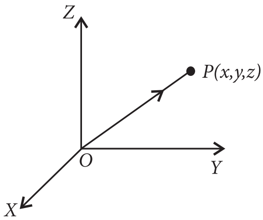
Vector Algebra

QUICK RECAP

VECTOR

- A physical quantity having magnitude as well as direction is called a vector. A vector is represented by a line segment, denoted as $\overrightarrow{AB}$ or $\vec{a}$. Here, point A is the initial point and B is the terminal point of the vector $\overrightarrow{AB}$.
- **Magnitude :** The distance between the points A and B is called the magnitude of the directed line segment $\overrightarrow{AB}$. It is denoted by $|\overrightarrow{AB}|$.
- **Position Vector :** Let P be any point in space, having coordinates (x, y, z) with respect to some fixed point O (0, 0, 0) as origin, then

the vector $\overrightarrow{OP}$ having O as its initial point and P as its terminal point is called the position vector of the point P with respect to O. The vector $\overrightarrow{OP}$ is usually denoted by $\vec{r}$.



Magnitude of $\overrightarrow{OP}$ is, $|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$

i.e., $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$.

In general, the position vectors of points A, B, C, etc. with respect to the origin O are denoted by $\vec{a}, \vec{b}, \vec{c}$, etc. respectively.

- **Direction Cosines and Direction Ratios :** The angles α, β, γ made by the vector $\vec{r}$ with the positive directions of x, y and z-axes respectively are called its direction angles. The cosine values of these angles, *i.e.*, $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called direction cosines of the vector $\vec{r}$, and usually denoted by l, m and n respectively.

  Direction cosines of $\vec{r}$ are given as

  $$l = \frac{x}{\sqrt{x^2 + y^2 + z^2}}, m = \frac{y}{\sqrt{x^2 + y^2 + z^2}},$$

  $$n = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

  The numbers lr, mr and nr, proportional to the direction cosines of vector $\vec{r}$ are called direction ratios of the vector $\vec{r}$ and denoted as a, b and c respectively.

  i.e., $a = lr$, $b = mr$ and $c = nr$

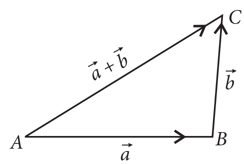
  Note : $l^2 + m^2 + n^2 = 1$ and $a^2 + b^2 + c^2 \neq 1$, (in general).

TYPES OF VECTORS

- **Zero vector :** A vector whose initial and terminal points coincide is called a zero (or null) vector. It cannot be assigned a definite direction as it has zero magnitude and it is denoted by the $\vec{0}$.
- **Unit Vector :** A vector whose magnitude is unity *i.e.*, $|\vec{a}| = 1$. It is denoted by $\hat{a}$.
- **Equal Vectors :** Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal, written as $\vec{a} = \vec{b}$, iff they have equal magnitudes and direction regardless of the positions of their initial points.
- **Coinitial Vectors :** Vectors having same initial point are called co-initial vectors.
- **Collinear Vectors :** Two or more vectors are called collinear if they have same or parallel supports, irrespective of their magnitudes and directions.
- **Negative of a Vector :** A vector having the same magnitude as that of a given vector but directed in the opposite sense is called negative of the given vector *i.e.*, $\overrightarrow{BA} = -\overrightarrow{AB}$.

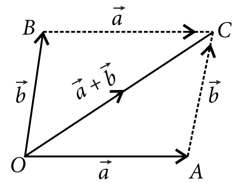
ADDITION OF VECTORS

- **Triangle law :** Let the vectors be $\vec{a}$ and $\vec{b}$ so positioned such that initial point of one coincides with terminal point of the other. If $\vec{a} = \overrightarrow{AB}$, $\vec{b} = \overrightarrow{BC}$. Then the vector $\vec{a} + \vec{b}$ is represented by the third side of ΔABC

  i.e., $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

- **Parallelogram law :** If the two vectors $\vec{a}$ and $\vec{b}$ are represented by the two adjacent sides OA and OB of a parallelogram $OACB$, then their sum $\vec{a} + \vec{b}$ is represented in magnitude and direction by the diagonal OC of parallelogram $OACB$ through their common point O *i.e.*, $\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$

Properties of Vector Addition

- Vector addition is commutative *i.e.*,
  $\vec{a} + \vec{b} = \vec{b} + \vec{a}$.
- Vector addition is associative *i.e.*,
  $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$.
- Existence of additive identity : The zero vector acts as additive identity *i.e.*,
  $\vec{a} + \vec{0} = \vec{a} = \vec{0} + \vec{a}$ for any vector $\vec{a}$.
- Existence of additive inverse : The negative of $\vec{a}$ *i.e.*, $-\vec{a}$ acts as additive inverse *i.e.*,
  $\vec{a} + (-\vec{a}) = \vec{0} = (-\vec{a}) + \vec{a}$ for any vector $\vec{a}$.

MULTIPLICATION OF A VECTOR BY A SCALAR

- Let $\vec{a}$ be a given vector and λ be a given scalar (a real number), then $\lambda\vec{a}$ is defined as the multiplication of vector $\vec{a}$ by the scalar λ. Its magnitude is $|\lambda|$ times the modulus of $\vec{a}$ *i.e.*, $|\lambda\vec{a}| = |\lambda|\,|\vec{a}|$.

  Direction of $\lambda\vec{a}$ is same as that of $\vec{a}$ if $\lambda > 0$ and opposite to that of $\vec{a}$ if $\lambda < 0$.

  Note : If $\lambda = \frac{1}{|\vec{a}|}$, provided that $\vec{a} \neq 0$, then $\lambda\vec{a}$ represents the unit vector in the direction of $\vec{a}$ *i.e.* $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$

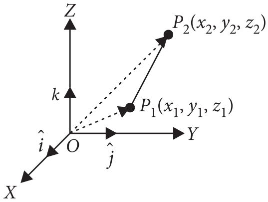
COMPONENTS OF A VECTOR

- Let O be the origin and $P(x, y, z)$ be any point in space. Let $\hat{i}, \hat{j}, \hat{k}$ be unit vectors along the X-axis, Y-axis and Z-axis respectively. Then $\overrightarrow{OP} = x\hat{i} + y\hat{j} + z\hat{k}$, which is called the component form of $\overrightarrow{OP}$. Here x, y and z are scalar components of $\overrightarrow{OP}$ and $x\hat{i}, y\hat{j}, z\hat{k}$ are vector components of $\overrightarrow{OP}$.

- If $\vec{a}$ and $\vec{b}$ are two given vectors as $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and λ be any scalar, then
  - $\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$
  - $\vec{a} - \vec{b} = (a_1 - b_1)\hat{i} + (a_2 - b_2)\hat{j} + (a_3 - b_3)\hat{k}$
  - $\lambda\vec{a} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$
  - $\vec{a} = \vec{b} \Leftrightarrow a_1 = b_1, a_2 = b_2$ and $a_3 = b_3$
  - $\vec{a}$ and $\vec{b}$ are collinear iff $\frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3} = \lambda$.

VECTOR JOINING TWO POINTS

- If $P_1(x_1, y_1, z_1)$ and P_2 (x_2, y_2, z_2) are any two points in the space then the vector joining P_1 and P_2 is the vector $\overrightarrow{P_1P_2}$.



  Applying triangle law in ΔOP_1P_2, we get
  $\overrightarrow{OP_1} + \overrightarrow{P_1P_2} = \overrightarrow{OP_2}$
  $\Rightarrow \overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$
  $= (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$
  $= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$
  $\therefore |\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

SECTION FORMULA

- Let A, B be two points such that $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$.
  - The position vector $\vec{r}$ of the point P which divides the line segment AB internally in the ratio $m : n$ is given by $\vec{r} = \frac{m\vec{b} + n\vec{a}}{m + n}$.
  - The position vector $\vec{r}$ of the point P which divides the line segment AB externally in the ratio $m : n$ is given by $\vec{r} = \frac{m\vec{b} - n\vec{a}}{m - n}$.
  - The position vector $\vec{r}$ of the mid-point of the line segment AB is given by $\vec{r} = \frac{\vec{a} + \vec{b}}{2}$.

PRODUCT OF TWO VECTORS

- **Scalar (or dot) product :** The scalar (or dot) product of two (non-zero) vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a} \cdot \vec{b}$ (read as $\vec{a}$ dot $\vec{b}$), is defined as $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta = ab\cos\theta$,
  where, $a = |\vec{a}|, b = |\vec{b}|$ and $\theta(0 \le \theta \le \pi)$ is the angle between $\vec{a}$ and $\vec{b}$.
  - **Properties of Scalar Product :**
    (i) Scalar product is commutative : $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
    (ii) $\vec{a} \cdot \vec{0} = 0$
    (iii) Scalar product is distributive over addition :
    - $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
    - $(\vec{a} + \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}$

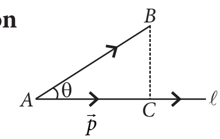
    (iv) $\lambda(\vec{a} \cdot \vec{b}) = (\lambda\vec{a}) \cdot \vec{b} = \vec{a} \cdot (\lambda\vec{b})$, λ be any scalar.
    (v) If $\hat{i}, \hat{j}, \hat{k}$ are three unit vectors along three mutually perpendicular lines, then $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$ and $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$
    (vi) Angle between two non-zero vectors $\vec{a}$ and $\vec{b}$ is given by $\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$
    i.e., $\theta = \cos^{-1}\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}\right)$

(vii) Two non-zero vectors $\vec{a}$ and $\vec{b}$ are mutually perpendicular if and only if $\vec{a}\cdot\vec{b}=0$

(viii) If $\theta = 0$, then $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|$

If $\theta = \pi$, then $\vec{a}\cdot\vec{b}=-|\vec{a}||\vec{b}|$

- **Projection of a vector on a line :** Let the vector $\overrightarrow{AB}$ makes an angle θ with directed line ℓ.



Projection of $\overrightarrow{AB}$ on $\ell=|\overrightarrow{AB}|\cos\theta=\overrightarrow{AC}=\vec{p}$.
The vector $\vec{p}$ is called the projection vector. Its magnitude is $|\vec{p}|$, which is known as projection of vector $\overrightarrow{AB}$.

Projection of a vector $\vec{a}$ on $\vec{b}$, is given as $\vec{a}\cdot\hat{b}$ *i.e.*, $\dfrac{1}{|\vec{b}|}(\vec{a}\cdot\vec{b})$.

- **Vector (or Cross) Product :** The vector (or cross) product of two (non-zero) vectors $\vec{a}$ and $\vec{b}$ (in an assigned order), denoted by $\vec{a}\times\vec{b}$ (read as $\vec{a}$ cross $\vec{b}$), is defined as $\vec{a}\times\vec{b}=|\vec{a}|\,|\vec{b}|\sin\theta\,\hat{n}$ where $\theta(0\le\theta\le\pi)$ is the angle between $\vec{a}$ and $\vec{b}$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$.

- **Properties of Vector Product :**

(i) Non-commutative : $\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}$

(ii) Vector product is distributive over addition :

$$\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$$

(iii) $\lambda(\vec{a}\times\vec{b})=(\lambda\vec{a})\times\vec{b}=\vec{a}\times(\lambda\vec{b})$, λ be any scalar.

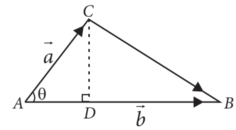
(iv) $(\lambda_1\vec{a})\times(\lambda_2\vec{b})=\lambda_1\lambda_2(\vec{a}\times\vec{b})$

(v) $\hat{i}\times\hat{j}=\hat{k},\ \hat{j}\times\hat{k}=\hat{i},\ \hat{k}\times\hat{i}=\hat{j}$

(vi) Two non-zero vectors $\vec{a}$, $\vec{b}$ are collinear if and only if $\vec{a}\times\vec{b}=\vec{0}$

Similarly, $\vec{a}\times\vec{a}=\vec{0}$ and $\vec{a}\times(-\vec{a})=\vec{0}$, since in the first situation $\theta = 0$ and in the second one, $\theta=\pi$, making the value of $\sin\theta$ to be 0.

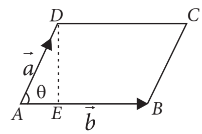
(vii) If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a triangle as given in the figure. Then,



Area of triangle $ABC=\dfrac{1}{2}AB\cdot CD$

$=\dfrac{1}{2}|\vec{b}||\vec{a}|\sin\theta\ =\dfrac{1}{2}|\vec{a}\times\vec{b}|$

(viii) If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a parallelogram as given in the figure.



Then, area of parallelogram $ABCD = AB\cdot DE$

$=|\vec{b}||\vec{a}|\sin\theta=|\vec{a}\times\vec{b}|$

(ix) If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}, \vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$,

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

$$=(a_2b_3-a_3b_2)\hat{i}+(a_3b_1-a_1b_3)\hat{j}+(a_1b_2-a_2b_1)\hat{k}$$

(x) Angle between two vectors $\vec{a}$ and $\vec{b}$ is given by $\sin\theta=\dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$

i.e., $\theta=\sin^{-1}\left(\dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}\right)$

- **Scalar Triple Product :** The scalar triple product of any three vectors $\vec{a},\vec{b}$ and $\vec{c}$ is written as $\vec{a}\cdot(\vec{b}\times\vec{c})$ or $[\vec{a}\ \vec{b}\ \vec{c}]$.

- **Coplanarity of Three Vectors :** Three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar iff $\vec{a}\cdot(\vec{b}\times\vec{c})=0$.

- Volume of parallelopiped formed by adjacent sides given by the three vectors $\vec{a}=(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})$, $\vec{b}=(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})$, and $\vec{c}=(c_1\hat{i}+c_2\hat{j}+c_3\hat{k})$, is $|\vec{a}\cdot(\vec{b}\times\vec{c})|$.

i.e., $|\vec{a}\cdot(\vec{b}\times\vec{c})|=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$

- For any three vectors $\vec{a},\vec{b}$ and $\vec{c}$,

(i) $[\vec{a}\ \vec{b}\ \vec{c}]=[\vec{b}\ \vec{c}\ \vec{a}]=[\vec{c}\ \vec{a}\ \vec{b}]$

(ii) $[\vec{a}\ \vec{b}\ \vec{c}]=-[\vec{a}\ \vec{c}\ \vec{b}]$ (iii) $[\vec{a}\ \vec{a}\ \vec{b}]=0$

Previous Years' CBSE Board Questions

10.2 Some Basic Concepts

VSA (1 mark)

1. Find a vector $\vec{a}$ of magnitude $5\sqrt{2}$, making an angle of $\frac{\pi}{4}$ with x-axis, $\frac{\pi}{2}$ with y-axis and an acute angle θ with z-axis. *(AI 2014)*

2. If a unit vector $\vec{a}$ makes angles $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find the value of θ. *(Delhi 2013)*

3. Find the magnitude of the vector $\vec{a} = 3\hat{i} - 2\hat{j} + 6\hat{k}$. *(AI 2011C)*

10.3 Types of Vectors

VSA (1 mark)

4. The value of p for which $p(\hat{i} + \hat{j} + \hat{k})$ is a unit vector is
   (a) 0 (b) $\frac{1}{\sqrt{3}}$
   (c) 1 (d) $\sqrt{3}$ *(2020)*

10.4 Addition of Vectors

VSA (1 mark)

5. $ABCD$ is a rhombus, whose diagonals intersect at E. Then $\overrightarrow{EA} + \overrightarrow{EB} + \overrightarrow{EC} + \overrightarrow{ED}$ equals
   (a) $\vec{0}$ (b) $\overrightarrow{AD}$
   (c) $2\overrightarrow{BC}$ (d) $2\overrightarrow{AD}$ *(2020)*

6. Find the sum of the vectors $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = -2\hat{i} + 4\hat{j} + 5\hat{k}$ and $\vec{c} = \hat{i} - 6\hat{j} - 7\hat{k}$. *(Delhi 2012)*

7. Find the sum of the following vectors :
   $\vec{a} = \hat{i} - 3\hat{k}$, $\vec{b} = 2\hat{j} - \hat{k}$, $\vec{c} = 2\hat{i} - 3\hat{j} + 2\hat{k}$ *(Delhi 2012)*

8. Find the sum of the following vectors :
   $\vec{a} = \hat{i} - 2\hat{j}, \vec{b} = 2\hat{i} - 3\hat{j}, \vec{c} = 2\hat{i} + 3\hat{k}$ *(Delhi 2012)*

9. If A, B and C are the vertices of a triangle ABC, then what is the value of $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$? *(Delhi 2011C)*

10.5 Multiplication of a Vector by a Scalar

VSA (1 mark)

10. The position vector of two points A and B are $\overrightarrow{OA} = 2\hat{i} - \hat{j} - \hat{k}$ and $\overrightarrow{OB} = 2\hat{i} - \hat{j} + 2\hat{k}$, respectively. The position vector of a point P which divides the line segment joining A and B in the ratio 2 : 1 is ________. *(2020)*

11. Find the position vector of a point which divides the join of points with position vectors $\vec{a} - 2\vec{b}$ and $2\vec{a} + \vec{b}$ externally in the ratio 2 : 1. *(Delhi 2016)*

12. Write the position vector of the point which divides the join of points with position vectors $3\vec{a} - 2\vec{b}$ and $2\vec{a} + 3\vec{b}$ in the ratio 2 : 1. *(AI 2016)*

13. Find the unit vector in the direction of the sum of the vectors $2\hat{i} + 3\hat{j} - \hat{k}$ and $4\hat{i} - 3\hat{j} + 2\hat{k}$. *(Foreign 2015)*

14. Find a vector in the direction of $\vec{a} = \hat{i} - 2\hat{j}$ that has magnitude 7 units. *(Delhi 2015C)*

15. Write the direction ratios of the vector $3\vec{a} + 2\vec{b}$ where $\vec{a} = \hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = 2\hat{i} - 4\hat{j} + 5\hat{k}$. *(AI 2015C)*

16. Write a unit vector in the direction of the sum of the vectors $\vec{a} = 2\hat{i} + 2\hat{j} - 5\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} - 7\hat{k}$. *(Delhi 2014)*

17. Find the value of 'p' for which the vectors $3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\hat{i} - 2p\hat{j} + 3\hat{k}$ are parallel. *(AI 2014)*

18. Find a vector in the direction of vector $2\hat{i} - 3\hat{j} + 6\hat{k}$ which has magnitude 21 units. *(Foreign 2014)*

19. Write a unit vector in the direction of vector $\overrightarrow{PQ}$, where P and Q are the points(1, 3, 0) and (4, 5, 6) respectively. *(Foreign 2014)*

20. Write a vector in the direction of the vector $\hat{i}-2\hat{j}+2\hat{k}$ that has magnitude 9 units. *(Delhi 2014C)*

21. If $\vec{a}=x\hat{i}+2\hat{j}-z\hat{k}$ and $\vec{b}=3\hat{i}-y\hat{j}+\hat{k}$ are two equal vectors, then write the value of $x+y+z$. *(Delhi 2013)*

22. Write a unit vector in the direction of the sum of vectors $\vec{a}=2\hat{i}-\hat{j}+2\hat{k}$ and $\vec{b}=-\hat{i}+\hat{j}+3\hat{k}$. *(Delhi 2013)*

23. P and Q are two points with position vectors $3\vec{a}-2\vec{b}$ and $\vec{a}+\vec{b}$ respectively. Write the position vector of a point R which divides the line segment PQ in the ratio 2 : 1 externally. *(AI 2013)*

24. A and B are two points with position vectors $2\vec{a}-3\vec{b}$ and $6\vec{b}-\vec{a}$ respectively. Write the position vector of a point P which divides the line segment AB internally in the ratio 1 : 2. *(AI 2013)*

25. L and M are two points with position vectors $2\vec{a}-\vec{b}$ and $\vec{a}+2\vec{b}$ respectively. Write the position vector of a point N which divides the line segment LM in the ratio 2 : 1 externally. *(AI 2013)*

26. Find the scalar components of the vector $\overrightarrow{AB}$ with initial point $A(2, 1)$ and terminal point $B(-5, 7)$. *(AI 2012)*

27. Find a unit vector parallel to the sum of the vectors $\hat{i}+\hat{j}+\hat{k}$ and $2\hat{i}-3\hat{j}+5\hat{k}$. *(Delhi 2012C)*

28. Find a unit vector in the direction of the vector $\vec{a}=3\hat{i}-2\hat{j}+6\hat{k}$. *(AI 2012C)*

29. Write the direction cosines of the vector $-2\hat{i}+\hat{j}-5\hat{k}$. *(Delhi 2011)*

30. For what value of 'a', the vectors $2\hat{i}-3\hat{j}+4\hat{k}$ and $a\hat{i}+6\hat{j}-8\hat{k}$ are collinear ? *(Delhi 2011)*

31. Write a unit vector in the direction of the vector $\vec{a}=2\hat{i}+\hat{j}+2\hat{k}$. *(AI 2011)*

32. Find a unit vector in the direction of $\vec{a}=2\hat{i}-3\hat{j}+6\hat{k}$. *(Delhi 2011C)*

33. Find a unit vector in the direction of the vector $\vec{a}=2\hat{i}+3\hat{j}+6\hat{k}$. *(AI 2011C)*

SA (2 marks)

34. X and Y are two points with position vectors $3\vec{a}+\vec{b}$ and $\vec{a}-3\vec{b}$ respectively. Write the position vector of a point Z which divides the line segment XY in the ratio 2 : 1 externally. *(AI 2019)*

LA 1 (4 marks)

35. The two vectors $\hat{j}+\hat{k}$ and $3\hat{i}-\hat{j}+4\hat{k}$ represent the two sides AB and AC, respectively of a ΔABC. Find the length of the median through A. *(Delhi 2016, Foreign 2015)*

36. Find a vector of magnitude 5 units and parallel to the resultant of the vectors $\vec{a}=2\hat{i}+3\hat{j}-\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}+\hat{k}$. *(Delhi 2011)*

10.6 Product of Two Vectors

VSA (1 mark)

37. If the projection of $\vec{a}=\hat{i}-2\hat{j}+3\hat{k}$ on $\vec{b}=2\hat{i}+\lambda\hat{k}$ is zero, then the value of λ is
(a) 0 (b) 1
(c) $\frac{-2}{3}$ (d) $\frac{-3}{2}$ *(2020)*

38. The area of the parallelogram whose diagonals are $2\hat{i}$ and $-3\hat{k}$ is ________ square units.

39. The value of λ for which the vectors $2\hat{i}-\lambda\hat{j}+\hat{k}$ and $\hat{i}+2\hat{j}-\hat{k}$ are orthogonal is ________. *(2020)*

40. If $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along three mutually perpendicular directions, then
(a) $\hat{i}\cdot\hat{j}=1$ (b) $\hat{i}\times\hat{j}=1$
(c) $\hat{i}\cdot\hat{k}=0$ (d) $\hat{i}\times\hat{k}=0$ *(2020)*

41. Find the magnitude of each of the two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude such that the angle between them is 60° and their scalar product is $\frac{9}{2}$. *(2018)*

42. Write the number of vectors of unit length perpendicular to both the vectors $\vec{a} = 2\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = \hat{j} + \hat{k}$. *(AI 2016)*

43. If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, then write the value of $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$. *(Foreign 2016)*

44. If $\left|\vec{a} \times \vec{b}\right|^2 + \left|\vec{a} \cdot \vec{b}\right|^2 = 400$ and $|\vec{a}| = 5$ then write the value of $|\vec{b}|$. *(Foreign 2016)*

45. If $\vec{a} = 7\hat{i} + \hat{j} - 4\hat{k}$ and $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$, then find the projection of $\vec{a}$ on $\vec{b}$. *(Delhi 2015, 2013C)*

46. If $\hat{a}$, $\hat{b}$ and $\hat{c}$ are mutually perpendicular unit vectors, then find the value of $|2\hat{a} + \hat{b} + \hat{c}|$. *(AI 2015)*

47. Write a unit vector perpendicular to both the vectors $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + \hat{j}$. *(AI 2015)*

48. Find the area of a parallelogram whose adjacent sides are represented by the vectors $2\hat{i} - 3\hat{k}$ and $4\hat{j} + 2\hat{k}$. *(Foreign 2015)*

49. If $\vec{a}$ and $\vec{b}$ are unit vectors, then what is the angle between $\vec{a}$ and $\vec{b}$ so that $\sqrt{2}\,\vec{a} - \vec{b}$ is a unit vector ? *(Delhi 2015C)*

50. Find the projection of the vector $\vec{a} = 2\hat{i} + 3\hat{j} + 2\hat{k}$ on the vector $\vec{b} = 2\hat{i} + 2\hat{j} + \hat{k}$. *(AI 2015C)*

51. Find the projection of vector $\hat{i} + 3\hat{j} + 7\hat{k}$ on the vector $2\hat{i} - 3\hat{j} + 6\hat{k}$. *(Delhi 2014)*

52. If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $\vec{a} + \vec{b}$ is also a unit vector, then find the angle between $\vec{a}$ and $\vec{b}$. *(Delhi 2014)*

53. If vectors $\vec{a}$ and $\vec{b}$ are such that, $|\vec{a}| = 3, |\vec{b}| = \frac{2}{3}$ and $\vec{a} \times \vec{b}$ is a unit vector, then write the angle between $\vec{a}$ and $\vec{b}$. *(Delhi 2014)*

54. If $\vec{a}$ and $\vec{b}$ are perpendicular vectors, $\left|\vec{a} + \vec{b}\right| = 13$ and $|\vec{a}| = 5$, find the value of $|\vec{b}|$. *(AI 2014)*

55. Write the projection of the vector $\hat{i} + \hat{j} + \hat{k}$ along the vector $\hat{j}$. *(Foreign 2014)*

56. Write the value of
$\hat{i} \times (\hat{j} + \hat{k}) + \hat{j} \times (\hat{k} + \hat{i}) + \hat{k} \times (\hat{i} + \hat{j})$. *(Foreign 2014)*

57. Write the projection of the vector $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ on the vector $\vec{b} = \hat{i} + 2\hat{j} + 2\hat{k}$. *(Delhi 2014C)*

58. If $\vec{a}$ and $\vec{b}$ are unit vectors, then find the angle between $\vec{a}$ and $\vec{b}$, given that $(\sqrt{3}\,\vec{a} - \vec{b})$ is a unit vector. *(Delhi 2014C)*

59. Write the value of cosine of the angle which the vector $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ makes with y-axis. *(Delhi 2014C)*

60. If $|\vec{a}| = 8, |\vec{b}| = 3$ and $\left|\vec{a} \times \vec{b}\right| = 12$, find the angle between $\vec{a}$ and $\vec{b}$. *(AI 2014C)*

1. Find the angle between x-axis and the vector $\hat{i} + \hat{j} + \hat{k}$. *(AI 2014C)*

62. Find $|\vec{x}|$, if for a unit vector $\vec{a}$, $(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 15$ *(AI 2013)*

63. Write the value of λ so that the vectors $\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$ are perpendicular to each other. *(Delhi 2013C, AI 2012C)*

64. For what value of λ are the vectors $\hat{i} + 2\lambda\hat{j} + \hat{k}$ and $2\hat{i} + \hat{j} - 3\hat{k}$ are perpendicular? *(AI 2013C, 2011C, Delhi 2012C)*

65. Write the projection of $\vec{b} + \vec{c}$ on $\vec{a}$, where $\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}, \vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$. *(AI 2013C)*

66. Find 'λ' when the projection of $\vec{a} = \lambda\hat{i} + \hat{j} + 4\hat{k}$ on $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$ is 4 units. *(Delhi 2012)*

67. Write the value of $(\hat{i} \times \hat{j}) \cdot \hat{k} + \hat{i} \cdot \hat{j}$. *(AI 2012)*

68. Write the value of $(\hat{k}\times\hat{j})\cdot\hat{i}+\hat{j}\cdot\hat{k}$. *(AI 2012)*

69. Write the value of $(\hat{k}\times\hat{i})\cdot\hat{j}+\hat{i}\cdot\hat{k}$. *(AI 2012)*

70. Write the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes $\sqrt{3}$ and 2 respectively having $\vec{a}\cdot\vec{b}=\sqrt{6}$. *(AI 2011)*

71. Write the projection of the vector $\hat{i}-\hat{j}$ on the vector $\hat{i}+\hat{j}$. *(AI 2011)*

72. If $|\vec{a}|=\sqrt{3}, |\vec{b}|=2$ and angle between $\vec{a}$ and $\vec{b}$ is 60°, find $\vec{a}\cdot\vec{b}$. *(Delhi 2011C)*

SA (2 marks)

73. Find a unit vector perpendicular to each of the vectors $\vec{a}$ and $\vec{b}$ and where $\vec{a}=5\hat{i}+6\hat{j}-2\hat{k}$ and $\vec{b}=7\hat{i}+6\hat{j}+2\hat{k}$. *(2020)*

74. Show that for any two non-zero vectors $\vec{a}$ and $\vec{b}$, $|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|$ iff $\vec{a}$ and $\vec{b}$ are perpendicular vectors. *(2020)*

75. Show that the vectors $2\hat{i}-\hat{j}+\hat{k}$, $3\hat{i}+7\hat{j}+\hat{k}$ and $5\hat{i}+6\hat{j}+2\hat{k}$ form the sides of a right-angled triangle. *(2020)*

76. If the sum of two unit vectors is a unit vector, prove that the magnitude of their difference is $\sqrt{3}$. *(Delhi 2019)*

77. Let $\vec{a}=\hat{i}+2\hat{j}-3\hat{k}$ and $\vec{b}=3\hat{i}-\hat{j}+2\hat{k}$ be two vectors. Show that the vectors $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ are perpendicular to each other. *(AI 2019)*

78. If θ is the angle between two vectors $\hat{i}-2\hat{j}+3\hat{k}$ and $3\hat{i}-2\hat{j}+\hat{k}$, find sin θ. *(2018)*

LA 1 (4 marks)

79. If $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$ and $\vec{b}=2\hat{i}+4\hat{j}-5\hat{k}$ represent two adjacent sides of a parallelogram, find unit vectors parallel to the diagonals of the parallelogram. *(2020)*

80. Using vectors, find the area of the triangle ABC with vertices $A(1, 2, 3)$, $B(2, -1, 4)$ and $C(4, 5, -1)$ *(2020, Delhi 2013, AI 2013)*

81. If $\hat{i}+\hat{j}+\hat{k}$, $2\hat{i}+5\hat{j}$, $3\hat{i}+2\hat{j}-3\hat{k}$ and $\hat{i}-6\hat{j}-\hat{k}$ respectively are the position vectors of points A, B, C and D, then find the angle between the straight lines AB and CD. Find whether $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear or not. *(Delhi 2019)*

82. Let $\vec{a}=4\hat{i}+5\hat{j}-\hat{k}, \vec{b}=\hat{i}-4\hat{j}+5\hat{k}$ and $\vec{c}=3\hat{i}+\hat{j}-\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{c}$ and $\vec{b}$ and $\vec{d}\cdot\vec{a}=21$. *(2018)*

83. If $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors of equal magnitudes, show that the vector $\vec{a}+\vec{b}+\vec{c}$ is equally inclined to $\vec{a}, \vec{b}$ and $\vec{c}$. Also, find the angle which $\vec{a}+\vec{b}+\vec{c}$ makes with $\vec{a}$ or $\vec{b}$ or $\vec{c}$. *(Delhi 2017)*

84. Show that the points A, B, C with position vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ respectively, are the vertices of a right-angled triangle. Hence find the area of the triangle. *(AI 2017)*

85. The two adjacent sides of a parallelogram are $2\hat{i}-4\hat{j}-5\hat{k}$ and $2\hat{i}+2\hat{j}+3\hat{k}$. Find the two unit vectors parallel to its diagonals. Using the diagonal vectors, find the area of the parallelogram. *(AI 2016)*

86. If $\vec{a}\times\vec{b}=\vec{c}\times\vec{d}$ and $\vec{a}\times\vec{c}=\vec{b}\times\vec{d}$, show that $\vec{a}-\vec{d}$ is parallel to $\vec{b}-\vec{c}$, where $\vec{a}\neq\vec{d}$ and $\vec{b}\neq\vec{c}$. *(Foreign 2016)*

87. If $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, find $(\vec{r}\times\hat{i})\cdot(\vec{r}\times\hat{j})+xy$. *(Delhi 2015)*

88. If $\vec{a}=\hat{i}+2\hat{j}+\hat{k}, \vec{b}=2\hat{i}+\hat{j}$ and $\vec{c}=3\hat{i}-4\hat{j}-5\hat{k}$, then find a unit vector perpendicular to both of the vectors $(\vec{a}-\vec{b})$ and $(\vec{c}-\vec{b})$. *(AI 2015)*

89. Vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ and $|\vec{a}|=3$, $|\vec{b}|=5$ and $|\vec{c}|=7$. Find the angle between $\vec{a}$ and $\vec{b}$. *(Delhi 2014)*

90. The scalar product of the vector $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of vectors $\vec{b}=2\hat{i}+4\hat{j}-5\hat{k}$ and $\vec{c}=\lambda\hat{i}+2\hat{j}+3\hat{k}$ is

equal to one. Find the value of λ and hence find the unit vector along $\vec{b}+\vec{c}$. *(AI 2014)*

91. Find a unit vector perpendicular to both of the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ where $\vec{a}=\hat{i}+\hat{j}+\hat{k}, \vec{b}=\hat{i}+2\hat{j}+3\hat{k}$. *(Foreign 2014)*

92. If $\vec{a}=2\hat{i}-3\hat{j}+\hat{k}, \vec{b}=-\hat{i}+\hat{k}$, $\vec{c}=2\hat{j}-\hat{k}$ are three vectors, find the area of the parallelogram having diagonals $(\vec{a}+\vec{b})$ and $(\vec{b}+\vec{c})$. *(Delhi 2014C)*

93. Find the vector $\vec{p}$ which is perpendicular to both $\vec{\alpha}=4\hat{i}+5\hat{j}-\hat{k}$ and $\vec{\beta}=\hat{i}-4\hat{j}+5\hat{k}$ and $\vec{p}\cdot\vec{q}=21$, where $\vec{q}=3\hat{i}+\hat{j}-\hat{k}$. *(AI 2014C)*

94. If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}+\vec{b}|=|\vec{a}|$, then prove that vector $2\vec{a}+\vec{b}$ is perpendicular to vector $\vec{b}$. *(Delhi 2013)*

95. If $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=\hat{j}-\hat{k}$, find a vector $\vec{c}$, such that $\vec{a}\times\vec{c}=\vec{b}$ and $\vec{a}\cdot\vec{c}=3$. *(Delhi 2013)*

96. If $\vec{a}=\hat{i}-\hat{j}+7\hat{k}$ and $\vec{b}=5\hat{i}-\hat{j}+\lambda\hat{k}$, then find the value of λ, so that $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are perpendicular vectors. *(AI 2013)*

97. If $\vec{a},\vec{b},\vec{c}$ are three mutually perpendicular vectors of the same magnitude, prove that $\vec{a}+\vec{b}+\vec{c}$ is equally inclined with the vectors $\vec{a},\vec{b}$ and $\vec{c}$. *(Delhi 2013C)*

98. Dot product of a vector with vectors $\hat{i}-\hat{j}+\hat{k}$, $2\hat{i}+\hat{j}-3\hat{k}$ and $\hat{i}+\hat{j}+\hat{k}$ are respectively 4, 0 and 2. Find the vector. *(Delhi 2013C)*

99. Find the values of λ for which the angle between the vectors $\vec{a}=2\lambda^2\hat{i}+4\lambda\hat{j}+\hat{k}$ and $\vec{b}=7\hat{i}-2\hat{j}+\lambda\hat{k}$ is obtuse. *(AI 2013C)*

100. If $\vec{a},\vec{b}$ and $\vec{c}$ are three vectors such that each one is perpendicular to the vector obtained by sum of the other two and $|\vec{a}|=3, |\vec{b}|=4$ and $|\vec{c}|=5$, then prove that $|\vec{a}+\vec{b}+\vec{c}|=5\sqrt{2}$. *(AI 2013C)*

101. If $\vec{a}=3\hat{i}-\hat{j}$ and $\vec{b}=2\hat{i}+\hat{j}-3\hat{k}$ then express $\vec{b}$ in the form $\vec{b}=\vec{b}_1+\vec{b}_2$ where $\vec{b}_1 \parallel \vec{a}$ and $\vec{b}_2 \perp \vec{a}$. *(AI 2013C)*

102. If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $|\vec{a}|=5, |\vec{b}|=12$ and $|\vec{c}|=13$ and $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, find the value of $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$. *(Delhi 2012)*

103. Let $\vec{a}=\hat{i}+4\hat{j}+2\hat{k}, \vec{b}=3\hat{i}-2\hat{j}+7\hat{k}$ and $\vec{c}=2\hat{i}-\hat{j}+4\hat{k}$. Find a vector $\vec{p}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{p}\cdot\vec{c}=18$. *(AI 2012)*

104. If the sum of two unit vectors $\hat{a}$ and $\hat{b}$ is a unit vector, show that the magnitude of their difference is $\sqrt{3}$. *(Delhi 2012C)*

105. Find a unit vector perpendicular to each of the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$, where $\vec{a}=3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$. *(Delhi 2011)*

106. If two vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}|=2, |\vec{b}|=1$ and $\vec{a}\cdot\vec{b}=1$, then find the value of $(3\vec{a}-5\vec{b})\cdot(2\vec{a}+7\vec{b})$. *(Delhi 2011)*

107. Using vectors, find the area of the triangle with vertices $A(1, 1, 2)$, $B(2, 3, 5)$ and $C(1, 5, 5)$. *(AI 2011)*

108. If $\vec{a},\vec{b}$ and $\vec{c}$ are three mutually perpendicular vectors of equal magnitude, show that $\vec{a}+\vec{b}+\vec{c}$ is equally inclined to $\vec{a},\vec{b}$ and $\vec{c}$. Also find the angle. *(Delhi 2011C)*

109. If $\vec{a},\vec{b}$ and $\vec{c}$ are three vectors such that $|\vec{a}|=3, |\vec{b}|=4$ and $|\vec{c}|=5$ and each one of them is perpendicular to the sum of the other two, then find $|\vec{a}+\vec{b}+\vec{c}|$. *(AI 2011C)*

10.7 Scalar Triple Product

VSA (1 mark)

110. Find λ, if the vectors $\vec{a}=\hat{i}+3\hat{j}+\hat{k}$, $\vec{b}=2\hat{i}-\hat{j}-\hat{k}$ and $\vec{c}=\lambda\hat{j}+3\hat{k}$ are coplanar. *(Delhi 2015)*

111. Find $\vec{a}\cdot(\vec{b}\times\vec{c})$, if $\vec{a}=2\hat{i}+\hat{j}+3\hat{k}, \vec{b}=-\hat{i}+2\hat{j}+\hat{k}$ and $\vec{c}=3\hat{i}+\hat{j}+2\hat{k}$. *(AI 2014)*

SA (2 marks)

112. Find the volume of the parallelepiped whose adjacent edges are represented by $2\vec{a}, -\vec{b}$ and $3\vec{c}$, where $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$, $\vec{b} = 3\hat{i} + 4\hat{j} - 5\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 3\hat{k}$. *(2020)*

113. If $\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k}, \vec{b} = \hat{i} - 2\hat{j} + \hat{k}$ and $\vec{c} = -3\hat{i} + \hat{j} + 2\hat{k}$, find $[\vec{a}\ \vec{b}\ \vec{c}]$. *(Delhi 2019)*

LA 1 (4 marks)

114. Find the value of x, for which the four points $A(x, -1, -1)$, $B(4, 5, 1)$, $C(3, 9, 4)$ and $D(-4, 4, 4)$ are coplanar. *(AI 2019)*

115. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$, then

(a) Let $c_1 = 1$ and $c_2 = 2$, find c_3 which makes $\vec{a}$, $\vec{b}$ and $\vec{c}$ coplanar.

(b) If $c_2 = -1$ and $c_3 = 1$, show that no value of c_1 can make $\vec{a}$, $\vec{b}$ and $\vec{c}$ coplanar. *(Delhi 2017)*

116. Find the value of λ, if four points with position vectors $3\hat{i} + 6\hat{j} + 9\hat{k}$, $\hat{i} + 2\hat{j} + 3\hat{k}$, $2\hat{i} + 3\hat{j} + \hat{k}$ and $4\hat{i} + 6\hat{j} + \lambda\hat{k}$ are coplanar. *(AI 2017)*

117. Show that the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar if $\vec{a} + \vec{b}, \vec{b} + \vec{c}$ and $\vec{c} + \vec{a}$ are coplanar. *(Delhi 2016)*

118. Find the value of λ so that the four points A, B, C and D with position vectors $4\hat{i} + 5\hat{j} + \hat{k}$, $-\hat{j} - \hat{k}$, $3\hat{i} + \lambda\hat{j} + 4\hat{k}$ and $-4\hat{i} + 4\hat{j} + 4\hat{k}$ respectively are coplanar. *(Delhi 2015C)*

119. Prove that : $[\vec{a}, \vec{b} + \vec{c}, \vec{d}] = [\vec{a}, \vec{b}, \vec{d}] + [\vec{a}, \vec{c}, \vec{d}]$. *(AI 2015C)*

120. Prove that, for any three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$
$[\vec{a} + \vec{b}\ \ \vec{b} + \vec{c}\ \ \vec{c} + \vec{a}] = 2[\vec{a}\ \vec{b}\ \vec{c}]$ *(Delhi 2014)*

121. Show that the four points A, B, C and D with position vectors $4\hat{i} + 5\hat{j} + \hat{k}, -\hat{j} - \hat{k}$, $3\hat{i} + 9\hat{j} + 4\hat{k}$ and $4(-\hat{i} + \hat{j} + \hat{k})$ respectively are coplanar. *(AI 2014)*

122. Show that the vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar if and only if $\vec{a} + \vec{b}, \vec{b} + \vec{c}$ and $\vec{c} + \vec{a}$ are coplanar. *(Foreign 2014)*

123. If the three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar, prove that the vectors $\vec{a} + \vec{b}, \vec{b} + \vec{c}$ and $\vec{c} + \vec{a}$ are also coplanar. *(Delhi 2014C, 2013C)*

Detailed Solutions

1. Here, $l = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, $m = \cos\frac{\pi}{2} = 0$, $n = \cos\theta$

Since, $l^2 + m^2 + n^2 = 1$

$$\Rightarrow \frac{1}{2} + 0 + \cos^2\theta = 1 \Rightarrow \cos^2\theta = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\Rightarrow \cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4} \quad \therefore n = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$\therefore$ The vector of magnitude $5\sqrt{2}$ is

$$\vec{a} = 5\sqrt{2}(l\hat{i} + m\hat{j} + n\hat{k})$$

$$= 5\sqrt{2}\left(\frac{1}{\sqrt{2}}\hat{i} + 0\hat{j} + \frac{1}{\sqrt{2}}\hat{k}\right) = 5(\hat{i} + \hat{k})$$

2. $l = \cos\frac{\pi}{3} = \frac{1}{2}, m = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $n = \cos\theta$

Now, $l^2 + m^2 + n^2 = 1$

$$\Rightarrow \left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + n^2 = 1$$

$$\Rightarrow \frac{1}{4} + \frac{1}{2} + n^2 = 1 \Rightarrow n^2 = \frac{1}{4} \Rightarrow n = \pm\frac{1}{2}$$

$$\Rightarrow \cos\theta = \pm\frac{1}{2}$$

But θ is an acute angle (given).

$$\therefore \quad \theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

3. Here, $\vec{a} = 3\hat{i} - 2\hat{j} + 6\hat{k}$

$\therefore$ Its magnitude $= |\vec{a}|$

$$= \sqrt{3^2 + (-2)^2 + 6^2} = \sqrt{9 + 4 + 36} = \sqrt{49} = 7.$$

4. **(b) :** Let $a=(\hat{i}+\hat{j}+\hat{k})$

So, unit vector of $\vec{a}=\dfrac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{1+1+1}}=\dfrac{1}{\sqrt{3}}(\hat{i}+\hat{j}+\hat{k})$

$\therefore$ The value of p is $\dfrac{1}{\sqrt{3}}$.

5. **(a) :** $\overrightarrow{EA}+\overrightarrow{EB}+\overrightarrow{EC}+\overrightarrow{ED}$

$=\overrightarrow{EA}+\overrightarrow{EB}-\overrightarrow{EA}-\overrightarrow{EB}$

[As diagonals of a rhombus bisect each other]

$=\vec{0}$

6. The given vectors are

$\vec{a}=\hat{i}-2\hat{j}+\hat{k},\ \vec{b}=-2\hat{i}+4\hat{j}+5\hat{k},\ \vec{c}=\hat{i}-6\hat{j}-7\hat{k}$

$\therefore$ Their sum $=\vec{a}+\vec{b}+\vec{c}$

$=(\hat{i}-2\hat{j}+\hat{k})+(-2\hat{i}+4\hat{j}+5\hat{k})+(\hat{i}-6\hat{j}-7\hat{k})$

$=-4\hat{j}-\hat{k}$.

7. Required sum $=\vec{a}+\vec{b}+\vec{c}$

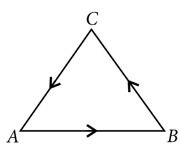
$=(\hat{i}-3\hat{k})+(2\hat{j}-\hat{k})+(2\hat{i}-3\hat{j}+2\hat{k})$

$=3\hat{i}-\hat{j}-2\hat{k}$.

8. Required sum $=\vec{a}+\vec{b}+\vec{c}$

$=(\hat{i}-2\hat{j})+(2\hat{i}-3\hat{j})+(2\hat{i}+3\hat{k})=5\hat{i}-5\hat{j}+3\hat{k}$.

9. Let ABC be the given triangle.



Now $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$

(By Triangle law)

$\Rightarrow \overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=\overrightarrow{AC}+\overrightarrow{CA}=\vec{0}$

10. Required position vector of point P

$$=\frac{1(2\hat{i}-\hat{j}-\hat{k})+2(2\hat{i}-\hat{j}+2\hat{k})}{2+1}$$

$$=\frac{2\hat{i}-\hat{j}-\hat{k}+4\hat{i}-2\hat{j}+4\hat{k}}{3}$$

$$=\frac{1}{3}(6\hat{i}-3\hat{j}+3\hat{k})=2\hat{i}-\hat{j}+\hat{k}$$

11. Required position vector

$$=\frac{2\cdot(2\vec{a}+\vec{b})-1(\vec{a}-2\vec{b})}{2-1}=\frac{4\vec{a}+2\vec{b}-\vec{a}+2\vec{b}}{1}$$

$=3\vec{a}+4\vec{b}$

12. Required position vector

$$=\frac{2(2\vec{a}+3\vec{b})+1(3\vec{a}-2\vec{b})}{2+1}=\frac{7\vec{a}+4\vec{b}}{3}=\frac{7}{3}\vec{a}+\frac{4}{3}\vec{b}$$

13. Let $a=2\hat{i}+3\hat{j}-\hat{k}$ and $b=4\hat{i}-3\hat{j}+2\hat{k}$.

Then, the sum of the given vectors is

$\vec{c}=\vec{a}+\vec{b}=(2+4)\hat{i}+(3-3)\hat{j}+(-1+2)\hat{k}=6\hat{i}+\hat{k}$

and $|\vec{c}|=|\vec{a}+\vec{b}|=\sqrt{6^2+1^2}=\sqrt{36+1}=\sqrt{37}$

$\therefore$ Unit vector, $\hat{c}=\dfrac{\vec{c}}{|\vec{c}|}=\dfrac{6\hat{i}+\hat{k}}{\sqrt{37}}=\dfrac{6}{\sqrt{37}}\hat{i}+\dfrac{1}{\sqrt{37}}\hat{k}$

14. A unit vector in the direction of $\vec{a}=\hat{i}-2\hat{j}$

is $\hat{a}=\dfrac{\vec{a}}{|\vec{a}|}=\dfrac{\hat{i}-2\hat{j}}{\sqrt{1^2+(-2)^2}}=\dfrac{1}{\sqrt{5}}(\hat{i}-2\hat{j})$

$\therefore$ The required vector of magnitude 7 in the direction of $\vec{a}=7\cdot\hat{a}=\dfrac{7}{\sqrt{5}}(\hat{i}-2\hat{j})$.

15. $\vec{a}=\hat{i}+\hat{j}-2\hat{k};\ \vec{b}=2\hat{i}-4\hat{j}+5\hat{k}$

$\therefore\ 3\vec{a}+2\vec{b}=3(\hat{i}+\hat{j}-2\hat{k})+2(2\hat{i}-4\hat{j}+5\hat{k})$

$=(3\hat{i}+3\hat{j}-6\hat{k})+(4\hat{i}-8\hat{j}+10\hat{k})=7\hat{i}-5\hat{j}+4\hat{k}$

$\therefore$ The direction ratios of the vector $3\vec{a}+2\vec{b}$ are $7,-5,4$.

16. *Refer to answer 13.*

17. Let $\vec{a}=3\hat{i}+2\hat{j}+9\hat{k}$ and $\vec{b}=\hat{i}-2p\hat{j}+3\hat{k}$

For $\vec{a}$ and $\vec{b}$ to be parallel, $\vec{b}=\lambda\vec{a}$.

$\Rightarrow \hat{i}-2p\hat{j}+3\hat{k}=\lambda(3\hat{i}+2\hat{j}+9\hat{k})=3\lambda\hat{i}+2\lambda\hat{j}+9\lambda\hat{k}$

$\Rightarrow 1=3\lambda;\ -2p=2\lambda,\ 3=9\lambda$

$\Rightarrow \lambda=\dfrac{1}{3}$ and $p=-\lambda=-\dfrac{1}{3}$

18. Let $\vec{a}=2\hat{i}-3\hat{j}+6\hat{k}$.

The vector in the direction of $\vec{a}$ with a magnitude of $21=21\times\hat{a}$

$\therefore$ Required vector $=21\times\dfrac{2\hat{i}-3\hat{j}+6\hat{k}}{\sqrt{2^2+(-3)^2+6^2}}$

$=21\times\dfrac{2\hat{i}-3\hat{j}+6\hat{k}}{7}=6\hat{i}-9\hat{j}+18\hat{k}$

19. We have $\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}$

$=(4\hat{i}+5\hat{j}+6\hat{k})-(\hat{i}+3\hat{j})=3\hat{i}+2\hat{j}+6\hat{k}$

Required unit vector $=\dfrac{3\hat{i}+2\hat{j}+6\hat{k}}{\sqrt{3^2+2^2+6^2}}=\dfrac{3\hat{i}+2\hat{j}+6\hat{k}}{7}$

20. *Refer to answer 18.*

21. Given, $\vec{a} = \vec{b}$

$\Rightarrow \quad x\hat{i} + 2\hat{j} - z\hat{k} = 3\hat{i} - y\hat{j} + \hat{k}$

$\therefore \quad x = 3, y = -2, z = -1$

Hence, the value of $x + y + z = 0$

22. *Refer to answer 13.*

23. *Refer to answer 11.*

24. *Refer to answer 12.*

25. *Refer to answer 11.*

26. Vector $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (-5\hat{i} + 7\hat{j}) - (2\hat{i} + \hat{j}) = -7\hat{i} + 6\hat{j}$

So, its scalar components are $(-7, 6)$.

27. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} - 3\hat{j} + 5\hat{k}$

$\therefore \vec{a} + \vec{b} = (\hat{i} + \hat{j} + \hat{k}) + (2\hat{i} - 3\hat{j} + 5\hat{k})$

$= 3\hat{i} - 2\hat{j} + 6\hat{k}$

Any vector parallel to $\vec{a} + \vec{b}$

$= \lambda(\vec{a} + \vec{b}) = \lambda(3\hat{i} - 2\hat{j} + 6\hat{k})$

$\therefore$ The unit vector in this direction

$$= \frac{\lambda(3\hat{i} - 2\hat{j} + 6\hat{k})}{\sqrt{(3\lambda)^2 + (-2\lambda)^2 + (6\lambda)^2}}$$

$$= \frac{\lambda(3\hat{i} - 2\hat{j} + 6\hat{k})}{|\lambda| \cdot 7} = \pm\frac{1}{7}(3\hat{i} - 2\hat{j} + 6\hat{k})$$

28. The given vector is $\vec{a} = 3\hat{i} - 2\hat{j} + 6\hat{k}$

$\Rightarrow |\vec{a}| = \sqrt{3^2 + (-2)^2 + 6^2} = 7$

$\therefore$ A unit vector in the direction of vector $\vec{a}$ is

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{7}(3\hat{i} - 2\hat{j} + 6\hat{k})$$

29. We have, $\vec{a} = -2\hat{i} + \hat{j} - 5\hat{k}$

Direction cosines of the given vector are

$$\left(\frac{-2}{\sqrt{(-2)^2 + (1)^2 + (-5)^2}}, \frac{1}{\sqrt{(-2)^2 + (1)^2 + (-5)^2}}, \frac{-5}{\sqrt{(-2)^2 + (1)^2 + (-5)^2}}\right)$$

$$= \left(\frac{-2}{\sqrt{4+1+25}}, \frac{1}{\sqrt{4+1+25}}, \frac{-5}{\sqrt{4+1+25}}\right)$$

$\therefore$ Direction cosines are $\left(\dfrac{-2}{\sqrt{30}}, \dfrac{1}{\sqrt{30}}, \dfrac{-5}{\sqrt{30}}\right)$

30. We have, $2\hat{i} - 3\hat{j} + 4\hat{k}$ and $a\hat{i} + 6\hat{j} - 8\hat{k}$

Two vectors are collinear if and only if,

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \lambda \Rightarrow \frac{2}{a} = \frac{-3}{6} = \frac{4}{-8} = \frac{-1}{2} = \lambda$$

$$\Rightarrow \frac{2}{a} = \frac{-1}{2} \Rightarrow a = -4$$

31. $\vec{a} = 2\hat{i} + \hat{j} + 2\hat{k}$

$|\vec{a}| = \sqrt{(2)^2 + (1)^2 + (2)^2} = \sqrt{4+1+4} = \sqrt{9} = 3.$

Required unit vector is $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|} = \dfrac{2\hat{i} + \hat{j} + 2\hat{k}}{3}$

$= \dfrac{2}{3}\hat{i} + \dfrac{1}{3}\hat{j} + \dfrac{2}{3}\hat{k}$

32. *Refer to answer 31.*

33. *Refer to answer 31.*

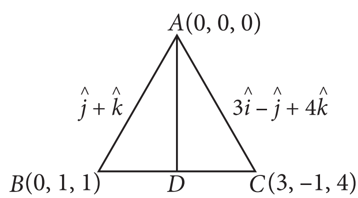
34. Position vector which divides the line segment joining points with position vectors $3\vec{a} + \vec{b}$ and $\vec{a} - 3\vec{b}$ in the ratio 2 : 1 externally is given by

$$\frac{2(\vec{a} - 3\vec{b}) - 1(3\vec{a} + \vec{b})}{2-1} = \frac{2\vec{a} - 6\vec{b} - 3\vec{a} - \vec{b}}{1}$$

$$= -\vec{a} - 7\vec{b}$$

35. Take A to be as origin (0, 0, 0).

$\therefore$ Coordinates of B are (0, 1, 1) and coordinates of C are (3, –1, 4).



Let D be the mid point of BC and AD is a median of ΔABC.

$\therefore$ Coordinates of D are $\left(\dfrac{3}{2}, 0, \dfrac{5}{2}\right)$

So, length of $AD = \sqrt{\left(\dfrac{3}{2} - 0\right)^2 + (0)^2 + \left(\dfrac{5}{2} - 0\right)^2}$

$= \sqrt{\dfrac{9}{4} + \dfrac{25}{4}} = \dfrac{\sqrt{34}}{2}$ units

36. $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$, $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$

$\therefore \quad \vec{a} + \vec{b} = (2\hat{i} + 3\hat{j} - \hat{k}) + (\hat{i} - 2\hat{j} + \hat{k}) = 3\hat{i} + \hat{j}$

$|\vec{a}+\vec{b}| = \sqrt{3^2+1^2} = \sqrt{10}$

$\therefore$ A vector of magnitude 5 in the direction of

$\vec{a}+\vec{b}$ is $\frac{5(\vec{a}+\vec{b})}{|\vec{a}+\vec{b}|} = \frac{5(3\hat{i}+\hat{j})}{\sqrt{10}}$

37. **(c) :** Here, $\vec{a}=\hat{i}-2\hat{j}+3\hat{k}, \vec{b}=2\hat{i}+\lambda\hat{k}$

Since, projection of $\vec{a}$ on $\vec{b}=0$

$$\Rightarrow \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}=0 \Rightarrow \frac{(\hat{i}-2\hat{j}+3\hat{k})\cdot(2\hat{i}+\lambda\hat{k})}{\sqrt{2^2+\lambda^2}}=0$$

$$\Rightarrow \frac{2+3\lambda}{\sqrt{4+\lambda^2}}=0 \Rightarrow 2+3\lambda=0 \Rightarrow \lambda=-\frac{2}{3}$$

38. Given, two diagonals $\vec{d}_1$ and $\vec{d}_2$ are $2\hat{i}$ and $-3\hat{k}$ respectively.

$$\therefore\ \vec{d}_1\times\vec{d}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 0 & 0\\ 0 & 0 & -3\end{vmatrix}=\hat{i}(0)-\hat{j}(-6-0)+\hat{k}(0)=6\hat{j}$$

So, area of the parallelogram = $\frac{1}{2}|\vec{d}_1\times\vec{d}_2|$

$=\frac{1}{2}\times 6 = 3$ sq. units

39. Let $\vec{a}=2\hat{i}-\lambda\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}-\hat{k}$

We know, $\vec{a}$ and $\vec{b}$ are orthogonal iff $\vec{a}\cdot\vec{b}=0$

$\Rightarrow (2\hat{i}-\lambda\hat{j}+\hat{k})\cdot(\hat{i}+2\hat{j}-\hat{k})=0$

$\Rightarrow 2-2\lambda-1=0 \Rightarrow 1-2\lambda=0 \Rightarrow \lambda=\frac{1}{2}$

40. **(c) :** Since, $\hat{i}, \hat{j}, \hat{k}$ are mutually perpendicular.

$\therefore\ \hat{i}\cdot\hat{k}=0$

41. Given, $|\vec{a}|=|\vec{b}|, \theta=60°$ and $\vec{a}\cdot\vec{b}=\frac{9}{2}$

Now, $\cos\theta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

$\Rightarrow \cos 60° = \frac{9/2}{|\vec{a}|^2} \Rightarrow \frac{1}{2}=\frac{9/2}{|\vec{a}|^2}$

$\Rightarrow |\vec{a}|^2=9 \Rightarrow |\vec{a}|=3 \therefore |\vec{a}|=|\vec{b}|=3$

42. Given, $\vec{a}=2\hat{i}+\hat{j}+2\hat{k}$ and $\vec{b}=\hat{j}+\hat{k}$

Unit vectors perpendicular to $\vec{a}$ and $\vec{b}$ are

$\pm\left(\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}\right)$.

Now, $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 1 & 2\\ 0 & 1 & 1\end{vmatrix}=-\hat{i}-2\hat{j}+2\hat{k}$

$\therefore$ Unit vectors perpendicular to $\vec{a}$ and $\vec{b}$ are

$$\pm\frac{(-\hat{i}-2\hat{j}+2\hat{k})}{\sqrt{(-1)^2+(-2)^2+(2)^2}}=\pm\left(-\frac{1}{3}\hat{i}-\frac{2}{3}\hat{j}+\frac{2}{3}\hat{k}\right)$$

So, there are two unit vectors perpendicular to the given vectors.

43. We have $\vec{a},\vec{b},\vec{c}$ are unit vectors.

Therefore, $|\vec{a}|=1, |\vec{b}|=1$ and $|\vec{c}|=1$

Also, $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ (given)

$\Rightarrow |\vec{a}+\vec{b}+\vec{c}|^2=0$

$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0$

$\Rightarrow 1+1+1+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0$

$\Rightarrow 3+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0$

$\Rightarrow (\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=-\frac{3}{2}$

44. $|\vec{a}\times\vec{b}|^2+|\vec{a}\cdot\vec{b}|^2=400$

$\Rightarrow \{|\vec{a}||\vec{b}|\sin\theta\}^2+\{|\vec{a}||\vec{b}|\cos\theta\}^2=400$

$\Rightarrow |\vec{a}|^2|\vec{b}|^2\sin^2\theta+|\vec{a}|^2|\vec{b}|^2\cos^2\theta=400$

$\Rightarrow |\vec{a}|^2|\vec{b}|^2=400 \Rightarrow 25\times|\vec{b}|^2=400$ $[\because |\vec{a}|=5]$

$\Rightarrow |\vec{b}|^2=16 \Rightarrow |\vec{b}|=4$

45. Projection of $\vec{a}$ on $\vec{b}=\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

$$=\frac{(7\hat{i}+\hat{j}-4\hat{k})\cdot(2\hat{i}+6\hat{j}+3\hat{k})}{\sqrt{(2)^2+(6)^2+(3)^2}}=\frac{14+6-12}{7}=\frac{8}{7}$$

46. Here $\hat{a},\hat{b}$ and $\hat{c}$ are mutually perpendicular unit vectors.

$\Rightarrow |\hat{a}|=|\hat{b}|=|\hat{c}|=1$ and $\hat{a}\cdot\hat{b}=\hat{b}\cdot\hat{c}=\hat{c}\cdot\hat{a}=0$ ...(1)

$\therefore |2\hat{a}+\hat{b}+\hat{c}|^2=(2\hat{a}+\hat{b}+\hat{c}).(2\hat{a}+\hat{b}+\hat{c})$

$=4\hat{a}\cdot\hat{a}+2\hat{a}\cdot\hat{b}+2\hat{a}\cdot\hat{c}+2\hat{b}\cdot\hat{a}+\hat{b}\cdot\hat{b}+\hat{b}\cdot\hat{c}+2\hat{c}\cdot\hat{a}$
$+\hat{c}\cdot\hat{b}+\hat{c}\cdot\hat{c}$

$=4|\hat{a}|^2+|\hat{b}|^2+|\hat{c}|^2+4\hat{a}\cdot\hat{b}+2\hat{b}\cdot\hat{c}+4\hat{a}\cdot\hat{c}$

$(\because b\cdot a=a\cdot b, c\cdot a=a\cdot c, c\cdot b=b\cdot c)$

$= 4\cdot1^2 + 1^2 + 1^2$ [Using (1)]
$= 6$

$\therefore\ |2\hat{a}+\hat{b}+\hat{c}| = \sqrt{6}.$

47. Here, $\vec{a} = \hat{i}+\hat{j}+\hat{k}$ and $\vec{b} = \hat{i}+\hat{j}$

Vector perpendicular to both $\vec{a}$ and $\vec{b}$ is

$$\vec{a}\times\vec{b} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & 1\\ 1 & 1 & 0\end{vmatrix} = -\hat{i}+\hat{j}+0\hat{k} = -\hat{i}+\hat{j}$$

$\therefore$ Unit vector perpendicular to both $\vec{a}$ and $\vec{b}$

$$= \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|} = \frac{-\hat{i}+\hat{j}}{\sqrt{(-1)^2+1^2}} = \frac{1}{\sqrt{2}}\left(-\hat{i}+\hat{j}\right).$$

48. Let $\vec{a} = 2\hat{i}-3\hat{k}$ and $\vec{b} = 4\hat{j}+2\hat{k}$

The area of a parallelogram with $\vec{a}$ and $\vec{b}$ as its adjacent sides is given by $|\vec{a}\times\vec{b}|$.

$$\text{Now, } \vec{a}\times\vec{b} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 0 & -3\\ 0 & 4 & 2\end{vmatrix} = 12\hat{i}-4\hat{j}+8\hat{k}$$

$$\therefore\ |\vec{a}\times\vec{b}| = \sqrt{(12)^2+(-4)^2+(8)^2} = \sqrt{144+16+64}$$

$= \sqrt{224} = 4\sqrt{14}$ sq. units.

49. Let θ be the angle between the unit vectors $\vec{a}$ and $\vec{b}$.

$$\therefore\ \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \vec{a}\cdot\vec{b} \qquad (\because\ |\vec{a}| = 1 = |\vec{b}|)\ \ldots(1)$$

Now $1 = \left|\sqrt{2}\,\vec{a}-\vec{b}\right|$

$$\Rightarrow\ 1 = \left|\sqrt{2}\,\vec{a}-\vec{b}\right|^2 = \left(\sqrt{2}\,\vec{a}-\vec{b}\right)\cdot\left(\sqrt{2}\,\vec{a}-\vec{b}\right)$$

$$= 2|\vec{a}|^2 - \sqrt{2}\,\vec{a}\cdot\vec{b} - \vec{b}\cdot\sqrt{2}\,\vec{a} + |\vec{b}|^2 = 2-2\sqrt{2}\,\vec{a}\cdot\vec{b}+1$$

$(\because\ \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a})$

$= 3-2\sqrt{2}\ \vec{a}\cdot\vec{b}$

$$\Rightarrow\ \vec{a}\cdot\vec{b} = \frac{1}{\sqrt{2}}\ \Rightarrow\ \cos\theta = \frac{1}{\sqrt{2}} \qquad \text{[By using (1)]}$$

$\therefore\ \theta = \pi/4$

50. *Refer to answer 45.*

51. *Refer to answer 45.*

52. Given $|\vec{a}| = 1 = |\vec{b}|$, $|\vec{a}+\vec{b}| = 1$

$\Rightarrow |\vec{a}+\vec{b}|^2 = 1 \Rightarrow (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = 1$

$\Rightarrow \vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b} = 1$

$\Rightarrow |\vec{a}|^2+2\vec{a}\cdot\vec{b}+|\vec{b}|^2 = 1 \Rightarrow 1+2\vec{a}\cdot\vec{b}+1 = 1$

$\Rightarrow 2\vec{a}\cdot\vec{b} = -1 \Rightarrow 2|\vec{a}|\cdot|\vec{b}|\cos\theta = -1$

$\Rightarrow 2\cdot1\cdot1\cos\theta = -1$

$$\Rightarrow\ \cos\theta = -\frac{1}{2} \Rightarrow \theta = 120°$$

53. Given, $|\vec{a}| = 3, |\vec{b}| = \frac{2}{3}, |\vec{a}\times\vec{b}| = 1$

$$\Rightarrow |\vec{a}||\vec{b}|\sin\theta = 1\ \Rightarrow 3.\frac{2}{3}\sin\theta = 1$$

$$\Rightarrow \sin\theta = \frac{1}{2}\ \Rightarrow \theta = \frac{\pi}{6}$$

54. Given: $\vec{a}\perp\vec{b}\ \Rightarrow \vec{a}\cdot\vec{b} = 0$

Also, $|\vec{a}| = 5$ and $|\vec{a}+\vec{b}| = 13$

$\Rightarrow |\vec{a}+\vec{b}|^2 = 13^2 \Rightarrow (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = 169$

$\Rightarrow \vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b} = 169$

$\Rightarrow |\vec{a}|^2+0+0+|\vec{b}|^2 = 169$

$\Rightarrow |\vec{b}|^2 = 169-|\vec{a}|^2 = 169-5^2 = 144$

$\Rightarrow |\vec{b}| = 12$

55. The projection of the vector $\left(\hat{i}+\hat{j}+\hat{k}\right)$ along the vector $\hat{j}$ is $(\hat{i}+\hat{j}+\hat{k})\cdot\left(\dfrac{\hat{j}}{\sqrt{0^2+1^2+0^2}}\right) = 1$

56. We have,

$\hat{i}\times\left(\hat{j}+\hat{k}\right)+\hat{j}\times\left(\hat{k}+\hat{i}\right)+\hat{k}\times\left(\hat{i}+\hat{j}\right)$

$= \hat{i}\times\hat{j}+\hat{i}\times\hat{k}+\hat{j}\times\hat{k}+\hat{j}\times\hat{i}+\hat{k}\times\hat{i}+\hat{k}\times\hat{j}$

$= \hat{k}-\hat{j}+\hat{i}-\hat{k}+\hat{j}-\hat{i} = \vec{0}.$

57. *Refer to answer 45.*

58. *Refer to answer 49.*

59. Let θ be the angle between the vector $\vec{a} = \hat{i}+\hat{j}+\hat{k}$ and y-axis *i.e.*, $\vec{b} = \hat{j}$

$$\therefore \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{\left(\hat{i}+\hat{j}+\hat{k}\right).\hat{j}}{\left|\hat{i}+\hat{j}+\hat{k}\right|\left|\hat{j}\right|}$$

$$= \frac{1}{\sqrt{1^2+1^2+1^2}\sqrt{1^2}} = \frac{1}{\sqrt{3}}$$

60. Let angle between the vectors $\vec{a}$ and $\vec{b}$ be θ.

Given: $|\vec{a}| = 8, |\vec{b}| = 3$ and $|\vec{a}\times\vec{b}| = 12$

$\Rightarrow |\vec{a}|\cdot|\vec{b}|\cdot|\sin\theta| = 12\ \Rightarrow 8\times3\sin\theta = 12$

$$\Rightarrow \sin\theta = \frac{12}{24} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}.$$

61. Here, $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and vector along x-axis is $\hat{i}$.

$\therefore$ Angle between $\vec{a}$ and $\hat{i}$ is given by

$$\cos\theta=\frac{\vec{a}.\hat{i}}{\sqrt{1^2+1^2+1^2}.\sqrt{1^2}}=\frac{(\hat{i}+\hat{j}+\hat{k})\cdot\hat{i}}{\sqrt{3}\cdot 1}=\frac{1}{\sqrt{3}}$$

$$\Rightarrow\theta=\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

62. Here $(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a})=15$, where $\vec{a}$ is unit vector.

$\Rightarrow\vec{x}\cdot\vec{x}+\vec{x}\cdot\vec{a}-\vec{a}\cdot\vec{x}-\vec{a}\cdot\vec{a}=15$

$\Rightarrow|\vec{x}|^2-|\vec{a}|^2=15$ $\quad(\because\vec{x}\cdot\vec{a}=\vec{a}\cdot\vec{x})$

$\Rightarrow|\vec{x}|^2-1=15$ $\quad(\because|\vec{a}|=1)$

$\Rightarrow|\vec{x}|^2=16=4^2\Rightarrow|\vec{x}|=4$

63. Here, $\vec{a}=2\hat{i}+\lambda\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}+3\hat{k}$

For $\vec{a}$ is perpendicular to $\vec{b}$, $\vec{a}\cdot\vec{b}=0$

$\Rightarrow(2\hat{i}+\lambda\hat{j}+\hat{k})\cdot(\hat{i}-2\hat{j}+3\hat{k})=0$

$\Rightarrow 2\times 1+\lambda(-2)+1\times 3=0$

$\Rightarrow 2-2\lambda+3=0$

$\Rightarrow\lambda=\frac{5}{2}$

64. *Refer to answer 63.*

65. Here, $\vec{a}=2\hat{i}-2\hat{j}+\hat{k}$

$\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$ and $\vec{c}=2\hat{i}-\hat{j}+4\hat{k}$

$\Rightarrow\quad\vec{b}+\vec{c}=3\hat{i}+\hat{j}+2\hat{k}$

$\therefore$ Projection of $\vec{b}+\vec{c}$ on $\vec{a}$

$$=\frac{(\vec{b}+\vec{c}).\vec{a}}{|\vec{a}|}=\frac{(3\hat{i}+\hat{j}+2\hat{k}).(2\hat{i}-2\hat{j}+\hat{k})}{|2\hat{i}-2\hat{j}+\hat{k}|}$$

$$=\frac{3\times 2+1\times(-2)+2\times 1}{\sqrt{2^2+(-2)^2+1^2}}=\frac{6}{3}=2$$

66. Here, $\vec{a}=\lambda\hat{i}+\hat{j}+4\hat{k}$, $\vec{b}=2\hat{i}+6\hat{j}+3\hat{k}$

Given : Projection of $\vec{a}$ on $\vec{b}=4$

$$\Rightarrow\quad\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}=4\Rightarrow\frac{(\lambda\hat{i}+\hat{j}+4\hat{k})\cdot(2\hat{i}+6\hat{j}+3\hat{k})}{|2\hat{i}+6\hat{j}+3\hat{k}|}=4$$

$$\Rightarrow\quad\frac{2\lambda+6+12}{\sqrt{2^2+6^2+3^2}}=4$$

$\Rightarrow\quad 2\lambda+18=4\times 7$

$\Rightarrow\quad 2\lambda=28-18=10\Rightarrow\lambda=5.$

67. $(\hat{i}\times\hat{j})\cdot\hat{k}+\hat{i}\cdot\hat{j}=\hat{k}\cdot\hat{k}+\hat{i}\cdot\hat{j}=1+0=1$

68. $(\hat{k}\times\hat{j})\cdot\hat{i}+\hat{j}\cdot\hat{k}=-\hat{i}\cdot\hat{i}+\hat{j}\cdot\hat{k}=-1+0=-1$

69. $(\hat{k}\times\hat{i})\cdot\hat{j}+\hat{i}\cdot\hat{k}=\hat{j}\cdot\hat{j}+0=1+0=1$

70. Let θ be the angle between $\vec{a}$ and $\vec{b}$, then

$$\cos\theta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}=\frac{\sqrt{6}}{\sqrt{3}\cdot 2}\times\frac{\sqrt{3}}{\sqrt{3}}$$

$$=\frac{\sqrt{18}}{3\times 2}=\frac{3\sqrt{2}}{3\times 2}=\frac{\sqrt{2}}{2}=\frac{1}{\sqrt{2}}$$

$\therefore\quad\theta=\cos^{-1}\frac{1}{\sqrt{2}}\Rightarrow\theta=\frac{\pi}{4}.$

71. Let $\vec{a}=\hat{i}-\hat{j}$ and $\vec{b}=\hat{i}+\hat{j}$

$|\vec{b}|=\sqrt{1^2+1^2}=\sqrt{1+1}=\sqrt{2}$

Also, $\vec{a}\cdot\vec{b}=(\hat{i}-\hat{j})\cdot(\hat{i}+\hat{j})=\hat{i}\cdot\hat{i}+\hat{i}\cdot\hat{j}-\hat{j}\cdot\hat{i}-\hat{j}\cdot\hat{j}=1-1=0$

$\therefore\quad$ Projection of $\vec{a}$ on $\vec{b}=\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}=\frac{0}{\sqrt{2}}=0$

72. Here, $|\vec{a}|=\sqrt{3},|\vec{b}|=2$ and angle between $\vec{a}$ and $\vec{b}$ is 60°.

Now $\vec{a}\cdot\vec{b}=|\vec{a}|\cdot|\vec{b}|\cdot\cos 60^\circ=\sqrt{3}\times 2\times\frac{1}{2}=\sqrt{3}.$

73. Here, $\vec{a}=5\hat{i}+6\hat{j}-2\hat{k}$ and $\vec{b}=7\hat{i}+6\hat{j}+2\hat{k}$

Vector perpendicular to both $\vec{a}$ and $\vec{b}$ is

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\5&6&-2\\7&6&2\end{vmatrix}$$

$=\hat{i}(12+12)-\hat{j}(10+14)+\hat{k}(30-42)$

$=24\hat{i}-24\hat{j}-12\hat{k}=12(2\hat{i}-2\hat{j}-\hat{k})$

$\therefore\quad$ Unit vector perpendicular to both $\vec{a}$ and $\vec{b}$

$$=\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}=\frac{24\hat{i}-24\hat{j}-12\hat{k}}{\sqrt{576+576+144}}=\frac{12(2\hat{i}-2\hat{j}-\hat{k})}{\sqrt{1296}}$$

$$=\frac{12(2\hat{i}-2\hat{j}-\hat{k})}{36}=\frac{1}{3}(2\hat{i}-2\hat{j}-\hat{k})$$

74. For any two non-zero vectors $\vec{a}$ and $\vec{b}$, we have

$|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|\Rightarrow|\vec{a}+\vec{b}|^2=|\vec{a}-\vec{b}|^2$

$\Rightarrow\quad\vec{a}^2+\vec{b}^2+2\vec{a}\cdot\vec{b}=\vec{a}^2+\vec{b}^2-2\vec{a}\cdot\vec{b}$

$\Rightarrow\quad 4\vec{a}\cdot\vec{b}=0\Rightarrow\vec{a}\cdot\vec{b}=0$

So, $\vec{a}$ and $\vec{b}$ are perpendicular vectors.

75. Let $A(2\hat{i}-\hat{j}+\hat{k})$, $B(3\hat{i}+7\hat{j}+\hat{k})$ and $C(5\hat{i}+6\hat{j}+2\hat{k})$

Then, $\overrightarrow{AB}=(3-2)\hat{i}+(7+1)\hat{j}+(1-1)\hat{k}=\hat{i}+8\hat{j}$

$\overrightarrow{AC} = (5-2)\hat{i} + (6+1)\hat{j} + (2-1)\hat{k} = 3\hat{i} + 7\hat{j} + \hat{k}$

$\overrightarrow{BC} = (5-3)\hat{i} + (6-7)\hat{j} + (2-1)\hat{k} = 2\hat{i} - \hat{j} + \hat{k}$

Now, angle between $\overrightarrow{AC}$ and $\overrightarrow{BC}$ is given by

$$\Rightarrow \quad \cos\theta = \frac{\overrightarrow{AC}\cdot\overrightarrow{BC}}{|\overrightarrow{AC}||\overrightarrow{BC}|} = \frac{6-7+1}{\sqrt{9+49+1}\sqrt{4+1+1}}$$

$\Rightarrow \quad \cos\theta = 0 \Rightarrow AC \perp BC$

So, A, B, C are the vertices of right angled triangle.

76. Given, $\hat{a} + \hat{b} = \hat{c}$

$\Rightarrow (\hat{a}+\hat{b})\cdot(\hat{a}+\hat{b}) = \hat{c}\cdot\hat{c}$

$\Rightarrow \hat{a}\cdot\hat{a} + \hat{a}\cdot\hat{b} + \hat{b}\cdot\hat{b} + \hat{b}\cdot\hat{a} = \hat{c}\cdot\hat{c}$

$\Rightarrow 1 + \hat{a}\cdot\hat{b} + 1 + \hat{a}\cdot\hat{b} = 1$

$\Rightarrow 2\,\hat{a}\cdot\hat{b} = -1$ ...(i)

Now $(\hat{a}-\hat{b})^2 = (\hat{a}-\hat{b})\cdot(\hat{a}-\hat{b})$

$= \hat{a}\cdot\hat{a} - \hat{a}\cdot\hat{b} - \hat{b}\cdot\hat{a} + \hat{b}\cdot\hat{b} = 1 - \hat{a}\cdot\hat{b} - \hat{a}\cdot\hat{b} + 1$

$= 2 - 2\hat{a}\cdot\hat{b} = 2 - (-1)$ [Using(i)]

$= 3$

$\therefore\ |\hat{a} - \hat{b}| = \sqrt{3}$

77. Given, $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{b} = 3\hat{i} - \hat{j} + 2\hat{k}$

Now, $\vec{a} + \vec{b} = 4\hat{i} + \hat{j} - \hat{k}$

Also, $\vec{a} - \vec{b} = -2\hat{i} + 3\hat{j} - 5\hat{k}$

Now, $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = (4\hat{i} + \hat{j} - \hat{k})\cdot(-2\hat{i} + 3\hat{j} - 5\hat{k})$

$= (4)(-2) + (1)(3) + (-1)(-5) = -8 + 3 + 5 = 0$

Hence, $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ are perpendicular to each other.

78. Let $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k},\ \vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$

Now, $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$$\Rightarrow \quad (\hat{i}-2\hat{j}+3\hat{k})\cdot(3\hat{i}-2\hat{j}+\hat{k}) = \sqrt{(1)^2+(-2)^2+(3)^2} \times \sqrt{(3)^2+(-2)^2+(1)^2}\cos\theta$$

$\Rightarrow \quad 3+4+3 = \sqrt{14}\times\sqrt{14}\cos\theta$

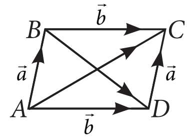
$\Rightarrow \quad \cos\theta = \dfrac{10}{14}$

$\therefore \quad \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\dfrac{100}{196}} = \sqrt{\dfrac{96}{196}}$

$\Rightarrow \quad \sin\theta = \dfrac{4\sqrt{6}}{14} = \dfrac{2\sqrt{6}}{7}$

79. Let $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$

Then diagonal $\overrightarrow{AC}$ of the parallelogram is



$\vec{p} = \vec{a} + \vec{b}$

$= \hat{i} + 2\hat{j} + 3\hat{k} + 2\hat{i} + 4\hat{j} - 5\hat{k}$

$= 3\hat{i} + 6\hat{j} - 2\hat{k}$

Therefore unit vector parallel to it is

$$\frac{\vec{p}}{|\vec{p}|} = \frac{3\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{9+36+4}} = \frac{1}{7}(3\hat{i}+6\hat{j}-2\hat{k})$$

Now, diagonal $\overrightarrow{BD}$ of the parallelogram is

$$\vec{p}' = \vec{b} - \vec{a} = 2\hat{i} + 4\hat{j} - 5\hat{k} - \hat{i} - 2\hat{j} - 3\hat{k} = \hat{i} + 2\hat{j} - 8\hat{k}$$

Therefore unit vector parallel to it is

$$\frac{\vec{p}'}{|\vec{p}'|} = \frac{\hat{i}+2\hat{j}-8\hat{k}}{\sqrt{1+4+64}} = \frac{1}{\sqrt{69}}(\hat{i}+2\hat{j}-8\hat{k})$$

80. Given, ΔABC with vertices

$A(1,2,3) \equiv \hat{i} + 2\hat{j} + 3\hat{k},\ B(2,-1,4) \equiv 2\hat{i} - \hat{j} + 4\hat{k}$,

$C(4,5,-1) \equiv 4\hat{i} + 5\hat{j} - \hat{k}$

Now $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (2\hat{i} - \hat{j} + 4\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$

$= \hat{i} - 3\hat{j} + \hat{k}$.

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (4\hat{i} + 5\hat{j} - \hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$

$= 3\hat{i} + 3\hat{j} - 4\hat{k}$.

$$\therefore \quad (\overrightarrow{AB}\times\overrightarrow{AC}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 1 \\ 3 & 3 & -4 \end{vmatrix} = 9\hat{i} + 7\hat{j} + 12\hat{k}$$

Hence, area of ΔABC

$= \dfrac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}| = \dfrac{1}{2}|9\hat{i} + 7\hat{j} + 12\hat{k}|$

$= \dfrac{1}{2}\sqrt{9^2 + 7^2 + 12^2} = \dfrac{1}{2}\sqrt{81 + 49 + 144}$

$= \dfrac{1}{2}\sqrt{274}$ sq. units

81. Given, position vector of $A = \hat{i} + \hat{j} + \hat{k}$

Position vector of $B = 2\hat{i} + 5\hat{j}$

Position vector of $C = 3\hat{i} + 2\hat{j} - 3\hat{k}$

Position vector of $D = \hat{i} - 6\hat{j} - \hat{k}$

$\therefore\ \overrightarrow{AB} = (2\hat{i} + 5\hat{j}) - (\hat{i} + \hat{j} + \hat{k}) = \hat{i} + 4\hat{j} - \hat{k}$ and

$\overrightarrow{CD} = (\hat{i} - 6\hat{j} - \hat{k}) - (3\hat{i} + 2\hat{j} - 3\hat{k}) = -2\hat{i} - 8\hat{j} + 2\hat{k}$

Now $|\overrightarrow{AB}| = \sqrt{(1)^2 + (4)^2 + (1)^2} = \sqrt{18}$

$$|\overrightarrow{CD}| = \sqrt{(-2)^2+(-8)^2+(2)^2} = \sqrt{4+64+4}$$
$$= \sqrt{72} = 2\sqrt{18}$$

Let θ be the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$.

$$\therefore \cos\theta = \frac{\overrightarrow{AB}\cdot\overrightarrow{CD}}{|\overrightarrow{AB}||\overrightarrow{CD}|} = \frac{(\hat{i}+4\hat{j}-\hat{k})\cdot(-2\hat{i}-8\hat{j}+2\hat{k})}{(\sqrt{18})(2\sqrt{18})}$$
$$= \frac{-2-32-2}{36} = \frac{-36}{36} = -1$$

$\Rightarrow \quad \cos\theta = -1 \quad \Rightarrow \quad \theta = \pi$

Since, angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$ is 180°.

$\therefore \quad \overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear.

82. Let $\vec{d} = x\hat{i}+y\hat{j}+z\hat{k}$

Now, it is given that, $\vec{d}$ is perpendicular to $\vec{b} = \hat{i}-4\hat{j}+5\hat{k}$ and $\vec{c} = 3\hat{i}+\hat{j}-\hat{k}$

$\therefore \quad \vec{d}\cdot\vec{b} = 0$ and $\vec{d}\cdot\vec{c} = 0$

$\Rightarrow \quad x - 4y + 5z = 0$ ...(i)

and $3x + y - z = 0$ ...(ii)

Also, $\vec{d}\cdot\vec{a} = 21$, where $\vec{a} = 4\hat{i} + 5\hat{j} - \hat{k}$

$\Rightarrow \quad 4x + 5y - z = 21$ ...(iii)

Eliminating z from (i) and (ii), we get

$16x + y = 0$ ...(iv)

Eliminating z from (ii) and (iii), we get

$x + 4y = 21$ ...(v)

Solving (iv) and (v), we get

$x = \frac{-1}{3}, y = \frac{16}{3}$

Putting the values of x and y in (i), we get $z = \frac{13}{3}$

$\therefore \quad \vec{d} = \frac{-1}{3}\hat{i}+\frac{16}{3}\hat{j}+\frac{13}{3}\hat{k}$ is the required vector.

83. $|\vec{a}| = |\vec{b}| = |\vec{c}|$ (Given) ...(i)

and $\vec{a}\cdot\vec{b} = 0, \vec{b}\cdot\vec{c} = 0, \vec{c}\cdot\vec{a} = 0$ ...(ii)

Let $(\vec{a}+\vec{b}+\vec{c})$ be inclined to vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ by angles α, β and γ respectively. Then

$$\cos\alpha = \frac{(\vec{a}+\vec{b}+\vec{c})\cdot\vec{a}}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|} = \frac{\vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{a}+\vec{c}\cdot\vec{a}}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|}$$
$$= \frac{|\vec{a}|^2+0+0}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|} \quad [\text{Using (ii)}]$$
$$= \frac{|\vec{a}|}{|\vec{a}+\vec{b}+\vec{c}|} \quad \text{...(iii)}$$

Similarly, $\cos\beta = \frac{|\vec{b}|}{|\vec{a}+\vec{b}+\vec{c}|}$ ...(iv)

and $\cos\gamma = \frac{|\vec{c}|}{|\vec{a}+\vec{b}+\vec{c}|}$ ...(v)

From (i), (iii), (iv) and (v), we get

$\cos\alpha = \cos\beta = \cos\gamma \Rightarrow \alpha = \beta = \gamma$

Hence, the vector $\vec{a}+\vec{b}+\vec{c}$ is equally inclined to the vector $\vec{a}$, $\vec{b}$ and $\vec{c}$.

Also the angle between them is given as

$$\alpha = \cos^{-1}\left(\frac{|\vec{a}|}{|\vec{a}+\vec{b}+\vec{c}|}\right), \beta = \cos^{-1}\left(\frac{|\vec{b}|}{|\vec{a}+\vec{b}+\vec{c}|}\right),$$
$$\gamma = \cos^{-1}\left(\frac{|\vec{c}|}{|\vec{a}+\vec{b}+\vec{c}|}\right)$$

84. We have, $A(2\hat{i}-\hat{j}+\hat{k})$, $B(\hat{i}-3\hat{j}-5\hat{k})$ and $C(3\hat{i}-4\hat{j}-4\hat{k})$

Then, $\overrightarrow{AB} = (1-2)\hat{i}+(-3+1)\hat{j}+(-5-1)\hat{k}$
$$= -\hat{i}-2\hat{j}-6\hat{k}$$

$\overrightarrow{AC} = (3-2)\hat{i}+(-4+1)\hat{j}+(-4-1)\hat{k} = \hat{i}-3\hat{j}-5\hat{k}$

and $\overrightarrow{BC} = (3-1)\hat{i}+(-4+3)\hat{j}+(-4+5)\hat{k} = 2\hat{i}-\hat{j}+\hat{k}$

Now angle between $\overrightarrow{AC}$ and $\overrightarrow{BC}$ is given by

$$\cos\theta = \frac{(\overrightarrow{AC})(\overrightarrow{BC})}{|\overrightarrow{AC}||\overrightarrow{BC}|} = \frac{2+3-5}{\sqrt{1+9+25}\,.\,\sqrt{4+1+1}}$$

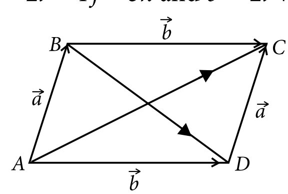
$\Rightarrow \quad \cos\theta = 0 \Rightarrow BC \perp AC$

So, A, B, C are vertices of right angled triangle.

Now area of $\Delta ABC = \frac{1}{2}|\overrightarrow{AC}\times\overrightarrow{BC}|$

$$= \frac{1}{2}\left\|\begin{matrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -3 & -5\\ 2 & -1 & 1\end{matrix}\right\| = \frac{1}{2}|(-3-5)\hat{i}-(1+10)\hat{j}+(-1+6)\hat{k}|$$
$$= \frac{1}{2}|-8\hat{i}-11\hat{j}+5\hat{k}|$$
$$= \frac{1}{2}\sqrt{64+121+25} = \frac{\sqrt{210}}{2} \text{ sq. units.}$$

85. Let $\vec{a} = 2\hat{i} - 4\hat{j} - 5\hat{k}$ and $\vec{b} = 2\hat{i} + 2\hat{j} + 3\hat{k}$



Then diagonal $\overrightarrow{AC}$ of the parallelogram is

$\vec{p} = \vec{a} + \vec{b}$

$= 2\hat{i} - 4\hat{j} - 5\hat{k} + 2\hat{i} + 2\hat{j} + 3\hat{k} = 4\hat{i} - 2\hat{j} - 2\hat{k}$

Therefore, unit vector parallel to it is

$$\frac{\vec{p}}{|\vec{p}|} = \frac{4\hat{i} - 2\hat{j} - 2\hat{k}}{\sqrt{16+4+4}} = \frac{2\hat{i} - \hat{j} - \hat{k}}{\sqrt{6}}$$

Now, diagonal $\overrightarrow{BD}$ of the parallelogram is

$\vec{p}\,' = \vec{b} - \vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k} - 2\hat{i} + 4\hat{j} + 5\hat{k} = 6\hat{j} + 8\hat{k}$

Therefore, unit vector parallel to it is

$$\frac{\vec{p}\,'}{|\vec{p}\,'|} = \frac{6\hat{j} + 8\hat{k}}{\sqrt{36+64}} = \frac{6\hat{j} + 8\hat{k}}{10} = \frac{3\hat{j} + 4\hat{k}}{5}$$

$$\text{Now, } \vec{p} \times \vec{p}\,' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -2 & -2 \\ 0 & 6 & 8 \end{vmatrix}$$

$= \hat{i}(-16+12) - \hat{j}(32-0) + \hat{k}(24-0)$

$= -4\hat{i} - 32\hat{j} + 24\hat{k}$

$\therefore$ Area of parallelogram $= \dfrac{|\vec{p} \times \vec{p}\,'|}{2}$

$$= \frac{\sqrt{16+1024+576}}{2} = 2\sqrt{101} \text{ sq. units.}$$

86. Two non zero vectors are parallel if and only if their cross product is zero vector.

So, we have to prove that cross product of $\vec{a} - \vec{d}$ and $\vec{b} - \vec{c}$ is zero vector.

$(\vec{a} - \vec{d}) \times (\vec{b} - \vec{c}) = (\vec{a} \times \vec{b}) - (\vec{a} \times \vec{c}) - (\vec{d} \times \vec{b}) + (\vec{d} \times \vec{c})$

Since, it is given that $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ and $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$

And, $\vec{d} \times \vec{b} = -\vec{b} \times \vec{d}$, $\vec{d} \times \vec{c} = -\vec{c} \times \vec{d}$

Therefore,

$(\vec{a} - \vec{d}) \times (\vec{b} - \vec{c}) = (\vec{c} \times \vec{d}) - (\vec{b} \times \vec{d}) + (\vec{b} \times \vec{d}) - (\vec{c} \times \vec{d}) = \vec{0}$

Hence, $\vec{a} - \vec{d}$ is parallel to $\vec{b} - \vec{c}$, where

$\vec{a} \neq \vec{d}$ and $\vec{b} \neq \vec{c}$.

87. $(\vec{r} \times \hat{i}) \cdot (\vec{r} \times \hat{j}) + xy$

$= [(x\hat{i} + y\hat{j} + z\hat{k}) \times \hat{i}] \cdot [(x\hat{i} + y\hat{j} + z\hat{k}) \times \hat{j}] + xy$

$= (-y\hat{k} + z\hat{j}) \cdot (x\hat{k} - z\hat{i}) + xy \ = -xy + xy = 0$

88. Here, $\vec{a} = \hat{i} + 2\hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} + \hat{j}$, $\vec{c} = 3\hat{i} - 4\hat{j} - 5\hat{k}$

$\therefore\ \vec{a} - \vec{b} = (\hat{i} + 2\hat{j} + \hat{k}) - (2\hat{i} + \hat{j}) = -\hat{i} + \hat{j} + \hat{k}$

$\vec{c} - \vec{b} = (3\hat{i} - 4\hat{j} - 5\hat{k}) - (2\hat{i} + \hat{j}) = \hat{i} - 5\hat{j} - 5\hat{k}$

Vector perpendicular to both $\vec{a} - \vec{b}$ and $\vec{c} - \vec{b}$ is

$$(\vec{a} - \vec{b}) \times (\vec{c} - \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 1 \\ 1 & -5 & -5 \end{vmatrix}$$

$= (-5+5)\hat{i} - (5-1)\hat{j} + (5-1)\hat{k} = -4\hat{j} + 4\hat{k}$

$\therefore$ Unit vector perpendicular to both $\vec{a} - \vec{b}$ and $\vec{c} - \vec{b}$

$$= \frac{-4\hat{j} + 4\hat{k}}{|-4\hat{j} + 4\hat{k}|} = \frac{-4\hat{j} + 4\hat{k}}{\sqrt{(-4)^2 + 4^2}} = \frac{-4\hat{j} + 4\hat{k}}{4\sqrt{2}} = \frac{1}{\sqrt{2}}(-\hat{j} + \hat{k}).$$

89. Given $\vec{a} + \vec{b} + \vec{c} = 0$ and $|\vec{a}| = 3$, $|\vec{b}| = 5$, $|\vec{c}| = 7$

We have $\vec{a} + \vec{b} + \vec{c} = 0$

$\Rightarrow \vec{a} + \vec{b} = -\vec{c} \Rightarrow |\vec{a} + \vec{b}|^2 = |-\vec{c}|^2$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2(\vec{a} \cdot \vec{b}) = |\vec{c}|^2$

$\Rightarrow 9 + 25 + 2|\vec{a}||\vec{b}|\cos\theta = 49$

$\Rightarrow 2 \times 3 \times 5 \times \cos\theta = 49 - 34 = 15$

$\Rightarrow \cos\theta = \dfrac{15}{30} = \dfrac{1}{2} \Rightarrow \theta = \dfrac{\pi}{3} = 60°$

90. Here, $\vec{a} = \hat{i} + \hat{j} + \hat{k}$; $\vec{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and

$\vec{c} = \lambda\hat{i} + 2\hat{j} + 3\hat{k}$

$\Rightarrow \vec{b} + \vec{c} = (2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}$

The unit vector along $\vec{b} + \vec{c}$ is $\vec{p} = \dfrac{\vec{b} + \vec{c}}{|\vec{b} + \vec{c}|}$

$$= \frac{(2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{(2+\lambda)^2 + 6^2 + (-2)^2}} = \frac{(2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{\lambda^2 + 4\lambda + 44}}$$

Also, $\vec{a} \cdot \vec{p} = 1$ (Given)

$$\Rightarrow \frac{(2+\lambda) + 6 - 2}{\sqrt{\lambda^2 + 4\lambda + 44}} = 1$$

$\Rightarrow \sqrt{\lambda^2 + 4\lambda + 44} = \lambda + 6$

$\Rightarrow \lambda^2 + 4\lambda + 44 = \lambda^2 + 12\lambda + 36$

$\Rightarrow 8\lambda = 8 \Rightarrow \lambda = 1$

$\therefore$ The required unit vector

$$\vec{p} = \frac{(2+1)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{1+4+44}} = \frac{1}{7}(3\hat{i} + 6\hat{j} - 2\hat{k}).$$

91. We have $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$

Let $\vec{r} = \vec{a} + \vec{b} = 2\hat{i} + 3\hat{j} + 4\hat{k}$

and $\vec{p} = \vec{a} - \vec{b} = -\hat{j} - 2\hat{k}$

A unit vector perpendicular to both $\vec{r}$ and $\vec{p}$ is given as $\pm\dfrac{\vec{r} \times \vec{p}}{|\vec{r} \times \vec{p}|}$.

Now, $\vec{r}\times\vec{p}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 4\\ 0 & -1 & -2\end{vmatrix}=-2\hat{i}+4\hat{j}-2\hat{k}$

So, the required unit vector is

$$=\pm\frac{\left(-2\hat{i}+4\hat{j}-2\hat{k}\right)}{\sqrt{(-2)^2+4^2+(-2)^2}}=\mp\frac{\left(\hat{i}-2\hat{j}+\hat{k}\right)}{\sqrt{6}}.$$

92. Here, $\vec{a}=2\hat{i}-3\hat{j}+\hat{k}; \vec{b}=-\hat{i}+\hat{k}; \vec{c}=2\hat{j}-\hat{k}$

$\therefore \quad \vec{a}+\vec{b}=\left(2\hat{i}-3\hat{j}+\hat{k}\right)+\left(-\hat{i}+\hat{k}\right)=\hat{i}-3\hat{j}+2\hat{k},$

$\vec{b}+\vec{c}=\left(-\hat{i}+\hat{k}\right)+\left(2\hat{j}-\hat{k}\right)=-\hat{i}+2\hat{j}$

$$\therefore \left(\vec{a}+\vec{b}\right)\times\left(\vec{b}+\vec{c}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -3 & 2\\ -1 & 2 & 0\end{vmatrix}=-4\hat{i}-2\hat{j}-\hat{k}$$

$\therefore$ Area of a parallelogram whose diagonals are $\vec{a}+\vec{b}$ and $\vec{b}+\vec{c}$

$$=\frac{1}{2}\left|\left(\vec{a}+\vec{b}\right)\times\left(\vec{b}+\vec{c}\right)\right|=\frac{1}{2}\left|-4\hat{i}-2\hat{j}-\hat{k}\right|$$

$$=\frac{1}{2}\sqrt{(-4)^2+(-2)^2+(-1)^2}=\frac{\sqrt{21}}{2}\text{ sq.units}.$$

93. *Refer to answer 82.*

94. Here $|\vec{a}+\vec{b}|=|\vec{a}|$

$\Rightarrow |\vec{a}+\vec{b}|^2=|\vec{a}|^2 \Rightarrow \left(\vec{a}+\vec{b}\right)\cdot\left(\vec{a}+\vec{b}\right)=\vec{a}\cdot\vec{a}$

$\Rightarrow \vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}=\vec{a}\cdot\vec{a}$

$\Rightarrow 2\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{b}=0 \qquad \left[\because \vec{b}\cdot\vec{a}=\vec{a}\cdot\vec{b}\right]$

$\Rightarrow \left(2\vec{a}+\vec{b}\right)\cdot\vec{b}=0 \Rightarrow \left(2\vec{a}+\vec{b}\right)\perp\vec{b}$

95. Given $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=\hat{j}-\hat{k}$

Let $\vec{c}=x\hat{i}+y\hat{j}+z\hat{k}$

Now we have, $\vec{a}\times\vec{c}=\vec{b}$

$\Rightarrow \quad (\hat{i}+\hat{j}+\hat{k})\times(x\hat{i}+y\hat{j}+z\hat{k})=\hat{j}-\hat{k}$

$$\Rightarrow \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & 1\\ x & y & z\end{vmatrix}=\hat{j}-\hat{k}$$

$\Rightarrow \quad \hat{i}(z-y)-\hat{j}(z-x)+\hat{k}(y-x)=\hat{j}-\hat{k}$

$\Rightarrow \quad z-y=0, x-z=1$ and $y-x=-1$

$\Rightarrow \quad y=z, x-z=1, x-y=1 \qquad$ .....(i)

Also, we have $\vec{a}\cdot\vec{c}=3$

$\Rightarrow \quad (\hat{i}+\hat{j}+\hat{k})\cdot(x\hat{i}+y\hat{j}+z\hat{k})=3$

$\Rightarrow \quad x+y+z=3$

$\Rightarrow \quad x+x-1+x-1=3 \qquad$ [Using (i)]

$\Rightarrow \quad 3x-2=3 \Rightarrow x=\frac{5}{3}, y=\frac{2}{3}, z=\frac{2}{3}$

Hence, $\vec{c}=\frac{5}{3}\hat{i}+\frac{2}{3}\hat{j}+\frac{2}{3}\hat{k}$

96. Here $\vec{a}=\hat{i}-\hat{j}+7\hat{k}$; $\vec{b}=5\hat{i}-\hat{j}+\lambda\hat{k}$

$\therefore \vec{a}+\vec{b}=6\hat{i}-2\hat{j}+(7+\lambda)\hat{k}$

$\vec{a}-\vec{b}=-4\hat{i}+(7-\lambda)\hat{k}$

For $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ to be perpendicular,

$\left(\vec{a}+\vec{b}\right)\cdot\left(\vec{a}-\vec{b}\right)=0$

$\Rightarrow \left[6\hat{i}-2\hat{j}+(7+\lambda)\hat{k}\right]\cdot\left[-4\hat{i}+(7-\lambda)\hat{k}\right]=0$

$\Rightarrow \quad 6\times(-4)+(7+\lambda)\times(7-\lambda)=0$

$\Rightarrow \quad -24+49-\lambda^2=0 \Rightarrow \lambda^2=25 \Rightarrow \lambda=\pm 5$

97. *Refer to answer 83.*

98. Let the required vector be $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$.

Also let,

$\vec{a}=\hat{i}-\hat{j}+\hat{k}, \vec{b}=2\hat{i}+\hat{j}-3\hat{k}$ and $\vec{c}=\hat{i}+\hat{j}+\hat{k}$

$\vec{r}\cdot\vec{a}=4,\ \vec{r}\cdot\vec{b}=0,\ \vec{r}\cdot\vec{c}=2 \qquad$ (Given)

$\Rightarrow \quad x-y+z=4 \qquad$ ...(i)

$2x+y-3z=0 \qquad$ ...(ii)

$x+y+z=2 \qquad$ ...(iii)

Now (iii) – (i) $\Rightarrow 2y=-2 \Rightarrow y=-1$

From (ii) and (iii)

$2x-3z-1=0, x+z-3=0 \Rightarrow x=2, z=1$

$\therefore \quad$ The required vector is $\vec{r}=2\hat{i}-\hat{j}+\hat{k}$.

99. Here, $\vec{a}=2\lambda^2\hat{i}+4\lambda\hat{j}+\hat{k}$ and

$\vec{b}=7\hat{i}-2\hat{j}+\lambda\hat{k}$

If θ is the angle between the vectors $\vec{a}$ and $\vec{b}$,

then $\cos\theta=\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

For θ to be obtuse, $\cos\theta<0 \Rightarrow \vec{a}\cdot\vec{b}<0$

$\Rightarrow \quad \left(2\lambda^2\hat{i}+4\lambda\hat{j}+\hat{k}\right)\cdot\left(7\hat{i}-2\hat{j}+\lambda\hat{k}\right)<0$

$\Rightarrow 2\lambda^2\cdot 7+4\lambda\cdot(-2)+1\cdot\lambda<0$

$\Rightarrow 14\lambda^2-7\lambda<0 \Rightarrow \lambda(2\lambda-1)<0$

$\Rightarrow \quad$ Either $\lambda<0, 2\lambda-1>0$ or $\lambda>0, 2\lambda-1<0$

$\Rightarrow$ Either $\lambda<0, \lambda>\frac{1}{2}$ or $\lambda>0, \lambda<\frac{1}{2}$

First alternative is impossible.

$\therefore \quad \lambda>0, \lambda<\frac{1}{2}$ *i.e.*, $0<\lambda<\frac{1}{2}$ *i.e.*, $\lambda\in\left]0,\frac{1}{2}\right[$

100. Given, $|\vec{a}|=3, |\vec{b}|=4, |\vec{c}|=5 \qquad$ ...(i)

and $\vec{a}\cdot\left(\vec{b}+\vec{c}\right)=0,\ \vec{b}\cdot\left(\vec{c}+\vec{a}\right)=0,\ \vec{c}\cdot\left(\vec{a}+\vec{b}\right)=0$

$\therefore \quad \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c} + \vec{b}\cdot\vec{a} + \vec{c}\cdot\vec{a} + \vec{c}\cdot\vec{b}$
$= 0 + 0 + 0 = 0$

$\Rightarrow 2(\vec{a}\cdot\vec{b}) + 2(\vec{b}\cdot\vec{c}) + 2(\vec{c}\cdot\vec{a}) = 0 \qquad \text{...(ii)}$

Now $|\vec{a}+\vec{b}+\vec{c}|^2 = (\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})$

$= (\vec{a}+\vec{b}+\vec{c})\cdot\vec{a} + (\vec{a}+\vec{b}+\vec{c})\cdot\vec{b} + (\vec{a}+\vec{b}+\vec{c})\cdot\vec{c}$

$= \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{a} + \vec{c}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b} + \vec{c}\cdot\vec{b} + \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{c}$

$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}\cdot\vec{b}) + 2(\vec{b}\cdot\vec{c}) + 2(\vec{c}\cdot\vec{a})$

$= 3^2 + 4^2 + 5^2 + 0$ [Using (i) and (ii)]

$= 50$

$\therefore |\vec{a}+\vec{b}+\vec{c}| = 5\sqrt{2}$.

101. Here $\vec{a} = 3\hat{i} - \hat{j}$, $\vec{b} = 2\hat{i} + \hat{j} - 3\hat{k}$

We have to express : $\vec{b} = \vec{b}_1 + \vec{b}_2$, where

$\vec{b}_1 \parallel \vec{a}$ and $\vec{b}_2 \perp \vec{a}$

Let $\vec{b}_1 = \lambda\vec{a} = \lambda(3\hat{i} - \hat{j})$ and $\vec{b}_2 = x\hat{i} + y\hat{j} + z\hat{k}$

Now $\vec{b}_2 \perp \vec{a} \Rightarrow \vec{b}_2 \cdot \vec{a} = 0$

$\Rightarrow (x\hat{i} + y\hat{j} + z\hat{k})\cdot(3\hat{i} - \hat{j}) = 0$

$\Rightarrow \quad 3x - y = 0 \qquad \text{...(i)}$

Now, $\vec{b} = \vec{b}_1 + \vec{b}_2$

$\Rightarrow 2\hat{i} + \hat{j} - 3\hat{k} = \lambda(3\hat{i} - \hat{j}) + (x\hat{i} + y\hat{j} + z\hat{k})$

On comparing, we get

$\left.\begin{matrix} 2 = 3\lambda + x \\ 1 = -\lambda + y \end{matrix}\right] \Rightarrow x + 3y = 5 \qquad \text{...(ii)}$

and $-3 = z \Rightarrow z = -3$

Solving (i) and (ii), we get $x = \dfrac{1}{2}, y = \dfrac{3}{2}$

$\therefore 1 = -\lambda + y = -\lambda + \dfrac{3}{2} \Rightarrow \lambda = \dfrac{1}{2}$

Hence, $\vec{b}_1 = \lambda(3\hat{i} - \hat{j}) = \dfrac{3}{2}\hat{i} - \dfrac{1}{2}\hat{j}$

and $\vec{b}_2 = \dfrac{1}{2}\hat{i} + \dfrac{3}{2}\hat{j} - 3\hat{k}$

102. We have,

$|\vec{a}| = 5, |\vec{b}| = 12, |\vec{c}| = 13$ and $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

$\Rightarrow (\vec{a}+\vec{b}+\vec{c})^2 = |\vec{0}|^2$ (Squaring on both sides)

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2[\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}] = 0$

$\Rightarrow 25 + 144 + 169 + 2[\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}] = 0$

$\Rightarrow 2[\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}] = -338$

$\Rightarrow \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a} = \dfrac{-338}{2} = -169$

103. *Refer to answer 82.*

104. *Refer to answer 76.*

105. *Refer to answer 91.*

106. We have $|\vec{a}| = 2$, $|\vec{b}| = 1$ and $\vec{a}\cdot\vec{b} = 1$

Now, $(3\vec{a} - 5\vec{b})\cdot(2\vec{a} + 7\vec{b})$

$= 6|\vec{a}|^2 + 21\,\vec{a}\cdot\vec{b} - 10\vec{a}\cdot\vec{b} - 35|\vec{b}|^2$

$= 6|\vec{a}|^2 + 11\vec{a}\cdot\vec{b} - 35|\vec{b}|^2$

$= 6(2)^2 + 11(1) - 35(1)^2 = 24 + 11 - 35 = 0$

107. *Refer to answer 80.*

108. *Refer to answer 97.*

Also the angle between them is given as

$$\alpha = \cos^{-1}\left(\frac{|\vec{a}|}{|\vec{a}+\vec{b}+\vec{c}|}\right),$$

$$\beta = \cos^{-1}\left(\frac{|\vec{b}|}{|\vec{a}+\vec{b}+\vec{c}|}\right),$$

$$\gamma = \cos^{-1}\left(\frac{|\vec{c}|}{|\vec{a}+\vec{b}+\vec{c}|}\right)$$

109. *Refer to answer 100.*

110. Since the vectors are coplanar.

$$\therefore \quad \begin{vmatrix} 1 & 3 & 1 \\ 2 & -1 & -1 \\ 0 & \lambda & 3 \end{vmatrix} = 0$$

$\Rightarrow \quad 1(-3 + \lambda) - 3(6 - 0) + 1(2\lambda - 0) = 0$

$\Rightarrow \quad -3 + \lambda - 18 + 2\lambda = 0$

$\Rightarrow \quad 3\lambda - 21 = 0 \;\Rightarrow\; \lambda = 7$

111. Here $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$, $\vec{c} = 3\hat{i} + \hat{j} + 2\hat{k}$

$$\text{Now, } \vec{b}\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix} = 3\hat{i} + 5\hat{j} - 7\hat{k}$$

$\therefore \vec{a}\cdot(\vec{b}\times\vec{c}) = (2\hat{i} + \hat{j} + 3\hat{k})\cdot(3\hat{i} + 5\hat{j} - 7\hat{k})$

$= 2\times 3 + 1\times 5 + 3\times(-7)$

$= 6 + 5 - 21 = -10$

112. Given, $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$, $\vec{b} = 3\hat{i} + 4\hat{j} - 5\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 3\hat{k}$

$\therefore \quad 2\vec{a} = 2\hat{i} - 2\hat{j} + 4\hat{k}$

$\quad -\vec{b} = -3\hat{i} - 4\hat{j} + 5\hat{k}$

$\quad 3\vec{c} = 6\hat{i} - 3\hat{j} + 9\hat{k}$

Now, $2\vec{a}\cdot(-\vec{b}\times 3\vec{c}) = \begin{vmatrix} 2 & -2 & 4 \\ -3 & -4 & 5 \\ 6 & -3 & 9 \end{vmatrix}$

$= 2(-36 + 15) + 2(-27 - 30) + 4(9 + 24)$
$= 2(-21) - 2(57) + 4(33)$
$= -42 - 114 + 132 = -24$

$\therefore$ Volume of parallelepiped

$|2\vec{a}\cdot(-\vec{b}\times 3\vec{c})| = |-24| = 24$ cubic units

113. Given, $\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k}$, $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$ and $\vec{c} = -3\hat{i} + \hat{j} + 2\hat{k}$

Now, $\vec{b}\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 1 \\ -3 & 1 & 2 \end{vmatrix}$

$= \hat{i}(-4-1) - \hat{j}(2+3) + \hat{k}(1-6)$

$= -5\hat{i} - 5\hat{j} - 5\hat{k}$

$\therefore\ \vec{a}\cdot(\vec{b}\times\vec{c}) = (2\hat{i} + 3\hat{j} + \hat{k})\cdot(-5\hat{i} - 5\hat{j} - 5\hat{k})$

$= -10 - 15 - 5 = -30$

114. Given points are $A(x, -1, -1)$, $B(4, 5, 1)$, $C(3, 9, 4)$ and $D(-4, 4, 4)$.

$\overrightarrow{AB} = (4-x)\hat{i} + (5+1)\hat{j} + (1+1)\hat{k}$

$= (4-x)\hat{i} + 6\hat{j} + 2\hat{k}$

$\overrightarrow{AC} = (3-x)\hat{i} + (9+1)\hat{j} + (4+1)\hat{k}$

$= (3-x)\hat{i} + 10\hat{j} + 5\hat{k}$

$\overrightarrow{AD} = (-4-x)\hat{i} + (4+1)\hat{j} + (4+1)\hat{k}$

$= -(4+x)\hat{i} + 5\hat{j} + 5\hat{k}$

The given points will be coplanar iff

$[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}] = 0$

Now, $[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}] = 0 \Rightarrow \begin{vmatrix} 4-x & 6 & 2 \\ 3-x & 10 & 5 \\ -(4+x) & 5 & 5 \end{vmatrix} = 0$

$\Rightarrow\ (4 - x)(50 - 25) - 6(15 - 5x + 20 + 5x) + 2(15 - 5x + 40 + 10x) = 0$

$\Rightarrow\ (4 - x)(25) - 6(35) + 2(55 + 5x) = 0$

$\Rightarrow\ 100 - 25x - 210 + 110 + 10x = 0$

$\Rightarrow\ -15x = 0$

$\Rightarrow\ x = 0$

115. We have, $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i}$, $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

(a) $c_1 = 1, c_2 = 2\ \therefore\ \vec{c} = \hat{i} + 2\hat{j} + c_3\hat{k}$

Given that $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar

$\therefore\ \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 2 & c_3 \end{vmatrix} = 0$

$\Rightarrow\ -1(c_3) + 1(2) = 0 \Rightarrow c_3 = 2$

(b) $c_2 = -1, c_3 = 1\ \therefore\ \vec{c} = c_1\hat{i} - \hat{j} + \hat{k}$

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar.

$\therefore\ \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ c_1 & -1 & 1 \end{vmatrix} = 0$

$\Rightarrow\ -1(1) + 1(-1) = 0 \Rightarrow -2 = 0$, which is false.

So, no value of c_1 can make $\vec{a}$, $\vec{b}$ and $\vec{c}$ coplanar.

116. Let A, B, C, D be the given points. The given points will be coplanar iff any one of the following triads of vectors are coplanar.

$\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$; $\overrightarrow{BC}, \overrightarrow{BA}, \overrightarrow{BD}$ *etc.*

If $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ are coplanar, then their scalar triple product $[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}] = 0$ where, $A(3\hat{i} + 6\hat{j} + 9\hat{k})$, $B(\hat{i} + 2\hat{j} + 3\hat{k})$, $C(2\hat{i} + 3\hat{j} + \hat{k})$ and $D(4\hat{i} + 6\hat{j} + \lambda\hat{k})$.

Now, $\overrightarrow{AB} = (\hat{i} + 2\hat{j} + 3\hat{k}) - (3\hat{i} + 6\hat{j} + 9\hat{k})$

$= -2\hat{i} - 4\hat{j} - 6\hat{k}$

$\overrightarrow{AC} = (2\hat{i} + 3\hat{j} + \hat{k}) - (3\hat{i} + 6\hat{j} + 9\hat{k}) = -\hat{i} - 3\hat{j} - 8\hat{k}$

$\overrightarrow{AD} = (4\hat{i} + 6\hat{j} + \lambda\hat{k}) - (3\hat{i} + 6\hat{j} + 9\hat{k}) = \hat{i} + (\lambda - 9)\hat{k}$

$\therefore\ [\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}] = \begin{vmatrix} -2 & -4 & -6 \\ -1 & -3 & -8 \\ 1 & 0 & \lambda - 9 \end{vmatrix} = 0$

$\Rightarrow\ -2(-3\lambda + 27) + 4(-\lambda + 9 + 8) - 6(0 + 3) = 0$

$\Rightarrow\ 2(3\lambda - 27) - 4(\lambda - 17) - 6(3) = 0$

$\Rightarrow\ 6\lambda - 54 - 4\lambda + 68 - 18 = 0$

$\Rightarrow\ 2\lambda - 4 = 0 \Rightarrow \lambda = 2$

117. Since, $\vec{a} + \vec{b}$, $\vec{b} + \vec{c}$ and $\vec{c} + \vec{a}$ are coplanar.

$\therefore\ (\vec{a} + \vec{b})\cdot\left[(\vec{b} + \vec{c})\times(\vec{c} + \vec{a})\right] = 0$

$\Rightarrow\ (\vec{a} + \vec{b})\cdot\left[\vec{b}\times\vec{c} + \vec{b}\times\vec{a} + \vec{c}\times\vec{c} + \vec{c}\times\vec{a}\right] = 0$

$\Rightarrow\ (\vec{a} + \vec{b})\cdot(\vec{b}\times\vec{c} + \vec{b}\times\vec{a} + \vec{c}\times\vec{a}) = 0\ \left[\because\ \vec{c}\times\vec{c} = 0\right]$

$\Rightarrow \quad \vec{a}\cdot(\vec{b}\times\vec{c})+\vec{a}\cdot(\vec{b}\times\vec{a})+\vec{a}\cdot(\vec{c}\times\vec{a})+\vec{b}\cdot(\vec{b}\times\vec{c})$
$+\vec{b}\cdot(\vec{b}\times\vec{a})+\vec{b}\cdot(\vec{c}\times\vec{a})=0$

$\Rightarrow \quad 2\left[\vec{a}\cdot(\vec{b}\times\vec{c})\right]=0 \Rightarrow \vec{a}\cdot(\vec{b}\times\vec{c})=0$

$\Rightarrow \quad \vec{a}, \vec{b}$ and $\vec{c}$ are coplanar.

118. Here position vectors of A, B, C and D are $4\hat{i}+5\hat{j}+\hat{k}, -\hat{j}-\hat{k}, 3\hat{i}+\lambda\hat{j}+4\hat{k}$ and $-4\hat{i}+4\hat{j}+4\hat{k}$ respectively.

$\Rightarrow \overrightarrow{AB} = -\hat{j}-\hat{k}-\left(4\hat{i}+5\hat{j}+\hat{k}\right) = -4\hat{i}-6\hat{j}-2\hat{k}$

$\overrightarrow{AC}=\left(3\hat{i}+\lambda\hat{j}+4\hat{k}\right)-\left(4\hat{i}+5\hat{j}+\hat{k}\right)=-\hat{i}+(\lambda-5)\hat{j}+3\hat{k}$

$\overrightarrow{AD} = \left(-4\hat{i}+4\hat{j}+4\hat{k}\right)-\left(4\hat{i}+5\hat{j}+\hat{k}\right) = -8\hat{i}-\hat{j}+3\hat{k}$

$\therefore$ For points A, B, C, D to be coplanar

$\Leftrightarrow$ Vectors $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ will be coplanar

$\Leftrightarrow [\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}]=0$

$$\Rightarrow \begin{vmatrix} -4 & -6 & -2 \\ -1 & \lambda-5 & 3 \\ -8 & -1 & 3 \end{vmatrix}=0$$

$\Rightarrow -4(3\lambda-15+3)+6(-3+24)-2(1+8\lambda-40)=0$
$\Rightarrow -12\lambda + 48 + 126 + 78 - 16\lambda = 0$
$\Rightarrow 28\lambda = 252 \Rightarrow \lambda = 9.$

119. We know that $[\vec{a}, \vec{b}, \vec{c}]=\vec{a}\cdot(\vec{b}\times\vec{c})$

$\therefore$ L.H.S. $=[\vec{a},\vec{b}+\vec{c},\vec{d}] = \vec{a}\cdot[(\vec{b}+\vec{c})\times\vec{d}]$

$=\vec{a}\cdot[\vec{b}\times\vec{d}+\vec{c}\times\vec{d}] = \vec{a}\cdot(\vec{b}\times\vec{d})+\vec{a}\cdot(\vec{c}\times\vec{d})$

$=[\vec{a},\vec{b},\vec{d}]+[\vec{a},\vec{c},\vec{d}]=$ R.H.S.

120. We have, $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}]$
$=(\vec{a}+\vec{b})\cdot\{(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})\}$
$=(\vec{a}+\vec{b})\cdot\{(\vec{b}\times\vec{c})+(\vec{b}\times\vec{a})+(\vec{c}\times\vec{c})+(\vec{c}\times\vec{a})\}$
$=(\vec{a}+\vec{b})\cdot\{(\vec{b}\times\vec{c})+(\vec{b}\times\vec{a})+(\vec{c}\times\vec{a})\}$ $[\because\ \vec{c}\times\vec{c}=0]$
$=\vec{a}\cdot(\vec{b}\times\vec{c})+\vec{a}\cdot(\vec{b}\times\vec{a})+\vec{a}\cdot(\vec{c}\times\vec{a})+$
$\vec{b}\cdot(\vec{b}\times\vec{c})+\vec{b}\cdot(\vec{b}\times\vec{a})+\vec{b}\cdot(\vec{c}\times\vec{a})$
$=[\vec{a}\ \vec{b}\ \vec{c}]+[\vec{b}\ \vec{c}\ \vec{a}]$
[$\because$ Scalar triple product with two equal vectors is 0]
$=[\vec{a}\ \vec{b}\ \vec{c}]+[\vec{a}\ \vec{b}\ \vec{c}]$ $\quad (\because\ [\vec{b}\ \vec{c}\ \vec{a}]=[\vec{a}\ \vec{b}\ \vec{c}])$
$= 2[\vec{a}\ \vec{b}\ \vec{c}]$

121. *Refer to answer 118.*

122. If the vectors $\vec{a}+\vec{b}, \vec{b}+\vec{c}$ and $\vec{c}+\vec{a}$ are coplanar, then $\left[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}\right]=0$

$\Leftrightarrow (\vec{a}+\vec{b})\cdot\left[(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})\right]=0$
$\Leftrightarrow (\vec{a}+\vec{b})\cdot\left[\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{c}+\vec{c}\times\vec{a}\right]=0$
$\Leftrightarrow (\vec{a}+\vec{b})\cdot\left[\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{0}+\vec{c}\times\vec{a}\right]=0$
$\Leftrightarrow \vec{a}\cdot(\vec{b}\times\vec{c})+\vec{a}\cdot(\vec{b}\times\vec{a})+\vec{a}\cdot(\vec{c}\times\vec{a})$
$+\vec{b}\cdot(\vec{b}\times\vec{c})+\vec{b}\cdot(\vec{b}\times\vec{a})+\vec{b}\cdot(\vec{c}\times\vec{a})=0$
$\Leftrightarrow \left[\vec{a}\ \vec{b}\ \vec{c}\right]+\left[\vec{a}\ \vec{b}\ \vec{a}\right]+\left[\vec{a}\ \vec{c}\ \vec{a}\right]+$
$\left[\vec{b}\ \vec{b}\ \vec{c}\right]+\left[\vec{b}\ \vec{b}\ \vec{a}\right]+\left[\vec{b}\ \vec{c}\ \vec{a}\right]=0$
$\Leftrightarrow \left[\vec{a}\ \vec{b}\ \vec{c}\right]+0+0+0+0+\left[\vec{a}\ \vec{b}\ \vec{c}\right]=0$
$\Leftrightarrow 2\left[\vec{a}\ \vec{b}\ \vec{c}\right]=0 \Leftrightarrow \left[\vec{a}\ \vec{b}\ \vec{c}\right]=0$

$\therefore$ The vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar.

Hence the vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar if and only if $\vec{a}+\vec{b}, \vec{b}+\vec{c}$ and $\vec{c}+\vec{a}$ are coplanar.

123. *Refer to answer 122.*