quad P = 2x + 4y$

$\Rightarrow \quad P = \frac{4yr^2}{y^2 - r^2} + 4y \quad$ (From (i))

Differentiating w.r.t. y, we get

$\Rightarrow \quad \frac{dP}{dy} = \frac{(y^2 - r^2).4r^2 - 4yr^2(2y - 0)}{(y^2 - r^2)^2} + 4$

$\Rightarrow \quad \frac{dP}{dy} = \frac{4y^2r^2 - 4r^4 - 8y^2r^2}{(y^2 - r^2)^2} + 4$

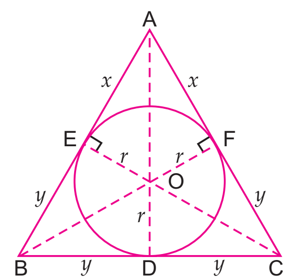
$\Rightarrow \quad \frac{dP}{dy} = \frac{-4r^2(r^2 + y^2)}{(y^2 - r^2)^2} + 4$

For critical point $\frac{dP}{dy} = 0$

$\Rightarrow \quad \frac{-4r^2(r^2 + y^2)}{(y^2 - r^2)^2} + 4 = 0$

$\Rightarrow \quad -4r^2(r^2 + y^2) + 4(y^2 - r^2)^2 = 0$

$\Rightarrow \quad -r^4 - r^2y^2 + y^4 + r^4 - 2y^2r^2 = 0$



$\Rightarrow \quad y^4 - 3r^2y^2 = 0$

$\Rightarrow \quad y^2[y^2 - 3r^2] = 0$

$\Rightarrow \quad y = \sqrt{3}r \quad [\because y \neq 0]$

Also $\left.\dfrac{d^2P}{dr^2}\right]_{\sqrt{3}r} = +\text{ve}$

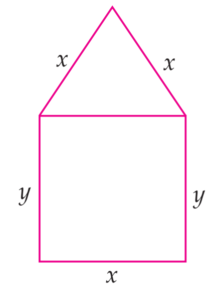
$\Rightarrow$ when $y = \sqrt{3}r$, the value of P is minimum.

$\therefore$ Least perimeter $= 4y + \dfrac{4r^2y}{y^2 - r^2} = 4\sqrt{3}r + \dfrac{4r^2.\sqrt{3}r}{3r^2 - r^2} = 4\sqrt{3}r + \dfrac{4\sqrt{3}r^3}{2r^2} = 6\sqrt{3}r$ units

20. A window has the shape of a rectangle surmounted by an equilateral triangle. If the perimeter of the window is 12 m, find the dimensions of the rectangle that will produce the largest area of the window. **[*CBSE(AI) 2011*] [*HOTS*]**

Sol. Let x and y be the dimensions of rectangular part of window and x be side of equilateral part.

If A be the total area of window, then $A = x.y + \dfrac{\sqrt{3}}{4}x^2$ ...(*i*)



Also, $x + 2y + 2x = 12 \quad \Rightarrow \quad 3x + 2y = 12$

$\Rightarrow \quad y = \dfrac{12 - 3x}{2}$

$\therefore \quad A = x.\dfrac{(12 - 3x)}{2} + \dfrac{\sqrt{3}}{4}x^2$ [From (*i*)]

$\Rightarrow \quad A = 6x - \dfrac{3x^2}{2} + \dfrac{\sqrt{3}}{4}x^2$

$\Rightarrow \quad A' = 6 - 3x + \dfrac{\sqrt{3}}{2}x$ [Differentiating with respect to x]

Now, for maxima or minima

$A' = 0 \quad \Rightarrow \quad 6 - 3x + \dfrac{\sqrt{3}}{2}x = 0 \quad \Rightarrow \quad x = \dfrac{12}{6 - \sqrt{3}}$

Again $\quad A'' = -3 + \dfrac{\sqrt{3}}{2} < 0$ (for any value of x) $\Rightarrow A'']_{x = \frac{12}{6-\sqrt{3}}} < 0$ *i.e.*, is maximum if

$$x = \frac{12}{6 - \sqrt{3}} \text{ and } y = \frac{12 - 3\left(\frac{12}{6 - \sqrt{3}}\right)}{2}.$$

i.e., for largest area of window, dimensions of rectangle are

$$x = \frac{12}{6 - \sqrt{3}} \text{ and } y = \frac{18 - 6\sqrt{3}}{6 - \sqrt{3}}.$$

PROFICIENCY EXERCISE

■ **Objective Type Questions:** **[1 mark each]**

1. Choose and write the correct option in each of the following questions.

(*i*) The points at which the tangents to the curve $y = x^3 - 12x + 18$ are parallel to x-axis are

(*a*) (2, –2), (–2, –34) (*b*) (2, 34), (–2, 0) (*c*) (0, 34), (–2, 0) (*d*) (2, 2), (–2, 34)

(*ii*) The sides of an equilateral triangle are increasing at the rate of 2 cm/sec. The rate at which the area increases, when side is 10 cm is

(*a*) 10 cm^2/s (*b*) $\sqrt{3}$ cm^2/s (*c*) $10\sqrt{3}$ cm^2/s (*d*) $\dfrac{10}{3}$ cm^2/s

(*iii*) The maximum value of slope of the curve $y = -x^3 + 3x^2 + 12x - 5$ is [CBSE 2020 (65/3/1)]

(*a*) 15 (*b*) 12 (*c*) 9 (*d*) 0

(*iv*) If the function $f(x) = 2x^2 - kx + 5$ is increasing on [1, 2], then k lies in the interval

(*a*) $(-\infty, 4)$ (*b*) $(4, \infty)$ (*c*) $(-\infty, 8)$ (*d*) $(8, \infty)$

(*v*) If the curve $ay + x^2 = 7$ and $x^3 = y$, cut orthogonally at (1, 1), then the value of a is

(*a*) 1 (*b*) 0 (*c*) – 6 (*d*) 6

(*vi*) The approximate value of $(33)^{\frac{1}{5}}$ is

(*a*) 2.0125 (*b*) 2.1 (*c*) 2.01 (*d*) none of these

(*vii*) The equation of the normal to the curve $y = x(2 - x)$ at the point (2, 0) is

(*a*) $x - 2y = 2$ (*b*) $x - 2y + 2 = 0$ (*c*) $2x + y = 4$ (*d*) $2x + y - 4 = 0$

(*viii*) The angle of intersection of the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ at the origin, is

(*a*) $\frac{\pi}{6}$ (*b*) $\frac{\pi}{3}$ (*c*) $\frac{\pi}{2}$ (*d*) $\frac{\pi}{4}$

2. Fill in the blanks.

(*i*) The slope of the tangent to the curve $y = x^3 - x$ at the point (2, 6) is ______________ .
[CBSE 2020 (65/4/1)]

(*ii*) The maximum value of $f(x) = x + \frac{1}{x}, x < 0$ is ______________ .

(*iii*) The rate of change of the area of a circle with respect to its radius r, when r = 3 cm, is ______________ . [CBSE 2020 (65/4/1)]

(*iv*) If $f(x) = \dfrac{1}{4x^2 + 2x + 1}$, then its maximum value is ______________ .

■ Very Short Answer Questions: [1 mark each]

3. If the rate of change of volume of a sphere is equal to the rate of change of its radius, find the radius of the sphere.

4. Find the interval in which the function f given by $f(x) = 7 - 4x - x^2$ is strictly increasing.
[CBSE 2020 (65/3/1)]

5. At what points on the curve $x^2 + y^2 - 2x - 4y + 1 = 0$, the tangents are parallel to y-axis?

6. It is given that at $x = 1$ the function $x^4 - 62x^2 + ax + 9$ attains the maximum value on the interval [0, 2]. Find the value of a.

7. Find the least value of λ such that the function $(x^2 + \lambda x + 1)$ is increasing on [1, 2].

■ Short Answer Questions–I: [2 marks each]

8. The contentment obtained after eating x-units of a new dish at a trial function is given by the function $C(x) = x^3 + 6x^2 + 5x + 3$. If the marginal contentment is defined as rate of change of $C(x)$ with respect to the number of units consumed at an instant, then find the marginal contentment when three units of dish are consumed. [CBSE (F) 2013]

9. Prove that the function $f(x) = \tan x - 4x$ is strictly decreasing on $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$.

10. Find the value of a for which the function $f(x) = \sin x - ax + b$ increasing on R.

11. Show that the function $f(x) = 4x^3 - 18x^2 + 27x - 7$ is always increasing on $\mathbb{R}$. [CBSE Delhi 2017]

12. Prove that $f(x) = \sin x + \sqrt{3} \cos x$ has maximum value at $x = \frac{\pi}{6}$.

13. Show that the function f defined by $f(x) = (x - 1)e^x + 1$ is an increasing function for all $x > 0$.
[CBSE 2020 (65/4/1)]

■ Short Answer Questions–II: [3 marks each]

14. Find the equations of the tangent and the normal to the curve $y = \dfrac{x-7}{(x-2)(x-3)}$ at the point where it cuts the x-axis. [*CBSE 2019 (65/3/1)*]

15. A ladder 13 m long is leaning against a vertical wall. The bottom of the ladder is dragged away from the wall along the ground at the rate of 2 cm/sec. How fast is the height on the wall decreasing when the foot of the ladder is 5 m away from the wall? [*CBSE 2019 (65/4/1)*]

16. Find the intervals in which the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is

(*a*) strictly increasing (*b*) strictly decreasing. [*CBSE Delhi 2014*]

17. Find the point on the curve $9y^2 = x^3$, where the normal to the curve makes equal intercepts on the axes. [*CBSE (F) 2015*]

18. Find the equations of the normals to the curve $y = x^3 + 2x + 6$ which are parallel to the line $x + 14y + 4 = 0$. [*CBSE Delhi 2010*]

19. Prove that the semi-vertical angle of the right circular cone of given volume and least curved surface area is $\cot^{-1}\sqrt{2}$. [*CBSE Delhi 2014*]

20. Find all the points of local maxima and local minima of the function

$$f(x) = -\frac{3}{4}x^4 - 8x^3 - \frac{45}{2}x^2 + 105$$

21. Using differentials, find the approximate value of $\sqrt{0.082}$.

■ Long Answer Questions: [5 marks each]

22. Find the dimensions of the rectangle of perimeter 36 cm which will sweep out a volume as large as possible, when revolved about one of its side. Also, find the maximum volume. [*CBSE 2020 (65/4/1)*]

23. Show that a right circular cylinder of the given volume open at the top has minimum total surface area, provided its height is equal to the radius of the base. [*CBSE (F) 2014*]

24. Show that the equation of normal at any point t on the curve $x = 3\cot t - \cos^3 t$ and $y = 3\sin t - \sin^3 t$ is $4(y\cos^3 t - x\sin^3 t) = 3\sin 4t$. [*CBSE Delhi 2016*]

25. Find the angle of intersection of the curve $y^2 = 4ax$ and $x^2 = 4by$. [*CBSE (F) 2016*]

26. The volume of a sphere is increasing at the rate of 3 cubic centimeter per second. Find the rate of increase of its surface area, when the radius is 2 cm. [*CBSE Delhi 2017*]

27. Find the local maxima and local minima, of the function $f(x) = \sin x - \cos x, 0 < x < 2\pi$, Also find the local maximum and local minimum values. [*CBSE Delhi 2015*]

28. Find the value of p for which the curves $x^2 = 9p(9-y)$ and $x^2 = p(y+1)$ cut each other at right angles. [*CBSE Allahabad 2015*]

29. Find the point on the curve $y = \dfrac{x}{1+x^2}$, where the tangent to the curve has the greatest slope. [*CBSE Chennai 2015*]

30. Find the absolute maximum and absolute minimum values of the function f given by $f(x) = \cos^2 x + \sin x, x \in [0, \pi]$. [*CBSE Guwahati 2015*]

31. Find the equation of tangents to the curve $y = \cos(x + y), -2\pi \le x \le 2\pi$, that are parallel to the line $x + 2y = 0$. [*CBSE (F) 2016*]

32. Determine the intervals in which the function $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$ is strictly increasing or strictly decreasing. [*CBSE (South) 2016*]

33. Find the equation of the normal at a point on the curve $x^2 = 4y$ which passes through the point (1, 2). Also find the equation of the corresponding tangent. [*CBSE Delhi 2013*]

34. A manufacturer can sell x items at a price of ₹ $\left(5 - \frac{x}{100}\right)$ each. The cost price of x items is ₹ $\left(\frac{x}{5} + 500\right)$. Find the number of items he should sell to earn maximum profit. [*CBSE (AI) 2009*]

35. A wire of length 34 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a rectangle whose length is twice its breadth. What should be the lengths of the two pieces, so that the combined area of the square and the rectangle is minimum? [*CBSE (F) 2017*]

36. Show that the rectangle of maximum perimeter which can be inscribed in a circle of radius r is the square of side $r\sqrt{2}$. [*CBSE Delhi 2011*]

37. Show that the rectangle of maximum area that can be inscribed in a circle is a square. [*CBSE Delhi 2008, 2011*]

38. Show that the normal at any point θ to the curve $x = a\cos\theta + a\theta\sin\theta, y = a\sin\theta - a\theta\cos\theta$ is at a constant distance from the origin. [*CBSE(AI) 2011*]

Answers

1. (*i*) (*d*) (*ii*) (*c*) (*iii*) (*a*) (*iv*) (*a*) (*v*) (*d*) (*vi*) (*a*) (*vii*) (*a*) (*viii*) (*c*)

2. (*i*) 11 (*ii*) –2 (*iii*) 6π cm^2/cm (*iv*) $\frac{4}{3}$

3. $\frac{1}{2\sqrt{\pi}}$ units **4.** $(-\infty, -2)$ **5.** $(-1, 2)$ and $(3, 2)$ **6.** $a = 120$ **7.** $\lambda = -2$

8. 68 units **10.** $(-\infty, -1)$ **14.** $x - 20y - 7 = 0$ and $20x + y - 140 = 0$ respectively

15. $\frac{5}{6}$ cm/sec **16.** (*a*) $(-1, 0) \cup (2, \infty)$ (*b*) $(-\infty, -1) \cup (0, 2)$ **17.** $\left(4, \frac{8}{3}\right)$ and $\left(4, \frac{-8}{3}\right)$

18. $x + 14y - 254 = 0$ and $x + 14y + 86 = 0$ **20.** Local maxima at 0, –5; and local minima at –3

21. 0.2867 **22.** Length = 12 cm, breadth = 6 cm and maximum volume = $\frac{216}{\pi}$ cm^3

25. 90° **26.** 3 cm^2/sec. **27.** Local maximum value = $\sqrt{2}$, local minimum value = $-\sqrt{2}$

28. $p = 0, 4$ **29.** (0, 0)

30. Absolute maximum value = $\frac{5}{4}$ at $x = \frac{\pi}{6}$ and $\frac{5\pi}{6}$, absolute minimum value = 1 at $x = 0$, $\frac{\pi}{2}$ and π

31. $x + 2y = 0$ **32.** $(1, 2) \cup (3, \infty)$; $(-\infty, 1) \cup (2, 3)$ **33.** $x + y - 3 = 0$; $x - y - 1 = 0$ **34.** 240 items

35. 16 m, 18 m

SELF-ASSESSMENT TEST

Time allowed: 1 hour **Max. marks: 30**

1. **Choose and write the correct option in the following questions.** **(4 × 1 = 4)**

(*i*) If $y = x^4 - 10$ and x changes from 2 to 1.99, then what is the change in y?

(*a*) 0.32 (*b*) 0.032 (*c*) 5.68 (*d*) 5.698

(*ii*) The maximum slope of curve $y = -x^3 + 3x^2 + 9x - 27$ is

(*a*) 0 (*b*) 12 (*c*) 16 (*d*) 32

(*iii*) The maximum value of $\frac{\log x}{x}$ in $[2, \infty)$ is

(*a*) 0 (*b*) 1 (*c*) $\frac{1}{e}$ (*d*) e

(*iv*) The function $f(x) = x^3 - 27x + 5$ is monotonically increasing when

(*a*) $x < -3$ (*b*) $|x| > 3$ (*c*) $x \le -3$ (*d*) $|x| \ge 3$

2. Fill in the blanks. **(2 × 1 = 2)**

(*i*) The equation of normal to the curve $2y + x^2 = 3$ at point (1, 1) is ____________ .

(*ii*) The maximum value of $\sin x \cdot \cos x$ is ____________ .

■ Solve the following questions. (2 × 1 = 2)

3. The maximum and minimum value of the function $f(x) = |x + 2| - 1$.

4. Show that $f(x) = e^x$ do not have maxima or minima.

■ Solve the following questions. (4 × 2 = 8)

5. Show that the tangent to the curve $y = 7x^3 + 11$ are parallel at the points $x = 2$ and $x = -2$.

6. Find two numbers whose sum is 24 and whose product is as large as possible.

7. Find the least value of λ such that the function $(x^2 + \lambda x + 1)$ is increasing on [1, 2].

8. Find the value of a for which the function $f(x) = \sin x - ax + b$ increasing on R.

■ Solve the following questions. (3 × 3 = 9)

9. Find the intervals in which the function f given by $f(x) = \dfrac{4\sin x - 2x - x\cos x}{2 + \cos x}$ is (*i*) increasing (*ii*) decreasing.

10. Find the equation of tangent to the curve $y = \sqrt{3x - 2}$, which is parallel to the line $4x - 2y + 5 = 0$.

11. The fuel cost for running a train is proportional to the square of the speed generated in km per hour. If the fuel costs ₹ 48 per hour at speed 16 km per hour and the fixed charges amount to ₹ 1200 per hour then find the most economical speed of train, when total distance covered by train is 5 km.

■ Solve the following question. (1 × 5 = 5)

12. A square piece of tin of side 18 cm is to be made into a box without top by cutting a square from each corner and folding up the flaps to form a box. Find the maximum volume of the box.

Answers

1. (*i*) (*a*) (*ii*) (*b*) (*iii*) (*c*) (*iv*) (*d*)

2. (*i*) $x - y = 0$ (*ii*) $\dfrac{1}{2}$

3. Min value = – 1 & maximum value does not exist

6. Both numbers are same and is 12. **7.** $\lambda = -2$ **8.** $(-\infty, -1)$

9. (*i*) $f(x)$ is increasing in the interval $\left(0, \dfrac{\pi}{2}\right)$ and $\left(\dfrac{3\pi}{2}, 2\pi\right)$ (*ii*) decreasing in the interval $\left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$

10. $48x - 24y - 23 = 0$ **11.** $v = 50$ km/hour **12.** 432 cm^3