M A T H E M A T I C S

HYPERBOLA

INTRODUCTION OF HYPERBOLA

NOTES

What you already know

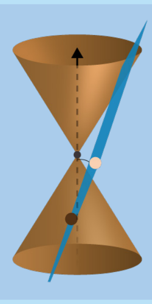
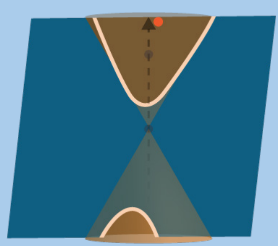
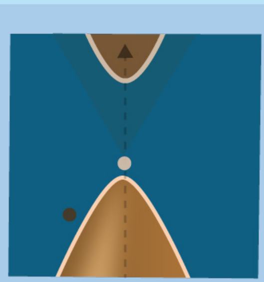
- Basics about conics
- Conditions for different conics
- Analytic interpretation of different conics
- Basic understanding of geometrical figures
- Basic algebraic formulas

What you will learn

- Mathematical interpretation of a hyperbola
- Standard equation of a hyperbola

Visualising Hyperbolas

Let us analyse the configuration of the following figures:

Plane and cone arrangement	Isometric view	Side view
		

Thus, we can say that when the cutting plane intersects both the nappes of a double-sided right cone, we get a conic hyperbola.

Result

- The cutting plane divides both the nappes into two unequal parts.
- We get a symmetric hyperbola when the cutting plane is perpendicular to the base of the cone.

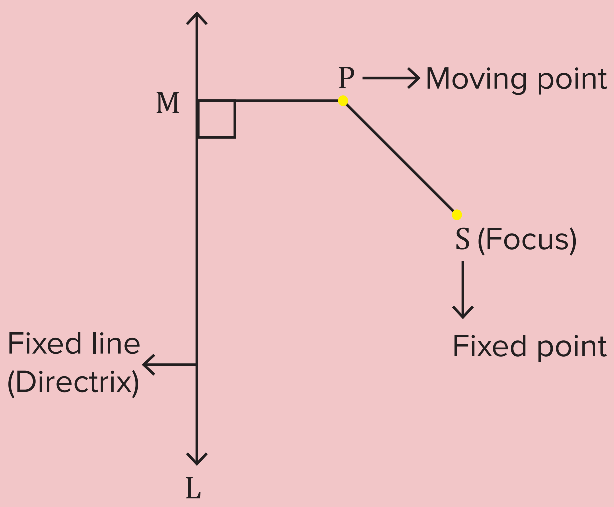
Analytic interpretation

A conic is the locus of a point such that the ratio of its distance from a fixed point to a fixed line is always a constant.

$$\frac{\text{Distance from a fixed point (focus)}}{\text{Distance from a fixed line (directrix)}} = \text{constant}$$

$$= \text{eccentricity } (e) = \frac{PS}{PM}$$

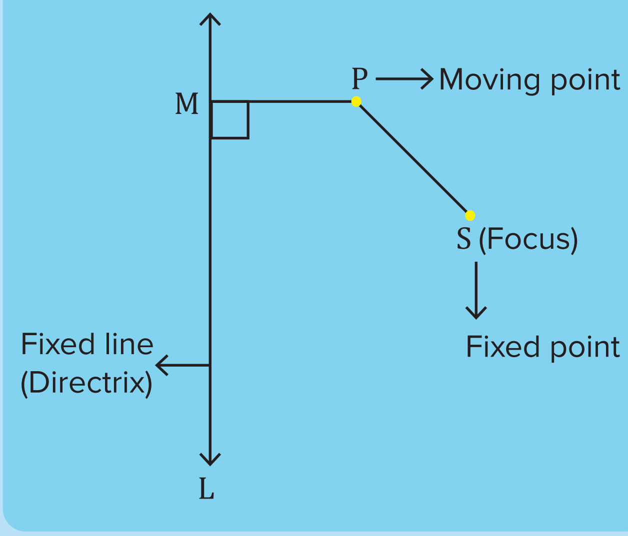
For hyperbola eccentricity $(e) > 1$

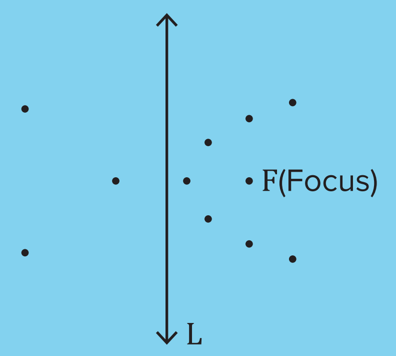


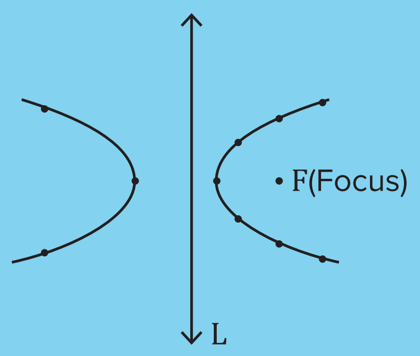
Hyperbola

A hyperbola is the locus of a moving point such that the ratio of its distance from a fixed point (focus) and a fixed line (directrix) is a constant that is always greater than 1. $(e > 1)$

For hyperbola $\Rightarrow \frac{PS}{PM} = e;\ e > 1$







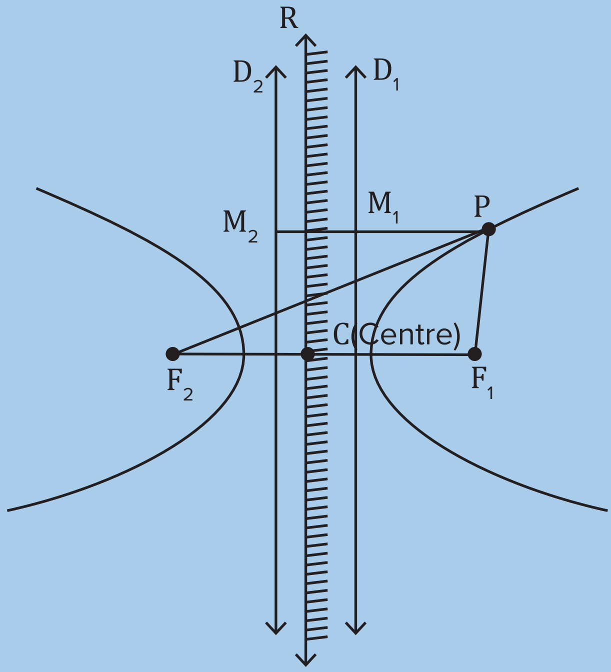
On tracing the trajectory of a moving P, such that its distance from the fixed point, focus (F) and perpendicular distance from the line L is greater than 1, we get a hyperbola as shown in the figure above.

Result

Consider a plane mirror line (R) and then take the image of focus (F_1), which will be F_2. Now, mirror image of line D_1 by taking line (R) as mirror, we get D_2.

Furthermore,

Conditions $\frac{PF_1}{PM_1} = e$ and $\frac{PF_2}{PM_2} = e$ will be valid for any point P on the hyperbola. Hence, for a hyperbola, there are two foci (F_1 and F_2) and two directrices (D_1 and D_2). The two directrices are parallel to each other. The midpoint of the line segment joining its foci is the centre of the hyperbola.



Find the equation of the hyperbola whose focus is (1, 2), directrix is 2x + y - 1 = 0, and eccentricity is $\sqrt{3}$.

Solution

Step 1:

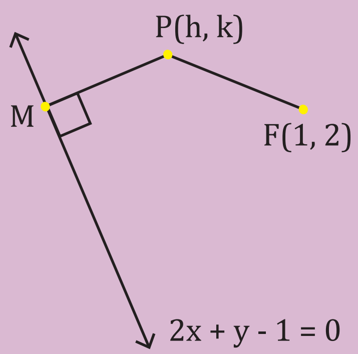
Given, $F \equiv (1, 2), e = \sqrt{3}$

Directrix, $L : 2x + y - 1 = 0$

Let us consider P(h, k) to be a moving point.

Now, by definition, $\frac{PF}{PM} = e \Rightarrow PF = ePM$

$$\Rightarrow \sqrt{(h-1)^2 + (k-2)^2} = \sqrt{3} \times \frac{|2h + k - 1|}{\sqrt{2^2 + 1^2}}$$



Step 2:

Now, after squaring both the sides, we get,

$\Rightarrow (h - 1)^2 + (k - 2)^2 = \frac{3}{5}(2h + k - 1)^2$

$\Rightarrow 5(h^2 + 1 - 2h + k^2 + 4 - 4k) = 3(4h^2 + k^2 + 1 + 4hk - 2k - 4h)$

$\Rightarrow 7h^2 - 2k^2 + 12hk - 2h + 14k - 22 = 0$

Now, to obtain equation of hyperbola replace $h \to x$ and $k \to y$

$\Rightarrow 7x^2 + 12xy - 2y^2 - 2x + 14y - 22 = 0$

As we can see after comparing with equation

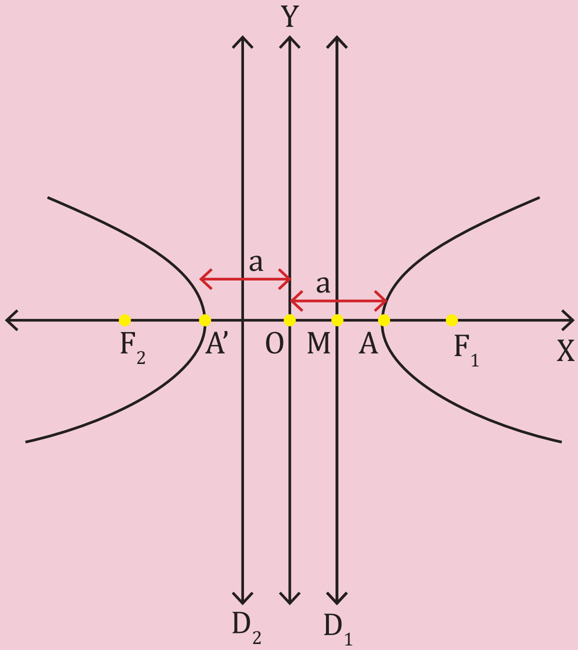
$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

We get, h = 6, a = 7 and b = -2

$\Rightarrow h^2 - ab = 50 > 0$, so the equation will represent a hyperbola

Standard equation of a hyperbola

Given, a hyperbola having two foci (F_1 and F_2), two directrices (D_1 and D_2), and vertices as A and A'. The line passing through foci of a hyperbola and perpendicular to the directrices is known as the axis of symmetry. Here, x - axis is the axis of symmetry.



Let us consider a point P(h, k) lying on the hyperbola.

Now, by the definition of hyperbola,

$$\frac{AF_1}{AM} = e \text{ and } \frac{A'F_1}{A'M} = e$$

By using basic interpolation from figure, we get,

$$\frac{AF_1}{AM} = \frac{OF_1 - OA}{OA - OM} \Rightarrow \frac{OF_1 - a}{a - OM} = e$$

Or $a - OF_1 = eOM - ea$ ---(i)

And,

$$\frac{A'F_1}{A'M} = \frac{OA' + OF_1}{OA' + OM} \Rightarrow \frac{a + OF_1}{a + OM} = e$$

Or $a + OF_1 = eOM + ea$ ---(ii)

Now, adding equations (i) and (ii),

$2a = 2eOM$

$$\Rightarrow OM = \frac{a}{e}$$

Now, after subtracting (ii) from (i), we get,

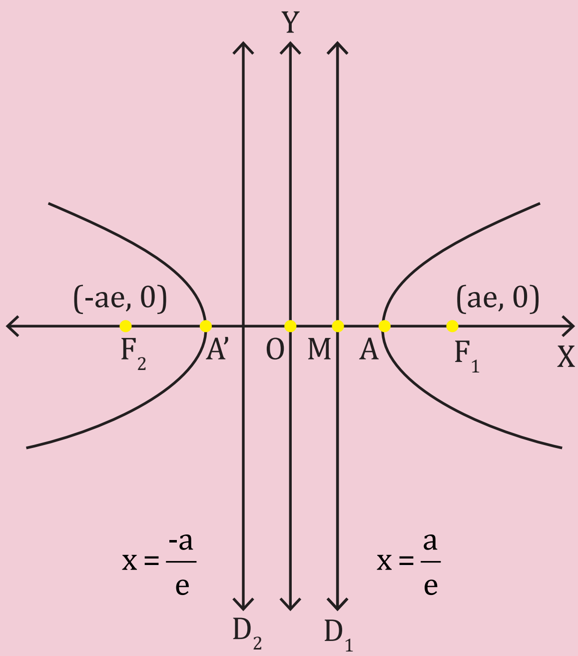
$-2OF_1 = -2ea \quad \Rightarrow OF_1 = ae$

Now, Let $A \equiv (a, 0)$ and $A' \equiv (-a, 0)$ be the vertices.

Then, $F_2 \equiv (-ae, 0)$ and $F_1 \equiv (ae, 0)$

Now, equations of directrices,

$$D_2 : x = \frac{-a}{e} \text{ and } D_1 : x = \frac{a}{e}$$



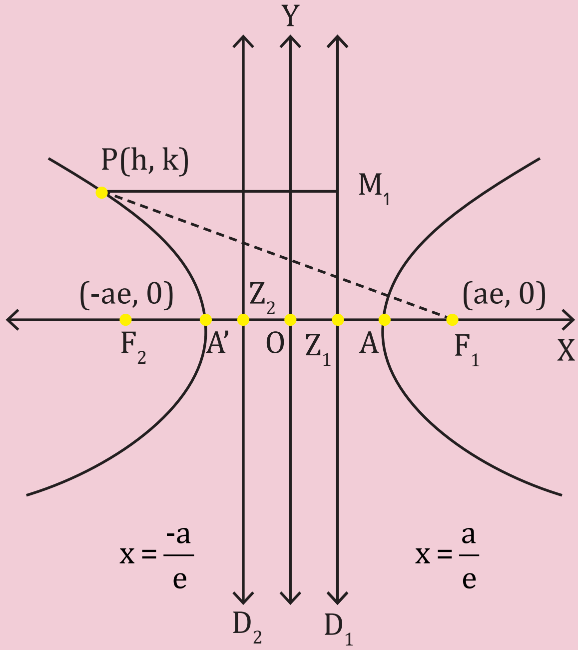
Now, for locus,
Let us consider $P \equiv (h, k)$ be any point on the hyperbola.
As we know by the definition of a hyperbola,

$$\frac{PF_1}{PM_1} = e$$

$$\Rightarrow PF_1 = ePM_1$$

$$\Rightarrow \sqrt{(h - ae)^2 + k^2} = e \times \frac{|eh - a|}{\sqrt{e^2}}$$

$$\Rightarrow \sqrt{(h - ae)^2 + k^2} = |eh - a|$$



Now, squaring both the sides, we get,

$$(h - ae)^2 + k^2 = (eh - a)^2$$

$$\Rightarrow h^2 + a^2e^2 - 2aeh + k^2 = e^2h^2 + a^2 - 2aeh$$

$$\Rightarrow h^2(1 - e^2) + k^2 = a^2(1 - e^2)$$

or $$\frac{h^2}{a^2} + \frac{k^2}{a^2(1 - e^2)} = 1$$

As we can see, $e > 1 \Rightarrow (1 - e^2) < 0$

Therefore, the equation becomes the following:

$$\frac{h^2}{a^2} - \frac{k^2}{a^2(e^2 - 1)} = 1$$

$$\Rightarrow \frac{h^2}{a^2} - \frac{k^2}{b^2} = 1; \text{where } b^2 = a^2(e^2 - 1)$$

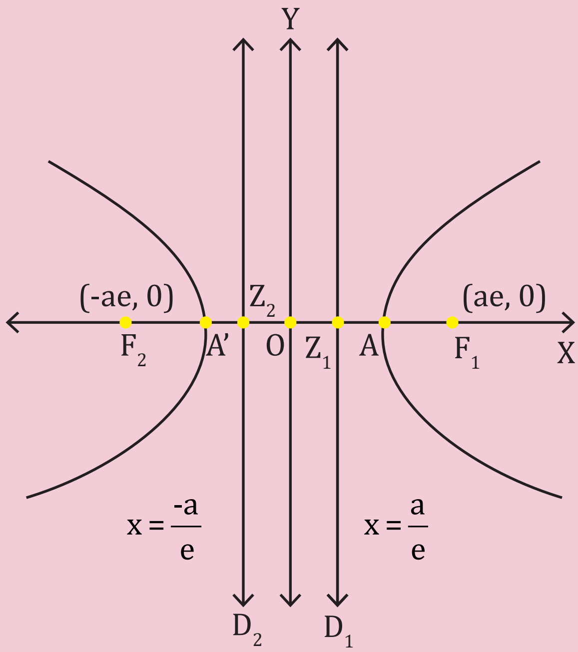
Now, replace $h \rightarrow x$ and $k \rightarrow y$

We get, $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Therefore, the standard equation of a hyperbola is the following:

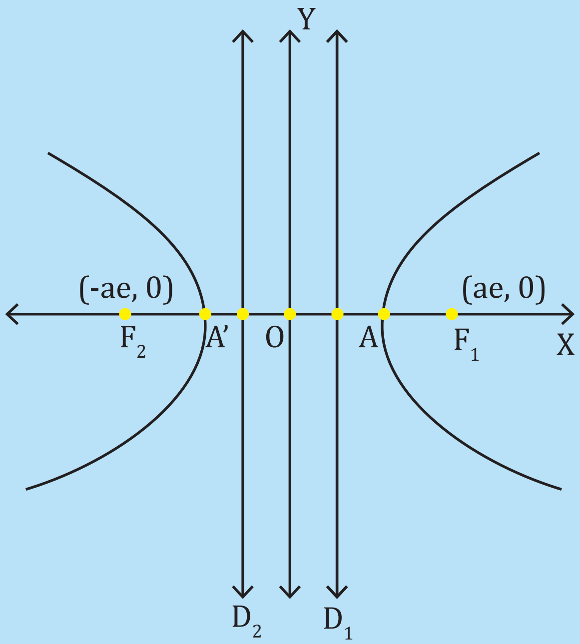
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Where $b^2 = a^2(e^2 - 1)$



Comparison of a hyperbola and an ellipse

<table>
<tr><th>Hyperbola</th><th>Ellipse</th></tr>
<tr><td>

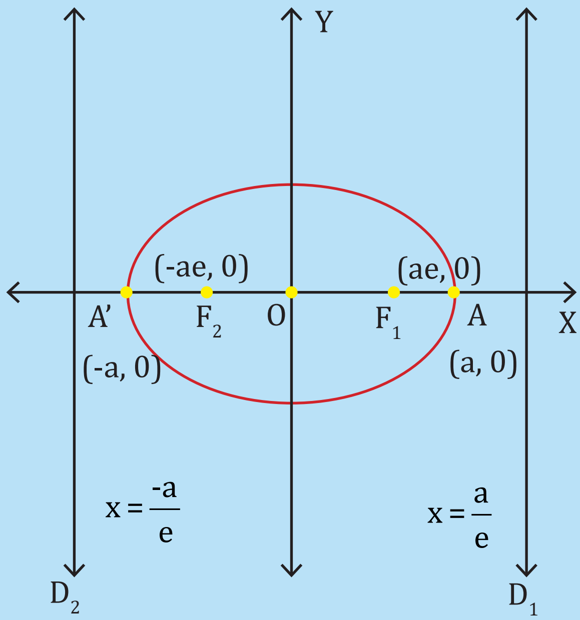


$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$b^2 = a^2\left(e^2 - 1\right)$$

When we replace b^2 by $-b^2$ in the equation of a hyperbola, then we get the equation of a standard ellipse. Similarly, if we replace b^2 by $-b^2$ in the relation between a, b and e , we get a similar relation for ellipse.

</td><td>



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$b^2 = a^2\left(1 - e^2\right)$$

When we replace b^2 by $-b^2$ in the equation of an ellipse, then we get the equation of a standard hyperbola. Similarly, if we replace b^2 by $-b^2$ in the relation between a, b and e, we get a similar relation for hyperbola

</td></tr>
<tr><td>

Let us consider the following:

$e = \sqrt{1.5} \Rightarrow b^2 = a^2\left(e^2 - 1\right) \Rightarrow b^2 = a^2 \times (0.5)$

$\Rightarrow b < a$

Now, $e = 2 \Rightarrow b^2 = a^2 \times 3 \Rightarrow b > a$

Therefore, we can say that the relationship between a and b is not impacting the configuration of the hyperbola.

⇒ There is no concept of major axis and minor axis in case of hyperbolas.

</td><td>

However, in case of an ellipse, when a > b, then it is a horizontal ellipse, and when b > a, then it is a vertical ellipse.

</td></tr>
<tr><td>

For a hyperbola, as we know that $e > 1$,

$\Rightarrow \frac{a}{e} < a < ae$

This means that for a hyperbola, the directrix $x = \frac{a}{e}$ comes first, followed by the

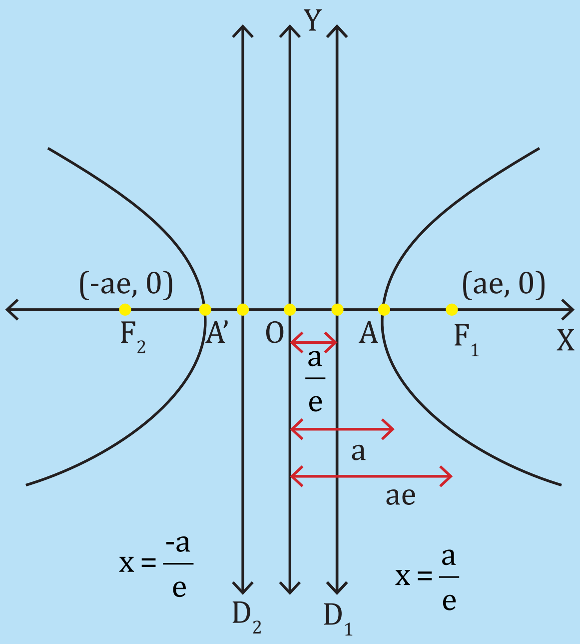
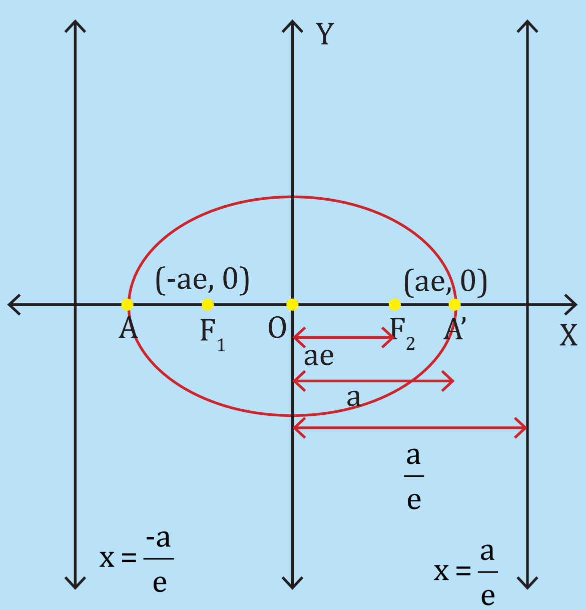
</td><td>

For an ellipse, as we know that $0 < e < 1$,

$\Rightarrow ae < a < \frac{a}{e}$

This means that for a ellipse , the focus $(ae, 0)$ comes first, followed by the vertex

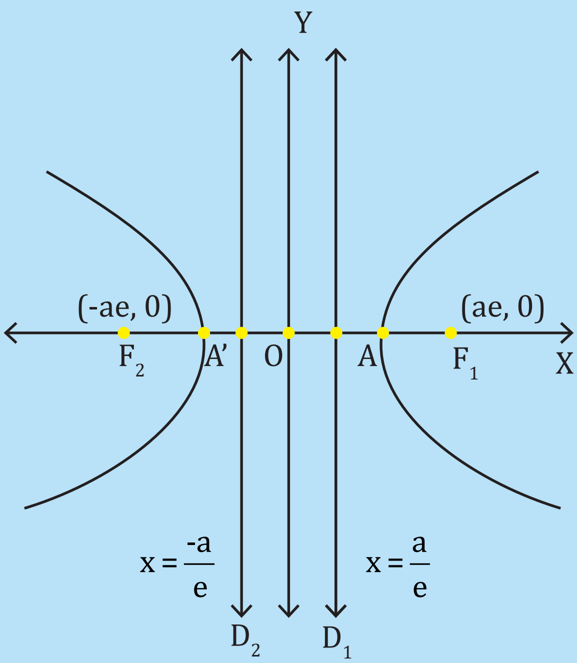
</td></tr>
</table>

vertex (a, 0), followed by the focus (ae, 0). Similarly, on the negative x-axis.	(a, 0), followed by the directrix $x = \frac{a}{e}$. Similarly, on the negative x-axis
	

Terms associated with the standard hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Centre

The centre of a hyperbola is the midpoint of the line segment joining its foci. The centre of the standard hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is (0, 0).



Foci

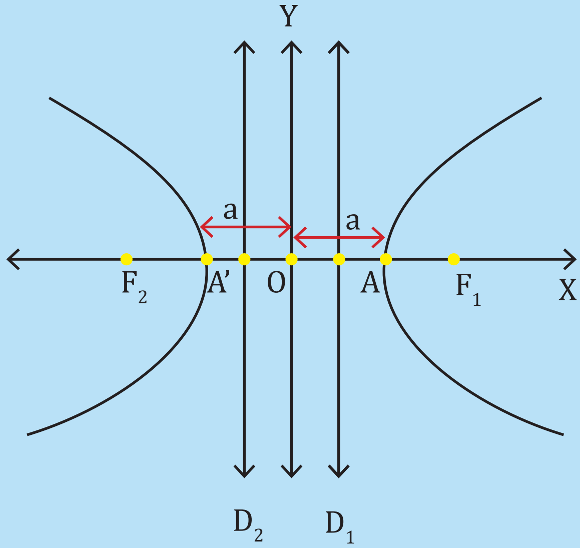
For hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ the foci are (± ae, 0).

Vertices

The points of intersection of the hyperbola with the line passing through the foci are known as vertices.

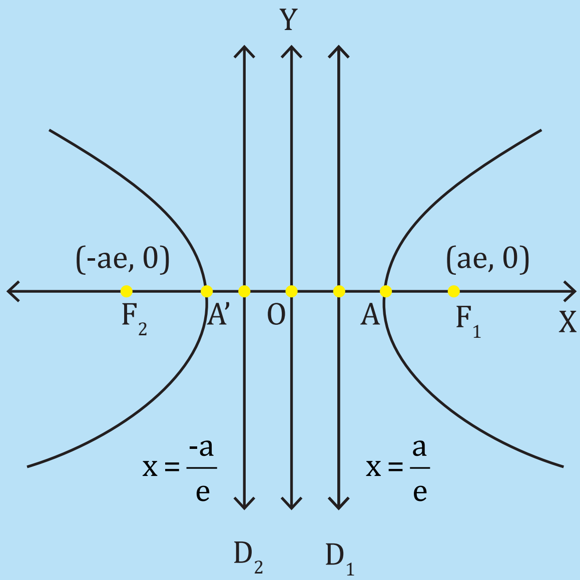
Here, in hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

vertices are $A'(-a, 0)$ and $A(a, 0)$.



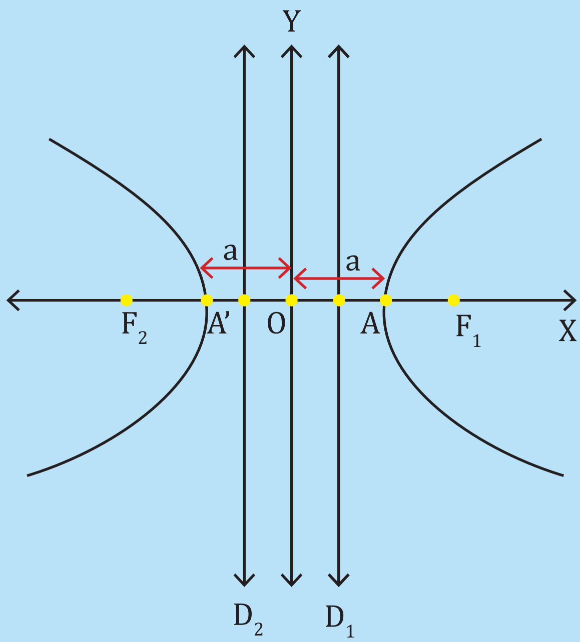
Directrices

In hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ the directrices are given by $x = \pm\frac{a}{e}$



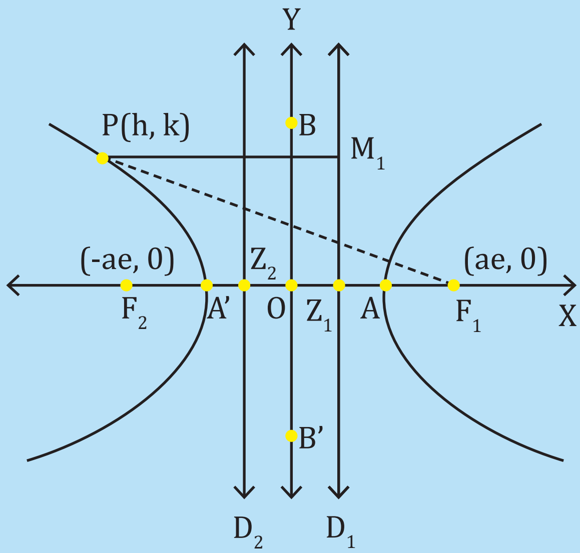
Transverse axis

The line joining the vertices A and A' is known as the transverse axis.
The length of the transverse axis is $2a$ (AA')



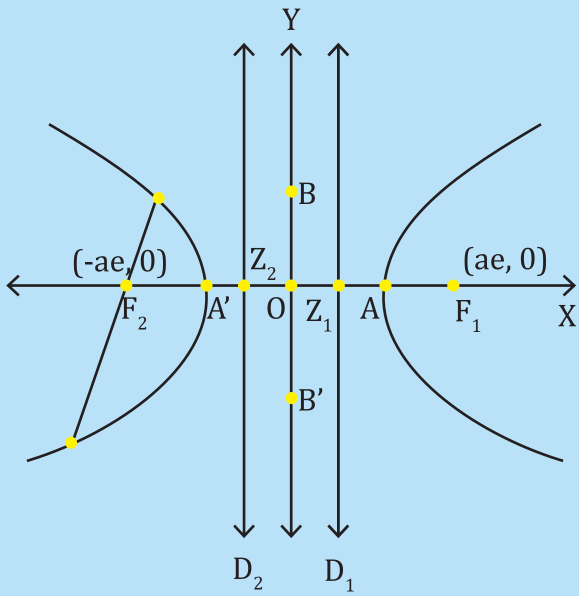
Conjugate axis

The line through the centre and perpendicular to the transverse axis is known as the conjugate axis.
The length of the conjugate axis is 2b (BB').



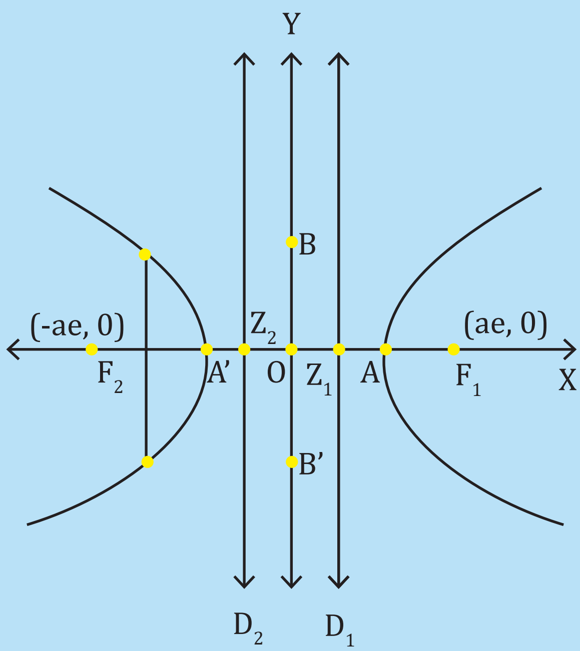
Focal chord

A chord passing through the focus is known as a focal chord.

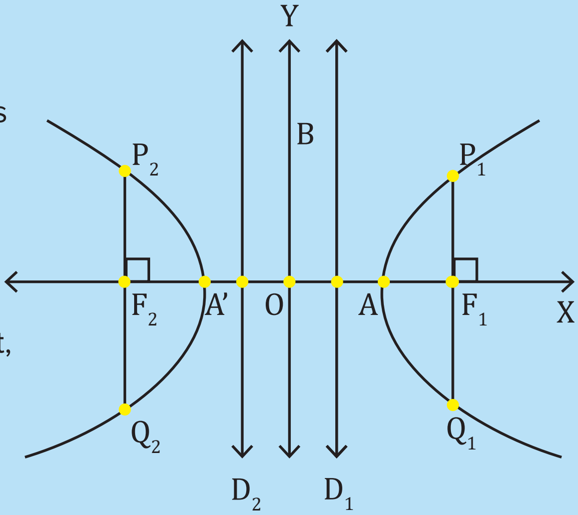


Double ordinate

A chord perpendicular to the transverse axis is known as the double ordinate.



Latus rectum



The focal chord perpendicular to the transverse axis is known as the latus rectum.

In the figure,

P_1Q_1 ; $P_2Q_2 \rightarrow$ LR = Latus rectum

As we know that the coordinates of point P_1 are (ae, y_0).

Now, by the standard equation of hyperbola, we get,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \frac{a^2e^2}{a^2} - \frac{y_0^{\ 2}}{b^2} = 1$$

Or $y_0^{\ 2} = b^2(e^2 - 1)$

Now, substitute $b^2 = a^2(e^2 - 1)$ in the given equation,

$$\Rightarrow y_0^{\ 2} = b^2 \times \frac{b^2}{a^2} \Rightarrow y_0 = \pm\frac{b^2}{a}$$

$$\Rightarrow P_1 \equiv \left(ae, \frac{b^2}{a}\right) \text{and } Q_1 \equiv \left(ae, -\frac{b^2}{a}\right)$$

And now, length of LR: $P_1Q_1 = \dfrac{2b^2}{a}$

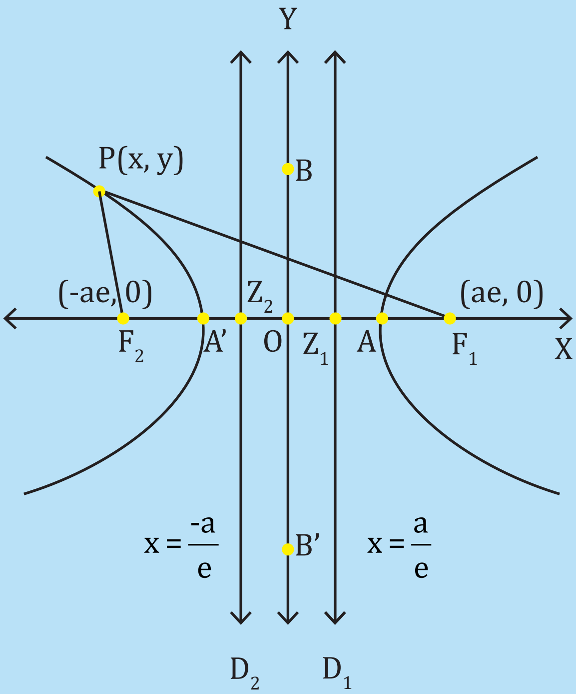
Now, same for LR: P_2Q_2

$$\Rightarrow P_2 \equiv \left(-ae, \frac{b^2}{a}\right) \text{and } Q_2 \equiv \left(-ae, -\frac{b^2}{a}\right)$$

Length of LR: $P_2Q_2 = \dfrac{2b^2}{a}$

Focal distance

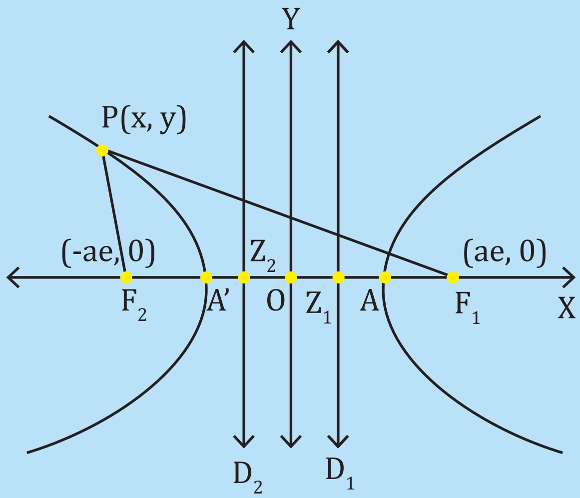
The distance from the focus to any point on the hyperbola is known as the focal distance or the focal radius.



Focal distance calculation

Let $P \equiv (x, y)$ be any point on the hyperbola.
Here, the focal distance is PF_1 and PF_2
As we know by the definition of hyperbola,

$$\Rightarrow \frac{PF_1}{PM_1} = e \text{ and } \frac{PF_2}{PM_2} = e$$



Here, we can see the following:

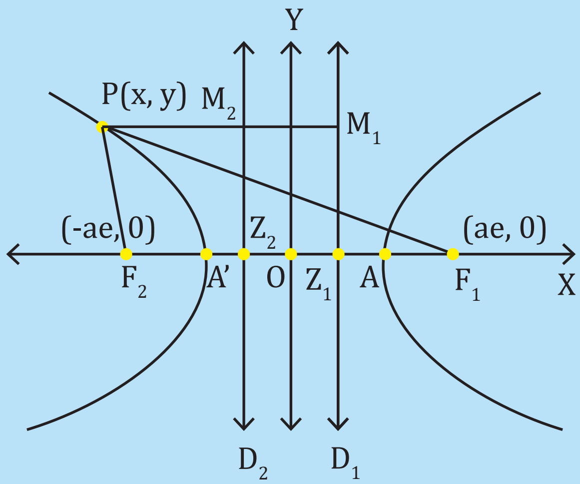
$$PM_1 = x + \frac{a}{e} \text{ and } PM_2 = x - \frac{a}{e}$$

$$\Rightarrow PF_1 = e(PM_1) \text{ and } PF_2 = e(PM_2)$$

$$\Rightarrow PF_1 = e\left(x + \frac{a}{e}\right) \text{ and } PF_2 = e\left(x - \frac{a}{e}\right)$$

Or $PF_1 = ex + a$ and $PF_2 = ex - a$

$\therefore$ Focal distance of $P(x, y) = ex \pm a$



Result

We can see that focal distance of a point $P(x, y)$ on a hyperbola is as follows:
$PF_1 = ex + a$ and $PF_2 = ex - a$
Now, as we can see that $|PF_1 - PF_2| = ex + a - (ex - a)$,
$\Rightarrow |PF_1 - PF_2| = 2a$ (For both the branches)
$\Rightarrow$ The difference of the focal distances of any point on the hyperbola is equal to the length of the transverse axis.

- **Alternative definition of hyperbola**
  Hyperbola is the set of all points in a plane, the difference of whose distances from two fixed points (foci) in the plane is a constant $(2a)$.

Concept Check

1. Let $0 < \theta < \frac{\pi}{2}$. If the eccentricity of the hyperbola $x^2 \sec^2\theta - y^2 \operatorname{cosec}^2\theta = 1$ is 2, which of the following is/are true?

   (a) $\theta = \frac{\pi}{6}$.
   (b) The length of LR is 3.
   (c) The distance between the foci is 2.
   (d) The equation of the directrices is $x = \pm 1$

Summary sheet

Key Result

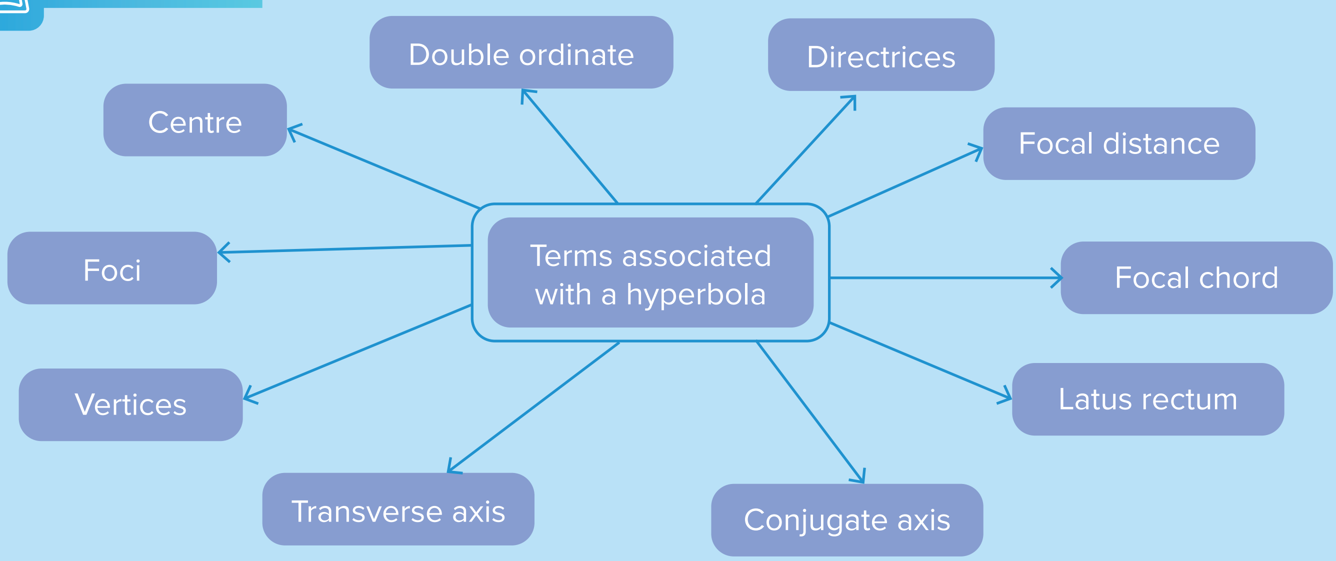
Hyperbola

- Hyperbola is the locus of a moving point such that the ratio of its distance from a fixed point (focus) and a fixed line (directrix) is a constant that is always greater than 1. $(e > 1)$

Terms associated with the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

- **Centre** - The centre of a hyperbola is the midpoint of the line segment joining its foci. For a standard hyperbola the centre is $(0, 0)$.
- **Foci** - The foci are $(\pm ae, 0)$.
- **Vertices** - The points of intersection of the hyperbola with the line passing through the foci are known as vertices. Here, the vertices are $A'(-a, 0)$ and $A(a, 0)$.
- **Directrices** - The directrices are given by $x = \pm \frac{a}{e}$
- **Transverse axis** -The line joining vertices A and A' is known as the transverse axis. The length of the transverse axis is $2a$ (AA').
- **Conjugate axis** - The line through the centre and perpendicular to the transverse axis is known as the conjugate axis. The length of the conjugate axis is $2b$ (BB').
- **Focal chord** - A chord passing through the focus is known as a focal chord.
- **Double ordinate** - A chord perpendicular to the transverse axis is known as the double ordinate.
- **Latus rectum** - The focal chord perpendicular to the transverse axis is known as the latus rectum.
- **Focal distance** - The distance from the focus to any point on the hyperbola is known as the focal distance or the focal radius.

Mind map



Self-Assessment

1. Find the equation of a hyperbola whose axes are the coordinates axes and the distances of one of its vertices from the foci are 3 and 1.

2. If the LR subtends a right angle at the centre of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then find its eccentricity.

Answers

Concept Check 1

Step 1:

Given, equation of hyperbola

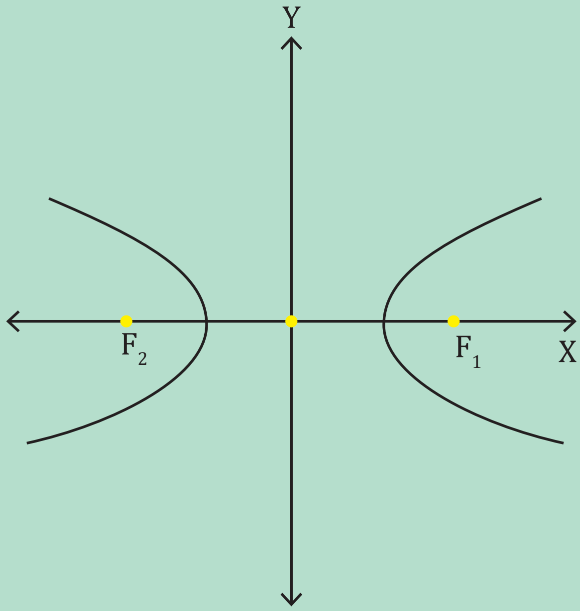
$x^2 \sec^2 \theta - y^2 \operatorname{cosec}^2 \theta = 1; 0 < \theta < \frac{\pi}{2}$

$\Rightarrow \frac{x^2}{\cos^2 \theta} - \frac{y^2}{\sin^2 \theta} = 1$, having eccentricity as 2

Now, comparing the given equation with standard equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get,

$\Rightarrow a^2 = \cos^2 \theta$ and $b^2 = \sin^2 \theta$

As we know, $e = \sqrt{1 + \frac{b^2}{a^2}}$



$\Rightarrow 2 = \sqrt{1 + \frac{\sin^2 \theta}{\cos^2 \theta}} \Rightarrow 2 = \sqrt{\frac{1}{\cos^2 \theta}}$

$\Rightarrow \frac{1}{|\cos \theta|} = 2$

Step 2:

As we know that for $\theta \in \left(0, \frac{\pi}{2}\right)$

$|\cos \theta| = \cos \theta$

Therefore, we get,

$\cos \theta = \frac{1}{2}$ or $\theta = \frac{\pi}{3}$

Now, $a^2 = \frac{1}{4}$ and $b^2 = \frac{3}{4}$

As we know, length of LR $= \frac{2b^2}{a} = 3$

Foci $\equiv (\pm ae, 0) = (\pm 1, 0)$

Equation of dirctrices, $x = \pm \frac{a}{e} = \pm \frac{1}{4}$

Hence, options (b) and (c) are the correct answers.

Self-Assessment 1

Step 1:

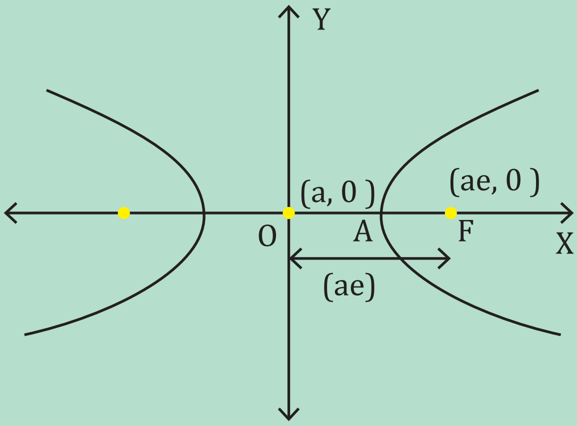
Given, axes of a hyperbola are the coordinate axes and distances of one of its vertices from the foci are 3 and 1.

$\Rightarrow ae - a = 1$ ---(i) and

$ae + a = 3$ ---(ii)

Now, dividing equations (ii) by (i),

$\Rightarrow \frac{e+1}{e-1} = 3 \Rightarrow e = 2$ and $a = 1$



Step 2:

Also, from $b^2 = a^2(e^2 - 1)$,

We get, $b^2 = 3$

Now, final equation,

$x^2 - \frac{y^2}{3} = 1$ or $\frac{x^2}{1} - \frac{y^2}{3} = 1$

Self-Assessment 2

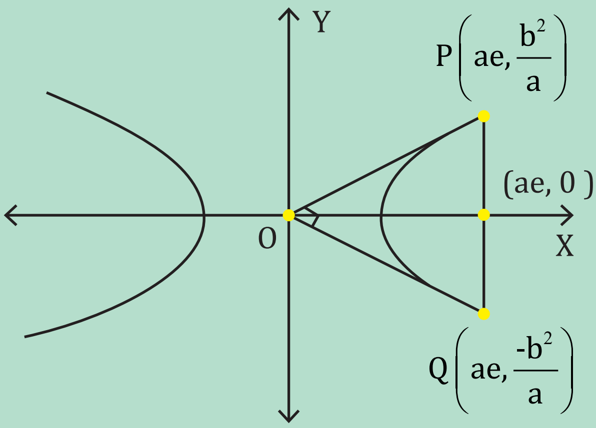
Step 1:

Given, LR of a hyperbola subtends a right angle at the centre of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

As given, $m_{OP} \times m_{OQ} = -1$

$$\Rightarrow \left(\frac{\frac{b^2}{a}}{ae}\right) \times \left(-\frac{\frac{b^2}{a}}{ae}\right) = -1$$

$$\Rightarrow \frac{b^4}{a^4e^2} = 1$$



Step 2:

As we know, $b^2 = a^2(e^2 - 1)$

$\Rightarrow (e^2 - 1)^2 = e^2$ or $e^4 - 3e^2 + 1 = 0$

Or we get,

$$e^2 = \frac{3 \pm \sqrt{5}}{2}$$

[∵ As we know that for a hyperbola, $e > 1$]

$$\Rightarrow e^2 = \frac{3+\sqrt{5}}{2}$$

Now, multiply and divide by 2

$$\Rightarrow e^2 = \frac{\left(6+2\sqrt{5}\right)}{2} \Rightarrow e^2 = \frac{\left(1+2\sqrt{5}+5\right)}{4}$$

$$\Rightarrow e = \frac{1+\sqrt{5}}{2}$$

MATHEMATICS

NOTES

HYPERBOLA

EQUATION OF VERTICAL, SHIFTED AND ROTATED HYPERBOLAS AND PARAMETRIC COORDINATES OF HYPERBOLA

What you already know

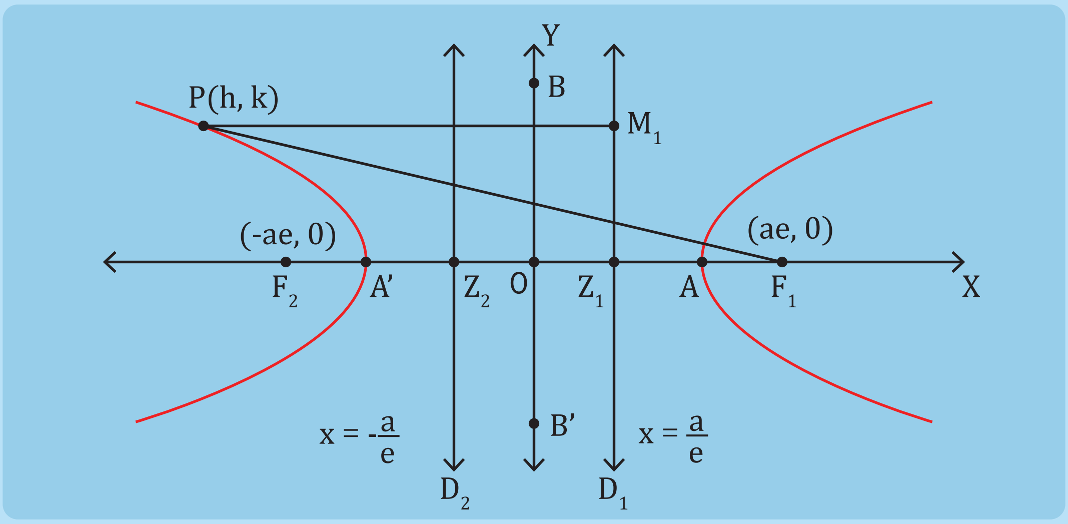
- Standard equation of hyperbola
- Vertices, directrices of hyperbola
- Transverse axis and conjugate axis of hyperbola
- Focal chord, double ordinate of hyperbola
- Latus rectum, focal distance of hyperbola

What you will learn

- Equation of vertical hyperbola
- Equation of shifted hyperbola
- Equation of rotated hyperbola
- Parametric coordinates of hyperbola

Quick Recap

1. The standard equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where a is the length of the semi-transverse axis and b is the length of the semi-conjugate axis. The relation between a, b, e is $b^2 = a^2 (e^2 - 1)$
2. **Foci:** The foci are F_1 (ae, 0) and F_2 (-ae, 0).



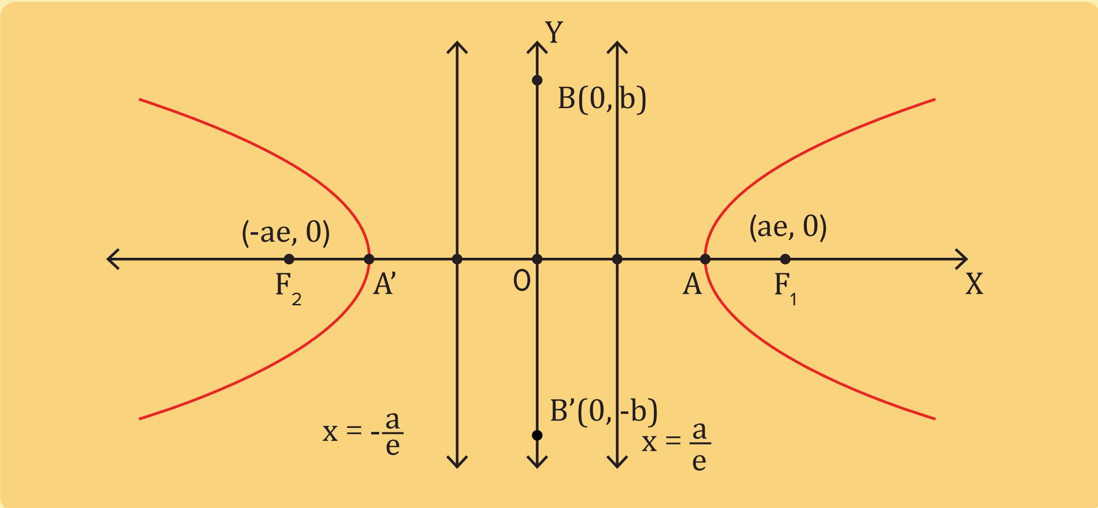
3. **Vertices:** The points of intersection of hyperbola with the line passing through the foci are known as vertices. Here, the vertices are A'(-a, 0) and A(a, 0)
4. **Directrices:** The equations of directrices are $x = -\frac{a}{e}$ and $x = \frac{a}{e}$
5. **Transverse axis:** The line joining vertices A and A' is known as the transverse axis. The equation of the transverse axis is y = 0. The length of the transverse axis is AA' = 2a.
6. **Conjugate axis:** The line through the centre and perpendicular to the transverse axis is known as the conjugate axis. The length of the conjugate axis is BB' = 2b.

7. **Focal chord:** The chord passing through the focus is known as the focal chord.
8. **Double ordinate:** A chord perpendicular to the transverse axis is known as double ordinate.
9. **Latus rectum:** The focal chord perpendicular to the transverse axis is known as latus rectum. The length of latus rectum is $\frac{2b^2}{a}$. The end points of latus rectum are $(ae, \frac{b^2}{a})$, $(ae, -\frac{b^2}{a})$ for the LR passing through (ae, 0); and $(-ae, \frac{b^2}{a})$, $(-ae, -\frac{b^2}{a})$ are the end points for the LR passing through (-ae, 0).
10. **Focal distance:** The distance between the focus to any point on the hyperbola is known as the focal distance or focal radii.

Standard Hyperbola

Horizontal hyperbola

If in the equation of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (where a > b), we replace b^2 with $-b^2$, then we get the equation, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. This is the equation of the standard hyperbola.



i) For an ellipse (a > b), the relation between a, b, and e is given by $b^2 = a^2(1 - e^2)$. For a hyperbola we replace b^2 by $-b^2$.
Hence, the relation between a, b, and e for a hyperbola is given by $b^2 = a^2(e^2 - 1)$

ii) For a hyperbola, e is a real quantity such that $e > 1$ and $\frac{a}{e} < a < ae$. So, in a hyperbola, the directrix comes first, followed by the vertex and the focus.

iii) The line (x-axis) passing through both the foci and perpendicular to both the directrices is known as the transverse axis of the hyperbola.

iv) The line (y-axis) passing through the centre and perpendicular to the transverse axis is known as the conjugate axis for the hyperbola.

v) The quantity 'a' represents the distance from the origin to each of the vertices. Hence, 'a' is also known as the length of the semi-transverse axis. Also, '2a' is the length of the transverse axis, which is the distance between the vertices.

vi) The quantity 'b' represents the distance from the origin to the points B(0, b) or B'(0, -b) . Hence, 'b' is also known as the length of the semi-conjugate axis. '2b' is the length of the conjugate axis, which is the distance between the points B(0, b) and B'(0, -b) .

Note

We have $b^2 = a^2(e^2 - 1)$

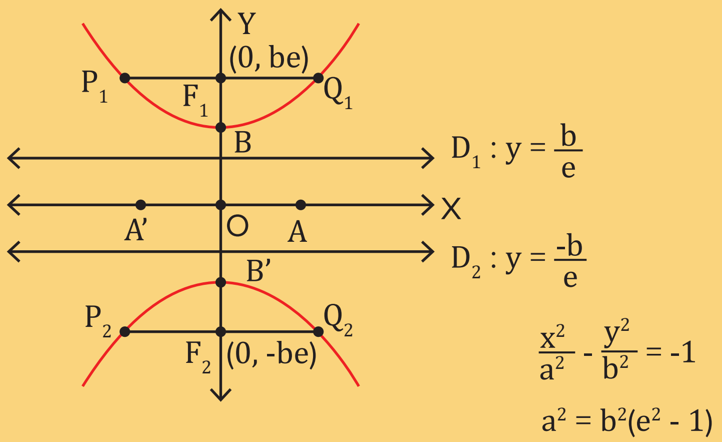
$\Rightarrow \frac{b^2}{a^2} = e^2 - 1 \Rightarrow e^2 = 1 + \frac{b^2}{a^2}$

$\Rightarrow e = \sqrt{1 + \frac{b^2}{a^2}}$ as e is a positive quantity.

$\Rightarrow e = \sqrt{1 + \frac{(2b)^2}{(2a)^2}}$ or $\Rightarrow e = \sqrt{1 + \frac{C^2}{T^2}}$, where C^2 is the square of the length of the conjugate axis and T^2 is the square of the length of the transverse axis.

Vertical hyperbola

Let us rotate the horizontal hyperbola by 90°, we get another hyperbola. The centre does not get disturbed by the rotation, so the centre is still at the origin. After the rotation, the foci get to the y-axis, and the directrices become parallel to the x-axis. Also, after the rotation, the branches start opening upward and downward along the y-axis and hence, we refer to the hyperbola obtained as **vertical hyperbola.**



i) The equation of the vertical hyperbola is given by $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

ii) The relation between a, b and e is given by $a^2 = b^2(e^2 - 1) \Rightarrow e = \sqrt{1 + \frac{a^2}{b^2}}$

iii) The line (y-axis) cutting both the branches of the hyperbolic curve is known as the transverse axis of the vertical hyperbola.

iv) The line (x-axis) that is not intersecting the hyperbolic curve is known as the conjugate axis of the vertical hyperbola.

v) The coordinates of both the foci are given by $F_1\ (0, be)$ and $F_2\ (0, -be)$.

vi) The equation of the directrices are given by $D_1 : y = \frac{b}{e}$ and $D_2 : y = \frac{-b}{e}$. Both the directrices are parallel to the x-axis.

vii) For a hyperbola, e is a real quantity such that $e > 1$ and $\frac{b}{e} < b < be$. So, in the hyperbola, the directrix comes first, followed by the vertex and the focus.

viii) The points where the hyperbolic curve intersects the transverse axis are known as vertices. The coordinates of the vertices are given by $B(0, b)$ and $B'(0, -b)$

ix) The quantity b represents the distance from the origin to each of the vertices. Hence, b is also known as the length of the semi-transverse axis. Also, $2b$ is the length of the transverse axis, which is the distance between the vertices.

x) The quantity 'a' represents the distance from the origin to points $A(a, 0)$ or $A'(-a, 0)$. Hence, 'a' is also known as the length of the semi-conjugate axis. Also, 2a is the length of the conjugate axis, which is the distance between points $A(a, 0)$ and $A'(-a, 0)$.

Note

We have $a^2 = b^2(e^2 - 1) \Rightarrow \frac{a^2}{b^2} = e^2 - 1 \Rightarrow e^2 = 1 + \frac{a^2}{b^2}$

$\Rightarrow e = \sqrt{1 + \frac{a^2}{b^2}}$ as e is a positive quantity.

$\Rightarrow e = \sqrt{1 + \frac{(2a)^2}{(2b)^2}}$ or $\Rightarrow e = \sqrt{1 + \frac{C^2}{T^2}}$, where C^2 is the square of the length of the conjugate axis and T^2 is the square of the length of the transverse axis.

xi) **Latus rectum :** It is a focal chord that is also a double ordinate. Let P_1Q_1 and P_2Q_2 be the latus rectums for a hyperbola. The length of a latus rectum denotes the spread of the curve at the focal points and is given by $\frac{2a^2}{b}$. The length of the semi-latus rectum is given by $\frac{a^2}{b}$. The coordinates of end-points of LR passing through $(0, be)$ are given by $\left(\frac{-a^2}{b}, be\right)$ and $\left(\frac{a^2}{b}, be\right)$.

The coordinates of end-points of LR passing through $(0, -be)$ are given by $\left(\frac{-a^2}{b}, -be\right)$ and $\left(\frac{a^2}{b}, -be\right)$.

xii) **Focal distance:** Let $P(x, y)$ be any point on the hyperbola. Then, focal distances will be the distance of P from each of the foci, i.e., PF_1 and PF_2 are the focal distances for point P, which in the combined form are given by $ey \pm b$

We can observe that difference of focal distances = $2b$

Note

The horizontal and vertical hyperbolas are known as conjugates of each other, i.e., the horizontal hyperbola is the conjugate of the vertical hyperbola and the vertical hyperbola is the conjugate of the horizontal hyperbola.

Let e_1, e_2 be the eccentricities of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and its conjugate hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$, respectively. Prove that $\frac{1}{(e_1)^2} + \frac{1}{(e_2)^2} = 1$

Solution

We know that for a horizontal hyperbola $\Rightarrow e_1 = \sqrt{1 + \frac{b^2}{a^2}}$

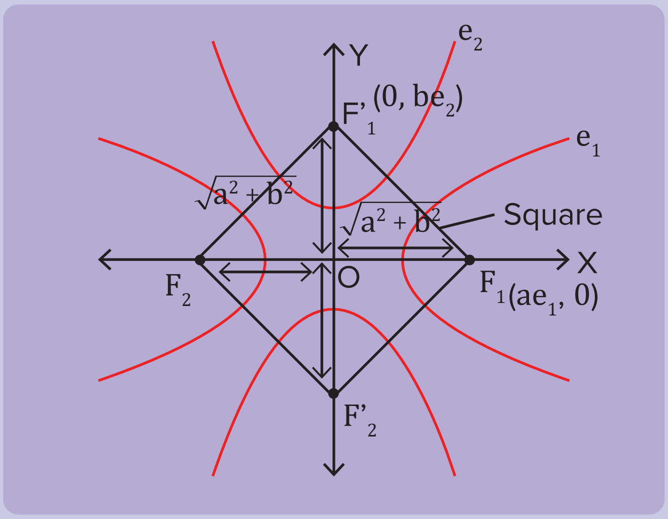
$\Rightarrow (e_1)^2 = \frac{a^2 + b^2}{a^2} \Rightarrow \frac{1}{(e_1)^2} = \frac{a^2}{a^2 + b^2}$

Similarly, $e_2 = \sqrt{1 + \frac{a^2}{b^2}} \Rightarrow \frac{1}{(e_2)^2} = \frac{b^2}{a^2 + b^2}$

Now, $\frac{1}{(e_1)^2} + \frac{1}{(e_2)^2} = \frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2} = \frac{a^2 + b^2}{a^2 + b^2} = 1;$ **Hence proved.**

Note

The foci of a hyperbola and its conjugate hyperbola are concyclic and form a square.



Proof

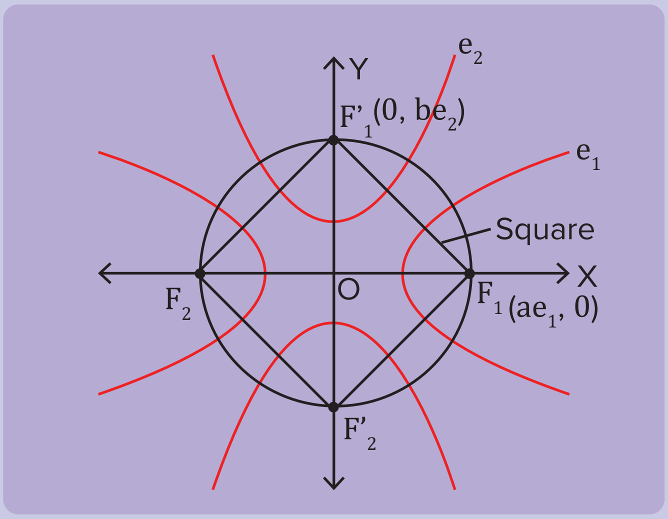
We know that the segments $F_1 F_2$ and $F_1' F_2'$ are perpendicular.

(i) Proof for square : Let e_1 be the eccentricity of horizontal hyperbola and e_2 be the eccentricity of vertical hyperbola.

$$e_1 = \sqrt{1 + \frac{b^2}{a^2}};\ e_2 = \sqrt{1 + \frac{a^2}{b^2}}$$

$$\Rightarrow ae_1 = a\sqrt{1 + \frac{b^2}{a^2}} = \sqrt{a^2 \frac{(a^2 + b^2)}{a^2}} = \sqrt{a^2 + b^2}$$

$$\Rightarrow be_2 = b\sqrt{1 + \frac{a^2}{b^2}} = \sqrt{b^2 \frac{(a^2 + b^2)}{b^2}} = \sqrt{a^2 + b^2}$$



Now, $OF_1 = ae_1$ and $OF_1' = be_2$; Since $ae_1 = be_2$, $F_1 F_2$ and $F_1' F_2'$ are perpendicular bisectors.

(ii) Since $OF_1 = OF_2 = (OF_1)' = (OF_2)' = \sqrt{a^2 + b^2}$, they are concyclic.

Rectangular Hyperbola

If the lengths of the transverse axis and conjugate axis of a hyperbola are equal, then it is known as **rectangular/equilateral hyperbola.**

$\Rightarrow 2a = 2b \Rightarrow a = b$

For $a = b$,

Horizontal hyperbola:

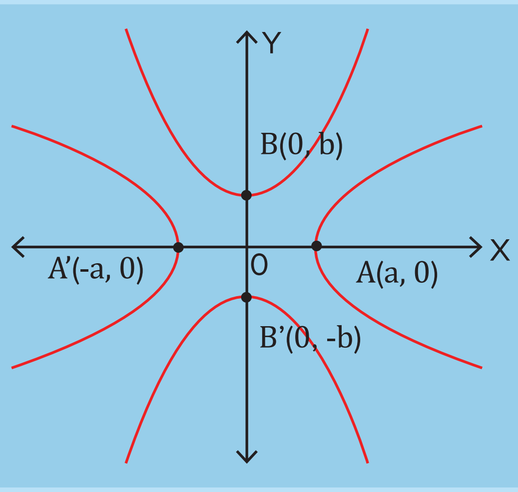
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \Rightarrow \frac{x^2}{a^2} - \frac{y^2}{a^2} = 1 \Rightarrow x^2 - y^2 = a^2$$

and $e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{2}$

Vertical hyperbola:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1, \Rightarrow \frac{x^2}{a^2} - \frac{y^2}{a^2} = -1 \Rightarrow x^2 - y^2 = -a^2$$

and $e = \sqrt{1 + \frac{a^2}{b^2}} = \sqrt{2}$



Note

If the axes of the hyperbola are rotated by an angle of $\frac{-\pi}{4}$ about the origin, then the equation of the rectangular hyperbola $x^2 - y^2 = a^2$ transforms to $xy = \frac{a^2}{2}$ or $xy = c^2$

Translated hyperbola

Consider the horizontal hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The centre of the hyperbola is shifted to $C(h, k)$.

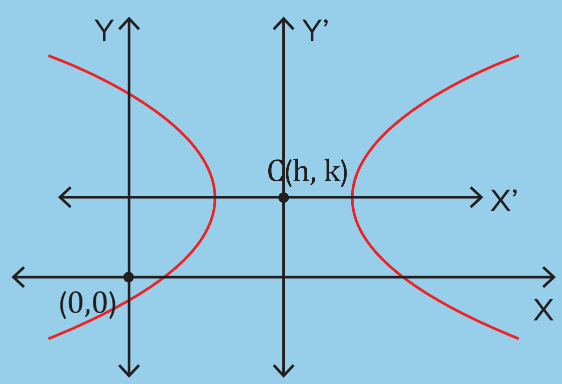
(i) Equation of the hyperbola with respect to the new coordinate plane X'Y'

$$\Rightarrow \frac{(x')^2}{a^2} - \frac{(y')^2}{b^2} = 1$$

(ii) Equation of the hyperbola with respect to the original coordinate plane XY

$$\Rightarrow \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Under shifting/translation, the lengths do not change. Hence, the relation between a, b, and e remains the same for both the hyperbolas, $b^2 = a^2 (e^2 - 1)$



Consider the hyperbola, $9x^2 - 16y^2 - 72x + 96y - 144 = 0$. Find the following:
Centre, Vertices, Eccentricity, Foci, Equation of directrices, Length of transverse axis, Equation of transverse axis, Length of conjugate axis, Equation of conjugate axis, Length of latus rectum.

Solution

Step 1:

Given, $9x^2 - 16y^2 - 72x + 96y - 144 = 0$

$\Rightarrow 9x^2 - 72x - 16y^2 + 96y = 144$

$\Rightarrow 9(x^2 - 8x) - 16(y^2 - 6y) = 144 \Rightarrow 9(x^2 - 8x + 16 - 16) - 16(y^2 - 6y + 9 - 9) = 144$

$\Rightarrow 9(x - 4)^2 - 16(y - 3)^2 = 144 \Rightarrow \frac{(x - 4)^2}{16} - \frac{(y - 3)^2}{9} = 1$

This is of the form $\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$, i.e., equation of translated horizontal hyperbola having its centre at (h, k).

$\Rightarrow a^2 = 16, b^2 = 9, h = 4$ and $k = 3$

Step 2:

Let $x - h = x'$ and $y - k = y'$

$\Rightarrow \frac{(x - 4)^2}{16} - \frac{(y - 3)^2}{9} = 1$

Centre $(x', y') \equiv (0, 0)$

$\Rightarrow x - h = 0$ and $y - k = 0 \Rightarrow x - 4 = 0$ and $y - 3 = 0 \Rightarrow x = 4$ & $y = 3$

∴ Centre (h, k) = (4, 3) and a = 4, b = 3

Step 3:

For an original standard hyperbola, the vertices are given by $(\pm a, 0)$.

$\Rightarrow (x', y') = (\pm a, 0)$

$\Rightarrow x - h = \pm a$ and $y - k = 0 \Rightarrow x = h \pm a$ and $y = k$

$\Rightarrow x = 4 \pm 4$ and $y = 3 \Rightarrow x = 8, 0$ and $y = 3$

$\Rightarrow$ A(0, 3), A'(8, 3) are the vertices for the translated hyperbola.

Step 4:

The coordinates of a foci standard hyperbola are given by $(x', y') = (\pm ae, 0)$

Where $e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{9}{16}} = \sqrt{\frac{25}{16}} = \frac{5}{4}$

$\Rightarrow e = \frac{5}{4}$

Foci: $(x', y') = (\pm ae, 0)$

$\Rightarrow x - h = \pm ae$ and $y - k = 0 \Rightarrow x = h \pm ae$ and $y = k$

$\Rightarrow x = 4 \pm 4\left(\frac{5}{4}\right)$ and $y = 3 \Rightarrow x = 4 \pm 5$ and $y = 3$

$\Rightarrow x = 9, -1$ and $y = 3$

$\Rightarrow$ Foci: $F_1(-1, 3)$ and $F_2(9, 3)$

Step 5:

End points of the latus rectum of a standard hyperbola are given by $\left(ae,\ \pm\frac{b^2}{a}\right), \left(-ae, \pm\frac{b^2}{a}\right)$.

⇒ Endpoints of the latus rectum of translated hyperbola are given by $\left(ae + h, \pm\frac{b^2}{a} + k\right)$,

$\left(-ae + h, \pm\frac{b^2}{a} + k\right)$.

⇒ End points of the latus rectum of a translated hyperbola are given by $\left(9, 3 \pm \frac{9}{4}\right), \left(-1, 3 \pm \frac{9}{4}\right)$.

Step 6:

Length of the latus rectum = $\frac{2b^2}{a} = 2\left(\frac{9}{4}\right) = \frac{9}{2}$

The equation of the transverse axis for a standard horizontal hyperbola having its centre at the origin is given by $y = 0$

⇒ The equation of the transverse axis for a translated hyperbola is given by $y' = 0$

⇒ $y - k = 0 \Rightarrow y = k \Rightarrow y = 3$

⇒ $y = 3$ is the equation of the transverse axis for the translated hyperbola.

Step 7:

The equation of the conjugate axis for a standard horizontal hyperbola having its centre at the origin is given by $x = 0$

⇒ The equation of the conjugate axis for a translated hyperbola is given by $x' = 0$

⇒ $x - h = 0 \Rightarrow x = h \Rightarrow x = 4$

⇒ $x = 4$ is the equation of the conjugate axis for the translated hyperbola.

Step 8:

The equation of directrices for a standard hyperbola is given by $x = \pm\frac{a}{e}$

⇒ The equation of directrices for a translated hyperbola is given by $x' = \pm\frac{a}{e}$

⇒ $x - h = \pm\frac{a}{e} \Rightarrow x - 4 = \pm\frac{16}{5} \Rightarrow x = 4 \pm \frac{16}{5}$

⇒ $x = 4 \pm \frac{16}{5}$ are the equations of directrices of the translated hyperbola.

Step 9:

Length of transverse axis of the translated hyperbola = $2a = 8$

Length of conjugate axis of the translated hyperbola = $2b = 6$

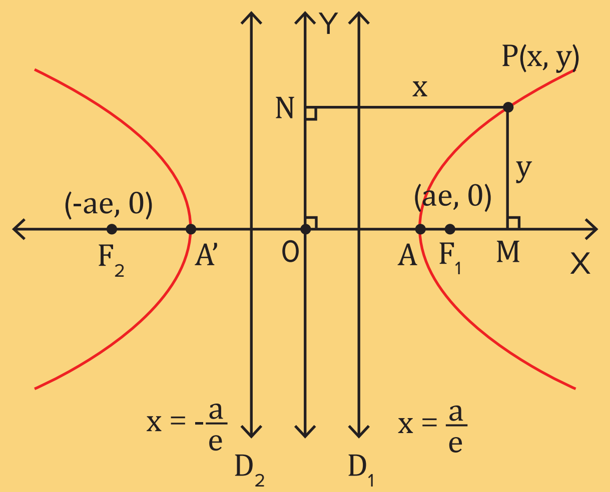
Axes of hyperbola parallel to the coordinate axes

Consider a standard horizontal hyperbola whose centre is at the origin, x-axis is the transverse axis, and y-axis is the conjugate axis. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where x and y are coordinates of any point on the hyperbola.

Consider a point in the first quadrant P(x, y) on this hyperbola and M and N are points on the x-axis and y-axis, respectively, such that PM is perpendicular to the x-axis and PN is perpendicular to the y-axis.

$\Rightarrow$ PM = y and PN = x, where PM is the perpendicular distance from the transverse axis and PN is the perpendicular distance from the conjugate axis of point P.

$$\Rightarrow \frac{(PN)^2}{a^2} - \frac{(PM)^2}{b^2} = 1$$

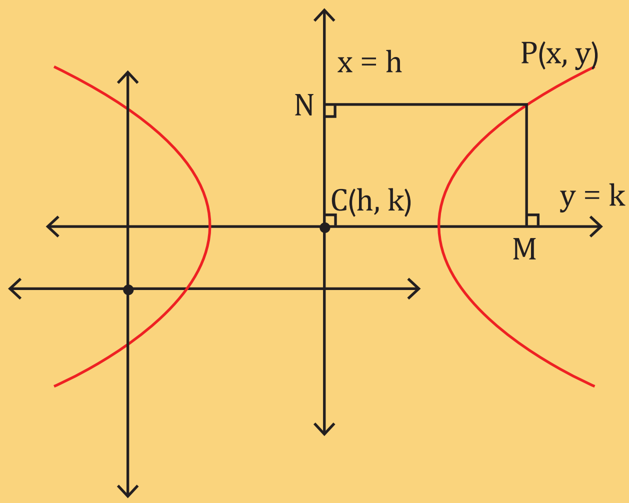


Now, the centre is shifted to (h, k). Hence, the new equation of the translated hyperbola becomes $\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$

Again, consider a point P(x, y) on this translated hyperbola, and M and N are points on the transverse axis and conjugate axis, respectively, such that PM is perpendicular to the transverse axis and PN is perpendicular to the conjugate axis.

$\Rightarrow$ PM = y - k and PN = x - h.

$$\Rightarrow \frac{(PN)^2}{a^2} - \frac{(PM)^2}{b^2} = 1$$



Note

For a hyperbola, it is not necessary for the coordinate axes or axes parallel to them, to be the axes of the hyperbola. Any two perpendicular lines can be the axes of a hyperbola.

Equation of a hyperbola referred to two perpendicular lines as axes

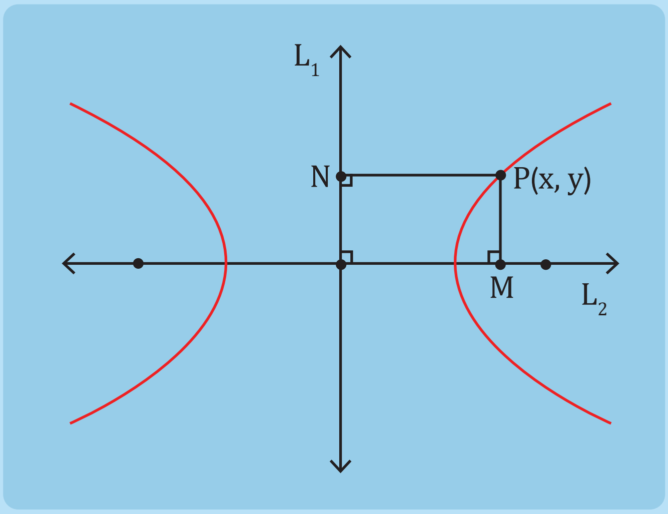
Consider lines $L_1 : a_1x + b_1y + c_1 = 0$ and $L_2 : b_1x - a_1y + c_2 = 0$ as two mutually perpendicular lines, which are the axes of the hyperbola such that the slope of $L_1(m_1) = \frac{-a_1}{b_1}$ and of $L_2(m_2) = \frac{b_1}{a_1}$

Let L_1 be the conjugate axis and L_2 be the transverse axis. Then their intersection will be the centre of this hyperbola. To find the equation of such hyperbola, consider a point P(x, y) on this translated and/or rotated hyperbola, and let M and N be the points on the transverse axis and conjugate axis, respectively, such that PM is perpendicular to the transverse axis and PN is perpendicular to the conjugate axis.

Now, using the equation derived for the standard hyperbola in terms of PM and PN, we get,

$$\frac{(PN)^2}{a^2} - \frac{(PM)^2}{b^2} = 1$$

Also, $PN = \frac{|a_1x + b_1y + c_1|}{\sqrt{(a_1)^2 + (b_1)^2}}$ and $PM = \frac{|b_1x - a_1y + c_2|}{\sqrt{(a_1)^2 + (b_1)^2}}$



Note

Eccentricity is still unchanged, i.e., $e = \sqrt{1 + \frac{b^2}{a^2}}$

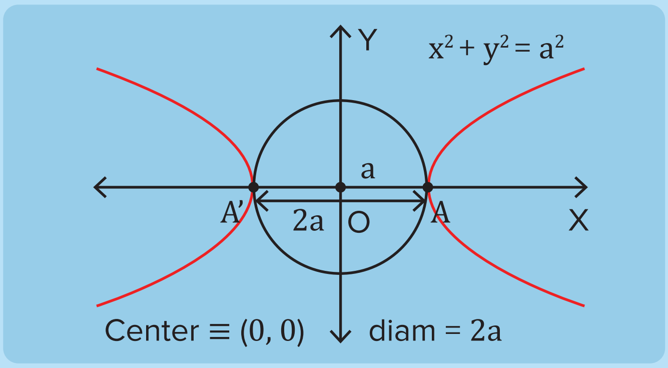
The centre of the hyperbola is the point of intersection of lines $L_1 = 0$ and $L_2 = 0$

Auxiliary Circle

Consider the standard horizontal hyperbola whose centre is at the origin, x-axis is the transverse axis, and y-axis is the conjugate axis. Its equation is given as follows:

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where x and y are coordinates of any point on the hyperbola.

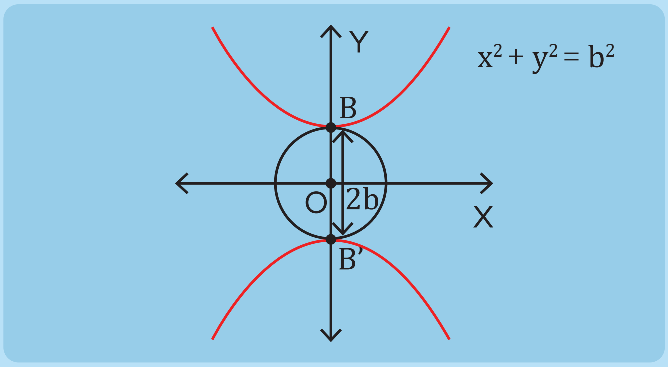
The circle described on the transverse axis as diameter is known as the auxiliary circle of the given hyperbola. Its equation is given as $x^2 + y^2 = a^2$.



Consider the standard vertical hyperbola whose centre is at the origin, y-axis is the transverse axis, and x-axis the conjugate axis. Its equation is given as follows:

$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$, where x and y are coordinates of any point on the hyperbola.

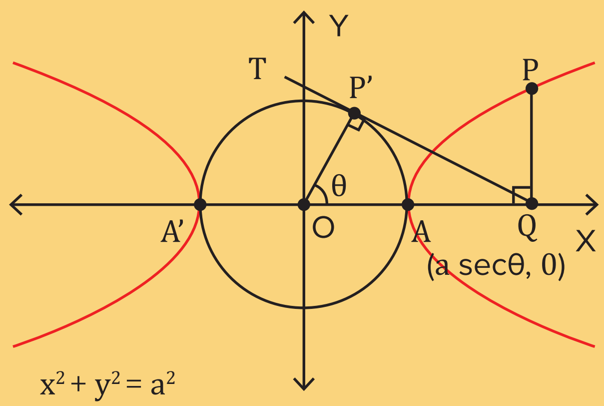
Then, the equation of the auxiliary circle of such hyperbola is $x^2 + y^2 = b^2$.



Parametric Equations

Consider the hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, with the centre at origin O.
Auxiliary circle: $x^2 + y^2 = a^2$
Let P(x, y) be a point on the hyperbola and Q be a point on the transverse axis such that PQ ⊥ x-axis.
QP' is known as the tangent from Q to the auxiliary circle.
OP' ⊥ QP'

Let $\theta \equiv P'OQ$, where $\theta \in [0,2\pi) - \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$ and the positive sense of θ be in the anti-clockwise direction, and is a real parameter.



Let us consider the auxiliary circle and the tangent QP'.
$P'(x_1, y_1)$ as a point on this circle and its parametric point can be written as (a cosθ, a sinθ)
Equation of tangent at P' is $T : xx_1 + yy_1 = a^2$
$\Rightarrow T : x(a\cos\theta) + y(a\sin\theta) = a^2$
When this tangent meets the x-axis, its ordinate is 0 and abscissa = $\frac{a^2}{a\cos\theta}$ = a secθ.

However, $\frac{1}{\cos\theta}$ exists only when θ excludes values $\left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$
Hence, θ cannot take values $\left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$ in the interval $[0, 2\pi)$
Also, Q is the point of the intersection of the tangent with the x-axis. Hence, Q ≡ (a secθ, 0)
However, PQ is ⊥ to the x-axis. Therefore, the abscissa of P will also be a secθ
Putting the value of abscissa of P in the equation of hyperbola and solving for its ordinate, we get point P as (a secθ, b tanθ), which is indeed the parametric form of any point of this hyperbola.
Parametric equations: x = a secθ and y = b tanθ

Note

θ is the eccentric angle of point P, which is not the angle measured between the x-axis and OP but the angle measured between the x-axis and OP' where P' is the point corresponding to point P on the auxiliary circle.
Similarly, following the same steps, the parametric equation for hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ is
x = a tanθ and y = b secθ, where $\theta \in [0, 2\pi) - \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$

Summary sheet

Key Takeaways

- The equation of a vertical hyperbola is given by $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$.
- The equation of a translated horizontal hyperbola with centre at (h, k) is $\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$.
- Parametric equations of standard horizontal hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, are $x = a \sec\theta$ and $y = b \tan\theta$.

Mind Map

Hyperbola

- Equation of vertical hyperbola
- Equation of shifted hyperbola
- Equation of rotated hyperbola
- Parametric coordinates of hyperbola

Self-Assessment

1. A circle cuts the rectangular hyperbola $xy = 1$ in the points (x_r, y_r), $r = 1, 2, 3, 4$. Find the value of $x_1 x_2 x_3 x_4$ & $y_1 y_2 y_3 y_4$.
2. The curve represented by $x = \sec\theta + \tan\theta$, $y = \sec\theta - \tan\theta$ is which among the following options?
   (a) A parabola (b) An ellipse (c) A circle (d) A rectangular hyperbola

Answers

Self-Assessment 1

Let the circle be $x^2 + y^2 = a^2$

Since $(t, \frac{1}{t})$ $(t \neq 0)$ lies on $xy = 1$, the points of intersection of the circle and the hyperbola are given by $t^2 + \frac{1}{t^2} = a^2 \Rightarrow t^4 - a^2t^2 + 1 = 0 \Rightarrow t^4 + 0(t^3) - a^2t^2 + 0(t) + 1 = 0$

If t_1, t_2, t_3, t_4 are the roots of the above biquadratic, then $t_1 t_2 t_3 t_4 = 1$

If $(x_r, y_r) = (t_r, \frac{1}{t_t})$, $r = 1, 2, 3, 4$, then $x_1 x_2 x_3 x_4 = t_1 t_2 t_3 t_4 = 1$

And $y_1 y_2 y_3 y_4 = \frac{1}{t_1 t_2 t_3 t_4} = 1$

Self-Assessment 2

$xy = (\sec\theta + \tan\theta)(\sec\theta - \tan\theta) = \sec^2\theta - \tan^2\theta = 1$

As $xy = 1$, the given curve is a rectangular hyperbola.

So, option (d) is the correct answer.

HYPERBOLA

TANGENT TO A HYPERBOLA

What you already know

- Equation of vertical hyperbola
- Equation of shifted hyperbola
- Equation of rotated hyperbola
- Parametric coordinates of hyperbola

What you will learn

- Position of a point
- Equation of tangent
  - (a) Point form
  - (b) Parametric form
  - (c) Slope form
- Point of contact of a tangent

(i) The equation of a vertical (conjugate) hyperbola is given by $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

(ii) The equation of a hyperbola, whose transverse axis is parallel to x-axis; and the centre is shifted to (h, k), is $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

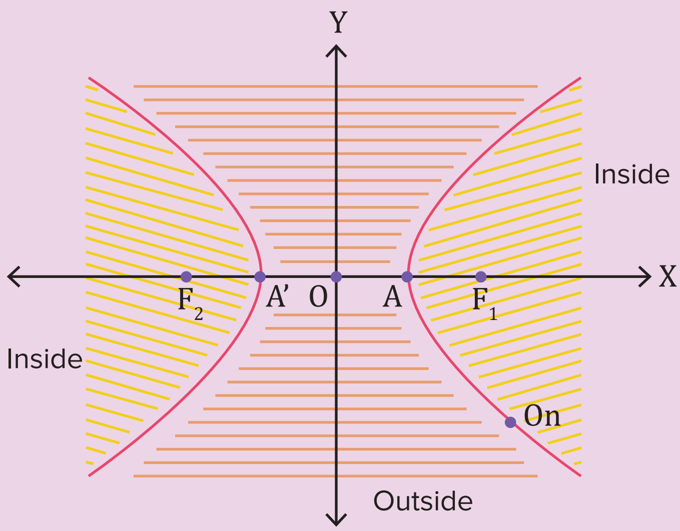
(iii) The equation of the auxiliary circle of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $x^2 + y^2 = a^2$

(iv) The parametric equations of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $x = a \sec \theta$ and $y = b \tan \theta$ where θ is parameter $(0 \le \theta < 2\pi)$

Position of a point $P(x_1, y_1)$ with respect to a hyperbola

Let $P(x_1, y_1)$ be a point in the plane of the hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

The hyperbola divides the Cartesian plane into three distinct regions, namely inside, on, and outside.



- **Interior:** The portion that contains foci F_1, F_2 constitutes the inside region of the hyperbola. Both these portions are disjoint from each other, but any point coming from either of these regions is termed as the interior point of the hyperbola.
- **Exterior:** The portion that contains the centre (O) constitutes the region outside the hyperbola, and any point coming from this region is known as the exterior point of the hyperbola.
- **On:** The infinitely many points on the two branches of this hyperbolic curve constitute the region on the hyperbola.

$S_1 > 0$	$P(x_1, y_1)$ lies inside the hyperbola
$S_1 < 0$	$P(x_1, y_1)$ lies outside the hyperbola
$S_1 = 0$	$P(x_1, y_1)$ lies on the hyperbola

Where, $S: \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1$, $S_1: \frac{(x_1)^2}{a^2} - \frac{(y_1)^2}{b^2} - 1$

Explanation

Case 1: Let us take $F_1(ae, 0)$.

Now, $S_{F_1} = \frac{a^2e^2}{a^2} - \frac{0^2}{b^2} - 1 = e^2 - 1 > 0$ (As $e > 1$)

Hence, $S_1 > 0 \Rightarrow$ The inside region of the hyperbola.

Case 2: Let us take O(0,0).

Now, $S_0 = \frac{0^2}{a^2} - \frac{0^2}{b^2} - 1 = -1 < 0$

Hence, $S_1 < 0 \Rightarrow$ The outside region of the hyperbola.

Case 3: If a point is on the hyperbola it will satisfy its equation.
Hence, $S_1 = 0$

Note

Circle, Parabola, Ellipse

Interior	$S_1 < 0$
On	$S_1 = 0$
Exterior	$S_1 > 0$

Hyperbola

Interior	$S_1 > 0$
On	$S_1 = 0$
Exterior	$S_1 < 0$

A Line and a hyperbola

If we are given a line and a hyperbola, we obtain 3 cases:

- **Intersecting:** The line will be an intersecting hyperbola at two distinct points
- **Tangent:** The line touches the hyperbola
- **Non-intersecting:** The line does not intersect the hyperbola

Consider a line, $L : y = mx + c$ ... (1)

and a hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ..... (2)

Substituting (1) in (2) and simplifying, we get:

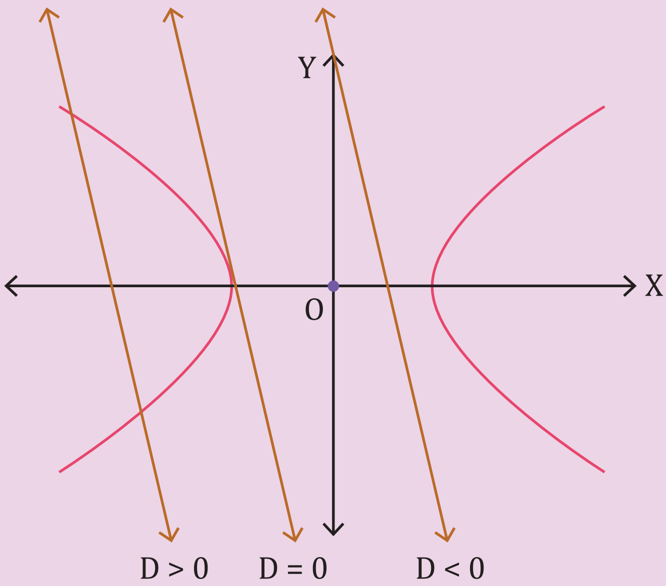
$(b^2 - a^2 m^2)x^2 - 2a^2 cmx - (a^2 c^2 + a^2b^2) = 0$

The discriminant of the above equation,

$D = (-2a^2 cm)^2 + 4(b^2 - a^2 m^2)(a^2 c^2 + a^2 b^2)$

$D = 4a^4 c^2 m^2 + 4b^2 a^2 c^2 + 4a^2 b^4 - 4a^4 m^2 c^2 - 4a^4 m^2 b^2$

$\Rightarrow D = 4a^2 b^2 (c^2 + b^2 - a^2 m^2)$

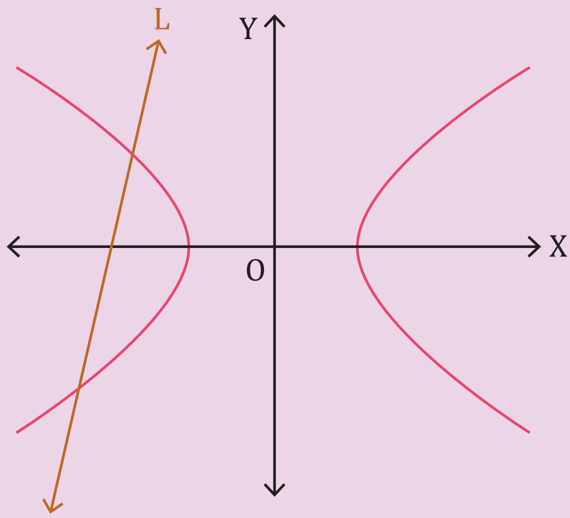


Case 1: When a line intersects hyperbola

As a hyperbola has a two-degree equation and a line has a linear equation. When we solve them simultaneously, we get a quadratic equation that can have at most two distinct real roots. Therefore, the number of points of intersection between a line and a hyperbola will be at-most two.

If line L meets the hyperbola at two distinct points, then the equation has two distinct real roots.

$\therefore D > 0$ or $c^2 > a^2 m^2 - b^2$

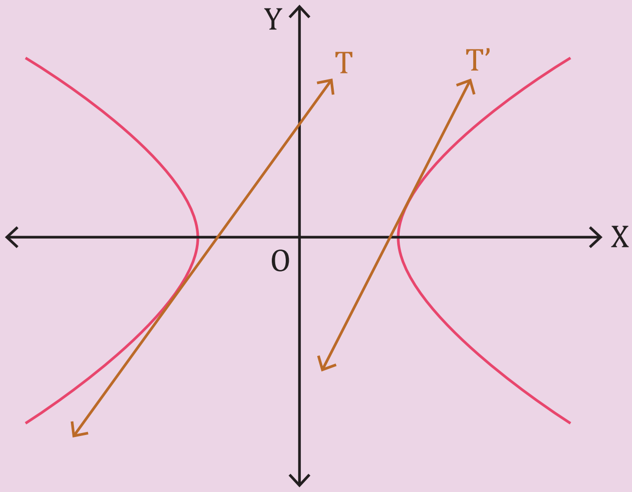


Case 2: When a line touches hyperbola

In this case, a tangent is a line that has a single point of contact also known as point of tangency. Therefore the line will touch exactly one of the branches, not two. From the figure, T, T' are two tangents on each branch of the hyperbola.

Reason: We get two real and repeated roots of the quadratic equation obtained by solving equations of the line and the hyperbola simultaneously.

$\therefore D = 0$ or $c^2 = a^2m^2 - b^2 \Rightarrow c = \pm\sqrt{a^2m^2 - b^2}$

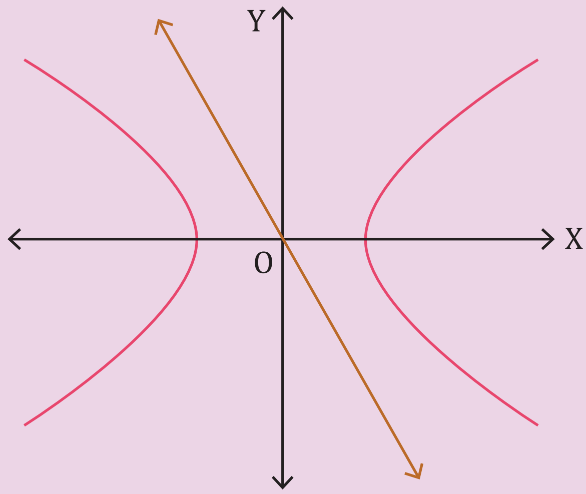


Case 3: When a line is not intersecting hyperbola

In this case, the line completely lies on the region that is outside the hyperbola.

If line L does not meet the hyperbola, then the equation has imaginary roots.

$\therefore D < 0$ or $c^2 < a^2m^2 - b^2$



Equation of Tangent

Point form

Consider a standard horizontal hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Line L is the tangent line to the hyperbola at point $P(x_1, y_1)$

Equation of tangent to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, at point $P(x_1, y_1)$ is $T = 0 \Rightarrow \frac{(xx_1)}{a^2} - \frac{(yy_1)}{b^2} - 1 = 0$

Which is obtained by taking all the terms to the LHS and replacing x^2 with xx_1 and y^2 with yy_1

Note

Equation of tangent to ellipse S: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ at point $P(x_1, y_1)$ is $T = 0$

$\Rightarrow \frac{(xx_1)}{a^2} + \frac{(yy_1)}{b^2} - 1 = 0$

Replace b^2 with $-b^2$ in the ellipse equation, we get equation of tangent to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at a point $P(x_1, y_1)$ as $\frac{(xx_1)}{a^2} - \frac{(yy_1)}{b^2} - 1 = 0$ i.e., $T = 0$.

Parametric form

The equation of the tangent to the standard horizontal hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at a point $P(\theta) \equiv (a \sec\theta, b \tan\theta)$ is $\frac{x \sec\theta}{a} - \frac{y \tan\theta}{b} - 1 = 0$

Proof

Equation of tangent at point $P(\theta) \equiv (a \sec\theta, b \tan\theta)$ is $T = 0 \Rightarrow \frac{(xx_1)}{a^2} - \frac{(yy_1)}{b^2} - 1 = 0$

Substitute $x_1 = a \sec\theta$ and $y_1 = b \tan\theta$

After simplifying, we get:

$$\frac{x \sec\theta}{a} - \frac{y \tan\theta}{b} - 1 = 0$$

Note

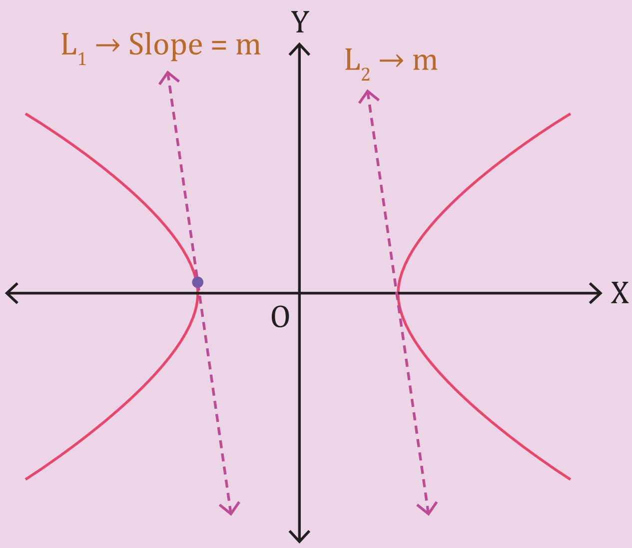
The equation of the tangent to the standard vertical hyperbola,

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$, at a point $P(\theta) \equiv (a \tan\theta, b \sec\theta)$ is $\frac{x \tan\theta}{a} - \frac{y \sec\theta}{b} + 1 = 0$.

Slope form

The equation of tangents of slope m to the standard horizontal hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are given by

$$y = mx \pm \sqrt{a^2m^2 - b^2}$$



Proof

$y = mx + c$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $c^2 = a^2 m^2 - b^2$

Therefore, $c = \pm\sqrt{a^2m^2 - b^2}$

The coordinates of the points of contact of tangents are

$\left(\frac{-a^2m}{c}, \frac{-b^2}{c}\right)$, where $c = \pm\sqrt{a^2m^2 - b^2}$

Note

The coordinates of the points of contact of tangents are $\left(\frac{-a^2m}{c}, \frac{b^2}{c}\right)$, where $c = \pm\sqrt{a^2m^2+b^2}$ for an ellipse.

Replace b^2 with $-b^2$, we get:

The coordinates of the points of contact of tangents are $\left(\frac{-a^2m}{c}, \frac{-b^2}{c}\right)$, where $c = \pm\sqrt{a^2m^2-b^2}$ for an hyperbola.

Find the point where the line, $5x + 12y + k = 0$ $(k < 0)$, touches the hyperbola, $x^2 - 9y^2 = 9$.

Solution

Step 1

Hyperbola: $\frac{x^2}{9} - \frac{y^2}{1} = 9 \qquad (a^2 = 9,\ b^2 = 1)$

Given line, $5x + 12y + k = 0$ $(k < 0)$

$\Rightarrow y = \left(\frac{-5}{12}\right)x - \frac{k}{12} \Rightarrow m = \frac{-5}{12},\ c = \frac{-k}{12}$

$y = mx + c$ touches $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow c^2 = a^2m^2 - b^2$

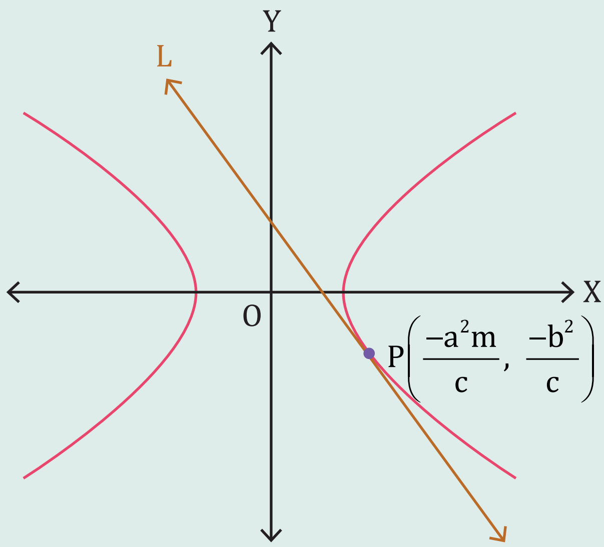
$\left(\frac{-k}{12}\right)^2 = 9\left(\frac{-5}{12}\right)^2 - 1$

After solving,

$k^2 = 81 \Rightarrow k = \pm 9$.

However, $k < 0$ (Given)

So, $k = -9 \Rightarrow L : 5x + 12y - 9 = 0$



Step 2

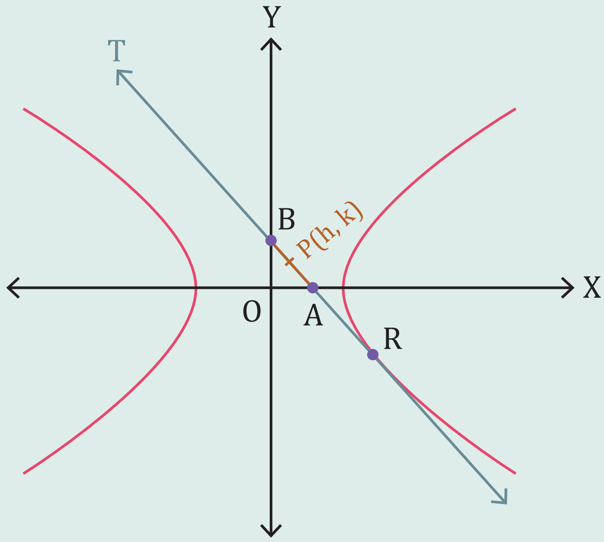
Point of contact $\equiv \left(\frac{-a^2m}{c}, \frac{-b^2}{c}\right) = \left(\frac{-9\left(\frac{-5}{12}\right)}{\left(\frac{9}{12}\right)}, \frac{-1}{\left(\frac{9}{12}\right)}\right) = \left(5, \frac{-4}{3}\right)$

Thus, the required point of contact is $P\left(5, \frac{-4}{3}\right)$

A tangent to hyperbola $4x^2 - 9y^2 = 36$ meets the coordinate axes at A and B, respectively. Find the locus of the midpoint of AB.

Solution

Step 1



Equation of the hyperbola is

$4x^2 - 9y^2 = 36 \Rightarrow \frac{x^2}{9} - \frac{y^2}{4} = 1$, which gives,

$a = 3,\ b = 2$

Equation of tangent in the parametric form is

$$\left(\frac{x}{a}\right)\sec\theta - \left(\frac{y}{b}\right)\tan\theta = 1$$

$$\Rightarrow \left(\frac{x}{3}\right)\sec\theta - \left(\frac{y}{2}\right)\tan\theta = 1$$

It meets the axes at A and B, where

$A \equiv (3\cos\theta,\ 0)$ and $B \equiv (0,\ -2\cot\theta)$

Step 2

Let the midpoint of AB be P(h, k), which is the locus point.
P is the midpoint of A & B

$$P \equiv \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \equiv \left(\frac{3\cos\theta + 0}{2}, \frac{0 - 2\cot\theta}{2}\right)$$

$$P \equiv \left(\left(\frac{3}{2}\right)\cos\theta,\ -\cot\theta\right)$$

$h = \left(\frac{3}{2}\right)\cos\theta$ and $k = -\cot\theta \Rightarrow \cos\theta = \frac{2h}{3}$ and

$\cot\theta = -k$

$\sec\theta = \frac{3}{2h}$ and $\tan\theta = \frac{-1}{k}$

Step 3

Since $\sec^2\theta - \tan^2\theta = 1$

$$\Rightarrow \left(\frac{3}{2h}\right)^2 - \left(\frac{-1}{k}\right)^2 = 1 \Rightarrow \frac{9}{4h^2} - \frac{1}{k^2} = 1$$

Locus of point P(h, k) : $\frac{9}{4x^2} - \frac{1}{y^2} = 1$

A pair of tangents are drawn from the point $(4, 3)$ to the hyperbola, $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

Let θ be the acute angle between them. Which of the following is/are true?

(a) $\tan\theta = \frac{4}{3}$

(b) One of the tangents is horizontal.

(c) One of the tangents is $3x - 4y = 0$.

(d) One of the tangents is vertical.

Solution

Step 1

Given,

Standard horizontal hyperbola,

$$\frac{x^2}{16} - \frac{y^2}{9} = 1 \Rightarrow a^2 = 16 \text{ and } b^2 = 9$$

Let the vertices of the given hyperbola be A and A' $\Rightarrow A(4, 0)$ and $A'(-4, 0)$

$$S_1 : \frac{x_1^{\,2}}{16} - \frac{y_1^{\,2}}{9} - 1$$

Substituting the value of point P in S_1, we get,

$$S_1 = -1 \Rightarrow S_1 < 0$$

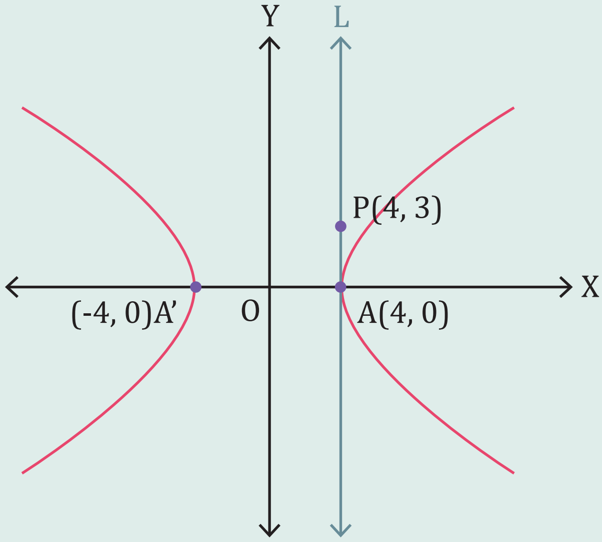
$\Rightarrow$ Point P is an exterior point.

Step 2

Observation: Point $P(4, 3)$ lies on the vertical line, $L : x = 4$, which also passes through point A.

$\Rightarrow$ Line $L : x = 4$ is one tangent to the hyperbola from point $P(4, 3)$, and the point of tangency is $A(4, 0)$.

$\Rightarrow$ One of the tangents is vertical and therefore, option (d) is correct.



Step 3

To find other tangent, let us consider the tangent equation for a hyperbola, $y = mx \pm \sqrt{a^2m^2 - b^2}$

Now, this tangent passes through point $P(4, 3)$ (given) and $a^2 = 16$, $b^2 = 9$.

Therefore, putting the values of x, y, a^2, and b^2, in the given tangent equation, we get,

$$3 = 4m \pm \sqrt{16m^2 - 9} \Rightarrow (3 - 4m)^2 = 16m^2 - 9$$

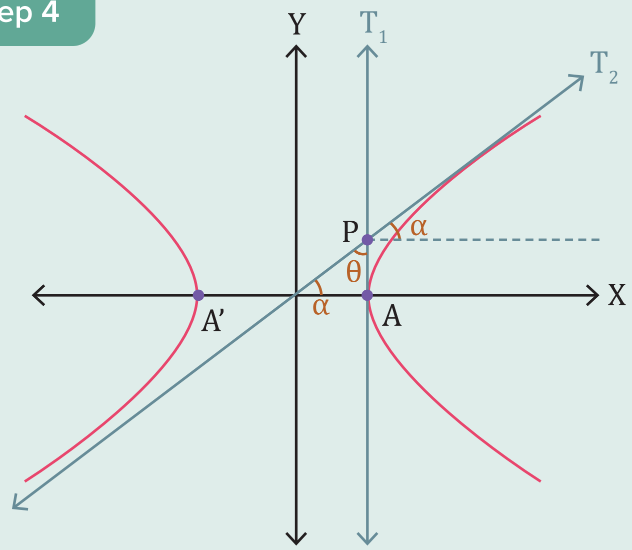
$$\Rightarrow 9 + 16m^2 - 24m = 16m^2 - 9 \Rightarrow 18 - 24m = 0 \Rightarrow m = \frac{3}{4}$$

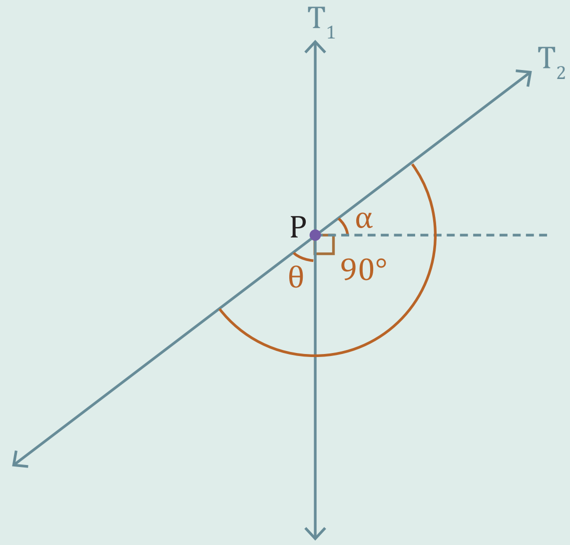
Hence, the equation of tangent at point $P(4, 3)$ with slope $(m) = \frac{3}{4}$ is given by:

$y - 3 = \frac{3}{4}(x - 4)$ or $4y - 12 = 3x - 12 \Rightarrow 3x - 4y = 0$

Hence, option (c) is also correct.

Step 4





Now, to find the acute angle (θ) between the two tangents, i.e., $x = 4$ and $3x - 4y = 0$, consider 'α' to be the angle made by line $3x - 4y = 0$ with positive direction of x- axis $\Rightarrow \tan\alpha = \frac{3}{4}$.

However, the angle between $x = 4$ and $3x - 4y = 0$ will be $90° - \alpha$

Hence, $\tan\theta = \tan(90° - \alpha) = \cot\alpha = \frac{4}{3}$

$\Rightarrow \tan\theta = \frac{4}{3}$

Hence, option (a) is also correct

Thus options (a), (c) & (d) are the correct answers.

If PQ is a double ordinate of hyperbola $\frac{x^2}{3} - \frac{y^2}{2} = 1$, ($P$ is in the first quadrant) such that OPQ is an equilateral triangle (O is the centre of the hyperbola), then find the area of $\triangle OPQ$.

Solution

Step 1

Given, $\frac{x^2}{3} - \frac{y^2}{2} = 1$

Comparing with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get, $a^2 = 3$ and $b^2 = 2$

The given hyperbola is a horizontal hyperbola whose centre is at origin $O(0,0)$.

Step 2

PQ is the double ordinate, which is a chord to the hyperbola and is perpendicular to the x-axis such that ΔOPQ is an equilateral triangle.

Area of an equilateral triangle = $\frac{\sqrt{3}}{4}$ (side)^2

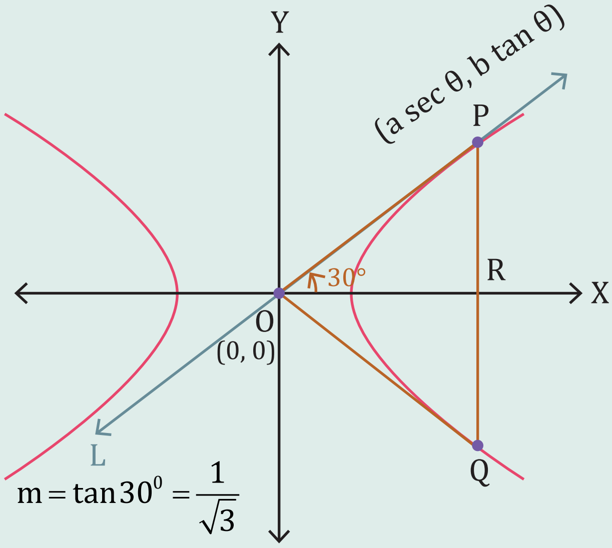
Let PQ cut the x-axis at R. Now, we see from the figure that ΔOPR and ΔOQR are congruent by S.A.S property.

Hence, $\angle POR = \angle QOR = \varphi(\text{say})$

However,

$\angle POQ = 60° \Rightarrow \angle POR + \angle QOR = \varphi + \varphi = 2\varphi$

$\Rightarrow 2\varphi = 60° \Rightarrow \varphi = 30°$



Step 3

Now, line OP makes angle $\varphi = 30^0$ and passes through $(0,0)$

Slope of line OP $= \tan 30^0 = \frac{1}{\sqrt{3}}$

Let P $\equiv \left(\sqrt{3}\sec\theta,\ \sqrt{2}\tan\theta\right)$, Q $\equiv \left(\sqrt{3}\sec\theta - \sqrt{2}\tan\theta\right)$

Slope of OP $= \frac{\sqrt{2}\tan\theta}{\sqrt{3}\sec\theta} = \frac{\sqrt{2}}{\sqrt{3}}\sin\theta$

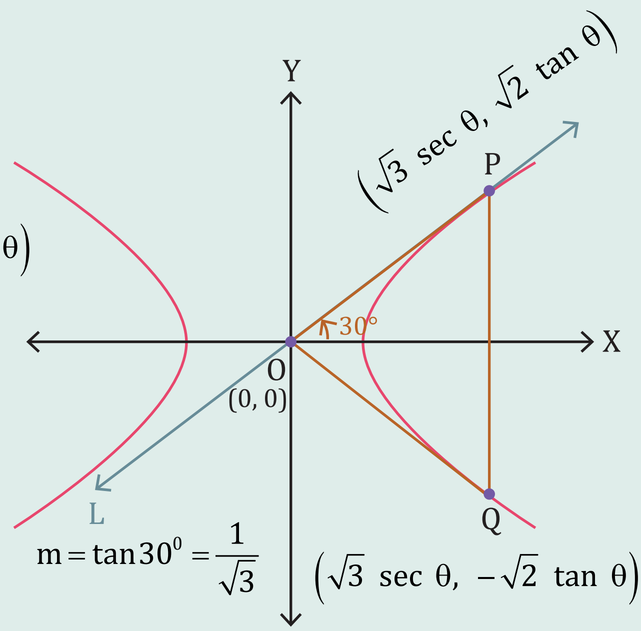
As m = slope of OP $\Rightarrow \frac{\sqrt{2}}{\sqrt{3}}\sin\theta = \frac{1}{\sqrt{3}}$

$\Rightarrow \sin\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$

Therefore,

$P \equiv \left(\sqrt{3}\sec\frac{\pi}{4},\ \sqrt{2}\tan\frac{\pi}{4}\right) = \left(\sqrt{3}\left(\sqrt{2}\right),\ \sqrt{2}(1)\right) = \left(\sqrt{6},\ \sqrt{2}\right)$

$\Rightarrow$ P is $\left(\sqrt{6},\ \sqrt{2}\right)$, and Q is $\left(\sqrt{6},\ -\sqrt{2}\right)$



Step 4

Length of OP $= \sqrt{6+2} = \sqrt{8}$

$\Rightarrow$ Area of $\Delta OPQ = \frac{\sqrt{3}}{4}\left(\sqrt{8}\right)^2 = 2\sqrt{3}$ sq. units.

Concept Check

Find the value of 'm' for which $y = mx + 6$ is a tangent to the hyperbola $\frac{x^2}{100} - \frac{y^2}{49} = 1$

Summary Sheet

- **Position of a point with respect to a hyperbola**

  i) $S_1 > 0 \Rightarrow$ Point lies inside the hyperbola

  ii) $S_1 < 0 \Rightarrow$ Point lies outside the hyperbola

  iii) $S_1 = 0 \Rightarrow$ Point lies on the hyperbola

- **Parametric form**

  Equation of tangent to a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, at a point $P(\theta) \equiv (a \sec\theta, b \tan\theta)$ is

  $$\frac{x \sec\theta}{a} - \frac{y \tan\theta}{b} - 1 = 0$$

- **Slope form**

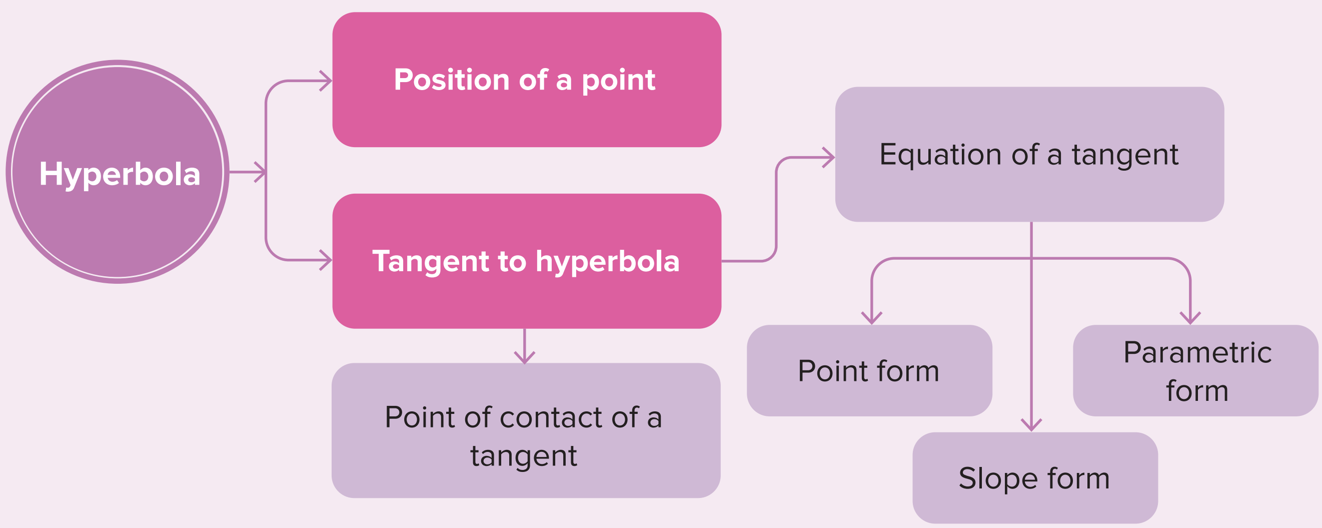
  Equation of tangent to hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, having slope m is $y = mx \pm \sqrt{a^2m^2 - b^2}$

- **Point form**

  Equation of tangent to a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, at a point $P(x_1, y_1)$ is $T = 0$

  $$\Rightarrow \frac{(xx_1)}{a^2} - \frac{(yy_1)}{b^2} - 1 = 0$$

Mind Map



Self-Assessment

The point of contact of $5x + 6y + 1 = 0$ to the hyperbola $2x^2 - 3y^2 = 2$

(a) (5, 4) (b) (-5, 4) (c) (-5, -4) (d) (5, -4)

Answers

Concept Check

If $y = mx + c$ touches $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ then $c^2 = a^2m^2 - b^2$

Given $y = mx + 6$ is a tangent to the hyperbola $\frac{x^2}{100} - \frac{y^2}{49} = 1$

Here, $c = 6$, $a^2 = 100$ and $b^2 = 49$.

Therefore $36 = 100m^2 - 49 \Rightarrow 100m^2 = 85 \Rightarrow m = \pm\sqrt{\frac{17}{20}}$

Self-Assessment

Given, $5x + 6y + 1 = 0$ is the tangent to the hyperbola $2x^2 - 3y^2 = 2$

$$2x^2 - 3y^2 = 2 \Rightarrow \frac{2x^2}{2} - \frac{3y^2}{2} = 1 \Rightarrow \frac{x^2}{1} - \frac{y^2}{\frac{2}{3}} = 1 \Rightarrow a^2 = 1 \text{ and } b^2 = \frac{2}{3}$$

Line: $5x + 6y + 1 = 0 \Rightarrow y = \frac{-5}{6}x - \frac{1}{6} \Rightarrow m = \frac{-5}{6}, c = \frac{-1}{6}$

Point of contact $\equiv \left(\frac{-a^2m}{c}, \frac{-b^2}{c}\right) = (-5, 4)$

So option (b) is the correct answer.

HYPERBOLA

NORMAL AND ASYMPTOTES OF HYPERBOLA

NOTES

What you already know

- Position of a point
- Equation of tangent
  - » Point form
  - » Parametric form
  - » Slope form
- Point of contact of a tangent

What you will learn

- Director circle
- Chord joining two points
- Chord with given midpoint
- Equation of normal to a hyperbola
  - » Point form
  - » Parametric form
  - » Slope form
- Asymptotes

Quick Recap

1. **Position of a point**

Position of a point $P(x_1, y_1)$ with respect to a hyperbola

(i) $S_1 > 0 \Rightarrow$ Point $P(x_1, y_1)$ lies inside the hyperbola

(ii) $S_1 < 0 \Rightarrow$ Point $P(x_1, y_1)$ lies outside the hyperbola

(iii) $S_1 = 0 \Rightarrow$ Point $P(x_1, y_1)$ lies on the hyperbola

2. **Parametric form:**

Equation of tangent to a hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, at a point $P(\theta) \equiv (a \sec\theta, b \tan\theta)$ is

$$\frac{x \sec\theta}{a} - \frac{y \tan\theta}{b} - 1 = 0$$

3. **Slope form:**

Equation of tangent to a hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, having slope 'm' is $y = mx \pm \sqrt{a^2m^2 - b^2}$

4. **Point form:**

Equation of tangent to a hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, at a point $P(x_1, y_1)$ is $T = 0$

$$\Rightarrow \frac{(xx_1)}{a^2} - \frac{(yy_1)}{b^2} - 1 = 0$$

Director Circle

The locus of the point of intersection of perpendicular tangents to a hyperbola is a circle concentric with the centre of the hyperbola and that circle is known as the director circle.

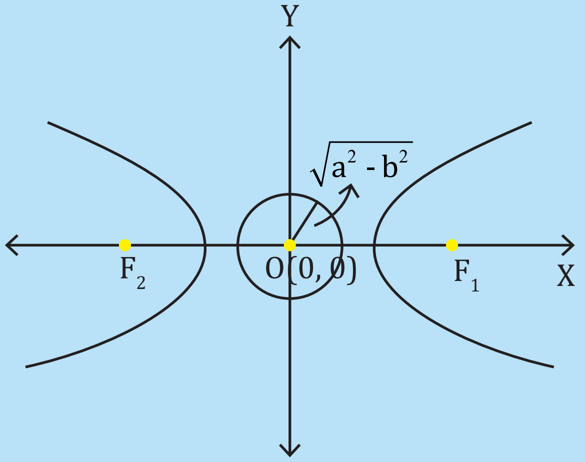
Equation of the hyperbola,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Equation of the director circle of hyperbla,

$x^2 + y^2 = a^2 - b^2$

$$\Rightarrow (x - 0)^2 + (y - 0)^2 = \left(\sqrt{a^2 - b^2}\right)^2$$



Note

- **For horizontal ellipse,**

  Director circle is $x^2 + y^2 = \left(\sqrt{a^2 + b^2}\right)^2$, where $a > b$ and $r = \sqrt{a^2 + b^2}$

- **For horizontal hyperbola,**

  Director circle is $x^2 + y^2 = a^2 - b^2 = \left(\sqrt{a^2 - b^2}\right)^2$, where $a > b$ and $r = \sqrt{a^2 - b^2}$

Proof

Equation of tangent to a hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$y = mx \pm \sqrt{a^2m^2 - b^2}$ ....(i)

Let P(h, k) be the locus of point of intersection of two tangents T_1 & T_2

Equation (i) is passing through the locus point P(h, k)

$$k = mh \pm \sqrt{a^2m^2 - b^2}$$

$$\Rightarrow k - mh = \pm \sqrt{a^2m^2 - b^2}$$

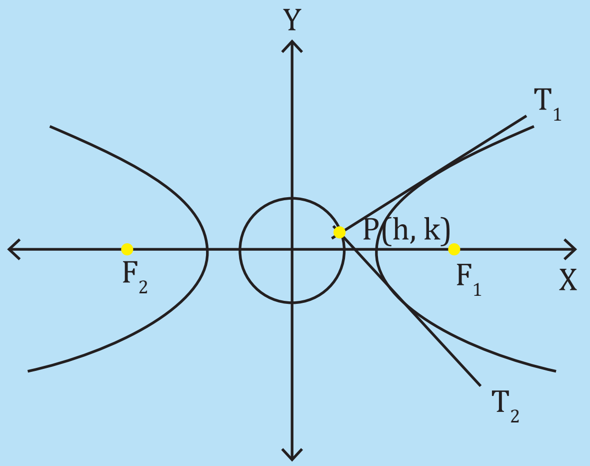
$$\Rightarrow (k - mh)^2 = \left(\pm \sqrt{a^2m^2 - b^2}\right)^2$$

$k^2 + m^2h^2 - 2mkh = a^2m^2 - b^2$

$m^2(h^2 - a^2) - 2mhk + (k^2 + b^2) = 0$

m_1 and m_2 are the roots of the given quadratic equation but m_1 and m_2 are the slopes of tangents that are distinct and real.

For director circle, $m_1.m_2 = -1$



$$\Rightarrow \frac{k^2 + b^2}{h^2 - a^2} = -1$$

$$\Rightarrow k^2 + b^2 = a^2 - h^2$$

$$\Rightarrow h^2 + k^2 = a^2 - b^2$$

Replace (h, k) with (x, y), we get

$x^2 + y^2 = a^2 - b^2$

Centre (0, 0) and $r = \sqrt{a^2 - b^2}$

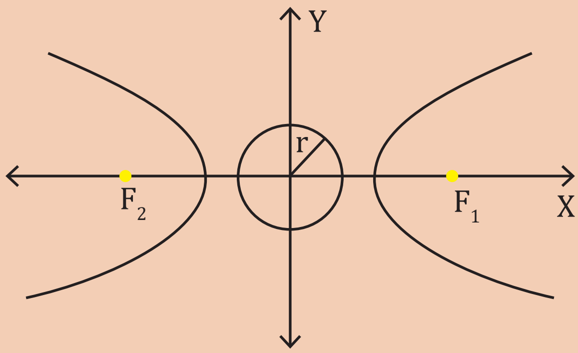
Note

For director circle, $r = \sqrt{a^2 - b^2}$

If $a^2 > b^2$, then it is a real director circle.

If $a^2 = b^2$, then it is a point circle. This point circle is at the origin and the hyperbola will be rectangular hyperbola.

If $a^2 < b^2$, then it is an imaginary circle.



Note

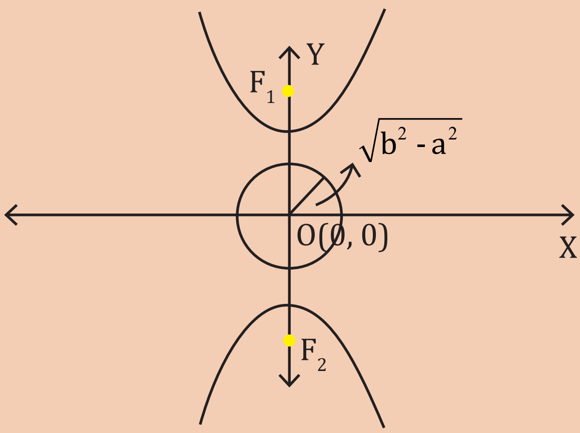
Equation of the director circle for the vertical hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ is $x^2 + y^2 = b^2 - a^2$

If a < b, then it is a real circle.

If a > b, then it is an imaginary circle.

If a = b, then it is a point circle.



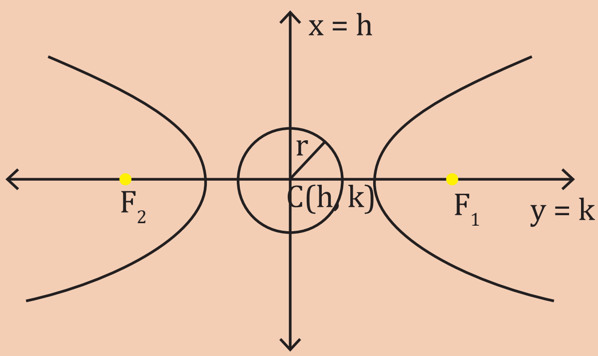
Note

Equation of director circle of shifted hyperbola

$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ is

$(x-h)^2 + (y-k)^2 = \left(\sqrt{a^2 - b^2}\right)^2$

As lengths don't change while shifting, the radius will remain the same.



Chord Joining Two Points

Equation of the chord joining the two points $P(\alpha)$ & $Q(\beta)$ on the hyperbola

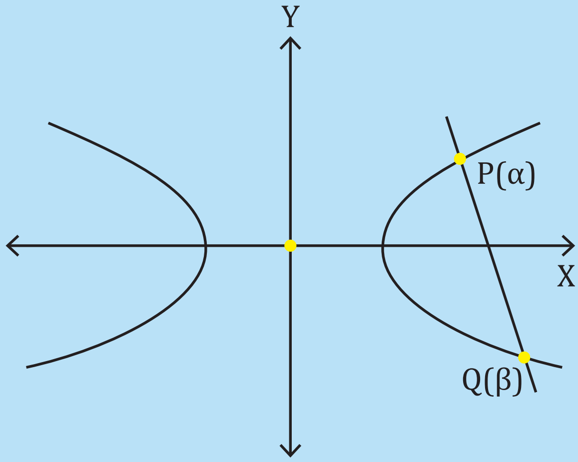
Let α be the eccentric angle of P then the coordinates of P wil be $(a\sec\alpha, b\tan\alpha)$ and similarly let β be the eccentric angle of Q then the coordinates of Q will be $(a\sec\beta, b\tan\beta)$.

So, the equation of the line segment joining the two points P & Q is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

On substituting the values of x_1, y_1, x_2, y_2, we get

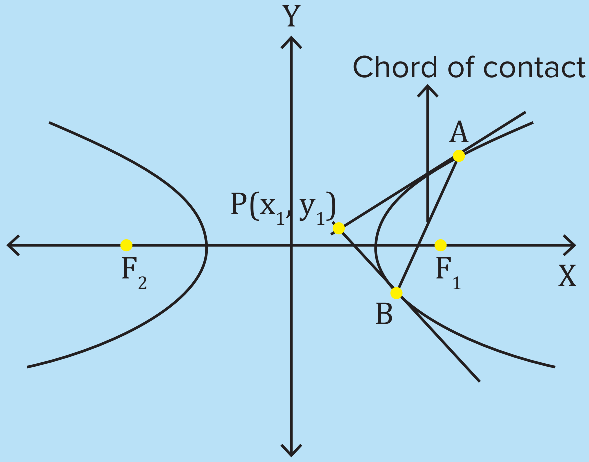
$$\frac{x}{a}\cos\left(\frac{\alpha-\beta}{2}\right) - \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha+\beta}{2}\right)$$



Chord of Contact

If the pair of tangents drawn from an external point $P(x_1, y_1)$ to the hyperbola $S = 0$ touch it at the points A & B, then AB is called the chord of contact with respect to $P(x_1, y_1)$;

Equation of AB is $T = 0$



Note

Point of intersection of the tangents at $A(\alpha)$, $B(\beta)$ on the hyperbola $S = 0$ is

$$\left(\left(\frac{a\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}\right), \left(\frac{b\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}\right)\right)$$

Proof

Equation of chord AB in terms of α & β : $\frac{x}{a}\cos\left(\frac{\alpha-\beta}{2}\right) - \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha+\beta}{2}\right)$ ....(i)

Equation of chord AB in terms of (x_1, y_1) is $T = 0$

$$\Rightarrow \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1 \text{ ....(ii)}$$

We know that the equations (i) & (ii) represents the same line. So their ratio of corresponding coefficients will be equal.

$$\frac{\frac{x_1}{a^2}}{\frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{a}} = \frac{\frac{-y_1}{b^2}}{\frac{-\sin\left(\frac{\alpha+\beta}{2}\right)}{b}} = \frac{1}{\cos\left(\frac{\alpha+\beta}{2}\right)}$$

$$x_1 = \frac{a\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)} \;\&\; y_1 = \frac{b\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}$$

Chord with given midpoint

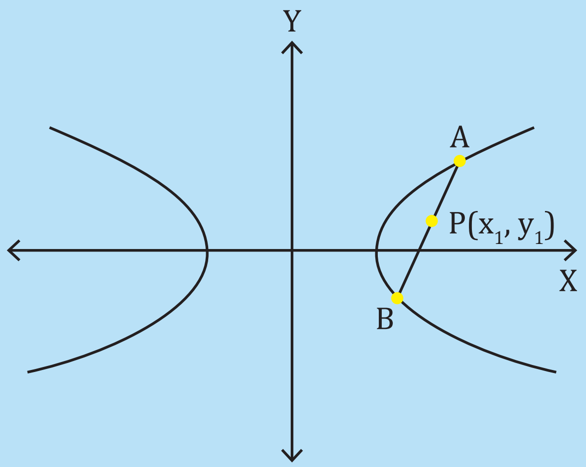
The equation of a chord of the hyperbola $S = 0$ whose midpoint is $P(x_1, y_1)$ is $T = S_1$, where P is the interior point of the hyperbola.

Now, $S: \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1$

$S_1: \frac{(x_1)^2}{a^2} - \frac{(y_1)^2}{b^2} - 1$

$T: \frac{(xx_1)}{a^2} - \frac{(yy_1)}{b^2} - 1$

Now, equate T with S_1 to get the equation of the chord with a given midpoint.



Tangents are drawn to the hyperbola $4x^2 - y^2 = 36$ at the points P and Q. If those tangents intersect at the point R(0, 3), then what is the area of $\triangle PQR$ (in square units)?

(a) $45\sqrt{5}$ (b) $54\sqrt{3}$ (c) $60\sqrt{3}$ (d) $36\sqrt{5}$

Solution

Step 1: Given, hyperbola,

$4x^2 - y^2 = 36 \Rightarrow \frac{x^2}{9} - \frac{y^2}{36} = 1$ ......(i)

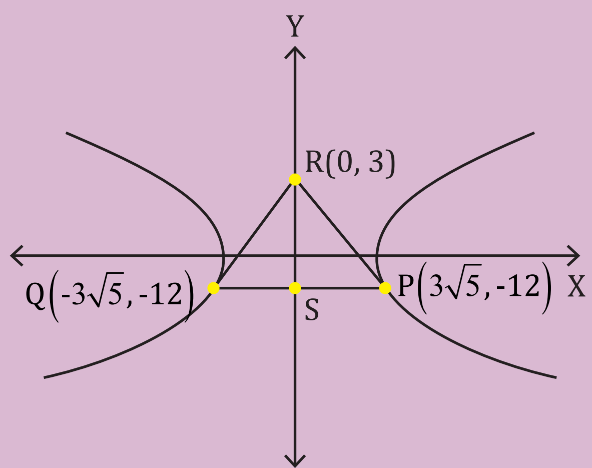
So, it is the standard horizontal hyperbola.

$R(0, 3)$ is the point of intersection of the tangents at P and Q.

So, PQ is the chord of contact of $R(0, 3)$.

$\Rightarrow$ The equation of PQ is $T = 0$

So, $\frac{x(0)}{9} - \frac{y(3)}{36} = 1 \Rightarrow y = -12$



$\Rightarrow$ y - coordinate of P and Q is -12

On putting this value of y in

$$\frac{x^2}{9} - \frac{y^2}{36} = 1 \Rightarrow \frac{x^2}{9} = 1 + \frac{144}{36} \Rightarrow x = \pm 3\sqrt{5}$$

Step 2:

$P \equiv (3\sqrt{5}, -12)$ and $Q \equiv (-3\sqrt{5}, -12)$

So, S is the mdipoint of the PQ chord,

$S \equiv (0, -12)$

$PQ = 3\sqrt{5} + 3\sqrt{5} = 6\sqrt{5}, RS = 3 + 12 = 15$

So, area of the triangle PQR

$= \frac{1}{2}(PQ)(RS) = \frac{1}{2}(6\sqrt{5})(15) = 45\sqrt{5}$ square units

Therefore, option (a) is the correct answer.

Equation of Normal to a Hyperbola

Point form

Equation of the normal to the hyperbola S = 0 at $P(x_1, y_1)$ is

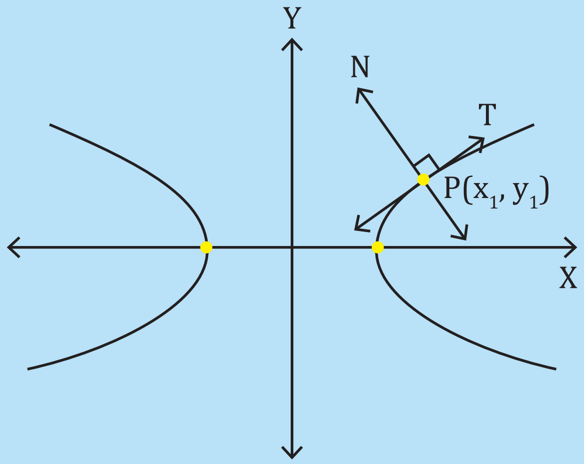
$$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$$

Equation of normal of horizontal ellipse

$$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$$

By changing b^2 to $-b^2$, we are getting the equation for normal to hyperbola as

$$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$$

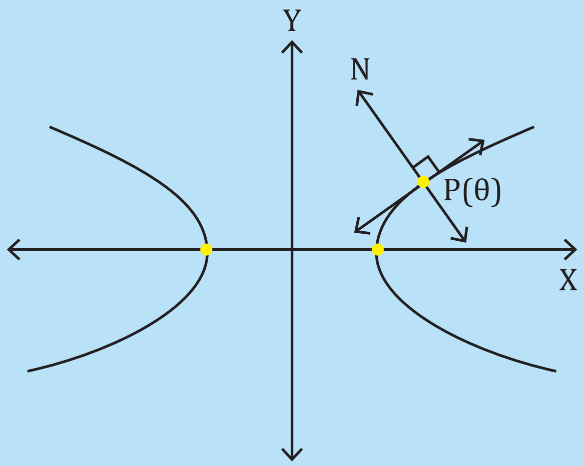


Parametric form

Equation of the normal to the hyperbola S = 0 at $P(\theta)$ is

$$\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2 \quad \left(\theta \neq 0, \frac{\pi}{2}, \pi\right)$$

Where parametric coordinates are $(a \sec\theta, b \tan\theta)$



Note

In parametric form comparison with ellipse equations is not possible as their parametric coordinates are not same. Parametric coordinates for ellipse are $(a\cos\theta, b\sin\theta)$ while parametric coordinates for hyperbola are $(a\sec\theta, b\tan\theta)$.

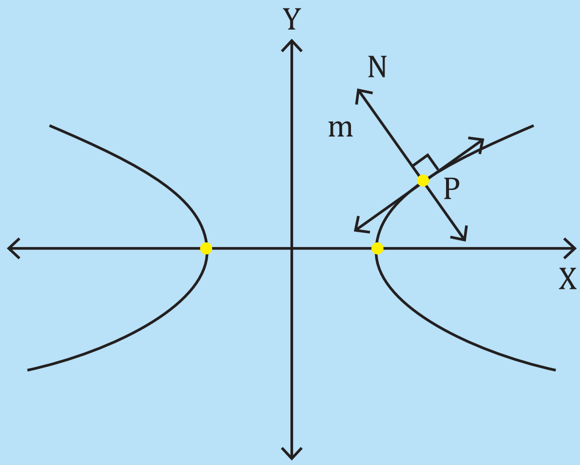
Slope form

Equation of the normal to the hyperbola $S = 0$ whose slope is m is

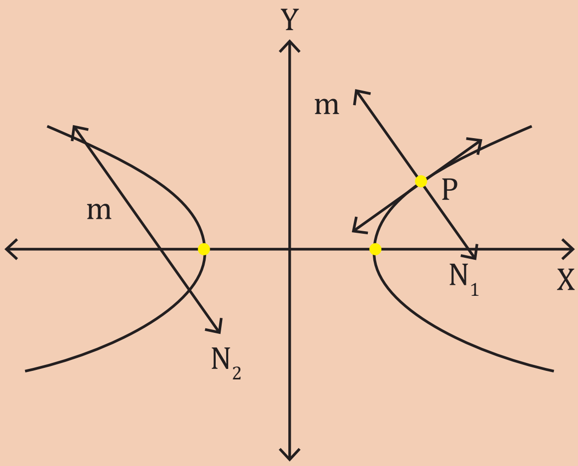
$$y = mx \pm \frac{(a^2 + b^2)m}{\sqrt{a^2 - b^2m^2}}, \text{ where } m \in \left(\frac{-a}{b}, \frac{a}{b}\right)$$

This can also be obtained by comparing it with equation of normal to horizontal ellipse,

$$y = mx \pm \frac{(a^2 - b^2)m}{\sqrt{a^2 + b^2m^2}}$$



Note



There will be two distinct normals parallel to one another.

JEE MAIN 2020

Let $P(3, 3)$ be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If the normal to it at P intersects the x-axis at $(9, 0)$ and 'e' is the eccentricity, then what is the ordered pair (a^2, e^2) equal to?

(a) $(9, 3)$ **(b)** $\left(\frac{9}{2}, 2\right)$ **(c)** $\left(\frac{9}{2}, 3\right)$ **(d)** $\left(\frac{3}{2}, 2\right)$

Solution

Step 1: Given, hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

We have a normal to the hyperbola at $(3, 3)$. Equation of the normal to the hyperbola $S = 0$ at $P(x_1, y_1)$ is

$$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$$

We get the equation of normal as follows:

$$N: \frac{a^2x}{3} + \frac{b^2y}{3} = a^2 + b^2 \text{ ....}(i)$$

Step 2: We know that,

$$e^2 = 1 + \frac{b^2}{a^2} \Rightarrow a^2 + b^2 = a^2e^2 \text{ ......}(ii)$$

From equations (i) & (ii), we get the following:

$$N: \frac{a^2x}{3} + \frac{b^2y^2}{3} = a^2e^2$$

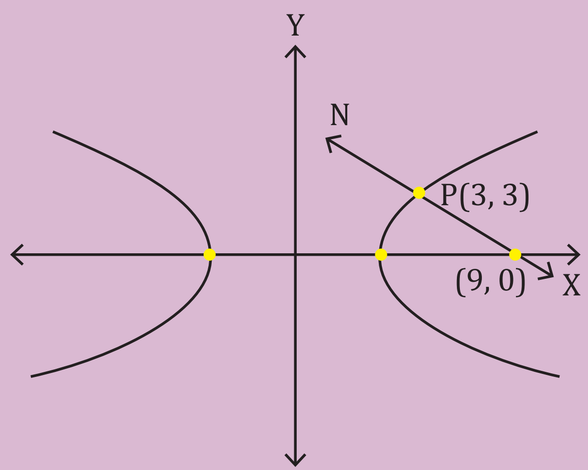
The normal intersects the x-axis at $(9, 0)$.

$$\Rightarrow \frac{9a^2}{3} + 0 = a^2e^2 \Rightarrow e^2 = 3$$

On substituting $e^2 = 3$ in $e^2 = 1 + \frac{b^2}{a^2}$, we get

the following:

$$\Rightarrow 3 = 1 + \frac{b^2}{a^2} \Rightarrow b^2 = 2a^2$$



Step 3: Now, P also lies on the hyperbola.

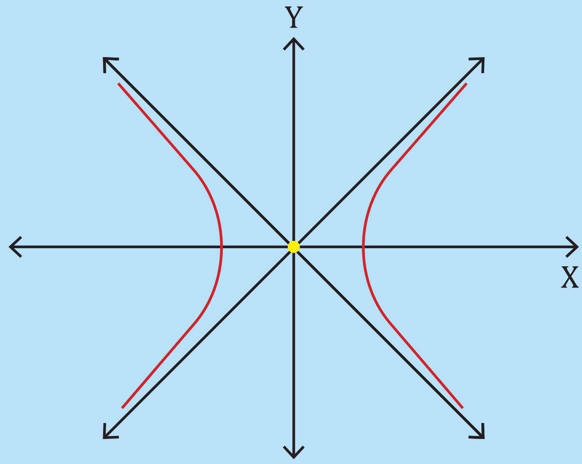
On substituting $(3, 3)$ in the equation of hyperbola we get the following

$$\frac{9}{a^2} - \frac{9}{b^2} = 1 \Rightarrow \frac{9}{a^2} - \frac{9}{2a^2} = 1 \Rightarrow a^2 = \frac{9}{2}$$

Therefore, the ordered pair is $\left(\frac{9}{2}, 3\right)$.

Asymptotes

If the distance between a line and a hyperbola tends to be zero when one or both of the x and y coordinates approach to infinity, then that line is known as 'asymptote' to the hyperbola. In short, an asymptote is a tangent to the hyperbola at infinity.



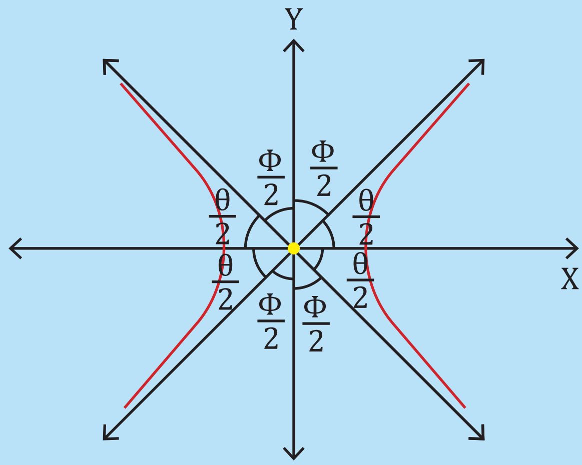
Equation of asymptotes to a horizontal hyperbola

We know that the line $y = mx + c$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

If $c^2 = a^2m^2 - b^2$ and the point of contact is $\left(\frac{-a^2m}{c}, \frac{-b^2}{c}\right)$, then this point of contact should represent infinity. For $y = mx + c$ to be asymptote of the hyperbola, c should be equal to 0.

So, $0 = a^2m^2 - b^2 \Rightarrow m = \pm\frac{b}{a}$

Hence, equations of asymptotes: $y = \pm\left(\frac{b}{a}\right)x$



Note

1. $y = \left(\frac{b}{a}\right)x$ and $y = -\left(\frac{b}{a}\right)x$ are the two asymptotes.
2. Both these lines pass through the centre (0, 0) of the hyperbola.
3. Both are inclined equally with the transverse x-axis and conjugate y-axis.
4. The x and y axes are acting as the angle bisectors of the angle between both the asymptotes.
5. For $y = \left(\frac{b}{a}\right)x \Rightarrow ay = bx \Rightarrow ay - bx = 0 \Rightarrow \frac{y}{b} - \frac{x}{a} = 0 \Rightarrow \frac{x}{a} - \frac{y}{b} = 0$

   For $y = -\left(\frac{b}{a}\right)x \Rightarrow ay + bx = 0 \Rightarrow \frac{x}{a} + \frac{y}{b} = 0$

   On multiplying both these equations, we get the combined form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$

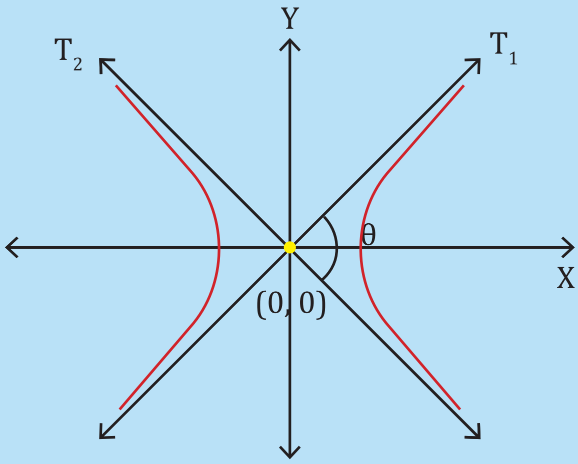
Observation

(1) Equation of both the asymptotes for this hyperbola, $y = \pm\left(\frac{b}{a}\right)x$

So, $m_1 = \frac{b}{a} > 0$ and $m_2 = \frac{-b}{a} < 0$,

slope of T_1 asymptote > 0 because line T_1 making a positive (anticlockwise) angle from the positive direction of x-axis and slope of $T_2 < 0$ because line T_2 making a negative (clockwise) angle from the positive direction of x-axis. So,

$T_1 : y = \left(\frac{b}{a}\right)x$ and $T_2 : y = -\left(\frac{b}{a}\right)x$



Results

(a) If θ is the acute angle between asymptotes,

then $m_1 = \frac{b}{a}, m_2 = \frac{-b}{a}$

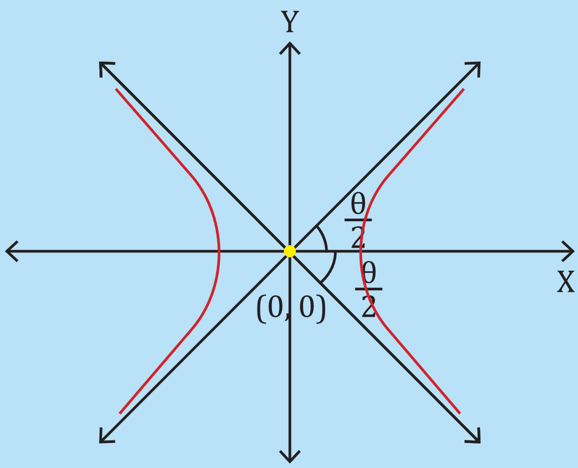
So, $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{\frac{b}{a} + \frac{b}{a}}{1 - \frac{b}{a} \times \frac{b}{a}}\right| = \left|\frac{2ab}{a^2 - b^2}\right|$

(b) $m_1 = \frac{b}{a} = \tan\left(\frac{\theta}{2}\right) \Rightarrow \frac{\theta}{2} = \tan^{-1}\left(\frac{b}{a}\right) \Rightarrow \theta = 2\tan^{-1}\left(\frac{b}{a}\right)$

(c) As we know $b^2 = a^2(e^2 - 1) \Rightarrow \frac{b}{a} = \sqrt{e^2 - 1}$

$\frac{b}{a} = \tan\left(\frac{\theta}{2}\right) = \sqrt{e^2 - 1} \Rightarrow e^2 = 1 + \tan^2\left(\frac{\theta}{2}\right) = \sec^2\left(\frac{\theta}{2}\right)$

$\Rightarrow e = \sec\left(\frac{\theta}{2}\right)$



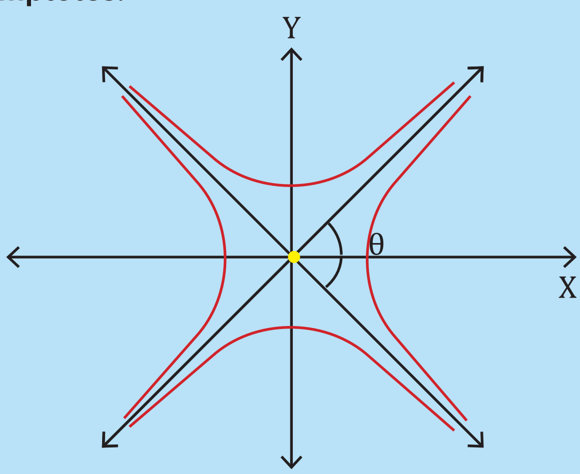
(2) A hyperbola and its conjugate have the same asymptotes.

Asymptotes of the horizontal hyperbola,

$H : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $y = \pm\left(\frac{b}{a}\right)x$ or $A : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$

Asymptotes of the conjugate hyperbola,

$C : \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ are $y = \pm\left(\frac{b}{a}\right)x$ or $A : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$



(3) $H: \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0, A: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0, C: \frac{x^2}{a^2} - \frac{y^2}{b^2} + 1 = 0$

$$H + C = \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 + \frac{x^2}{a^2} - \frac{y^2}{b^2} + 1 = 2\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) = 2A$$

Note

1. Hyperbola, pair of asymptotes, and conjugate hyperbola differ by a constant with $H + C = 2A$
2. If $H : ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$, then $A : ax^2 + by^2 + 2hxy + 2gx + 2fy + c + k = 0$

Find the equation of the asymptote of the hyperbola $xy - 3y - 2x = 0$.

Solution

Step 1: Given hyperbola is $xy - 3y - 2x = 0$, combined equation of pair of asymptote is
$xy - 3y - 2x + \lambda = 0$ (Represents a pair of lines)
General equation of a straight line in second degree is,
$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ and $\triangle = 0 \Rightarrow abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

Here we have, $0(x^2) + 2\left(\frac{1}{2}\right)xy + 0(y^2) + 2(-1)x + 2\left(\frac{-3}{2}\right)y + \lambda = 0$

Comparing it with the general equation, we get,

$a = 0, b = 0, c = \lambda, h = \frac{1}{2}, g = -1, f = \frac{-3}{2}$

$\Rightarrow 0 + 2\left(\frac{-3}{2}\right)(-1)\left(\frac{1}{2}\right) - 0 - 0 - \lambda\left(\frac{1}{2}\right)^2 = 0$

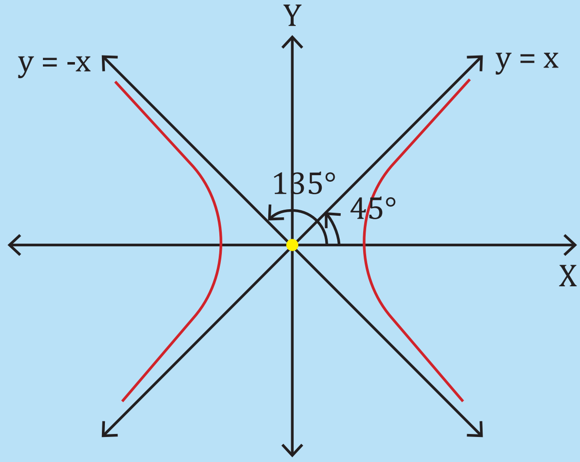
$\Rightarrow \lambda = 6$

Step 2: So, the equation of the pair of asymptotes is $xy - 3y - 2x + 6 = 0$
$x(y - 2) - 3(y - 2) = 0 \Rightarrow (x - 3)(y - 2) = 0 \Rightarrow L_1 : x - 3 = 0$ and $L_2 : y - 2 = 0$
So, $L_1 : x - 3 = 0$ and $L_2 : y - 2 = 0$ are the equations of asymptotes of the hyperbola.

Observation

We have observed that a pair of asymptotes for any standard hyperbola are $y = \pm\left(\frac{b}{a}\right)x$. If angle between the asymptotes is 90° then it is a rectangular hyperbola and that implies a = b. Its asymptotes will be, y = x and y= -x

So, their inclinations are 45° and 135°, respectively. The slopes are 1 and -1 and they both intersect at origin. Asymptotes are perpendicular to each other.
Note that any hyperbola with these two lines as their asymptotes will always be a rectangular hyperbola.



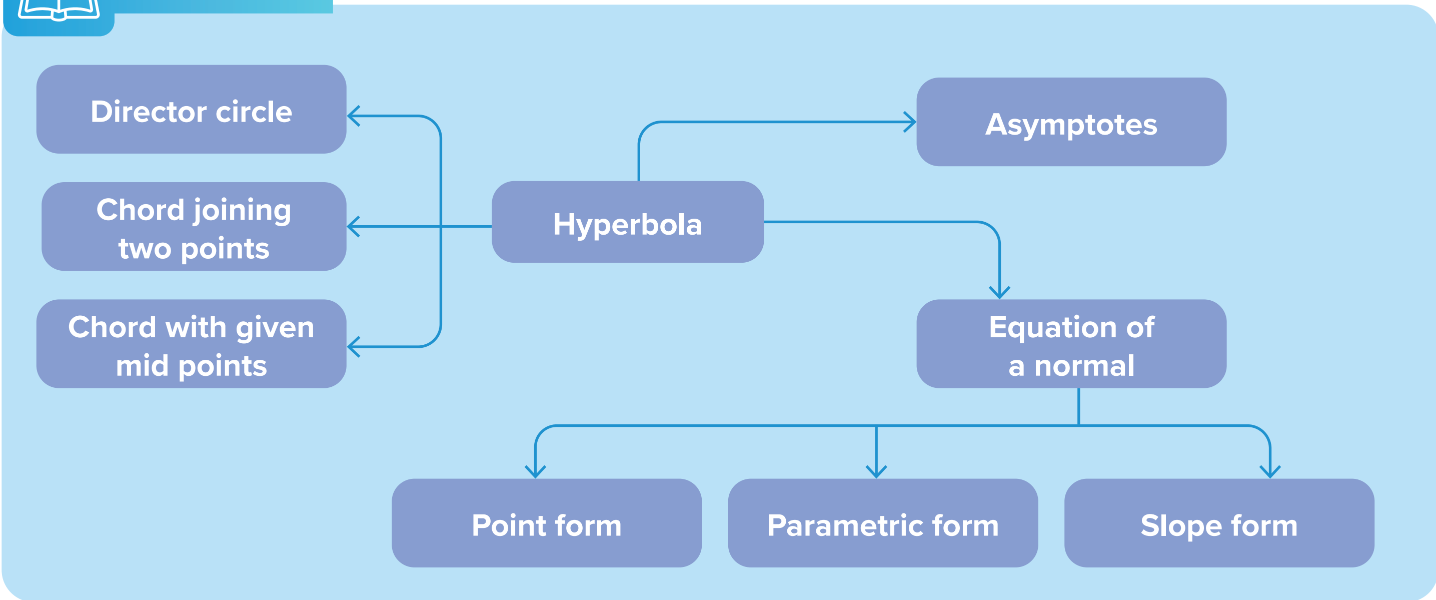
Concept Check

1. Find the equation of the hyperbola that has $3x - 4y + 7 = 0$ and $4x + 3y + 1 = 0$ as its asymptotes and passes through the origin.

Summary sheet

- Equation of the director circle for the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is
  $x^2 + y^2 = a^2 - b^2 = \left(\sqrt{a^2 - b^2}\right)^2$, where $a > b$ and $r = \sqrt{a^2 - b^2}$
- Equation of the director circle for the vertical hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ is $x^2 + y^2 = b^2 - a^2$
- **Point form:** Equation of the normal to the hyperbola $S = 0$ at $P(x_1, y_1)$ is $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$
- **Slope form:** Equation of the normal to the hyperbola $S = 0$ whose slope is m is
  $y = mx \pm \frac{(a^2 + b^2)m}{\sqrt{a^2 - b^2m^2}}$, where $m \in \left(\frac{-a}{b}, \frac{a}{b}\right)$
- **Parametric form:** Equation of the normal to the hyperbola $S = 0$ at $P(\theta)$ is
  $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2 \quad \left(\theta \neq 0, \frac{\pi}{2}, \pi\right)$
- Asymptotes of the horizontal hyperbola $H: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $y = \pm\left(\frac{b}{a}\right)x$ or $A: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$
- Asymptotes of the conjugate hyperbola $C: \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ are $y = \pm\left(\frac{b}{a}\right)x$ or $A: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$

Mind Map



Self-Assessment

1. Find the locus of the middle points of chords of hyperbola $3x^2 - 2y^2 + 4x - 6y = 0$ parallel to $y = 2x$.

Answers

Concept Check

Step 1:

Combined equation of the asymptotes is

$(3x - 4y + 7)(4x + 3y + 1) = 0$

$\Rightarrow 12x^2 - 7xy - 12y^2 + 31x + 17y + 7 = 0$.....(i)

Since the equation of hyperbola and combined equation of its asymptotes differ by a constant, the equation of hyperbola may be taken as follows:

$12x^2 - 7xy - 12y^2 + 31x + 17y + k = 0$ .....(ii)

Step 2:

As (ii) passes through origin $(0, 0)$, we have $k = 0$

Hence, the equation of the required hyperbola is

$12x^2 - 7xy - 12y^2 + 31x + 17y = 0$

Self-Assessment

Step 1:

Let the locus of the midpoint of the chord be (h, k).

We know that the equation of chord whose midpoint is known is given by $T = S_1$.

So the equation of chord whose midpoint is (h, k) is,

$3xh - 2yk + 2(x + h) - 3(y + k) = 3h^2 - 2k^2 + 4h - 6k$

$\Rightarrow x(3h + 2) - y(2k + 3) - 2h + 3k - 3h^2 + 2k^2 = 0$

Step 2:

Its slope is $\frac{3h+2}{2k+3}$ (As it is parallel to $y = 2x$)

$3h - 4k = 4$

$3x - 4y = 4$

Therefore, the locus of the middle points of chords of hyperbola $3x^2 - 2y^2 + 4x - 6y = 0$ parallel to $y = 2x$ is $3x - 4y = 4$.

M A T H E M A T I C S

HYPERBOLA

SPECIAL RECTANGULAR HYPERBOLA AND PROPERTIES OF HYPERBOLA

NOTES

What you already know

- Director circle
- Chord joining two points
- Chord with a given midpoint
- Equation of normal to a hyperbola
  (a) Point form
  (b) Parametric form
  (c) Slope form
- Asymptotes

What you will learn

- Rectangular hyperbola
- Rectangular hyperbola referred to its asymptotes as the axes of the coordinates
- Terms related to hyperbola

1. Equation of the director circle for the hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is the following:
   $x^2 + y^2 = a^2 - b^2 = (\sqrt{a^2 - b^2})^2$, where $a > b$ and $r = \sqrt{a^2 - b^2}$
2. Equation of the director circle for the vertical hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ is the following:
   $x^2 + y^2 = b^2 - a^2$
3. **Point form:** The equation of normal to hyperbola $S = 0$ at $P(x_1, y_1)$ is
   $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$
4. **Slope form:** The equation of normal to hyperbola $S = 0$, whose slope is m is the following; $y = mx \pm \frac{(a^2 + b^2)m}{\sqrt{a^2 - b^2m^2}}$, where $m \in \left(\frac{-a}{b}, \frac{a}{b}\right)$
5. **Parametric form:** The equation of normal to hyperbola $S = 0$ at $P(\theta)$ is the following: $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$
6. The asymptotes of horizontal hyperbola $H: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $y = \pm\left(\frac{b}{a}\right)x$ or $A: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$
7. The asymptotes of conjugate hyperbola $C: \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ are $y = \pm\left(\frac{b}{a}\right)x$ or $A: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$

Rectangular Hyperbola

If the lengths of the transverse axis and conjugate axis of a hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, are equal, then it is known as a rectangular(equilateral) hyperbola, i.e.,

$2a = 2b \Rightarrow a = b$

The equation of a rectangular hyperbola in the standard form is as follows:

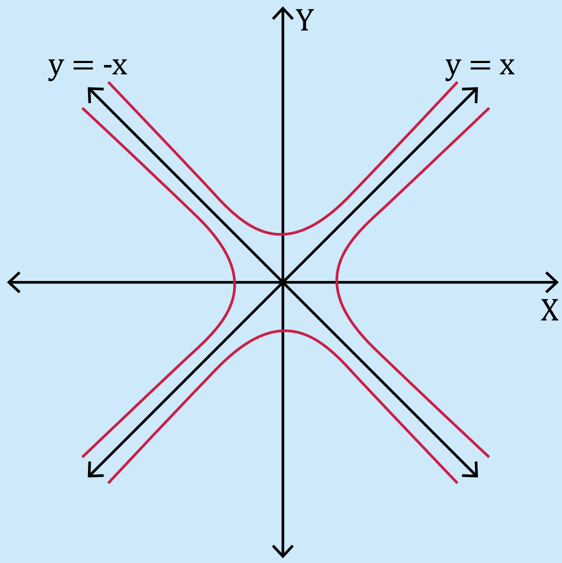
Case I: $x^2 - y^2 = a^2$ and $e = \sqrt{2}$, and the pair of asymptotes are given by $y = x$ and $y = -x$

Case II: $x^2 - y^2 = -a^2$ and $e = \sqrt{2}$, and the pair of asymptotes are given by $y = x$ and $y = -x$

(a) The equation of a pair of asymptotes is given by $x^2 - y^2 = 0$, where both $y = x$ and $y = -x$ are equally inclined with the coordinate axes.

(b) Both of them intersect at the origin, which is also the centre of both the hyperbolas.

(c) Also, they are mutually perpendicular to each other.



Deduction: For a hyperbola to be a rectangular hyperbola, both its asymptotes should not just intersect at the centre but also be mutually perpendicular to each other.

The given deduction will help us generate new rectangular hyperbolas other than the ones in cases I and II.

Rectangular hyperbola referred to its asymptotes as the axes of the coordinates

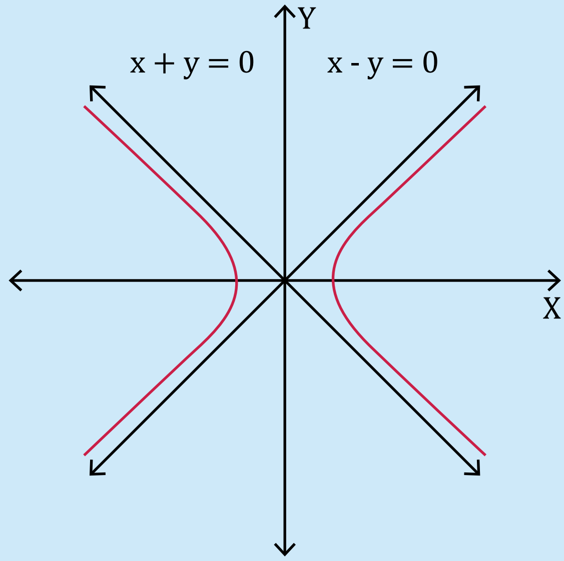
When the centre of any rectangular hyperbola is at the origin and its asymptotes coincide with the coordinate axes, the equation of this rectangular hyperbola is $xy = c^2$

1. In this hyperbola, the branches lie only in the **first and third quadrants**.
2. Coordinate axes, i.e., y-axis and x-axis, are the tangents to this hyperbola that meet at infinity and are therefore asymptotes to this hyperbola.
3. With the help of our deduction, we can see that this hyperbola is also a rectangular hyperbola, as the asymptotes are perpendicular to each other as well as they intersect at the centre of this hyperbola, which is also the origin.

Finding the equation of the given hyperbola. Claim: $xy = c^2$

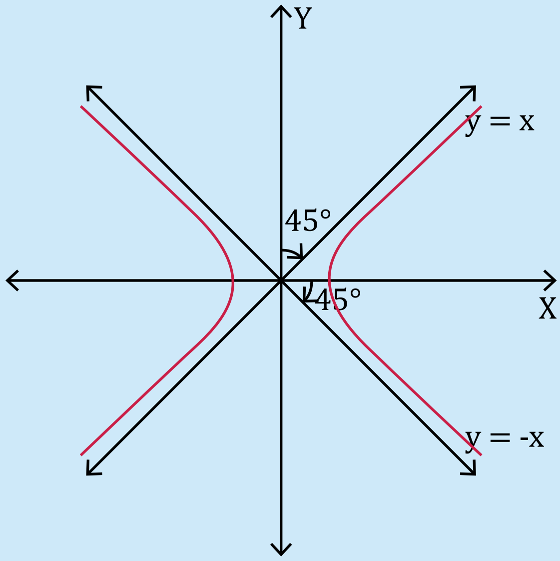
Proof

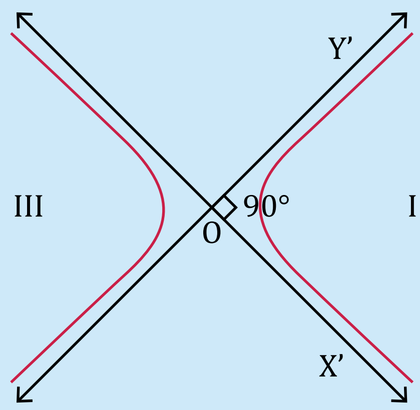
Consider the horizontal rectangular hyperbola, $x^2 - y^2 = a^2$, with the centre at the origin, whose asymptotes are given by $x - y = 0$ and $x + y = 0$



Exercise

Rotate the axes of the rectangular hyperbola by 45° in clockwise direction without changing the origin. Now you can notice that the new axes have coincided with asymptotes and the position of hyperbola with respect to the axes has changed. You can notice that one of the hyperbolic branches completely lies in the **first quadrant** now while the other completely lies in the **third quadrant** with respect to the new coordinate axes.



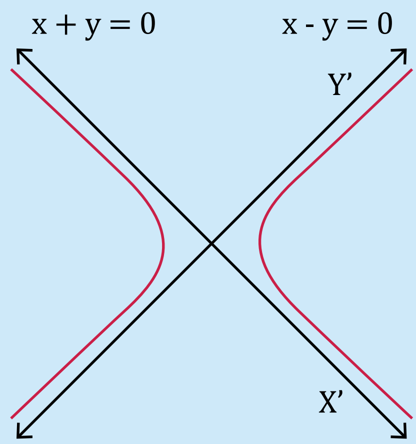


⇒ Now, under rotation, line $y = x$ becomes the new y-axis (Y') and line $y = -x$ becomes the new x-axis (X').
Original equation of the hyperbola was $x^2 - y^2 = a^2$, but now, we want to find a new equation of the hyperbola with respect to X' and Y', which are the new coordinate axes.

Recall

The rotated formulae for x & y axes are as follows:
$x' = x \cos\theta + y \sin\theta$ and $y' = -x \sin\theta + y \cos\theta$,
where θ is in the anti-clockwise sense.
⇒ For our required rotation, $\theta = -45°$
Putting the value of $\theta = -45°$, we get,
$x' = x\cos(-45°) + y\sin(-45°)$ and
$y' = -x\sin(-45°) + y\cos(-45°)$

$\Rightarrow x' = \frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}}$ and $y' = \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}$



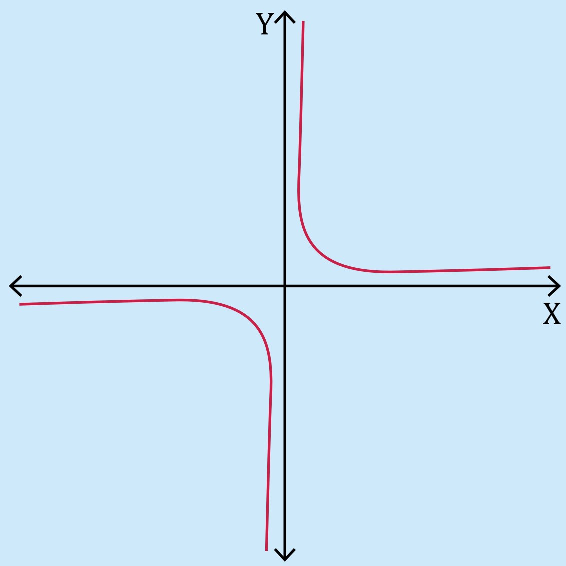
Therefore, the equation of hyperbola
$x^2 - y^2 = a^2$ reduces to $(x + y)(x - y) = a^2$
We can write the above equation as

$\left(\frac{x+y}{\sqrt{2}}\right)\left(\frac{x-y}{\sqrt{2}}\right) = \frac{a^2}{2}$ ....(i)

Substituting the values of $\frac{x+y}{\sqrt{2}}$ and $\frac{x-y}{\sqrt{2}}$ in (i),
we get,

$\Rightarrow y'x' = \frac{a^2}{2} = c^2 \Rightarrow x'y' = c^2$

Hence proved.
Therefore, when the centre of any rectangular hyperbola is at the origin and its asymptotes coincide with the coordinate axes, then its equation is $xy = c^2$



Note

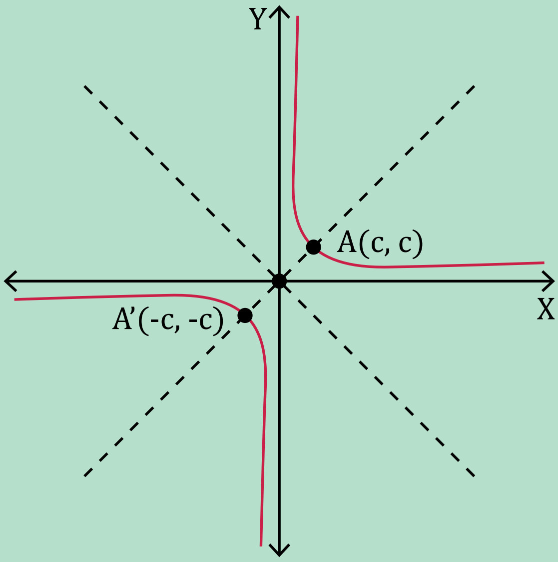
1. Some parameters will remain the same for all rectangular hyperbolas. They are as follows:
   (a) The distance between the centre and the foci is still ae.
   (b) The distance between the centre and each vertex is still a.
   (c) The distance between the centre and the directrix is still $\frac{a}{e}$.
   (d) The length of the latus rectum will still be $\frac{2b^2}{a} = 2a$, as $a = b$ for a rectangular hyperbola.
   (e) Eccentricity, $e = \sqrt{2}$
2. The asymptote equations for $xy = c^2$ are given by $x = 0$ and $y = 0$
3. The equations of the axes of hyperbola $xy = c^2$ are $y = x$ (transverse axis) and $y = -x$ (conjugate axis)

Terms related to hyperbola $xy = c^2$

(i) **Asymptotes:** Equation of the asymptotes are given by $x = 0$ and $y = 0$, i.e., x-axis and y-axis are the asymptotes.

(ii) **Eccentricity:** As the asymptotes are perpendicular to each other and intersect at the centre of the hyperbola, it is a rectangular hyperbola, and we know that the eccentricity of a rectangular hyperbola is $\sqrt{2}$.

We can look at this in another way. To get this rectangular hyperbola, we simply rotate the axes of the horizontal rectangular hyperbola by $45°$ in the clockwise direction without disturbing its shape and position . The rotation of the axes does not affect the eccentricity as eccentricity only measures the degree of flatness of the curve, so the eccentricity will remain $\sqrt{2}$.



(iii) **Centre:** The centre is $(0, 0)$.

(iv) **Equation of the transverse axis (T.A.):** $y = x$

(v) **Equation of the conjugate axis (C.A.):** $y = -x$

(vi) **Vertices:** We know that vertices are the points where the transverse axis cuts the hyperbolic curve. So, in order to get the coordinates of the vertices, we need to solve $xy = c^2$ and $y = x$ simultaneously.

$\Rightarrow x^2 = c^2 \Rightarrow x = \pm c$ and $y = \pm c$

Now, A lies in the first quadrant and its coordinates will therefore be $A(c, c)$, and A' is in the third quadrant and its coordinates will therefore be $A'(-c, -c)$.

Note

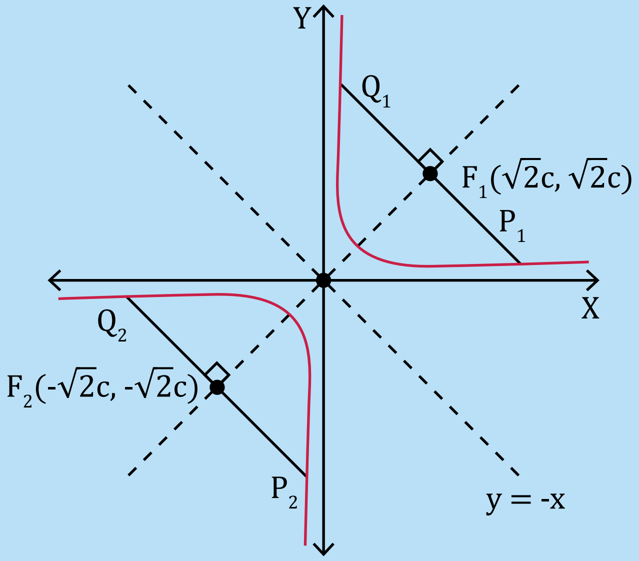
The distance from the origin to vertices OA and OA' is given by $a = \sqrt{2}c$

(vii) **Length of the T.A. (= C.A.):** The length of T.A. is the distance between the two vertices and is given by $2\sqrt{2}c$.

(viii) **Foci:** F_1 is lying on the line $y = x$. So both the coordinates are equal. i.e.(x, x)

$a = \sqrt{2}\,c \Rightarrow ae = \sqrt{2}c \times \sqrt{2} = 2c$ & $\sqrt{2}x = 2c \Rightarrow x = \sqrt{2}c$. The coordinates of the foci are given by $F_1(\sqrt{2}c, \sqrt{2}c)$ and $F_2(-\sqrt{2}c, -\sqrt{2}c)$.

(ix) **Equation of the latus rectum:** Both of the latus rectums are parallel to the line $y = -x$. One is passing through F_1 and other is passing through F_2. Therefore, the slopes of both latus rectums are the same, i.e., -1. The equation of latus rectum P_1Q_1 through $F_1(\sqrt{2}c, \sqrt{2}c)$ is given by $x + y = 2\sqrt{2}c$, and the equation of latus rectum P_2Q_2 through $F_2(-\sqrt{2}c, -\sqrt{2}c)$ is given by $x + y = -2\sqrt{2}c$



(x) **Length of the latus rectum:** As the length of latus does not change, the lengths of the latus rectums P_1Q_1 & P_2Q_2 is given by

$$\frac{2b^2}{a} = \frac{2a^2}{a} = 2a = 2\sqrt{2}c$$

(xi) **Equation of the directrices:** Let D_1 and D_2 be the two directrices. We will use the normal form to write the equations of D_1 and D_2. Let M be a point on transverse axis D_1.

Let $OM = p$

$\Rightarrow D_1: x \cos \alpha + y \sin \alpha = p$

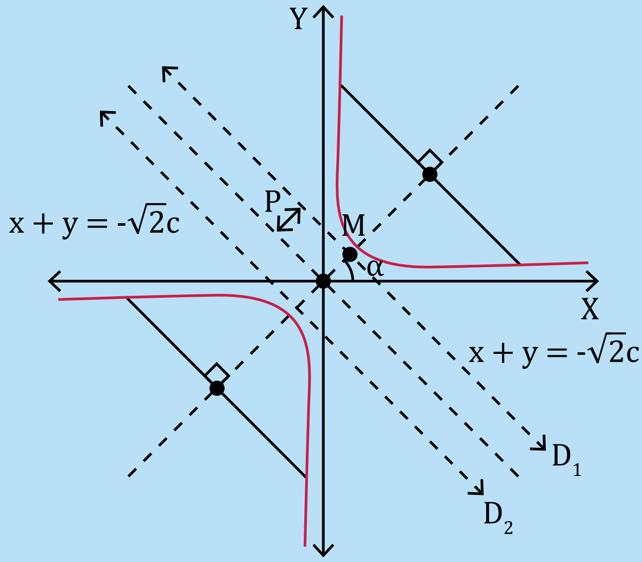
Now, p is the distance between the origin to directrix D_1 along the transverse axis.

$p = \frac{a}{e} = \frac{\sqrt{2}c}{\sqrt{2}} = c$ Also, $\alpha = 45°$

So, $D_1: x \cos(45°) + y \sin(45°) = c$

$\Rightarrow \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} = c \Rightarrow x + y = \sqrt{2}c$

Similarly, $D_2: x + y = -\sqrt{2}c$



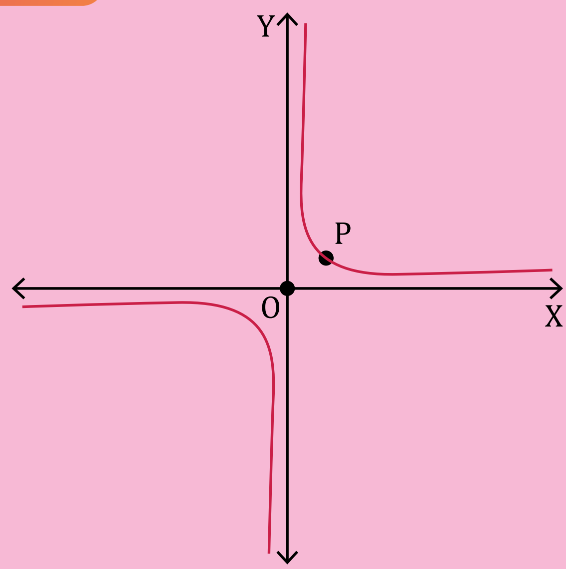
Parametric equation

We want to know how to represent any point on a rectangular hyperbola $xy = c^2$

Let P be any point on our hyperbola. Hence, it is represented by the following:

$P(ct, \frac{c}{t})$, where t is just a parameter such that $t \in R - \{0\}$

Hence, $x = ct$ and $y = \frac{c}{t}$ $(t \neq 0)$ are the parametric equations of this rectangular hyperbola.



Equation of the tangent for the rectangular hyperbola $xy = c^2$

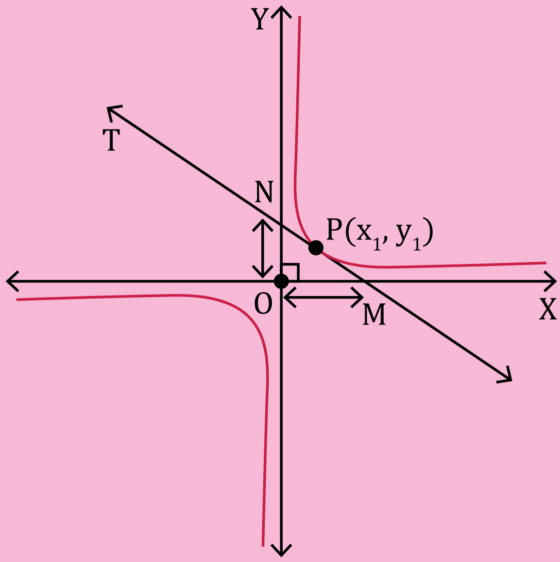
Consider a tangent (T) at point $P(x_1, y_1)$ to our hyperbola.
The equation of tangent with the help of the point form is given by $T = 0$
Using the rules for replacement in finding $T = 0$ with the help of the point form, we get,

$$T = 0: \frac{xy_1 + yx_1}{2} = c^2 \Rightarrow xy_1 + yx_1 = 2c^2$$

$\Rightarrow$ Hence, its slope $= \frac{-y_1}{x_1}$

Also, notice that $\frac{y_1}{x_1}$ is always positive as points (x_1, y_1) either lie in the first or the third quadrants. Hence, the slope of the tangent for our given hyperbola is always negative.

1. The slope of the tangent is always -ve.
2. The area formed by the tangent and the coordinate axes is always constant.



x-intercept $= \frac{2c^2}{y_1}$

y-intercept $= \frac{2c^2}{x_1}$

Area of $\triangle OMN = \frac{1}{2}(OM)(ON) = \frac{1}{2}\left(\frac{2c^2}{y_1}\right)\left(\frac{2c^2}{x_1}\right)$.

$= \frac{4c^4}{2x_1y_1} = \frac{4c^4}{2c^2} = 2c^2$ sq. units

Parametric form: $x = ct, y = \frac{c}{t}$
Equation of tangent in parametric form: $x + yt^2 = 2ct$

Equation of normal to the hyperbola: $xy = c^2$

$xy = c^2$
Equation of tangent at point $P(x_1, y_1)$ is
T: $xy_1 + yx_1 = 2c^2$

Slope of tangent $(m_T) = \frac{-y_1}{x_1}$

$\Rightarrow$ Slope of normal $(m_N) = \frac{x_1}{y_1}$

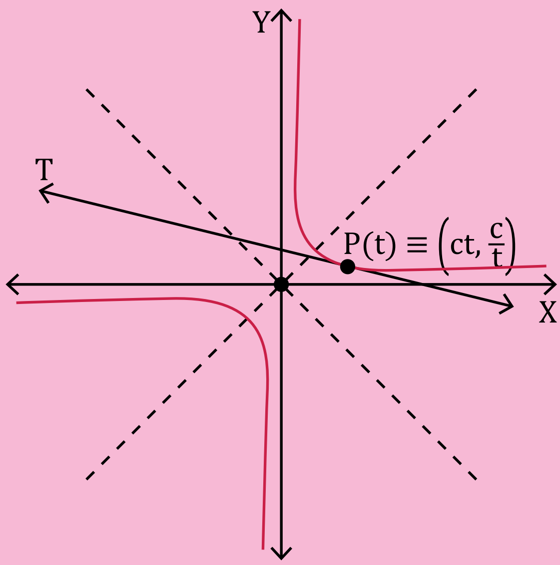
Equation of normal is $y - y_1 = \frac{x_1}{y_1}(x - x_1)$

$\Rightarrow yy_1 - (y_1)^2 = xx_1 - (x_1)^2$

$\Rightarrow xx_1 - yy_1 = (x_1)^2 - (y_1)^2$

The slope of a normal is always +ve.
Equation of normal in parametric form is :

$t^2x - y = c\left(t^3 - \frac{1}{t}\right)$



Solve

For hyperbola $xy = 16$, which of the following is/are true?

(a) The length of the latus rectum is $8\sqrt{2}$.

(b)The tangent at $(2, 8)$ is $2x + y - 12 = 0$

(c) The chord with midpoint $(5, 6)$ is $6x + 5y = 60$

(d) The distance between the foci is 18.

Solution

$xy = 16 \Rightarrow c^2 = 16$ $(\because xy = c^2)$

$c = 4$ $(c > 0)$

Option (a)

Length of the latus rectum $= \dfrac{2a^2}{a} = 2a = 2(\sqrt{2}c)$ $\{\because a = \sqrt{2}c\}$

$= 2\sqrt{2}(4) = 8\sqrt{2}$

Option (b)

Tangent at point (x_1, y_1) is $xy_1 + yx_1 = 2c^2$

At $P(2, 8) \Rightarrow 8x + 2y = 2(16) = 32 \Rightarrow 4x + y = 16$

Option (c)

The equation of chord with given midpoint is $T = S_1$

$\Rightarrow xy = c^2 \Rightarrow xy - c^2 = 0$ which represents S

The value of S at $P(5, 6)$is $5(6) - 16 = 14 = S_1$

$T: \dfrac{xy_1 + yx_1}{2} - c^2$

$\Rightarrow \dfrac{6x + 5y}{2} - 16 = 14 \Rightarrow 6x + 5y - 32 = 28$

$\Rightarrow 6x + 5y = 60$

Option (d)

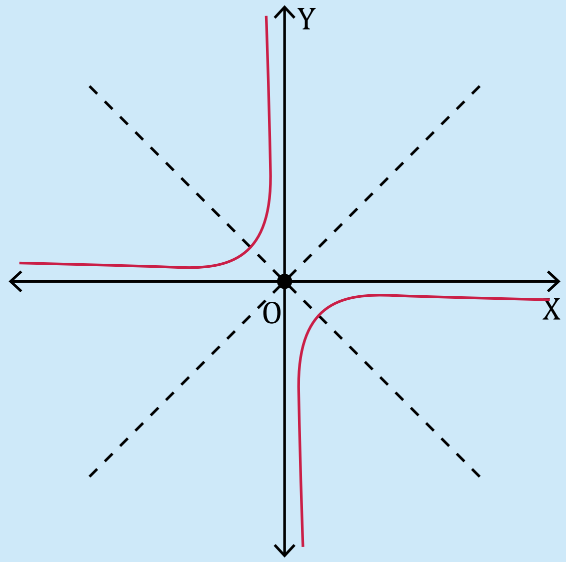
Distance between the foci $= 2ae$

$= 2(4\sqrt{2})\sqrt{2} = 16$

Options (a) and (c) are the correct answers.

Conjugate rectangular hyperbola: $xy = -c^2$

In order to obtain the equation of the conjugate hyperbola which has asymptotes as coordinate axis we rotate the axes of reference through an angle of -45°. Hence $xy = -c^2$ is the conjugate of the rectangular hyperbola $xy = c^2$. This is achieved by rotating the axes by 45° in anticlockwise sense.



Geometric properties of rectangular hyperbola

Property 1

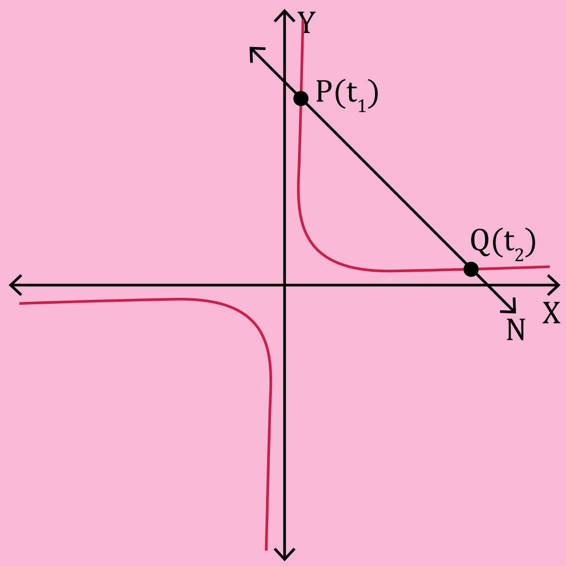
If the normal at point with parameter t_1 to the rectangular hyperbola, $xy = c^2$, meets it again at point with parameter t_2, then $(t_1)^3 t_2 = -1$

Proof

Given, $xy = c^2$

Normal N at point $P(t_1)$ intersects the curve again at $Q(t_2)$

> **Note**
>
> Equation of normal at (x_1, y_1)
> in point form: $xx_1 - yy_1 = (x_1)^2 - (y_1)^2$



Let $P(t_1) \equiv \left(ct_1, \frac{c}{t_1}\right)$

The equation of normal at $P(t_1)$,

$\Rightarrow x(ct_1) - y\left(\frac{c}{t_1}\right) = c^2(t_1)^2 - \frac{c^2}{(t_1)^2}$

$\Rightarrow (t_1)^3 x - t_1 y - c(t_1)^4 + c = 0$......(i)

$Q(t_2) \equiv \left(ct_2, \frac{c}{t_2}\right)$ lying on the normal

Equation (i) Passes through the point

$Q(t_2) \equiv \left(ct_2, \frac{c}{t_2}\right)$

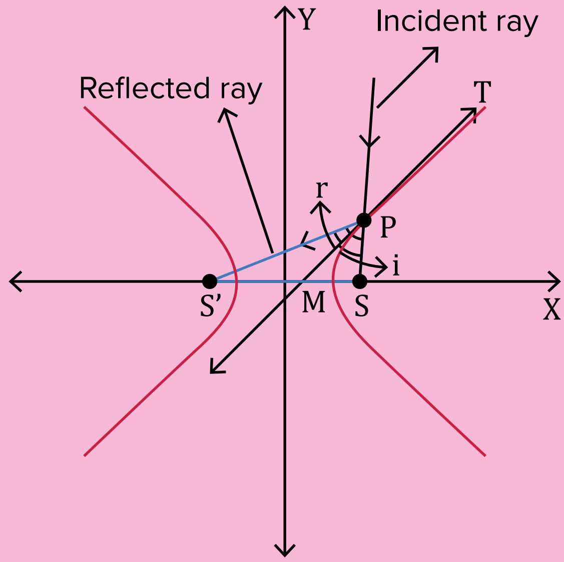
$\Rightarrow (t_1)^3(ct_2) - t_1\left(\frac{c}{t_2}\right) - c(t_1)^4 + c = 0 \quad \Rightarrow (t_1)^3(t_2)^2 - t_1 - (t_1)^4 t_2 + t_2 = 0 \quad \Rightarrow [(t_1)^3 t_2 + 1](t_2 - t_1) = 0$

Since P and Q are two distinct points, t_1 is not equal to $t_2 \Rightarrow t_1 \neq t_2$

$[(t_1)^3 t_2 + 1] = 0 \Rightarrow (t_1)^3 t_2 = -1$

Property 2

If an incoming light ray passing through one focus, S, strikes the convex side of the hyperbola, then it will get reflected towards the other focus, S'.



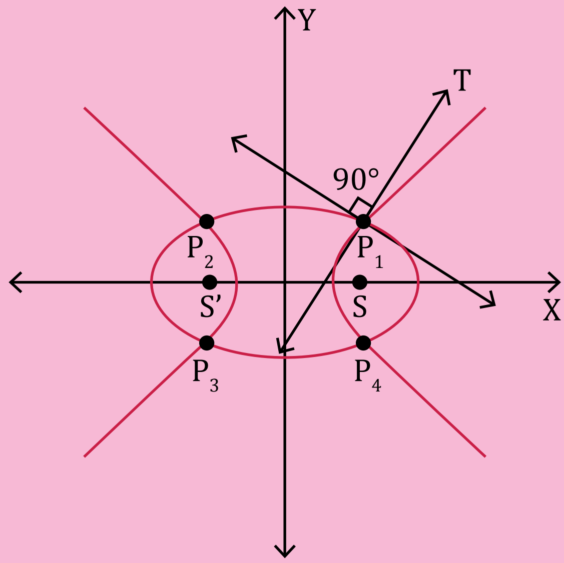
Proof

T is the tangent at point P. We must prove that ∠i = ∠r of PSS', with PM being the angle bisector of ∠SPS'

Property 3

If an ellipse and a hyperbola are confocal i.e., their foci are same, then they intersect orthogonally.

(The tangent of the ellipse at the intersection is the normal for the hyperbola.)



Property 4

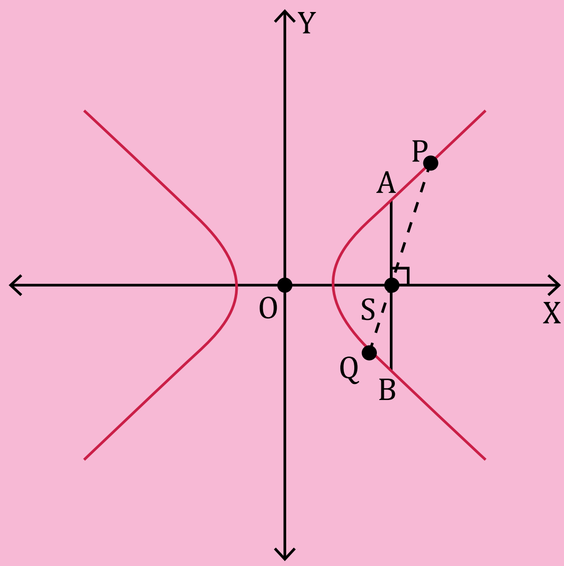
PSQ is a focal chord of hyperbola where S is the focus and P & Q are the points on the hyperbola $S = 0$ then $\frac{1}{SP} + \frac{1}{SQ} = \frac{2a}{b^2}$, or the semi-latus rectum is the harmonic mean of SP and SQ.

$\frac{b^2}{a}$ = Harmonic mean of SP and SQ

PS, $\frac{b^2}{a}$, QS are in HP.

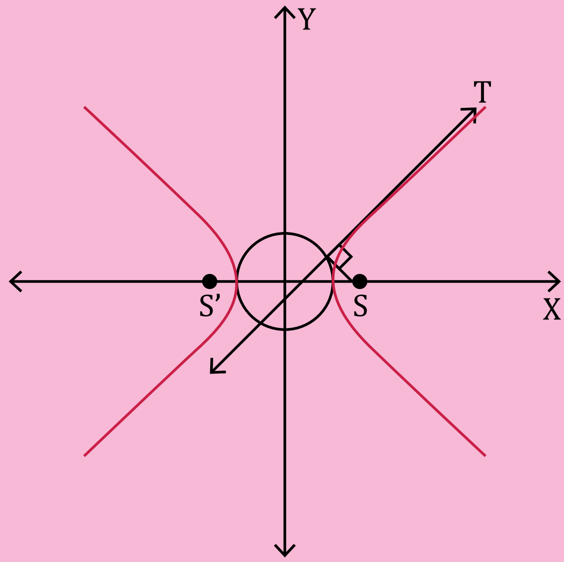
$\frac{1}{PS'}, \frac{a}{b^2}, \frac{1}{QS}$ are in AP.

$\frac{2a}{b^2} = \frac{1}{PS} + \frac{1}{QS}$ (This property is connecting a random focal chord to the latus rectum.)



Property 5

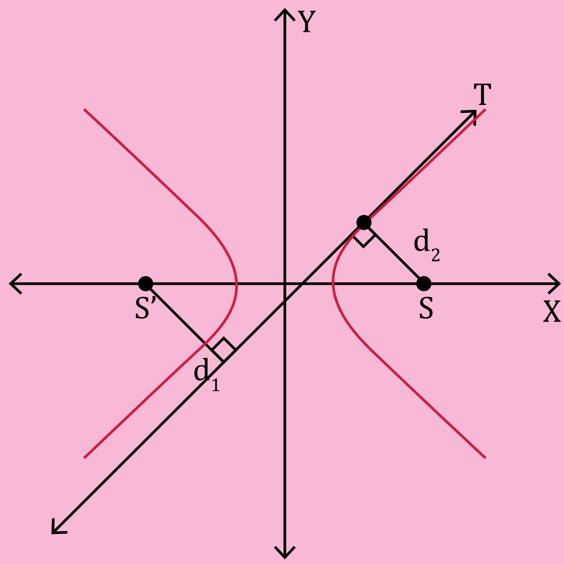
The feet of the perpendicular from the foci upon any tangent to a hyperbola lie on the auxiliary circle.



Property 6

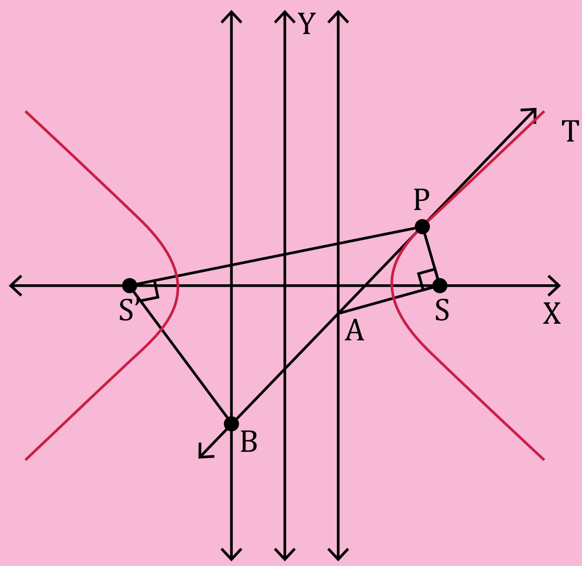
The product of the lengths of the perpendiculars from the foci upon any tangent of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is equal to the square of the semi-conjugate axis.

Let the perpendicular distance from focus S' to the tangent be d_1 and the perpendicular distance from focus S to the tangent be d_2, then $d_1 d_2 = b^2$



Property 7

The portion of the tangent to a hyperbola between the point of contact and the directrix subtends a right angle at the corresponding focus.



Property 8

If a triangle has its vertices on a rectangular hyperbola, then its orthocentre also lies on the same hyperbola.

Proof

Suppose that the parametric points on the hyperbola are $P\left(ct_1, \frac{c}{t_1}\right)$, $R\left(ct_2, \frac{c}{t_2}\right)$, and $Q\left(ct_3, \frac{c}{t_3}\right)$ respectively.

Now slope of QR is

$$\frac{\frac{c}{t_3} - \frac{c}{t_2}}{ct_3 - ct_2} = -\frac{1}{t_2 t_3}$$

Hence the slope of PM is $t_2 t_3$

The equation of altitude PM is

$$y - \frac{c}{t_1} = t_2 t_3 (x - ct_1)$$

$\Rightarrow t_1 y - c = x t_1 t_2 t_3 - c(t_1)^2 t_2 t_3$ ....(i)

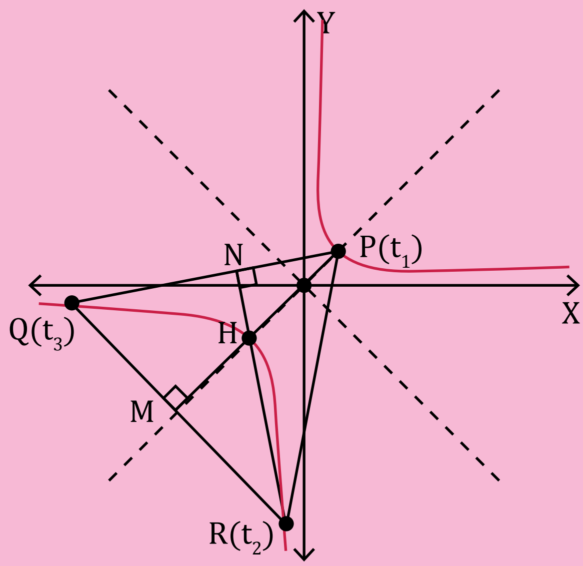
Similarly equation of altitude RN

Is $t_2 y - c = x t_1 t_2 t_3 - c(t_2)^2 t_1 t_3$ ....(ii)

Solving (i) & (ii)

We get the orthocentre of ΔPQR will be

$H \equiv \left(\frac{-c}{t_1 t_2 t_3}, -ct_1 t_2 t_3\right)$ which lies on $xy = c^2$



Concept Check

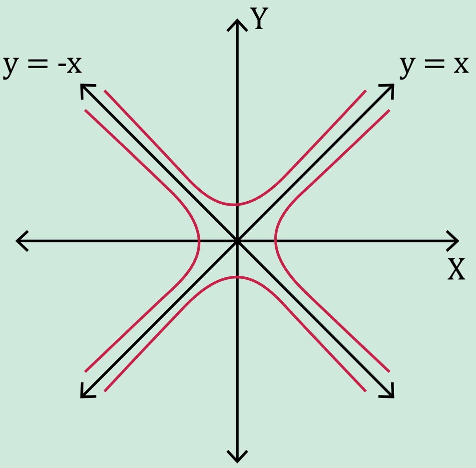
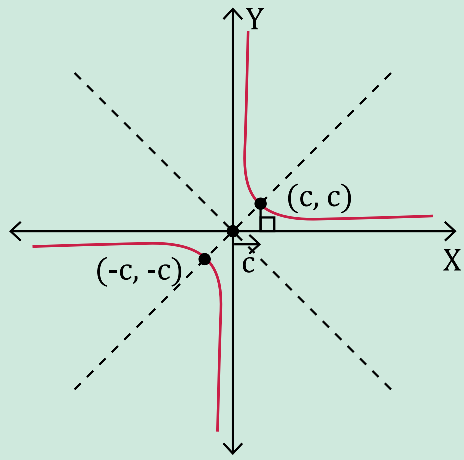
PQ and RS are two perpendicular chords of the rectangular hyperbola $xy = c^2$. If C is the centre of the rectangular hyperbola, then find the product of the slopes of CP, CQ, CR, and CS.

Summary Sheet

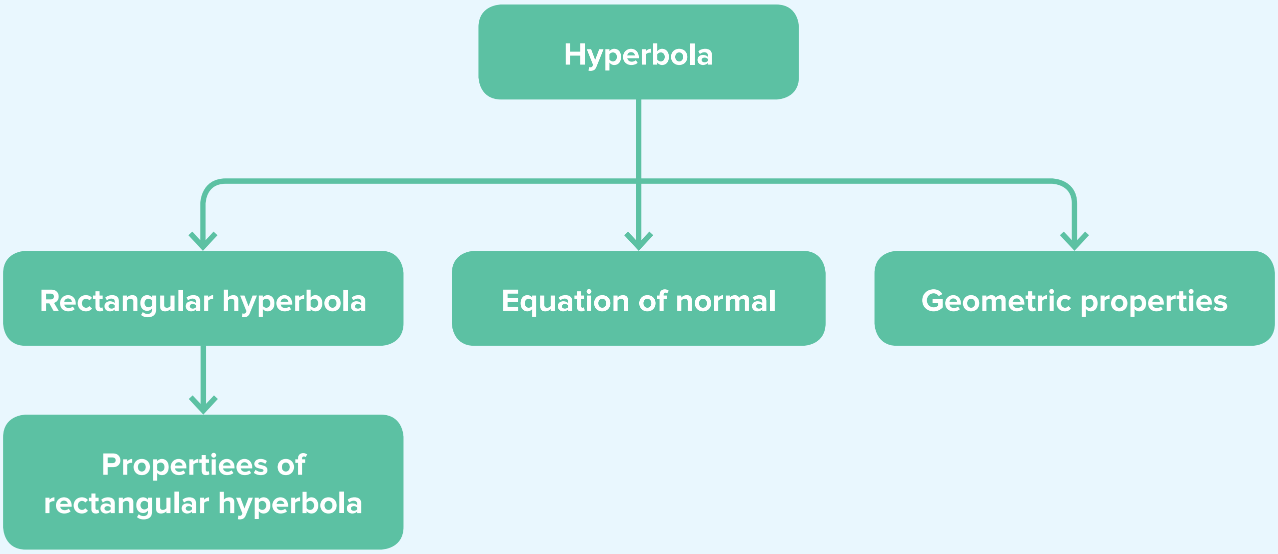
- Equation of a rectangular hyperbola in the standard form is as follows:

  Case I: $x^2 - y^2 = a^2$ and $e = \sqrt{2}$, and the pair of asymptotes are given by $y = x$ and $y = -x$

  Case II: $x^2 - y^2 = -a^2$ and $e = \sqrt{2}$, and the pair of asymptotes are given by $y = x$ and $y = -x$

	Rectangular hyperbola of the form $x^2 - y^2 = a^2$ 	Rectangular hyperbola of the form $xy = c^2$ 
Asymptotes	Asymptotes are perpendicular lines i.e. $x \pm y = 0$	Asymptotes are perpendicular lines i.e. $x = 0$ & $y = 0$
Eccentricity	$e = \sqrt{2}$	$e = \sqrt{2}$
Centre	$O(0, 0)$	$O(0, 0)$
Foci	$S(\sqrt{2}a, 0), S'(-\sqrt{2}a, 0)$	$S(\sqrt{2}c, \sqrt{2}c), S'(-\sqrt{2}c, -\sqrt{2}c)$
Directrices	$x = \pm \frac{a}{\sqrt{2}}$	$x + y = \pm \sqrt{2}c$
Latusrectum	$2a$	$2\sqrt{2}c$
Equation of tangent in point form	$xx_1 - yy_1 = a^2$	$xy_1 + yx_1 = 2c^2$ or $\frac{x}{x_1} + \frac{y}{y_1} = 2$
Equation of tangent in parametric form	$x \sec\theta - y \tan\theta = a$	$x + yt^2 = 2ct$
Equation of normal in point form	$\frac{x}{x_1} + \frac{y}{y_1} = 2$	$xx_1 - yy_1 = (x_1)^2 - (y_1)^2$
Equation of normal in parametric form	$\frac{x}{\sec\theta} + \frac{y}{\tan\theta} = 2a$	$t^2x - y = c\left(t^3 - \frac{1}{t}\right)$

Mind Map



Self-Assessment

A variable straight line of slope 4 intersects the hyperbola $xy = 1$ at two points. Find the locus of the point which divides the line segment between these points in the ratio 1 : 2

Answers

Concept Check

Step 1

Let the coordinates of

P, Q, R, and S be $\left(ct_i, \frac{c}{t_i}\right)$, respectively (where $i = 1, 2, 3, 4$)

Now, it is given that $PQ \perp RS$, so product of their slopes will be -1.

$$\Rightarrow \frac{\frac{c}{t_2} - \frac{c}{t_1}}{ct_2 - ct_1} \times \frac{\frac{c}{t_4} - \frac{c}{t_3}}{ct_4 - ct_3} = -1$$

$$\Rightarrow \frac{1}{t_1 t_2} \times \frac{1}{t_3 t_4} = -1$$

$$\Rightarrow \frac{1}{t_1 t_2 t_3 t_4} = -1$$

Step 2

Slope of CP $= \dfrac{\frac{c}{t_1}}{ct_1}$

Similarly, slope of CQ, CR, and CS is $\dfrac{\frac{c}{t_2}}{ct_2}, \dfrac{\frac{c}{t_3}}{ct_3}$, and $\dfrac{\frac{c}{t_4}}{ct_4}$ respectively.

Product of their slopes $= \dfrac{\frac{c}{t_1}}{ct_1} \times \dfrac{\frac{c}{t_2}}{ct_2} \times \dfrac{\frac{c}{t_3}}{ct_3} \times \dfrac{\frac{c}{t_4}}{ct_4}$

$= \left(\dfrac{1}{t_1t_2t_3t_4}\right)^2 = 1$

Self-Assessment

Step 1

Let the variable line of slope 4 cuts the hyperbola $xy = 1$at $P\left(t_1, \frac{1}{t_1}\right)$ and $Q\left(t_2, \frac{1}{t_2}\right)$. Let $R(h, k)$be the point dividing PQ in the ratio 1:2 then,

Slope of $PQ = 4$

$\dfrac{\frac{1}{t_2} - \frac{1}{t_1}}{t_2 - t_1} = 4 \Rightarrow t_1t_2 = -\frac{1}{4}$ ...(i)

Step 2

R divides PQ in the ratio 1:2. therefore,

$\dfrac{2t_1 + t_2}{3} = h$ and $\dfrac{\frac{2}{t_1} + \frac{1}{t_2}}{1 + 2} = k \Rightarrow 2t_1 + t_2 = 3h$ and $2t_2 + t_1 = 3kt_1t_2$

$2t_1 + t_2 = 3h$ and $t_1 + 2t_2 = -\frac{3k}{4}$ [using (i)]

$3(t_1 + t_2) = 3h - \frac{3k}{4}$ and $t_1 - t_2 = 3h + \frac{3k}{4}$[on adding and subtracting]

Step 3

$t_1 + t_2 = h - \frac{k}{4}$ and $t_1 - t_2 = 3\left(h + \frac{k}{4}\right)$

$\Rightarrow t_1 = 2h + \frac{k}{4}$ and $t_2 = -h - \frac{k}{2} \Rightarrow t_1t_2 = \left(\frac{8h + k}{4}\right)\left(\frac{-2h - k}{2}\right)$

$\Rightarrow \frac{-1}{4} = -\frac{(8h + k)(2h + k)}{8}$ [using (i)]

$\Rightarrow 2 = 16h^2 + 10hk + k^2$

$\Rightarrow 16h^2 + 10hk + k^2 - 2 = 0$

Hence the locus of (h, k) is $16h^2 + 10hk + k^2 - 2 = 0$