CHAPTER-6
APPLICATIONS OF DERIVATIVES

Topic-1 Rate of Change of Bodies

Concept covered: Interpretation of dy/dx as a rate measure

Revision Notes

➢ **Interpretation of $\frac{dy}{dx}$ as a rate measure :**

If two variables x and y are varying with respect to another variables say t, *i.e.*, if $x = f(t)$ and $y = g(t)$, then by the Chain Rule, we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \quad \frac{dx}{dt} \neq 0$$

Thus, the rate of change of y with respect to x can be calculated by using the rate of change of y and that of x both with respect to t.

Also, if y is a function of x and they are related as $y = f(x)$ then, $f'(\alpha)$, *i.e.*, represents the rate of change of y with respect to x at the instant when $x = \alpha$.

Topic-2 Tangents and Normals

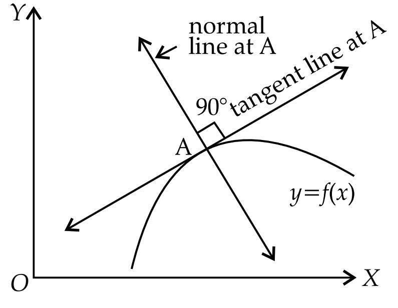
Concepts covered: Slope of a line, equation of tangent, equation of normal, acute angle between two curves

Revision Notes

➢ **Slope or gradient of a line :**

If a line makes an angle θ with the positive direction of X-axis in anti-clockwise direction, then $\tan\theta$ is called the slope or gradient of the line. [Note that θ is taken as positive or negative accordingly as it is measured in anti-clockwise (*i.e.*, from positive direction of X-axis to the positive direction of Y-axis) or clockwise direction respectively.]

➢ **Pictorial representation of tangent and normal :**



➢ **Facts about the slope of a line :**

(i) If a line is parallel to X-axis (or perpendicular to Y-axis), then its slope is 0 (zero) or $\frac{dy}{dx} = 0$.

(ii) If a line is parallel to Y-axis (or perpendicular to X-axis), then its slope is $\frac{1}{0}$, *i.e.*, not defined or $\frac{dx}{dy} = 0$.

(iii) If two lines are perpendicular, then product of their slopes equals – 1, *i.e.*, $m_1 \times m_2 = -1$. Whereas, for two parallel lines, their slopes are equal, *i.e.*, $m_1 = m_2$. (Here in both the cases m_1 and m_2 represent the slopes of respective lines).

- **Equation of tangent at (x_1, y_1) :**

  $(y - y_1) = m_T(x - x_1)$, where m_T is the slope of tangent such that $m_T = \left[\frac{dy}{dx}\right]_{at(x_1, y_1)}$

- **Equation of Normal at (x_1, y_1) :**

  $(y - y_1) = m_N(x - x_1)$, where m_N is the slope of such that $m_N = \dfrac{-1}{\left[\frac{dy}{dx}\right]_{at(x_1, y_1)}}$

  Note that $m_T \times m_N = -1$, which is obvious because tangent and normal are perpendicular to each other. In other words, the tangent and normal lines are inclined at right angle on each other.

- **Acute angle between the two curves whose slopes m_1 and m_2 are known :**

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 \cdot m_2}\right| \quad \Rightarrow \quad \theta = \tan^{-1}\left|\frac{m_2 - m_1}{1 + m_1 \cdot m_2}\right|$$

  It is absolutely sufficient to find one angle (generally the acute angle) between the two curves. Other angle between the curves is given by $\pi - \theta$.

  Note that if the curves cut orthogonally (*i.e.*, they cut each other at right angles), then it means $m_1 \times m_2 = -1$, where m_1 and m_2 represent the slopes of the tangent of curves at the intersection point.

- **Finding the slope of a line $ax + by + c = 0$:**

  Step 1 : Express the given line in the standard slope-intercept form $y = mx + c$, *i.e.*, $y = \left(-\frac{a}{b}\right)x - \frac{c}{b}$.

  Step 2 : By comparing to the standard form $y = mx + c$, we can conclude $-\frac{a}{b}$ is the slope of given line $ax + by + c = 0$.

Topic-3 Increasing / Decreasing Functions

Concepts covered: Increasing functions, Decreasing functions

Revision Notes

- A function $f(x)$ is said to be an increasing function in $[a, b]$, if as x increases, $f(x)$ also increases, *i.e.*, if $\alpha, \beta \in [a, b]$ and $\alpha > \beta \Rightarrow f(\alpha) > f(\beta)$.

  If $f'(x) \geq 0$ lies in (a, b), then $f(x)$ is an increasing function in $[a, b]$, provided $f(x)$ is continuous at $x = a$ and $x = b$.
- A function $f(x)$ is said to be a decreasing function in $[a, b]$, if as x increase, $f(x)$ decreases, *i.e.*, if $\alpha, \beta \in [a, b]$ and $\alpha > \beta \Rightarrow f(\alpha) < f(\beta)$.

  If $f'(x) \leq 0$ lies in (a, b), then $f(x)$ is a decreasing function in $[a, b]$, provided $f(x)$ is continuous at $x = a$ and $x = b$.

  (i) A function $f(x)$ is a constant function in $[a, b]$, if $f'(x) = 0$ for each $x \in (a, b)$.

  (ii) By monotonic function $f(x)$ in interval I, we mean that f is either only increasing in I or only decreasing in I.
- Finding the intervals of increasing and/or decreasing function:

  Algorithm

  Step 1 : Consider the function $y = f(x)$.

  Step 2 : Find $f'(x)$.

  Step 3 : Put $f'(x) = 0$ and solve to get the critical point(s).

  Step 4 : The value(s) of x for which $f'(x) > 0$, $f(x)$ is increasing and the value(s) of x for which $f'(x) < 0$, $f(x)$ is decreasing.

Mnemonics

Concept: Increasing and Decreasing Function

Mnemonics: Moving Immense Organs Insert Outdoor Extreme Operations

Interpretation:

Interpretation:

Dead → Find **Derivative** of function $f(x)$ *i.e.*, $f'(x)$

Zombies → Put Derivative equal to **zero** *i.e.*, $f'(x) = 0$

Consider → Get **Critical** points *i.e.*, values of x

Green → If $f'(x)$ is **Greater** than zero *i.e.*, $f'(x) > 0$, then $f(x)$ is increasing

Lemons → If $f'(x)$ is **Less** than zero *i.e.*, $f'(x) < 0$, then $f(x)$ is decreasing

Topic-4

Maxima and Minima

Concepts covered: Stationary points, absolute maxima/minima, local maxima/minima, first derivative test and second derivative test and application problems based on maxima/minima.

Revision Notes

- **Understanding maxima and minima :**
  Consider $y = f(x)$ be a well defined function on an interval I, then
  (a) f is said to have a maximum value in I, if there exists a point c in I such that $f(c) > f(x)$, for all $x \in I$
  The value corresponding to $f(c)$ is called the maximum value of f in I and the point c is called the **point of maximum value of f in I**.
  (b) f is said to have a minimum value in I, if there exists a point c in I such that $f(c) < f(x)$, for all $x \in I$.
  The value corresponding to $f(c)$ is called the minimum value of f in I and the point c is called the **point of minimum value of f in I**.
  (c) f is said to have an extreme value in I, if there exists a point c in I such that $f(c)$ is either a maximum value or a minimum value of f in I.
  The value $f(c)$ in this case, is called an extreme value of f in I and the point c called an **extreme point**.
- Let f be a real valued function and also take a point c from its domain. Then
  (i) c is called a point of **local maxima** if there exists a number $h > 0$ such that $f(c) > f(x)$, for all x in $(c - h, c + h)$. The value $f(c)$ is called the **local maximum value of f**.
  (ii) c is called a point of **local minima** if there exists a number $h > 0$ such that $f(c) < f(x)$, for all x in $(c - h, c + h)$. The value $f(c)$ is called the **local minimum value** of f.
- **Critical points :**
  It is a point c (say) in the domain of a function $f(x)$ at which either $f'(x)$ vanishes, *i.e.*, $f'(c) = 0$ or f is not differentiable.
- **First Derivative Test :**
  Consider $y = f(x)$ be a well defined function on an open interval I. Now procedure have been mentioned in the following algorithm :
  Step 1 : Find $\frac{dy}{dx}$.
  Step 2 : Find the critical point(s) or stationary point(s) by putting $\frac{dy}{dx} = 0$. Suppose $c \in I$ (where I is the interval) be any critical point and f be continuous at this point c. Then we may have the following situations :
  - $\frac{dy}{dx}$ changes sign from **positive to negative** as x increases through c, then the function attains a **local maximum** at $x = c$.
  - $\frac{dy}{dx}$ changes sign from **negative to positive** as x increases through c, then the function attains a **local minimum** at $x = c$.
- **Second Derivative Test :**
  Consider $y = f(x)$ be a well defined function on an open interval I and twice differentiable at a point c in the interval. Then we observe that :
  - $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$
    The value $f(c)$ is called the local maximum value of f.
  - $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$
    The value $f(c)$ is called the local minimum value of f.

  This test fails if $f'(c) = 0$ and $f''(c) = 0$. In such a case, we use **first derivative test** as discussed in the above.
- **Absolute maxima and absolute minima :**
  If f is a continuous function on a **closed interval** I, then f has the absolute maximum value and f attains it atleast once in I. Also f has the absolute minimum value and the function attains it atleast once in I.

  Algorithm
  Step 1 : Find all the critical points of f in the given interval, *i.e.*, find all the points x where either $f'(x) = 0$ or f is not differentiable.
  Step 2 : Take the end points of the given interval.
  Step 3 : At all these points (*i.e.*, the points found in Step 1 and Step 2) calculate the values of f.
  Step 4 : Identify the maximum and minimum value of f out of the values calculated in Step 3. This maximum value will be the **absolute maximum value** of f and the minimum value will be the **absolute minimum value** of the function f.

  Absolute maximum value is also called as **global maximum value** or **greatest value**. Similarly, absolute minimum value is called as **global minimum value** or the **least value**.