Binomial Theorem

"Obvious" is the most dangerous word in mathematics......... Bell, Eric Temple

Binomial expression :

Any algebraic expression which contains two dissimilar terms is called binomial expression.

For example : $x + y$, $x^2y + \frac{1}{xy^2}$, $3 - x$, $\sqrt{x^2+1} + \frac{1}{(x^3+1)^{1/3}}$ etc.

Terminology used in binomial theorem :

Factorial notation : $\underline{|n}$ or n! is pronounced as factorial n and is defined as

$$n! = \begin{cases} n(n-1)(n-2)........3 \;.\; 2 \;.\; 1 & ; \text{ if } \; n \in N \\ 1 & ; \text{ if } \; n = 0 \end{cases}$$

Note : $n! = n \,.\, (n-1)!$; $n \in N$

Mathematical meaning of ^nC_r : The term ^nC_r denotes number of combinations of r things choosen from n distinct things mathematically, $^nC_r = \frac{n!}{(n-r)!\; r!}$, n, r $\in$ W, $0 \le r \le n$

Note : Other symbols of of ^nC_r are $\binom{n}{r}$ and C(n, r).

Properties related to ^nC_r :

(i) $^nC_r = {^nC_{n-r}}$

Note : If $^nC_x = {^nC_y}$ $\Rightarrow$ Either $x = y$ or $x + y = n$

(ii) $^nC_r + {^nC_{r-1}} = {^{n+1}C_r}$

(iii) $\frac{^nC_r}{^nC_{r-1}} = \frac{n-r+1}{r}$

(iv) $^nC_r = \frac{n}{r}\, {^{n-1}C_{r-1}} = \frac{n(n-1)}{r(r-1)}\, {^{n-2}C_{r-2}} = ... = \frac{n(n-1)(n-2).........(n-(r-1))}{r\;(r-1)(r-2).......2\;.1}$

(v) If n and r are relatively prime, then ^nC_r is divisible by n. But converse is not necessarily true.

Statement of binomial theorem :

$(a + b)^n = {^nC_0}\, a^nb^0 + {^nC_1}\, a^{n-1}b^1 + {^nC_2}\, a^{n-2}b^2 + ... + {^nC_r}\, a^{n-r}b^r + ... + {^nC_n}\, a^0b^n$

where $n \in N$

or $(a + b)^n = \sum_{r=0}^{n} {^nC_r}\; a^{n-r}b^r$

Note : If we put a = 1 and b = x in the above binomial expansion, then

or $(1 + x)^n = {^nC_0} + {^nC_1}\, x + {^nC_2}\, x^2 + ... + {^nC_r}\, x^r + ... + {^nC_n}\, x^n$

or $(1 + x)^n = \sum_{r=0}^{n} {^nC_r}\; x^r$

Example # 1 : Expand the following binomials :

(i) $(x+\sqrt{2})^5$ (ii) $\left(1-\frac{3x^2}{2}\right)^4$

Solution : (i) $(x+\sqrt{2})^5 = {}^5C_0x^5 + {}^5C_1x^4\left(\sqrt{2}\right) + {}^5C_2x^3\left(\sqrt{2}\right)^2 + {}^5C_3\,x^2\left(\sqrt{2}\right)^3 + {}^5C_4x\left(\sqrt{2}\right)^4 + {}^5C_5\left(\sqrt{2}\right)^5$

$= x^5 + 5\sqrt{2}\,x^4 + 20x^3 + 20\sqrt{2}\,x^2 + 20x + 4\sqrt{2}$

(ii) $\left(1-\frac{3x^2}{2}\right)^4 = {}^4C_0 + {}^4C_1\left(-\frac{3x^2}{2}\right) + {}^4C_2\left(-\frac{3x^2}{2}\right)^2 + {}^4C_3\left(-\frac{3x^2}{2}\right)^3 + {}^4C_4\left(-\frac{3x^2}{2}\right)^4$

$= 1 - 6x^2\ \frac{27}{2} + x^4 - \frac{27}{2}\,x^6 + \frac{81}{16}\,x^8$

Example # 2 : Expand the binomial $\left(\frac{2}{x}+x\right)^{10}$ up to four terms

Solution : $\left(\frac{2}{x}+x\right)^{10} = {}^{10}C_0\left(\frac{2}{x}\right)^{10} + {}^{10}C_1\left(\frac{2}{x}\right)^9 x + {}^{10}C_2\left(\frac{2}{x}\right)^8 x^2 + {}^{10}C_3\left(\frac{2}{x}\right)^7 x^3 + ...$

Self practice problems :

(1) Write the first three terms in the expansion of $\left(2-\frac{y}{3}\right)^6$.

(2) Expand the binomial $\left(\frac{x^2}{3}+\frac{3}{x}\right)^5$.

Ans. (1) $64 - 64y + \frac{80}{3}\,y^2$ (2) $\frac{x^{10}}{243} + \frac{5}{27}\,x^7 + \frac{10}{3}x^4 + 30x + \frac{135}{x^2} + \frac{243}{x^5}$.

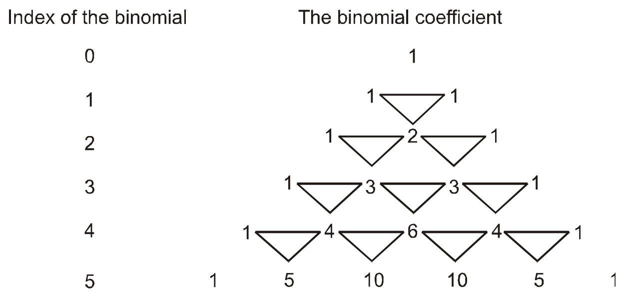
Observations :

(i) The number of terms in the binomial expansion $(a + b)^n$ is $n + 1$.

(ii) The sum of the indices of a and b in each term is n.

(iii) The binomial coefficients (${}^nC_0$, ${}^nC_1$ ....${}^nC_n$) of the terms equidistant from the beginning and the end are equal, i.e. ${}^nC_0 = {}^nC_n$, ${}^nC_1 = {}^nC_{n-1}$ etc. $\{\because\ {}^nC_r = {}^nC_{n-r}\}$

(iv) The binomial coefficient can be remembered with the help of the following pascal's Triangle (also known as Meru Prastra provided by Pingla)



Regarding Pascal's Triangle, we note the following :

(a) Each row of the triangle begins with 1 and ends with 1.

(b) Any entry in a row is the sum of two entries in the preceding row, one on the immediate left and the other on the immediate right.

Example # 3 : The number of dissimilar terms in the expansion of $(1 + x^4 - 2x^2)^{15}$ is

(A) 21 (B) 31 (C) 41 (D) 61

Solution : $(1 - x^2)^{30}$

Therefore number of dissimilar terms = **31.**

General term :

$(x + y)^n = {}^nC_0\, x^n y^0 + {}^nC_1\, x^{n-1} y^1 + \ldots\ldots\ldots + {}^nC_r\, x^{n-r} y^r + \ldots\ldots\ldots + {}^nC_n\, x^0 y^n$

$(r + 1)^{th}$ term is called general term and denoted by T_{r+1}.

$T_{r+1} = {}^nC_r\, x^{n-r} y^r$

Note : The r^{th} term from the end is equal to the $(n - r + 2)^{th}$ term from the begining, i.e. ${}^nC_{n-r+1}\, x^{r-1} y^{n-r+1}$

Example # 4 : Find (i) 15^{th} term of $(2x - 3y)^{20}$ (ii) 4^{th} term of $\left(\frac{3x}{5} - y\right)^7$

Solution : (i) $T_{14+1} = {}^{20}C_{14}\,(2x)^6\,(-3y)^{14} = {}^{20}C_{14}\, 2^6\, 3^{14}\, x^6.y^{14}$

(ii) $T_{3+1} = {}^7C_3 \left(\frac{3x}{5}\right)^4 (-y)^3 = -{}^7C_3 \left(\frac{3}{5}\right)^4 x^4y^3$

Example # 5 : Find the number of rational terms in the expansion of $\left(2^{\frac{1}{3}} + 3^{\frac{1}{5}}\right)^{600}$

Solution : The general term in the expansion of $\left(2^{\frac{1}{3}} + 3^{\frac{1}{5}}\right)^{600}$ is

$T_{r+1} = {}^{600}C_r \left(2^{\frac{1}{3}}\right)^{600-r} \left(3^{\frac{1}{5}}\right)^r = {}^{600}C_r\ 2^{\frac{600-r}{3}}\ 3^{\frac{r}{5}}$

The above term will be rational if exponent of 3 and 2 are integers

It means $\frac{600-r}{3}$ and $\frac{r}{5}$ must be integers.

The possible set of values of r is {0, 15, 30,45............,600}

Hence, number of rational terms is 41

Middle term(s) :

(a) If n is even, there is only one middle term, which is $\left(\frac{n+2}{2}\right)^{th}$ term.

(b) If n is odd, there are two middle terms, which are $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+1}{2}+1\right)^{th}$ terms.

Example # 6 : Find the middle term(s) in the expansion of

(i) $(1 + 2x)^{12}$ (ii) $\left(2y - \frac{y^2}{2}\right)^{11}$

Solution : (i) $(1 + 2x)^{12}$

Here, n is even, therefore middle term is $\left(\frac{12+2}{2}\right)^{th}$ term.

It means T_7 is middle term $T_7 = {}^{12}C_6\,(2x)^6$

(ii) $\left(2y - \frac{y^2}{2}\right)^{11}$

Here, n is odd therefore, middle terms are $\left(\frac{11+1}{2}\right)^{th}$ & $\left(\frac{11+1}{2}+1\right)^{th}$.

It means T_6 & T_7 are middle terms

$T_6 = {}^{11}C_5\,(2y)^6 \left(-\frac{y^2}{2}\right)^5 = -2\ {}^{11}C_5\, y^{16} \Rightarrow T_7 = {}^{11}C_6\,(2y)^5 \left(-\frac{y^2}{2}\right)^6 = \frac{{}^{11}C_6}{2} y^{17}$

Example # 7 : Find term which is independent of x in $\left(x^2 - \frac{1}{x^6}\right)^{16}$

Solution : $T_{r+1} = {}^{16}C_r\,(x^2)^{16-r}\left(-\frac{1}{x^6}\right)^r$

For term to be independent of x, exponent of x should be 0

$32 - 2r = 6r \quad \Rightarrow \quad r = 4 \quad \therefore\ T_5$ is independent of x.

Numerically greatest term in the expansion of $(a + b)^n$, $n \in N$

Binomial expansion of $(a + b)^n$ is as follows **: –**

$(a + b)^n = {}^nC_0\, a^n b^0 + {}^nC_1\, a^{n-1} b^1 + {}^nC_2\, a^{n-2} b^2 + ... + {}^nC_r\, a^{n-r} b^r + ... + {}^nC_n\, a^0 b^n$

If we put certain values of a and b in RHS, then each term of binomial expansion will have certain value. The term having numerically greatest value is said to be numerically greatest term.

Let T_r and T_{r+1} be the r^{th} and $(r + 1)^{th}$ terms respectively

$T_r = {}^nC_{r-1}\; a^{n-(r-1)}\, b^{r-1}$

$T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

Now, $\left|\frac{T_{r+1}}{T_r}\right| = \left|\frac{{}^nC_r}{{}^nC_{r-1}}\,\frac{a^{n-r}\;b^r}{a^{n-r+1}b^{r-1}}\right| = \frac{n-r+1}{r}\,.\left|\frac{b}{a}\right|$

Consider $\left|\frac{T_{r+1}}{T_r}\right| \geq 1$

$\left(\frac{n-r+1}{r}\right)\left|\frac{b}{a}\right| \geq 1 \quad \Rightarrow \quad \frac{n+1}{r} - 1 \geq \left|\frac{a}{b}\right| \quad \Rightarrow r \leq \frac{n+1}{1+\left|\frac{a}{b}\right|}$

Case - I When $\frac{n+1}{1+\left|\frac{a}{b}\right|}$ is an integer (say m), then

(i) $T_{r+1} > T_r$ when $r < m$ $(r = 1, 2, 3 ...., m – 1)$

i.e. $T_2 > T_1$, $T_3 > T_2$, ..., $T_m > T_{m-1}$

(ii) $T_{r+1} = T_r$ when $r = m$

i.e. $T_{m+1} = T_m$

(iii) $T_{r+1} < T_r$ when $r > m$ $(r = m + 1, m + 2, ...n)$

i.e. $T_{m+2} < T_{m+1}$, $T_{m+3} < T_{m+2}$, ...$T_{n+1} < T_n$

Conclusion :

When $\frac{n+1}{1+\left|\frac{a}{b}\right|}$ is an integer, say m, then T_m and T_{m+1} will be numerically greatest terms (both terms are equal in magnitude)

Case - II

When is not an integer (Let its integral part be m), then

(i) $T_{r+1} > T_r$ when $r <$ $(r = 1, 2, 3,........, m–1, m)$

i.e. $T_2 > T_1$, $T_3 > T_2$, ..., $T_{m+1} > T_m$

(ii) $T_{r+1} < T_r$ when $r >$ $(r = m + 1, m + 2, ...n)$

i.e. $T_{m+2} < T_{m+1}$, $T_{m+3} < T_{m+2}$, ..., $T_{n+1} < T_n$

Conclusion :

When is not an integer and its integral part is m, then T_{m+1} will be the numerically greatest term.

Note : (i) In any binomial expansion, the middle term(s) has greatest binomial coefficient.

In the expansion of $(a + b)^n$

If n	No. of greatest binomial coefficient	Greatest binomial coefficient
Even	1	${}^nC_{n/2}$
Odd	2	${}^nC_{(n-1)/2}$ and ${}^nC_{(n+1)/2}$

(Values of both these coefficients are equal)

(ii) In order to obtain the term having numerically greatest coefficient, put a = b = 1, and proceed as discussed above.

Example # 8 : Find the numerically greatest term in the expansion of $(7 - 3x)^{25}$ when $x = \frac{1}{3}$.

Solution : $m = \dfrac{n+1}{1+\left|\frac{a}{b}\right|} = \dfrac{25+1}{1+\left|\frac{7}{-1}\right|} = \dfrac{26}{8}$

$[m] = 3$ ([m] denotes GIF)

$\therefore$ T_4 is numerically greatest term

Self practice problems :

(3) Find the term independent of x in $\left(x^2 - \frac{3}{x}\right)^9$

(4) The sum of all rational terms in the expansion of $(3^{1/7} + 5^{1/2})^{14}$ is

(A) 3^2 (B) $3^2 + 5^7$ (C) $3^7 + 5^2$ (D) 5^7

(5) Find the coefficient of x^{-2} in $(1 + x^2 + x^4)\left(1-\frac{1}{x^2}\right)^{18}$

(6) Find the middle term(s) in the expansion of $(1 + 3x + 3x^2 + x^3)^{2n}$

(7) Find the numerically greatest term in the expansion of $(2 + 5x)^{21}$ when $x = \frac{2}{5}$.

Ans. (3) 28.3^7 (4) B (5) –681

(6) ${}^{6n}C_{3n} \cdot x^{3n}$ (7) $T_{11} = T_{12} = {}^{21}C_{10}\ 2^{21}$

Example # 9 : Show that $7^n + 5$ is divisible by 6, where n is a positive integer.

Solution : $7^n + 5 = (1 + 6)^n + 5 = {}^nC_0 + {}^nC_1 \cdot 6 + {}^nC_2 \cdot 6^2 + ... + {}^nC_n\, 6^n + 5.$

$= 6.\ C_1 + 6^2.\ C_2 + ... + C_n \cdot 6^n + 6.$

$= 6\lambda$, where λ is a positive integer

Hence, $7^n + 5$ is divisible by 6.

Example # 10 : What is the remainder when 7^{81} is divided by 5.

Solution : $7^{81} = 7.7^{80} = 7.\ (49)^{40} = 7\ (50 - 1)^{40}$

$= 7\ [{}^{40}C_0\ (50)^{40} - {}^{40}C_1\ (50)^{39} + ... - {}^{40}C_{39}\ (50)^1 + {}^{40}C_{40}\ (50)^0]$

$= 5(k) + 7$(where k is a positive integer) $= 5\ (k + 1) + 2$

Hence, remainder is 2.

Example # 11 : Find the last digit of the number $(13)^{12}$.

Solution : $(13)^{12} = (169)^6 = (170 - 1)^6$

$= {}^6C_0\,(170)^6 - {}^6C_1\,(170)^5 + \ldots\ldots.. - {}^6C_5\,(170)^1 + {}^6C_6\,(170)^0$

Hence, last digit is 1

Note : We can also conclude that last three digits are 481.

Example-12 : Which number is larger $(1.1)^{100000}$ or 10,000 ?

Solution : By Binomial Theorem

$(1.1)^{100000} = (1 + 0.1)^{100000} = 1 + {}^{100000}C_1\,(0.1)$ + other positive terms

$= 1 + 100000 \times 0.1$ + other positive terms

$= 1 + 10000$ + other positive terms

Hence $(1.1)^{100000} > 10,000$

Self practice problems :

(8) If n is a positive integer, then show that $6^n - 5n - 1$ is divisible by 25.

(9) What is the remainder when 3^{257} is divided by 80 .

(10) Find the last digit, last two digits and last three digits of the number $(81)^{25}$.

(11) Which number is larger $(1.3)^{2000}$ or 600

Ans. (9) 3 (10) 1, 01, 001 (11) $(1.3)^{2000}$.

Some standard expansions :

(i) Consider the expansion

$$(x + y)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} y^r = {}^nC_0\, x^n y^0 + {}^nC_1\, x^{n-1} y^1 + \ldots\ldots\ldots.. + {}^nC_r\, x^{n-r} y^r + \ldots\ldots\ldots. + {}^nC_n\, x^0 y^n \quad \ldots.(i)$$

(ii) Now replace $y \to -y$ we get

$$(x - y)^n = \sum_{r=0}^{n} {}^nC_r\, (-1)^r x^{n-r} y^r = {}^nC_0\, x^n y^0 - {}^nC_1\, x^{n-1} y^1 + \ldots + {}^nC_r\, (-1)^r x^{n-r} y^r + \ldots + {}^nC_n\, (-1)^n x^0 y^n \quad \ldots.(ii)$$

(iii) Adding (i) & (ii), we get

$(x + y)^n + (x - y)^n = 2[{}^nC_0\, x^n y^0 + {}^nC_2\, x^{n-2}\, y^2 + \ldots\ldots...]$

(iv) Subtracting (ii) from (i), we get

$(x + y)^n - (x - y)^n = 2[{}^nC_1\, x^{n-1} y^1 + {}^nC_3\, x^{n-3}\, y^3 + \ldots\ldots...]$

Properties of binomial coefficients :

$(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots... + C_rx^r + \ldots\ldots.... + C_nx^n \quad \ldots...(1)$

where C_r denotes ${}^nC_r$

(1) The sum of the binomial coefficients in the expansion of $(1 + x)^n$ is 2^n

Putting x = 1 in (1)

$${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots\ldots.. + {}^nC_n = 2^n \quad \ldots...(2)$$

or $$\sum_{r=0}^{n} {}^nC_r = 2^n$$

(2) Again putting x = –1 in (1), we get

$${}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \ldots\ldots\ldots... + (-1)^n\, {}^nC_n = 0 \quad \ldots...(3)$$

or $$\sum_{r=0}^{n} (-1)^r\, {}^nC_r = 0$$

(3) The sum of the binomial coefficients at odd position is equal to the sum of the binomial coefficients at even position and each is equal to 2^{n-1}.
from (2) and (3)

$${}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots\ldots\ldots\ldots\ldots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots\ldots\ldots\ldots\ldots = 2^{n-1}$$

(4) Sum of two consecutive binomial coefficients
$${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$$

L.H.S. $= {}^nC_r + {}^nC_{r-1} = \dfrac{n!}{(n-r)!\ r!} + \dfrac{n!}{(n-r+1)!\ (r-1)!}$

$$= \frac{n!}{(n-r)!\ (r-1)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right]$$

$$= \frac{n!}{(n-r)!\ (r-1)!}\ \frac{(n+1)}{r(n-r+1)}$$

$$= \frac{(n+1)!}{(n-r+1)!\ r!} = {}^{n+1}C_r = \text{R.H.S.}$$

(5) Ratio of two consecutive binomial coefficients

$$\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$$

(6) $${}^nC_r = \frac{n}{r}\ {}^{n-1}C_{r-1} = \frac{n(n-1)}{r(r-1)}\ {}^{n-2}C_{r-2} = \ldots\ldots\ldots\ldots = \frac{n(n-1)(n-2)\ldots\ldots\ldots(n-(r-1))}{r\ (r-1)(r-2)\ldots\ldots.2\ .1}$$

Example # 13 : If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots\ldots\ldots. + c_nx^n$, then show that

(i) $C_0 + 4C_1 + 4^2C_2 + \ldots\ldots\ldots. + 4^n\ C_n = 5^n$. (ii) $3C_0 + 5C_1 + 7.\ C_2 + \ldots\ldots.. + (2n + 3)\ C_n = 2^n(n + 3)$.

(iii) $$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \frac{C_3}{4} + \ldots\ldots\ldots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$$

Solution : (i) $(1 + x)^n = C_0 + C_1\ x + C_2x^2 + \ldots\ldots\ldots.. + C_nx^n$

put x = 4

$C_0 + 4C_1 + 4^2C_2 + \ldots\ldots\ldots. + 4^n\ C_n = 5^n$.

(ii) L.H.S. $= 3C_0 + 5C_1 + 7.\ C_2 + \ldots\ldots.. + (2n + 3)\ C_n$

$$= \sum_{r=0}^{n}(2r+3)\ .\ {}^nC_r = 2\sum_{r=0}^{n} r.\ {}^nC_r + 3\sum_{r=0}^{n}{}^nC_r$$

$$= 2n\sum_{r=1}^{n}{}^{n-1}C_{r-1} + 3\sum_{r=0}^{n}{}^nC_r = 2n\ .\ 2^{n-1} + 3.\ 2^n = 2^n(n + 3)\ \text{ RHS}$$

(iii) I **<u>Method : By Summation</u>**

L.H.S. $= C_0 + \dfrac{C_1}{2} + \dfrac{C_2}{3} + \dfrac{C_3}{4} + \ldots\ldots\ldots + \dfrac{C_n}{n+1}$

$$= \sum_{r=0}^{n} .\ \frac{{}^nC_r}{r+1} = \frac{1}{n+1}\sum_{r=0}^{n} .\ {}^{n+1}C_{r+1} \qquad \left\{\frac{n+1}{r+1}\ .\ {}^nC_r = {}^{n+1}C_{r+1}\right\} = \frac{2^{n+1}-1}{n+1}\ \text{ R.H.S.}$$

II **<u>Method : By Integration</u>**

$(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_n\ x^n$. Integrating both sides, within the limits 0 to 1.

$$\left[\frac{(1+x)^{n+1}}{n+1}\right]_0^1 = \left[C_0x + C_1\frac{x^2}{2} + C_2\frac{x^3}{3} + \ldots.. + C_n\frac{x^{n+1}}{n+1}\right]_0^1$$

$$\frac{2^{n+1}}{n+1} - \frac{1}{n+1} = \left(C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots.. + \frac{C_n}{n+1}\right) - 0$$

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \frac{C_3}{4} + \ldots\ldots\ldots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}\ \text{ Proved}$$

Example # 14 : If $(1 + x)^n = C_0 + C_1x + C_2x^2 + ...+ C_nx^n$, then prove that

(i) $C_0C_1 + C_1C_2 + C_2C_3 + ... + C_{n-1}\, C_n = {}^{2n}C_{n-1}$ or ${}^{2n}C_{n+1}$

(ii) $1^2.\ C_1^{\,2} + 2^2\ .\ C_2^{\,2} + 3^2.\ C_3^{\,2} + ... + n^2\,C_n^{\,2}\ . = n^2.\ {}^{2n-2}C_{n-1}$

Solution : (i) $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_n\, x^n.$ ....(i)

$(x + 1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + ... + C_n\, x^0$ ....(ii)

Multiplying (i) and (ii)

$(C_0 + C_1x + C_2x^2 + ... + C_nx^n)\ (C_0x^n + C_1x^{n-1} + ... + C_nx^0) = (1 + x)^{2n}$

Comparing coefficient of x^{n-1},

$C_0C_1 + C_1C_2 + C_2C_3 + ... + C_{n-1}\, C_n = {}^{2n}C_{n-1}$ or ${}^{2n}C_{n+1}$

(ii) $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_n\, x^n.$ ....(i)

differentiating w.r.t x......

$n(1 + x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + ... + nC_n\, x^{n-1}.$

multiplying by x.......

$n\, x(1 + x)^{n-1} = C_1x + 2C_2x^2 + 3C_3x^3 + ... + nC_n\, x^n$

Now differentiate w.r.t. x.....

$n(1 + x)^{n-1} + n\,(n-1)x.(1+x)^{n-2} = 1^2C_1 + 2^2C_2x + 3^2C_3x^2 + ... + n^2C_nx^{n-1}$ ....(ii)

$(x + 1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + ... + C_n\, x^0$ ....(iii)

multiplying (ii) & (iii) and comparing the cofficient of x^{n-1}

$1^2.\ C_1^{\,2} + 2^2\ .\ C_2^{\,2} + 3^2.\ C_3^{\,2} + ... + n^2\,C_n^{\,2}. = n\left({}^{2n-1}C_{n-1} - {}^{2n-2}C_{n-2}\right) + n^2\ {}^{2n-2}C_{n-2}$

$= n^2\ {}^{2n-2}C_{n-1}$ = R.H.S.

Example # 15 : Find the summation of the following series –

(i) ${}^mC_0 + {}^{m+1}C_1 + {}^{m+2}C_2 + ... + {}^nC_m$ (ii) ${}^nC_3 + 2\ .\ {}^{n+1}C_3 + 3.\ {}^{n+2}C_3 + ... + n\ .\ {}^{2n-1}C_3$

Solution : (i) I **Method :** Using property, ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

${}^mC_0 + {}^{m+1}C_1 + {}^{m+2}C_2 + ... + {}^nC_m$

${}^mC_m + {}^{m+1}C_m + {}^{m+2}C_m + ... + {}^nC_m$

$= \underbrace{{}^{m+1}C_{m+1} + {}^{m+1}C_m} + {}^{m+2}C_m + ... + {}^nC_m$ $\{\because {}^mC_m = {}^{m+1}C_{m+1}\}$

$= \underbrace{{}^{m+2}C_{m+1} + {}^{m+2}C_m} + ... + {}^nC_m = {}^{m+3}C_{m+1} + ... + {}^nC_m = {}^nC_{m+1} + {}^nC_m = \mathbf{{}^{n+1}C_{m+1}}$

II **Method**

${}^mC_m + {}^{m+1}C_m + {}^{m+2}C_m + ... + {}^nC_m$

The above series can be obtained by writing the coefficient of x^m in

$(1 + x)^m + (1 + x)^{m+1} + ... + (1 + x)^n$

Let $S = (1 + x)^m + (1 + x)^{m+1} + ... + (1 + x)^n$

$$= \frac{(1+x)^m\left[(1+x)^{n-m+1} - 1\right]}{x} = \frac{(1+x)^{n+1} - (1+x)^m}{x}$$

$$= \text{coefficient of } x^m \text{ in } \frac{(1+x)^{n+1}}{x} - \frac{(1+x)^m}{x} = {}^{n+1}C_{m+1} + 0 = {}^{n+1}C_{m+1}$$

(ii) ${}^{n}C_3 + 2 \cdot {}^{n+1}C_3 + 3 \cdot {}^{n+2}C_3 + ... + n \cdot {}^{2n-1}C_3$

The above series can be obatined by writing the coefficient of x^3 in

$(1 + x)^n + 2 \cdot (1 + x)^{n+1} + 3 \cdot (1 + x)^{n+2} + ... + n \cdot (1 + x)^{2n-1}$

Let $S = (1 + x)^n + 2 \cdot (1 + x)^{n+1} + 3 \cdot (1 + x)^{n+2} + ... + n (1 + x)^{2n-1}$ .....(i)

$(1 + x)S = (1 + x)^{n+1} + 2 (1 + x)^{n+2} + ... + (n - 1) (1 + x)^{2n-1} + n(1 + x)^{2n}$ ....(ii)

Subtracting (ii) from (i)

$-xS = (1 + x)^n + (1 + x)^{n+1} + (1 + x)^{n+2} + ... + (1 + x)^{2n-1} - n(1 + x)^{2n}$

$$= \frac{(1+x)^n\left[(1+x)^n - 1\right]}{x} - n(1+x)^{2n}$$

$$S = \frac{-(1+x)^{2n} + (1+x)^n}{x^2} + \frac{n(1+x)^{2n}}{x}$$

$x^3 : S$ (coefficient of x^3 in S)

$$x^3 : \frac{-(1+x)^{2n} + (1+x)^n}{x^2} + \frac{n(1+x)^{2n}}{x}$$

Hence, required summation of the series is $-{}^{2n}C_5 + {}^{n}C_5 + n \cdot {}^{2n}C_4$

Example # 16 : Prove that $C_1 - C_3 + C_5 - .... = 2^{n/2} \sin \frac{n\pi}{4}$.

Solution : Consider the expansion $(1 + x)^n = C_0 + C_1 x + C_2 x^2 + ... + C_n x^n$ ....(i)

putting $x = -i$ in (i) we get

$(1 - i)^n = C_0 - C_1 i - C_2 + C_3 i + C_4 + ...\ (-1)^n C_n i^n$

or $2^{n/2}\left[\cos\left(-\frac{n\pi}{4}\right) + i \sin\left(-\frac{n\pi}{4}\right)\right] = (C_0 - C_2 + C_4 - ....) - i (C_1 - C_3 + C_5 - .....)$ ....(ii)

Equating the imaginary part in (ii) we get $C_1 - C_3 + C_5 - .... = 2^{n/2} \sin \frac{n\pi}{4}$.

Self practice problems :

(12) Prove the following

(i) $5C_0 + 7C_1 + 9C_2 + ... + (2n + 5) C_n = 2^n (n + 5)$

(ii) $4C_0 + \frac{4^2}{2} \cdot C_1 + \frac{4^3}{3} C_2 + ... + \frac{4^{n+1}}{n+1} C_n = \frac{5^{n+1} - 1}{n+1}$

(iii) ${}^{n}C_0 \cdot {}^{n+1}C_n + {}^{n}C_1 \cdot {}^{n}C_{n-1} + {}^{n}C_2 \cdot {}^{n-1}C_{n-2} + ... + {}^{n}C_n \cdot {}^{1}C_0 = 2^{n-1} (n + 2)$

(iv) ${}^{2}C_2 + {}^{3}C_2 + ... + {}^{n}C_2 = {}^{n+1}C_3$

Binomial theorem for negative and fractional indices :

If $n \in R$, then $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2\,!} x^2 + \frac{n(n-1)(n-2)}{3\,!} x^3 + ...$

$................. + \frac{n(n-1)(n-2).......(n-r+1)}{r\,!} x^r + ... \infty.$

Remarks

(i) The above expansion is valid for any rational number other than a whole number if $| x | < 1$.

(ii) When the index is a negative integer or a fraction then number of terms in the expansion of $(1 + x)^n$ is infinite, and the symbol ${}^{n}C_r$ cannot be used to denote the coefficient of the general term.

(iii) The first term must be unity in the expansion, when index 'n' is a negative integer or fraction

$$(x+y)^n = \begin{cases} x^n\left(1+\frac{y}{x}\right)^n = x^n\left\{1+n\ .\ \frac{y}{x}+\frac{n\ (n-1)}{2\ !}\left(\frac{y}{x}\right)^2+.....\right\} & \text{if } \left|\frac{y}{x}\right|<1 \\ y^n\left(1+\frac{x}{y}\right)^n = y^n\left\{1+n\ .\ \frac{x}{y}+\frac{n\ (n-1)}{2\ !}\left(\frac{x}{y}\right)^2+.....\right\} & \text{if } \left|\frac{x}{y}\right|<1 \end{cases}$$

(iv) The general term in the expansion of $(1 + x)^n$ is $T_{r+1} = \frac{n(n-1)(n-2).........(n-r+1)}{r\ !}\ x^r$

(v) When 'n' is any rational number other than whole number then approximate value of $(1 + x)^n$ is $1 + nx$ (x^2 and higher powers of x can be neglected)

(vi) Expansions to be remembered ($|x| < 1$)
(a) $(1 + x)^{-1} = 1 - x + x^2 - x^3 + ... + (-1)^r x^r + ...\infty$
(b) $(1 - x)^{-1} = 1 + x + x^2 + x^3 + ... + x^r + ...\infty$
(c) $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + ... + (-1)^r (r + 1) x^r + ...\infty$
(d) $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + ... + (r + 1)x^r + ... \infty$

Example # 17 : Prove that the coefficient of x^r in $(1 - x)^{-n}$ is ${}^{n+r-1}C_r$

Solution: $(r + 1)^{th}$ term in the expansion of $(1 - x)^{-n}$ can be written as

$$T_{r+1} = \frac{-n(-n-1)(-n-2)......(-n-r+1)}{r\ !}\ (-x)^r$$

$$= (-1)^r\ \frac{n(n+1)(n+2)......(n+r-1)}{r\ !}\ (-x)^r = \frac{n(n+1)(n+2)......(n+r-1)}{r\ !}\ x^r$$

$$= \frac{(n-1)!\ \ n(n+1)......(n+r-1)}{(n-1)\ !\ r\ !}\ x^r$$ Hence, coefficient of x^r is $\frac{(n+r-1)!}{(n-1)!\ \ r!} = {}^{n+r-1}C_r$ Proved

Example-18 : If x is so small such that its square and higher powers may be neglected, then find the value of $\frac{(1-2x)^{1/3}+(1+5x)^{-3/2}}{(9+x)^{1/2}}$

Solution :

$$\frac{(1-2x)^{1/3}+(1+5x)^{-3/2}}{(9+x)^{1/2}} = \frac{1-\frac{2}{3}x+1-\frac{15x}{2}}{3\left(1+\frac{x}{9}\right)^{1/2}} = \frac{1}{3}\left(2-\frac{49}{6}x\right)\left(1+\frac{x}{9}\right)^{-1/2}$$

$$= \frac{1}{3}\left(2-\frac{49}{6}x\right)\ \left(1-\frac{x}{18}\right) = \frac{1}{2}\left(2-\frac{x}{9}-\frac{49}{6}x\right) = 1-\ \frac{x}{18}-\frac{49}{12}\ x = 1-\frac{149}{36}x$$

Self practice problems :

(13) Find the possible set of values of x for which expansion of $(3 - 2x)^{1/2}$ is valid in ascending powers of x.

(14) If $y = \frac{2}{5} + \frac{1.3}{2\ !}\left(\frac{2}{5}\right)^2 + \frac{1.3.5}{3\ !}\left(\frac{2}{5}\right)^3$ + ..., then find the value of $y^2 + 2y$

(15) The coefficient of x^{50} in $\frac{2-3x}{(1-x)^3}$ is

(A) 500 (B) 1000 (C) –1173 (D) 1173

Ans. (13) $x \in \left(-\frac{3}{2}, \frac{3}{2}\right)$ (14) 4 (15) C

Multinomial theorem : As we know the Binomial Theorem $(x + y)^n = \sum_{r=0}^{n} {}^nC_r \; x^{n-r} y^r = \sum_{r=0}^{n} \frac{n!}{(n-r)!\; r!} x^{n-r} y^r$

putting $n - r = r_1$, $r = r_2$ therefore, $(x + y)^n = \sum_{r_1+r_2=n} \frac{n!}{r_1!\; r_2!} x^{r_1} \cdot y^{r_2}$

Total number of terms in the expansion of $(x + y)^n$ is equal to number of non-negative integral solution of $r_1 + r_2 = n$ i.e. ${}^{n+2-1}C_{2-1} = {}^{n+1}C_1 = n + 1$

In the same fashion we can write the multinomial theorem

$$(x_1 + x_2 + x_3 + \ldots\ldots\ldots\ldots x_k)^n = \sum_{r_1+r_2+\ldots+r_k=n} \frac{n!}{r_1!\; r_2!\ldots\; r_k!} x_1^{r_1} \cdot x_2^{r_2} \ldots x_k^{r_k}$$

Here total number of terms in the expansion of $(x_1 + x_2 + \ldots\ldots\ldots + x_k)^n$ is equal to number of non-negative integral solution of $r_1 + r_2 + \ldots\ldots\ldots + r_k = n$ i.e. ${}^{n+k-1}C_{k-1}$

Example # 19 : Find the coefficient of $a^2 b^3 c^4 d$ in the expansion of $(a - b - c + d)^{10}$

Solution : $(a - b - c + d)^{10} = \sum_{r_1+r_2+r_3+r_4=10} \frac{(10)!}{r_1!\; r_2!\, r_3!\; r_4!} (a)^{r_1} \; (-b)^{r_2} \; (-c)^{r_3} \; (d)^{r_4}$

we want to get $a^2 b^3 c^4 d$ this implies that $r_1 = 2, r_2 = 3, r_3 = 4, r_4 = 1$

$\therefore$ coeff. of $a^2 b^3 c^4 d$ is $\frac{(10)!}{2!\; 3!\; 4!\; 1!} (-1)^3 (-1)^4 = -12600$

Example # 20 : In the expansion of $\left(1 + x + \frac{7}{x}\right)^{11}$, find the term independent of x.

Solution : $\left(1 + x + \frac{7}{x}\right)^{11} = \sum_{r_1+r_2+r_3=11} \frac{(11)!}{r_1!\; r_2!\; r_3!} (1)^{r_1} \; (x)^{r_2} \; \left(\frac{7}{x}\right)^{r_3}$

The exponent 11 is to be divided among the base variables 1, x and $\frac{7}{x}$ in such a way so that we get x^0. Therefore, possible set of values of (r_1, r_2, r_3) are (11, 0, 0), (9, 1, 1), (7, 2, 2), (5, 3, 3), (3, 4, 4), (1, 5, 5)

Hence the required term is

$$\frac{(11)!}{(11)!}(7^0) + \frac{(11)!}{9!\; 1\,!1\,!} 7^1 + \frac{(11)!}{7!\; 2\,!\; 2\,!} 7^2 + \frac{(11)!}{5!\; 3\,!\; 3\,!} 7^3 + \frac{(11)!}{3!\; 4\,!\; 4\,!} 7^4 + \frac{(11)!}{1\,!\; 5\,!\; 5\,!} 7^5$$

$$= 1 + \frac{(11)!}{9\,!\; 2\,!} \cdot \frac{2\,!}{1\,!\; 1\,!} 7^1 + \frac{(11)!}{7\,!\; 4\,!} \cdot \frac{4\,!}{2\,!\; 2\,!} 7^2 + \frac{(11)\,!}{5\,!\; 6\,!} \cdot \frac{6\,!}{3\,!\; 3\,!} 7^3$$

$$+ \frac{(11)\,!}{3\,!\; 8\,!} \cdot \frac{8\,!}{4\,!\; 4\,!} 7^4 + \frac{(11)\,!}{1\,!\; 10\,!} \cdot \frac{(10)\,!}{5\,!\; 5\,!} 7^5$$

$$= 1 + {}^{11}C_2 \cdot {}^2C_1 \cdot 7^1 + {}^{11}C_4 \cdot {}^4C_2 \cdot 7^2 + {}^{11}C_6 \cdot {}^6C_3 \cdot 7^3 + {}^{11}C_8 \cdot {}^8C_4 \cdot 7^4 + {}^{11}C_{10} \cdot {}^{10}C_5 \cdot 7^5 = 1 + \sum_{r=1}^{5} {}^{11}C_{2r} \cdot {}^{2r}C_r \cdot 7^r$$

Self practice problems :

(16) The number of terms in the expansion of $(a + b + c + d + e)^n$ is
(A) ${}^{n+4}C_4$ (B) ${}^{n+3}C_n$ (C) ${}^{n+5}C_n$ (D) $n + 1$

(17) Find the coefficient of $x^2 y^3 z^1$ in the expansion of $(x - 2y - 3z)^7$

(18) Find the coefficient of x^{17} in $(2x^2 - x - 3)^9$

Ans. (16) A (17) $\frac{7!}{2!\; 3!\; 1!}$ 24 (18) 2304

Exercise-1

✍ **Marked questions are recommended for Revision.**
✍ **चिन्हित प्रश्न दोहराने योग्य प्रश्न है।**

PART - I : SUBJECTIVE QUESTIONS

भाग - I : विषयात्मक प्रश्न (SUBJECTIVE QUESTIONS)

Section (A) : General Term & Coefficient of x^k in $(ax+b)^n$
खण्ड (A) : व्यापक पद एवं $(ax+b)^n$ में x^k का गुणांक

A-1. Expand the following :
निम्न का प्रसार करो :

(i) $\left(\frac{2}{x}-\frac{x}{2}\right)^5$, $(x \neq 0)$ (ii) $\left(y^2+\frac{2}{y}\right)^4$, $(y \neq 0)$

Ans. (i) $\left(\frac{2}{x}\right)^5 - 5\left(\frac{2}{x}\right)^3 + 10\left(\frac{2}{x}\right) - 10\left(\frac{x}{2}\right) + 5\left(\frac{x}{2}\right)^3 - \left(\frac{x}{2}\right)^5$

(ii) $y^8 + 8y^5 + 24y^2 + \frac{32}{y} + \frac{16}{y^4}$

Sol. **(i)** $\left(\frac{2^5}{x}\right) - {}^5C_1\left(\frac{2}{x}\right)^4\left(\frac{x}{2}\right)^1 + {}^5C_2\left(\frac{2}{x}\right)^3\left(\frac{x}{2}\right)^2 - {}^5C_3\left(\frac{2}{x}\right)^2\left(\frac{x}{2}\right)^3 + {}^5C_4\left(\frac{2}{x}\right)^1\left(\frac{x}{2}\right)^4 - {}^5C_5\left(\frac{x}{2}\right)^5$

$= \left(\frac{2}{x}\right)^5 - 5\left(\frac{2}{x}\right)^3 + 10\left(\frac{2}{x}\right) - 10\left(\frac{x}{2}\right) + 5\left(\frac{x}{2}\right)^3 - \left(\frac{x}{2}\right)^5$

(ii) $(y^2)^4 + {}^4C_1(y^2)^3(2/y) + {}^4C_2(y^2)^2(2/y)^2 + {}^4C_3(y^2)(2/y)^3 + {}^4C_4(2/y)^4$ **=** $y^8 + 8y^5 + 24y^2 + \frac{32}{y} + \frac{16}{y^4}$

A-2. In the binomial expansion of $\left(\sqrt[3]{2}+\frac{1}{\sqrt[3]{3}}\right)^n$, the ratio of the 7th term from the begining to the 7th term from the end is 1 : 6 ; find n.

$\left(\sqrt[3]{2}+\frac{1}{\sqrt[3]{3}}\right)^n$ के प्रसार में प्रारम्भ से 7 वें पद और अंत से 7 वें पद का अनुपात 1 : 6 है, तब n का मान ज्ञात करो।

Ans. n = 9

Sol. 7th term from beginning $T_7 = {}^nC_6(2)^{\frac{n-6}{3}}\left(\frac{1}{3}\right)^2$

7th term from the end $T_{n-5} = {}^nC_{n-6}(2)^2\left(\frac{1}{3}\right)^{\frac{n-6}{3}}$

$$\frac{T_7}{T_{n-5}} = \frac{1}{6} = \frac{2^{\left(\frac{n-6}{3}-2\right)}}{\left(\frac{1}{3}\right)^{\left(\frac{n-6}{3}-2\right)}} \Rightarrow \frac{1}{6} = (6)^{\frac{n-12}{3}} \Rightarrow \frac{n-12}{3} = -1 \Rightarrow n = 9$$

Hindi प्रारम्भ से 7वाँ पद $T_7 = {}^nC_6(2)^{\frac{n-6}{3}}\left(\frac{1}{3}\right)^2$

अन्त से 7वाँ पद $T_{n-5} = {}^nC_{n-6}(2)^2\left(\frac{1}{3}\right)^{\frac{n-6}{3}}$

$$\frac{T_7}{T_{n-5}} = \frac{1}{6} = \frac{2^{\left(\frac{n-6}{3}-2\right)}}{\left(\frac{1}{3}\right)^{\left(\frac{n-6}{3}-2\right)}} \Rightarrow \frac{1}{6} = (6)^{\frac{n-12}{3}} \quad \Rightarrow \quad \frac{n-12}{3} = -1 \Rightarrow n = 9$$

A-3. Find the degree of the polynomial $\left(x+(x^3-1)^{½}\right)^5 + \left(x-(x^3-1)^{½}\right)^5$.

बहुपद $\left(x+(x^3-1)^{½}\right)^5 + \left(x-(x^3-1)^{½}\right)^5$. की घात ज्ञात कीजिए।

Ans. 7

Sol. $= 2[x^5 + {}^5C_2 . x^3(x^3-1)^{1/2})^2 + {}^5C_4. x ((x^3-1)^{1/2})^4]$

$= 2[x^5 + 10x^3(x^3-1) + {}^5C_4.x (x^3-1)^2]$

$= 2[x^5 + 10x^6 - 10x^3 + 5x (x^6 - 2x^3 + 1)] \Rightarrow$ degree is 7 (घात 7 है)

A-4. Find the coefficient of

(i) x^6y^3 in $(x + y)^9$ (ii) $a^5 b^7$ in $(a - 2b)^{12}$

गुणांक का मान ज्ञात करो–

(i) $(x + y)^9$ में x^6y^3 का (ii) $(a - 2b)^{12}$ में $a^5 b^7$ का

Ans. (i) ${}^9C_3$ (ii) $-2^7 . {}^{12}C_7$

Sol. **(i)** $(x + y)^9 = \sum_{r=0}^{9} {}^9C_r x^{9-r} y^r$ $\therefore$ co-efficient of $x^6 y^3 = {}^9C_3$

(ii) $(a - 2b)^{12} = \sum_{r=0}^{12} {}^{12}C_r a^{12-r} (-2b)^r$ $\therefore$ Co-efficient of $a^5 b^7 = {}^{12}C_7 (-2)^7$

Hindi. **(i)** $(x + y)^9 = \sum_{r=0}^{9} {}^9C_r x^{9-r} y^r$ $\therefore$ $x^6 y^3$ का गुणांक $= {}^9C_3$

(ii) $(a - 2b)^{12} = \sum_{r=0}^{12} {}^{12}C_r a^{12-r} (-2b)^r$ $\therefore$ $a^5 b^7$ का गुणांक $= {}^{12}C_7 (-2)^7$

A-5. Find the co-efficient of x^7 in $\left(ax^2 + \frac{1}{b\ x}\right)^{11}$ and of x^{-7} in $\left(ax - \frac{1}{b\ x^2}\right)^{11}$ and find the relation between 'a' & 'b' so that these co-efficients are equal. (where a, b $\neq$ 0).

$\left(ax^2 + \frac{1}{b\ x}\right)^{11}$ के प्रसार में x^7 का गुणांक और $\left(ax - \frac{1}{b\ x^2}\right)^{11}$ के प्रसार में x^{-7} का गुणांक ज्ञात करो। यदि ये गुणांक परस्पर बराबर हो, तो 'a' एवं 'b' के बीच सम्बन्ध ज्ञात करो(जहाँ a, b $\neq$ 0)

Ans. ${}^{11}C_5 \frac{a^6}{b^5}$, ${}^{11}C_6 \frac{a^5}{b^6}$, $a b = 1$

Sol. Co-efficient of x^7 in $\left(ax^2 + \frac{1}{bx}\right)^{11}$

General Term $= {}^{11}C_r (ax^2)^{11-r} \left(\frac{1}{bx}\right)^r = {}^{11}C_r a^{11-r} b^{-r} x^{22-3r}$

Put $22 - 3r = 7$

$r = 5$ $\quad\therefore\quad$ Co-efficient of $x^7 = {}^{11}C_5\, a^6 . b^{-5}$

Co-efficient of x^{-7} in $\left(ax - \frac{1}{bx^2}\right)^{11}$

General Term = ${}^{11}C_r\, (ax)^{11-r}\left(-\frac{1}{bx^2}\right)^r = {}^{11}C_r\, a^{11-r} (b)^{-r} (-1)^r x^{11-3r}$

Put $11 - 3r = -7 \Rightarrow r = 6$ $\quad\therefore\quad$ Co-efficient of $x^{-7} = {}^{11}C_6\, a^5 b^{-6}$

Given that ${}^{11}C_5\, a^6 b^{-5} = {}^{11}C_6\, a^5 b^{-6} \Rightarrow ab = 1$

Hindi. $\left(ax^2 + \frac{1}{bx}\right)^{11}$ के प्रसार में x^7 का गुणांक

व्यापक पद $= {}^{11}C_r\, (ax^2)^{11-r}\left(\frac{1}{bx}\right)^r = {}^{11}C_r\, a^{11-r} b^{-r} x^{22-3r}$

$22 - 3r = 7$ रखने पर

$r = 5$ $\quad\therefore\quad$ x^7 का गुणांक $= {}^{11}C_5\, a^6 . b^{-5}$

$\left(ax - \frac{1}{bx^2}\right)^{11}$ के प्रसार में x^{-7} का गुणांक

व्यापक पद $= {}^{11}C_r\, (ax)^{11-r}\left(-\frac{1}{bx^2}\right)^r = {}^{11}C_r\, a^{11-r} (b)^{-r} (-1)^r x^{11-3r}$

$11 - 3r = -7$ रखने पर $\Rightarrow r = 6$ $\quad\therefore\quad$ x^{-7} का गुणांक $= {}^{11}C_6\, a^5 b^{-6}$

दिया गया है $\quad {}^{11}C_5\, a^6 b^{-5} = {}^{11}C_6\, a^5 b^{-6} \Rightarrow ab = 1$

A-6. Find the term independent of 'x' in the expansion of the expression,

$(1 + x + 2\,x^3)\left(\frac{3}{2}x^2 - \frac{1}{3\ \ x}\right)^9$.

$(1 + x + 2\,x^3)\left(\frac{3}{2}x^2 - \frac{1}{3\ \ x}\right)^9$ के प्रसार में x से स्वतंत्र पद ज्ञात कीजिए।

Ans $\frac{17}{54}$.

Sol. Co-efficient of x^0 in $(1 + x + 2x^3)\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$

= Co-efficient of x^0 in $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$ + Co-efficient of x^{-1} in $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$

+ 2 Co-efficient of x^{-3} in $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$

$= {}^9C_{r_1}\left(\frac{3}{2}x^2\right)^{9-r_1}\left(-\frac{1}{3x}\right)^{r_1} + {}^9C_{r_2}\left(\frac{3}{2}x^2\right)^{9-r_2}\left(-\frac{1}{3x}\right)^{r_2} + 2\ {}^9C_{r_3}\left(\frac{3}{2}x^2\right)^{9-r_3}\left(-\frac{1}{3x}\right)^{r_3}$

$= {}^9C_6\left(\frac{3}{2}\right)^3\left(-\frac{1}{3}\right)^6 + 2\ {}^9C_7\left(\frac{3}{2}\right)^2\left(-\frac{1}{3}\right)^7 = \frac{17}{54}$

Hindi. $(1 + x + 2x^3)\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$ के प्रसार में x^0 का गुणांक

$\Rightarrow \left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$ के प्रसार में x^0 का गुणांक $+\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$ के प्रसार में x^{-1} का गुणांक $+ 2\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$ के प्रसार में x^{-3} का गुणांक

$= {}^9C_{r_1}\left(\frac{3}{2}x^2\right)^{9-r_1}\left(-\frac{1}{3x}\right)^{r_1} + {}^9C_{r_2}\left(\frac{3}{2}x^2\right)^{9-r_2}\left(-\frac{1}{3x}\right)^{r_2} + 2\ {}^9C_{r_3}\left(\frac{3}{2}x^2\right)^{9-r_3}\left(-\frac{1}{3x}\right)^{r_3}$

$= {}^9C_6\left(\frac{3}{2}\right)^3\left(-\frac{1}{3}\right)^6 + 2\ {}^9C_7\left(\frac{3}{2}\right)^2\left(-\frac{1}{3}\right)^7 = \frac{17}{54}$

A-7. (i) Find the coefficient of x^5 in $(1 + 2x)^6(1 - x)^7$.
(ii) Find the coefficient of x^4 in $(1 + 2x)^4 (2 - x)^5$
(i) $(1 + 2x)^6(1 - x)^7$ में x^5 का गुणांक ज्ञात कीजिए।
(ii) $(1 + 2x)^4 (2 - x)^5$ में x^4 का गुणांक ज्ञात कीजिए।

Ans. (i) 171
(ii) –438

Sol. (i) $(1 + 2x)^6 (1 - x)^7$
$= (1 + {}^6C_1 (2x) + {}^6C_2 (2x)^2 + {}^6C_3 (2x)^3 + {}^6C_4 (2x)^4 + {}^6C_5 (2x)^5 + (2x)^6) (1 - x)^7$.
$= (1 + 12x + 60x^2 + 160 x^3 + 240x^4 + 192 x^5 + 64 x^6) (1 - x)^7$
= 1 × coeff of x^5 + 12 x coeff. of x^4 + 60 × coeff of x^3 + 160 × coeff of x^2 + 240 × coeff of x + 192 × constant term.
= 1 × x^5 का गुणांक + 12 (x^4 का गुणांक) + 60 (x^3 का गुणांक) + 160 (x^2 का गुणांक) + 240 (x^2 का गुणांक) + 192 × अचर पद

$= 1 \times (-1)^5 . {}^7C_5 + 12 \times {}^7C_4 - 60 \times {}^7C_3 + 160 \times {}^7C_2 - 240 \times {}^7C_1 + 192 \times 1$
$= -21 + 420 - 2100 + 3360 - 1680 + 192 = 171$

(ii) $(1 + 2x)^4 (2 - x)^5$
$[1 + {}^4C_1 (2x) + {}^4C_2 (2x)^2 + {}^4C_3 (2x)^3 + {}^4C_4(2x)^4] (2 -x)^5$
$= (1 + 8x + 24x^2 + 32x^3 + 16x^4) (2 - x)^5$
coefficient of x^4 = 1 × coefficient of x^4 + 8 × coefficient of x^3 + 24 × coefficient of x^2 + 32 × coefficient of x + 16 × constant term
x^4 का गुणांक = 1 × x^4 का गुणांक + 8 × x^3 का गुणांक + 24 × x^2 का गुणांक + 32 × x का गुणांक+ 16 × अचर पद
$T_{r+1} = {}^5C_r . 2^{5-r} (-x)^r = (-1)^r\ {}^5C_r\ 2^{5-r}\ x^r = 1 \times {}^5C_4 \times 2^1 - 8 \times {}^5C_3 \times 2^2 + 24 \times {}^5C_2 \times 2^3 - 32 \times {}^5C_1 \times 2^4 + 16 \times 2^5$

$= 10 - 320 + 1920 - 2560 + 512 = -438$

A-8. In the expansion of $\left(x^3 - \frac{1}{x^2}\right)^n$, $n \in N$, if the sum of the coefficients of x^5 and x^{10} is 0, then n is :

Ans. 15

यदि $\left(x^3 - \frac{1}{x^2}\right)^n$, $n \in N$ के प्रसार में x^5 और x^{10} के गुणांको का योग शून्य हो, तो n है–

Sol. $\left(x^3 - \frac{1}{x^2}\right)^n$

General term $= \frac{n!}{n!(n-r)!} (-1)^{n-r} x^{5r-2n}$

If $5r - 2n = 5$, then $5r = 2n + 5 \Rightarrow r = \frac{2n}{5} + 1$

If 5r – 2n = 10, then 5r = 2n + 10⇒ $r = \frac{2n}{5} + 2$

Let n = 5k

Now $\frac{5k!}{(2k+1)!(3k-1)!} - \frac{5k!}{(2k+2)!(3k-2)!} = 0$

⇒ $\frac{1}{3k-1} - \frac{1}{2k+2} = 0$ ⇒ k = 3 ⇒ n = 15

Hindi. $\left(x^3 - \frac{1}{x^2}\right)^n$

व्यापक पद = $\frac{n!}{n!(n-r)!}$ $(-1)^{n-r}\, x^{5r-2n}$

यदि 5r – 2n = 5, तब 5r = 2n + 5 ⇒ $r = \frac{2n}{5} + 1$

यदि 5r – 2n = 10, तब 5r = 2n + 10 ⇒ $r = \frac{2n}{5} + 2$

माना n = 5k

अब $\frac{5k!}{(2k+1)!(3k-1)!} - \frac{5k!}{(2k+2)!(3k-2)!} = 0$

⇒ $\frac{1}{3k-1} - \frac{1}{2k+2} = 0$ ⇒ k = 3 ⇒ n = 15

Section (B) : Middle term, Remainder & Numerically/Algebrically Greatest terms
खण्ड (B) : मध्य पद, शेषफल और संख्यात्मक/बीजगणितीय महत्तम पद

B-1. Find the middle term(s) in the expansion of
निम्न के प्रसार में मध्य पद ज्ञात करो–

(i) $\left(\frac{x}{y} - \frac{y}{x}\right)^7$ (ii) $(1 - 2x + x^2)^n$

Ans. (i) $-\frac{35x}{y}, \frac{35y}{x}$ (ii) $(-1)^n \frac{(2n)!}{n!\ n!} x^n$

Sol. **(i)** $\left(\frac{x}{y} - \frac{y}{x}\right)^7$ T_4 & T_5 are the middle term

(ii) $(1 - 2x + x^2)^n = (x - 1)^{2n}$
$T_{n+1} = {}^{2n}C_n (-1)^n x^n$

Hindi. **(i)** $\left(\frac{x}{y} - \frac{y}{x}\right)^7$ के प्रसार में T_4 और T_5 मध्य पद है

(ii) $(1 - 2x + x^2)^n = (x - 1)^{2n}$
$T_{n+1} = {}^{2n}C_n (-1)^n x^n$

B-2. Prove that the co-efficient of the middle term in the expansion of $(1 + x)^{2n}$ is equal to the sum of the co-efficients of middle terms in the expansion of $(1 + x)^{2n-1}$.
सिद्ध करो कि $(1 + x)^{2n}$ के प्रसार में मध्य पद का गुणांक, $(1 + x)^{2n-1}$ के प्रसार में मध्य पदों के गुणांकों के योगफल के बराबर है।

Sol. Co-efficient of middle term $(1 + x)^{2n} = {}^{2n}C_n$

${}^{2n}C_n = {}^{2n-1}C_{n-1} + {}^{2n-1}C_n$

Hindi. $(1 + x)^{2n}$ के प्रसार में मध्य पद का गुणांक $= {}^{2n}C_n$

${}^{2n}C_n = {}^{2n-1}C_{n-1} + {}^{2n-1}C_n$

B-3. (i) Find the remainder when 7^{98} is divided by 5

(ii) Using binomial theorem prove that $6^n - 5n$ always leaves the remainder 1 when divided by 25.

(iii) Find the last digit, last two digits and last three digits of the number $(27)^{27}$.

(i) यदि 7^{98} को 5 से विभाजित किया जाए, तो शेषफल ज्ञात करो।

(ii) द्विपद प्रमेय का उपयोग करते हुए सिद्ध कीजिए कि $6^n - 5n$ को 25 से विभाजित करने पर प्राप्त शेषफल सदैव 1 होता है।

(iii) $(27)^{27}$ का अन्तिम अंक, अन्तिम दो अंक व अन्तिम तीन अंक ज्ञात करो।

Ans. (i) 4

(iii) 3, 03, 803

Sol. (i) $7^{98} = (50 - 1)^{49} = {}^{49}C_0(50)^{49} - {}^{49}C_1(50)^{48} + \ldots\ldots - {}^{49}C_{49}$ $\Rightarrow$ Remainder शेषफल $= 5 - 1 = 4$

(ii) $6^n - 5n = (5 + 1)^n - 5n = 5^n + {}^nC_1.5^{n-1} + \ldots\ldots + {}^nC_{n-2}.5^2 + {}^nC_{n-1}.5 + 1 - 5n = 25\lambda + 1$

(iii) $(27)^{27} = 3^{81} = 3.(9)^{40}$

$= 3(10 - 1)^{40} = 3(10^{40} - {}^{40}C_1.10^{39} + \ldots\ldots + {}^{40}C_{38}.10^2 - {}^{40}C_{39}.10 + 1) = 3(1000\lambda - 400 + 1)$

Last 3 digits of this number = 803.

इस संख्या के अन्तिम 3 अंक = 803.

B-4. Which is larger : $(99^{50} + 100^{50})$ or $(101)^{50}$.

$(99^{50} + 100^{50})$ तथा $(101)^{50}$ में से कौनसा बड़ा है ?

Ans. 101^{50}

Sol. $(100 + 1)^{50} - 100^{50} - (100 - 1)^{50} = 2[{}^{50}C_1(100)^{49} + {}^{50}C_3(100)^{47} + \ldots\ldots + {}^{50}C_{49}(100)] - (100)^{50} > 0$

$(101)^{50} > (99^{50} + 100^{50})$

B-5. (i) Find numerically greatest term(s) in the expansion of $(3 - 5x)^{15}$ when $x = \frac{1}{5}$

(ii) Which term is the numerically greatest term in the expansion of $(2x + 5y)^{34}$, when x = 3 & y = 2 ?

(i) यदि $x = \frac{1}{5}$ तब $(3 - 5x)^{15}$ के प्रसार में महत्तम संख्यात्मक मान वाला(वाले) पद ज्ञात करो।

(ii) $(2x + 5y)^{34}$ के विस्तार में संख्यात्मक महत्तम पद होगा जब x = 3 तथा y = 2 ?

Ans. **(i)** $T_4 = -455 \times 3^{12}$ and $T_5 = 455 \times 3^{12}$

(ii) 22

Sol **(i)** For numerically greatest term in $(x + a)^n$ $r = \left[\dfrac{n+1}{1+\left|\frac{x}{a}\right|}\right] = \left[\dfrac{15+1}{1+\left|\frac{3}{1}\right|}\right] = 4$

Since value of $\dfrac{n+1}{1+\left|\frac{x}{a}\right|}$ is itself an integer. There are two terms, whose numerical values are greatest

T_4 and T_5

$T_4 = {}^{15}C_3 (3)^{12} (-1)^3 = -455 \times 3^{12}$

$T_5 = {}^{15}C_4 (3)^{11} (-1)^4 = 455 \times 3^{12}$

(i) $(x + a)^n$ के प्रसार में महत्तम संख्यात्मक मान वाले पद के लिए $r = \left[\dfrac{n+1}{1+\left|\dfrac{x}{a}\right|}\right] = \left[\dfrac{15+1}{1+\left|\dfrac{3}{1}\right|}\right] = 4$

चूँकि $\dfrac{n+1}{1+\left|\dfrac{x}{a}\right|}$ का मान एक पूर्णांक है अतः यहाँ दो पद है जिनका संख्यात्मक मान महत्तम है।

T_4 तथा T_5

$T_4 = {}^{15}C_3 (3)^{12} (-1)^3 = -455 \times 3^{12}$

$T_5 = {}^{15}C_4 (3)^{11} (-1)^4 = 455 \times 3^{12}$

(ii) For numerically greatest term महत्तम संख्यात्मक मान वाले पद के लिये $r = \left[\dfrac{n+1}{1+\left|\dfrac{x}{a}\right|}\right] = \left[\dfrac{34+1}{1+\left|\dfrac{6}{10}\right|}\right] \Rightarrow r = 21.$

B-6. Find the term in the expansion of $(2x - 5)^6$ which have

(i) Greatest binomial coefficient (ii) Greatest numerical coefficient

(iii) Algebrically greatest coefficient (iv) Algebrically least coefficient

$(2x - 5)^6$ के प्रसार में वह पद ज्ञात करो जो रखता है

(i) महत्तम द्विपद गुणांक (ii) महत्तम संख्यात्मक गुणांक

(iii) महत्तम बीजगणितीय गुणांक (iv) न्यूनतम बीजगणितीय गुणांक

Ans. (i) T_4 (ii) T_5, T_6 (iii) T_5 (iv) T_6

Sol. $(2x - 5)^6$

(i) Greatest binomical Co-efficient is of middle term $= T_{\frac{6}{2}+1} = T_4$

(ii) For greatest numerical term $r = \left[\dfrac{6+1}{1+\left|\dfrac{2}{5}\right|}\right] = \left[\dfrac{35}{7}\right] = 5$

Since $\dfrac{n+1}{1+\left|\dfrac{x}{a}\right|}$ itself is an integer.

$\therefore$ T_5 and T_6 both terms have are greatest numerical value

(iii) The positive term of greatest numerical value is Algebraically greatest i.e. T_5.

(iv) The negative term of greatest numerical value is algebraically least i.e. T_6

Hindi. $(2x - 5)^6$

(i) महत्तम द्विपद गुणांक वाला पद मध्य पद होता है अतः $= T_{\frac{6}{2}+1} = T_4$

(ii) महत्तम संख्यात्मक मान वाला पद $r = \left[\dfrac{6+1}{1+\left|\dfrac{2}{5}\right|}\right] = \left[\dfrac{35}{7}\right] = 5$

चूँकि $\frac{n+1}{1+\left|\frac{x}{a}\right|}$ एक पूर्णांक है अतः

T_5 तथा T_6 दोनों पद महत्तम संख्यात्मक मान वाले पद होंगे

(iii) बीजगणितीय महत्तम मान वाला पद धनात्मक महत्तम संख्यात्मक मान वाले पद के बराबर होता है, अर्थात् T_5

(iv) बीजगणितीय न्यूनतम मान वाला पद ऋणात्मक महत्तम संख्यात्मक मान वाले पद के बराबर होता है, अर्थात् T_6

Section (C) : Summation of series, Variable upper index & Product of binomial coefficients
खण्ड (C) : श्रेणी का योग, चर ऊपरी सूचकांक एवं द्विपद गुणांको का गुणन

C-1. If $C_0, C_1, C_2, \ldots C_n$ are the binomial coefficients in the expansion of $(1 + x)^n$ then prove that **:**
यदि $\mathbf{(1 + x)^n}$, $\mathbf{n \in N}$ के प्रसार में $\mathbf{C_0, C_1, C_2, .., C_n}$ द्विपद गुणांक है, तो सिद्ध करो **:**

(i)_ $C_0 - \frac{C_1}{\sqrt{2}} + \frac{C_2}{2} - \frac{C_3}{2\sqrt{2}}$ …….. upto (n + 1) terms equal to $\left(1-\frac{1}{\sqrt{2}}\right)^n$

(ii)_. $-C_1(3)^{n-1}(\sqrt{5})^1 + C_2(3)^{n-2}5 - C_3 3^{n-3}(5\sqrt{5})$ …..upto (n) terms equal to $(3-\sqrt{5})^n - 3^n$

(iii) $\frac{(3.2-1)}{2}C_1 + \frac{3^2.2^2-1}{2^2}C_2 + \frac{3^3.2^3-1}{2^3}C_3 + \ldots\ldots + \frac{3^n.2^n-1}{2^n}C_n = \frac{2^{3n}-3^n}{2^n}$

Sol. **(i)** Obvious

(ii) Obvious

(iii) $(C_0 + 3C_1 + 3^2C_2 + \ldots\ldots + 3^nC_n) - \left(C_0 + \frac{C_1}{2} + \frac{C_2}{2^2} + \ldots\ldots + \frac{C_n}{2^2}\right)$

$= 4^n - \left(\frac{3}{2}\right)^n = 2^{2n} - \frac{3^n}{2^n} = \frac{2^{3n}-3^n}{2^n}$

C-2. If $C_0, C_1, C_2, \ldots C_n$ are the binomial coefficients in the expansion of $(1 + x)^n$ then prove that **:**
यदि $\mathbf{(1 + x)^n}$, $\mathbf{n \in N}$ के प्रसार में $\mathbf{C_0, C_1, C_2, .., C_n}$ द्विपद गुणांक है, तो सिद्ध करो **:**

(i) $\frac{C_1}{C_0} + 2\frac{C_2}{C_1} + 3\frac{C_3}{C_2} + \ldots\ldots + n\frac{C_n}{C_{n-1}} = \frac{n(n+1)}{2}$

(ii) $(C_0 + C_1)(C_1 + C_2)(C_2 + C_3)(C_3 + C_4)\ldots\ldots(C_{n-1} + C_n) = \frac{C_0C_1C_2\ldots\ldots C_{n-1}(n+1)^n}{n!}$.

(iii) $C_o - 2C_1 + 3C_2 - 4C_3 + \ldots. + (-1)^n(n+1)C_n = 0$

(iv) $4C_0 + \frac{4^2}{2}.C_1 + \frac{4^3}{3}C_2 + \ldots\ldots\ldots + \frac{4^{n+1}}{n+1}C_n = \frac{5^{n+1}-1}{n+1}$

(v) $\frac{2^2.C_0}{1.2} + \frac{2^3.C_1}{2.3} + \frac{2^4.C_2}{3.4} + \ldots\ldots + \frac{2^{n+2}.C_n}{(n+1)(n+2)} = \frac{3^{n+2}-2n-5}{(n+1)(n+2)}$

(vi) $2.C_o + \frac{2^2.C_1}{2} + \frac{2^3.C_2}{3} + \frac{2^4.C_3}{4} + \ldots\ldots \frac{2^{n+1}.C_n}{n+1} = \frac{3^{n+1}-1}{n+1}$

Sol. **(i)** $\frac{C_1}{C_0}+2\frac{C_2}{C_1}+3\frac{C_3}{C_2}+ ....$

$= 1 + 2 + 3 + ...+ (n - 1) + n = \frac{n(n+1)}{2}$

(ii) $(C_0 + C_1)(C_1 + C_2)..........(C_{n-1} + C_n)$

$= ({}^{n+1}C_1)({}^{n+1}C_2)........({}^{n+1}C_n) = ((n + 1).C_0)\left(\frac{n+1}{2}.C_1\right) .... \left(\frac{n+1}{n}.C_{n-1}\right)$

$= \frac{(n+1)^n}{n\ !} C_0C_1........C_{n-1}$

(iii) $(1 + x)^n = C_0 + C_1x + ...+ C_nx^n$

$x(1 + x)^n = C_0x + C_1x^2 + ...+ C_nx^{n+1}$

Differentiating w.r.t. x x के सापेक्ष अवकलन करने पर

$(1 + x)^n + n\,x\,(1 + x)^{n-1} = C_0 + 2C_1x + ... + (n + 1)\,C_nx^n$

Putting x = – 1 रखने पर

$C_0 - 2C_1 + ...+ (-1)^n (n + 1)\,C_n = 0$

(iv) $S = \sum_{r=0}^{n}\frac{{}^nC_r}{r+1}4^{r+1} = \frac{1}{n+1}\sum_{r=0}^{n}{}^{n+1}C_{r+1}\quad 4^{r+1} = \frac{1}{n+1}\,[C_14 + C_24^2 + ...\ C_{n+1}4^{n+1}] = \frac{1}{n+1}\,[5^{n+1} - 1]$

Aliter

$(1 + x)^n = {}^nC_0 + {}^nC_1x + ... + {}^nC_n\,x^n$

$\frac{(1+x)^{n+1}}{n+1} - \frac{1}{n+1} = {}^nC_0\,x + \frac{{}^nC_1x^2}{2} + ...\ \frac{{}^nC_nx^{n+1}}{n+1}$

put x = 4

then $C_0 . 4 + \frac{4^2}{2}C_1 + ...+ \frac{4^{n+1}}{n+1}C_n = \frac{5^{n+1}-1}{n+1}$

(iv) $S = \sum_{r=0}^{n}\frac{{}^nC_r}{r+1}4^{r+1} = \frac{1}{n+1}\sum_{r=0}^{n}{}^{n+1}C_{r+1}\quad 4^{r+1} = \frac{1}{n+1}\,[C_14 + C_24^2 + ...\ C_{n+1}4^{n+1}]$

$= \frac{1}{n+1}\,[5^{n+1} - 1]$

वैकल्पिक :

$(1 + x)^n = {}^nC_0 + {}^nC_1x + ... + {}^nC_n\,x^n$

$\frac{(1+x)^{n+1}}{n+1} - \frac{1}{n+1} = {}^nC_0\,x + \frac{{}^nC_1x^2}{2} + ...\ \frac{{}^nC_nx^{n+1}}{n+1}$

x = 4 रखने पर

तब $C_0 . 4 + \frac{4^2}{2}C_1 + ...+ \frac{4^{n+1}}{n+1}C_n = \frac{5^{n+1}-1}{n+1}$

(v) $(1+x)^n = C_0 + C_1x + ...+ C_nx^n$

Integrating from 0 to x

$\frac{(1+x)^{n+1}-1}{n+1} = C_0x + \frac{C_1x^2}{2} +........+ \frac{C_nx^{n+1}}{n+1}$

Again integrate 0 to x

$\frac{(1+x)^{n+2}-1}{(n+1)(n+2)} - \frac{x}{n+1} = \frac{C_0x^2}{2} + \frac{C_1x^3}{2.3} + ... + \frac{C_nx^{n+1}}{(n+1)(n+2)}$

put x = 2

$$\frac{2^2.C_0}{2} + \frac{2^3.C_1}{2.3} + ... + \frac{2^{n+2}C_n}{(n+1)(n+2)} = \frac{3^{n+2}-2n-5}{(n+1)(n+2)}$$

(v) $(1+x)^n = C_0 + C_1x + ...+ C_nx^n$

0 से x तक समाकलन करने पर

$$\frac{(1+x)^{n+1}-1}{n+1} = C_0x + \frac{C_1x^2}{2} + ... + \frac{C_nx^{n+1}}{n+1}$$

पुनः 0 से x तक समाकलन करने पर

$$\frac{(1+x)^{n+2}-1}{(n+1)(n+2)} - \frac{x}{n+1} = \frac{C_0x^2}{2} + \frac{C_1x^3}{2.3} + ... + \frac{C_nx^{n+1}}{(n+1)(n+2)}$$

x = 2 रखने पर

$$\frac{2^2.C_0}{2} + \frac{2^3.C_1}{2.3} + ... + \frac{2^{n+2}C_n}{(n+1)(n+2)} = \frac{3^{n+2}-2n-5}{(n+1)(n+2)}$$

(vi) $(1 + x)^n = {}^nC_0 + {}^nC_1 x + ...+ {}^nC_nx^n$

$$\int_0^2 (1+x)^n dx = C_0x + \frac{C_1x^2}{2} + ... + \frac{C_nx^{n+1}}{n+1}\Bigg|_0^2$$

$$\frac{3^{n+1}-1}{n+1} = 2.C_0 + \frac{2^2C_1}{2} + ... + \frac{2^{n+1}C_n}{n+1}$$

C-3. Prove that सिद्ध करो

(i) ${}^nC_r + {}^{n-1}C_r + {}^{n-2}C_r + ... + {}^rC_r = {}^{n+1}C_{r+1}$

(ii) ${}^{10}C_2 + {}^{11}C_2 + {}^{12}C_2 + ...+ {}^{19}C_2 = 1020$

Sol. **(i)** ${}^nC_r + {}^{n-1}C_r + ... + {}^rC_r$ = Co-efficient of x^r in $(1 + x)^n + (1 + x)^{n-1} + ... + (1 + x)^r$

= Co-efficient of x^r in $(1 + x)^r \left[\frac{(1+x)^{n-r+1}-1}{x}\right]$ = Co-efficient of x^{r+1} in $(1 + x)^{n+1} = {}^{n+1}C_{r+1}$

Hindi. **(i)** ${}^nC_r + {}^{n-1}C_r + ... + {}^rC_r = (1 + x)^n + (1 + x)^{n-1} + ... + (1 + x)^r$ के प्रसार में x^r का गुणांक

$= (1 + x)^r \left[\frac{(1+x)^{n-r+1}-1}{x}\right]$ के प्रसार में x^r का गुणांक

$= (1 + x)^{n+1}$ के प्रसार में x^r का गुणांक $= {}^{n+1}C_{r+1}$

(ii) ${}^{20}C_3 - {}^{10}C_3 = \frac{20\times19\times18}{3\times2\times1} - \frac{10\times9\times8}{3\times2\times1} = 120 = 1140 - 120 = 1020$

C-4. If $(1 + x)^n = C_0 + C_1 x + C_2 x^2 + ... + C_n x^n$, prove that

यदि $(1 + x)^n = C_0 + C_1 x + C_2 x^2 + ... + C_n x^n$, सिद्ध करो

(i) $C_0 C_3 + C_1 C_4 + ... + C_{n-3} C_n = \frac{(2n)!}{(n+3)\ !\ (n-3)!}$

(ii) $C_0 C_r + C_1 C_{r+1} + ... + C_{n-r} C_n = \frac{(2n)!}{(n+r)\ !\ (n-r)!}$

(iii)✍ $C_0^2 - C_1^2 + C_2^2 - C_3^2 + ... + (-1)^n C_n^2 = 0$ or $(-1)^{n/2} C_{n/2}$ according as n is odd or even.

$C_0^2 - C_1^2 + C_2^2 - C_3^2 +...... + (-1)^n C_n^2 = 0$ या $(-1)^{n/2} C_{n/2}$ यदि n विषम या सम है।

[Revision Planner]

Sol. (i) $(1 + x)^n = C_0 + C_1 x + ... + C_n x^n$

$(x + 1)^n = C_0x^n + C_1 x^{n-1} + ... + C_n$

$C_0C_3 + C_1C_4 + ...+ C_{n-3}C_n$

= Co-efficient of x^{n-3} in $(1 + x)^{2n} = {}^{2n}C_{n-3}$

(ii) $C_0C_r + C_1C_{r+1} + ...+ C_{n-r}C_n$

= co-efficient of x^{n-r} in $(1+x)^{2n}$

$= {}^{2n}C_{n-r}$

(iii) $(1 + x)^n = C_0 + C_1 x + ... + C_n x^n$

$(x - 1)^n = C_0x^n - C_1 x^{n-1} + ... +(-1)^n C_n$

$C_0^2 - C_1^2 +............+(-1)^nC_n^2$

= co-efficient of x^n in $(x^2 - 1)^n = 0$ if n is odd

$= {}^nC_{n/2}(-1)^{n/2}$ if n is even

Hindi. (i) $(1 + x)^n = C_0 + C_1 x + ... + C_n x^n$

$(x + 1)^n = C_0x^n + C_1 x^{n-1} + ... + C_n$

$C_0C_3 + C_1C_4 + ...+ C_{n-3}C_n$

$= (1 + x)^{2n}$ में x^{n-3} का गुणांक $= {}^{2n}C_{n-3}$

(ii) $C_0C_r + C_1C_{r+1} + ...+ C_{n-r}C_n$

$= (1 + x)^{2n}$ में x^{n-r} का गुणांक

$= {}^{2n}C_{n-r}$

(iii) $(1 + x)^n = C_0 + C_1 x + ... + C_n x^n$

$(x - 1)^n = C_0x^n - C_1 x^{n-1} + ... +(-1)^n C_n$

$C_0^2 - C_1^2 +............+(-1)^nC_n^2$

$= (x^2 - 1)^n$ में x^n का गुणांक $= 0$ यदि n विषम है।

$= {}^nC_{n/2}(-1)^{n/2}$ यदि n सम है।

Section (D) : Negative & fractional index, Multinomial theorem
खण्ड (D) : ऋणात्मक व भिन्न घातांक, बहुपदीय प्रमेय

D-1. Find the co-efficient of x^6 in the expansion of $(1 - 2x)^{-5/2}$.

$(1 - 2x)^{-5/2}$ के प्रसार में x^6 का गुणांक ज्ञात कीजिए।

Ans. $\dfrac{15015}{16}$

Sol. In the expansion of $(1 - 2x)^{-5/2}$

$$T_{r+1} = \frac{\frac{5}{2}\left(\frac{5}{2}+1\right)\left(\frac{5}{2}+2\right).......\left(\frac{5}{2}+r-1\right)}{r\,!} \cdot (-1)^r (2)^r x^r$$

$$\therefore \text{ Coefficient of } x^6 = \frac{\frac{5}{2}\left(\frac{5}{2}+1\right)\left(\frac{5}{2}+2\right)\left(\frac{5}{2}+3\right)\left(\frac{5}{2}+4\right)\left(\frac{5}{2}+5\right)}{6\,!} = \frac{15015}{16}$$

Hindi. $(1 - 2x)^{-5/2}$ के प्रसार में

$$T_{r+1} = \frac{\frac{5}{2}\left(\frac{5}{2}+1\right)\left(\frac{5}{2}+2\right).......\left(\frac{5}{2}+r-1\right)}{r\,!} \cdot (-1)^r (2)^r x^r$$

$$\therefore\ x^6 \text{ का गुणांक } = \frac{\frac{5}{2}\left(\frac{5}{2}+1\right)\left(\frac{5}{2}+2\right)\left(\frac{5}{2}+3\right)\left(\frac{5}{2}+4\right)\left(\frac{5}{2}+5\right)}{6\,!} = \frac{15015}{16}$$

D-2. (i) Find the coefficient of x^{12} in $\dfrac{4+2x-x^2}{(1+x)^3}$

(ii) Find the coefficient of x^{100} in $\dfrac{3-5x}{(1-x)^2}$

(i) $\dfrac{4+2x-x^2}{(1+x)^3}$ में x^{12} का गुणांक ज्ञात कीजिए।

(ii) $\dfrac{3-5x}{(1-x)^2}$ में x^{100} का गुणांक ज्ञात कीजिए।

Ans. (i) 142
(ii) – 197

Sol. (i) $(4 + 2x - x^2)\ (1 + x)^{-3}$

$= 4 \times$ coeff of $x^{12} + 2 \times$ coeff of $x^{11} - 1 \times$ coeff of x^{10}

$= 4$ (x^{12} का गुणांक) $+ 2$ (x^{11} का गुणांक) $- 1$ (x^{10} का गुणांक)

In $(1 + x)^{-3}$

$T_{r+1} = (-1)^r\ {}^{3+r-1}C_r x^r$

$= (-1)^r\ {}^{r+2}C_r\ x^r$

$= 4 \times {}^{14}C_{12} - 2 \times {}^{13}C_{11} - {}^{12}C_{10}$

$= 4 \times \dfrac{14\times13}{2} - 2x\dfrac{13\times12}{2} - \dfrac{12\times11}{2}$

$= 364 - 156 - 66 = 142$

(ii) $(3 - 5x)\ (1 - x)^{-2}$

$= 3 \times$ coeff of $x^{100} - 5 \times$ coeff of x^{99}

$= 3$ (x^{100} का गुणांक) $- 5$ (x^{99} का गुणांक)

In $(1 - x)^{-2}$

$T_{r+1} = {}^{2+r-1}C_r\ x^r = {}^{r+1}C_r\ x^r$

$= 3 \times {}^{101}C_{100} - 5 \text{ x } {}^{100}C_{99}$

$= 3 \times 101 - 5 \text{ x } 100$

$= -197$

D-3. Assuming ' x ' to be so small that x^2 and higher powers of ' x ' can be neglected, show that, $\dfrac{\left(1+\frac{3}{4}x\right)^{-4}(16-3x)^{1/2}}{(8+x)^{2/3}}$ is approximately equal to, $1 - \dfrac{305}{96}x$.

यदि 'x' का मान इतना अल्प है कि x^2 और 'x' की उच्च घातों को नगण्य माना जा सकता है तो प्रदर्शित कीजिए कि $\dfrac{\left(1+\frac{3}{4}x\right)^{-4}(16-3x)^{1/2}}{(8+x)^{2/3}}$ का मान लगभग $1 - \dfrac{305}{96}x$ है।

Sol. $$\frac{\left(1+\frac{3}{4}x\right)^{-4}(16-3x)^{1/2}}{(8+x)^{2/3}} = \frac{(1-3x)\ .\ 4\left(1-\frac{3}{32}x\right)}{4\left(1+\frac{2x}{24}\right)} = \left(1-3x-\frac{3}{32}x\right)\left(1-\frac{x}{12}\right)$$

$$= 1 - \frac{x}{12} - 3x - \frac{3}{32}x = 1 - \frac{305}{96}x$$

D-4. (i) Find the coefficient of $a^5 b^4 c^7$ in the expansion of $(bc + ca + ab)^8$.
(ii) Sum of coefficients of odd powers of x in expansion of $(9x^2 + x - 8)^6$
(i) $(bc + ca + ab)^8$ के प्रसार में $a^5 b^4 c^7$ का गुणांक ज्ञात करो।
(ii) $(9x^2 + x - 8)^6$ के प्रसार में x की विषम घातों के गुणांको का योगफल है।
Ans. (i) 280 (ii) 2^5

Sol. (i) $(bc + ca + ab)^8$

$$\frac{8\,!}{r_1\,!\; r_2!\; r_3!}\,(bc)^{r_1}(ca)^{r_2}(ab)^{r_3}$$

$$\left.\begin{matrix} r_2 + r_3 = 5 \\ r_1 + r_3 = 4 \\ r_2 + r_1 = 7 \end{matrix}\right\} \Rightarrow \quad r_2 = 4, \quad r_1 = 3, \quad r_3 = 1$$

or या $\frac{8!}{4!3!1!} = 280$

(ii) $(9x^2 + x - 8)^6 = a_0 + a_1x + a_2x^2 + ... a_{12}x^{12}$

$2^6 = a_0 + a_1 + ... + a_{12} \quad (x = 1)$

$0 = a_0 - a_1 + ...+ a_{12} \quad (x = -1)$

$\Rightarrow \quad a_1 + a_3 + ... + a_{11} = 2^5$

D-5. Find the coefficient of x^7 in $(1 - 2x + x^3)^5$.
$(1 - 2x + x^3)^5$ में x^7 का गुणांक ज्ञात कीजिए।
Ans. 20

Sol. Co-efficient of x^7 in $(1 - 2x + x^3)^5$ के प्रसार में x^7 का गुणांक है

$= \frac{n\,!}{r_1!r_2!r_3!}(1)^{r_1}(-2x)^{r_2}(x^3)^{r_3} = r_2 + 3r_3 = 7 \;\&\; r_1, r_2, r_3 \le 5$

(i) $r_2 = 4, r_3 = 1, r_1 = 0$ (ii) $r_2 = 1, r_3 = 2, r_1 = 2 = \frac{5!}{4!1!}\,(2)^4 + \frac{5!}{2!2!1!} \times (-2)^1 = 20$

PART - II : ONLY ONE OPTION CORRECT TYPE

भाग - II : केवल एक सही विकल्प प्रकार (ONLY ONE OPTION CORRECT TYPE)

Section (A) : General Term & Coefficient of x^k in $(ax +b)^n$
खण्ड (A) : व्यापक पद एवं $(ax +b)^n$ में x^k का गुणांक

A-1. The $(m + 1)^{th}$ term of $\left(\frac{x}{y}+\frac{y}{x}\right)^{2m+1}$ is:
(A) independent of x (B) a constant
(C*) depends on the ratio x/y and m (D) none of these

$\left(\frac{x}{y}+\frac{y}{x}\right)^{2m+1}$ का (m + 1)वाँ पद
(A) x पर निर्भर नहीं है। (B) अचर है।
(C) अनुपात x/y और m पर निर्भर है। (D) इनमें से कोई नहीं

Sol. ${}^{2m+1}C_m \left(\frac{x}{y}\right)^{m+1}\left(\frac{y}{x}\right)^{m} = {}^{2m+1}C_m \left(\frac{x}{y}\right)$

Dependent upon the ratio $\frac{x}{y}$ and m.   अनुपात x/y और m पर निर्भर है।

A-2. The total number of distinct terms in the expansion of, $(x + a)^{100} + (x - a)^{100}$ after simplification is :
(A) 50   (B) 202   (C*) 51   (D) none of these
$(x + a)^{100} + (x - a)^{100}$ के प्रसार में सरल करने के बाद भिन्न–भिन्न पदों की कुल संख्या है :
(A) 50   (B) 202   (C) 51   (D) इनमें से कोई नहीं

Sol. $(x + a)^{100} + (x - a)^{100}$

$= 2\left({}^{100}C_0 \; x^{100} + {}^{100}C_2 \; x^{98}a^2 + ... + {}^{100}C_{100} \; a^{100}\right)$

Number of terms = 51 terms   (पदों की संख्या = 51 पद)

A-3. The value of, $\dfrac{18^3 + 7^3 + 3 \,.\, 18 \,.\, 7 \,.\, 25}{3^6 + 6 \,.\, 243 \,.\, 2 + 15 \,.\, 81 \,.\, 4 + 20 \,.\, 27 \,.\, 8 + 15 \,.\, 9 \,.\, 16 + 6 \,.\, 3 \,.\, 32 + 64}$ is :
(A*) 1   (B) 2   (C) 3   (D) none

$\dfrac{18^3 + 7^3 + 3 \,.\, 18 \,.\, 7 \,.\, 25}{3^6 + 6 \,.\, 243 \,.\, 2 + 15 \,.\, 81 \,.\, 4 + 20 \,.\, 27 \,.\, 8 + 15 \,.\, 9 \,.\, 16 + 6 \,.\, 3 \,.\, 32 + 64}$ का मान है –
(A) 1   (B) 2   (C) 3   (D) इनमें से कोई नहीं

Sol. $\dfrac{(18+7)^3}{(3+2)^6} = \dfrac{25^3}{5^6} = 1$

A-4. In the expansion of $\left(3 - \sqrt{\dfrac{17}{4} + 3\sqrt{2}}\right)^{15}$ the 11th term is a:
(A) positive integer   (B*) positive irrational number
(C) negative integer   (D) negative irrational number.

$\left(3 - \sqrt{\dfrac{17}{4} + 3\sqrt{2}}\right)^{15}$ के प्रसार में 11वाँ पद है –
(A) धनात्मक पूर्णांक   (B) धनात्मक अपरिमेय संख्या
(C) ऋणात्मक पूर्णांक   (D) ऋणात्मक अपरिमेय संख्या

Sol. $T_{11} = {}^{15}C_{10}\,(3)^5 \left(-\sqrt{\dfrac{17}{4} + 3\sqrt{2}}\right)^{10} = {}^{15}C_{10}(3)^5\left(\dfrac{17}{4} + 3\sqrt{2}\right)^5$ = a positive irrational number

Hindi $T_{11} = {}^{15}C_{10}\,(3)^5 \left(-\sqrt{\dfrac{17}{4} + 3\sqrt{2}}\right)^{10} = {}^{15}C_{10}(3)^5\left(\dfrac{17}{4} + 3\sqrt{2}\right)^5$ = धनात्मक अपरिमेय संख्या

A-5. If the second term of the expansion $\left[a^{1/13} + \dfrac{a}{\sqrt{a^{-1}}}\right]^n$ is $14a^{5/2}$, then the value of $\dfrac{{}^nC_3}{{}^nC_2}$ is:

यदि $\left[a^{1/13} + \dfrac{a}{\sqrt{a^{-1}}}\right]^n$ के प्रसार में द्वितीय पद $14a^{5/2}$ है, तो $\dfrac{{}^nC_3}{{}^nC_2}$ का मान है–
(A*) 4   (B) 3   (C) 12   (D) 6

Sol. $T_2 = {}^nC_1\,(a^{1/13})^{n-1}\,(a^{3/2}) = 14a^{5/2}$ $\Rightarrow$ n = 14 $\therefore$ $\dfrac{{}^nC_3}{{}^nC_2} = 4$

A-6. In the expansion of $(7^{1/3} + 11^{1/9})^{6561}$, the number of terms free from radicals is:
$(7^{1/3} + 11^{1/9})^{6561}$ के प्रसार में करणी चिन्ह (radical sign) से रहित पदों की संख्या है –
(A*) 730   (B) 729   (C) 725   (D) 750

Sol. $T_{r+1} = {}^{6561}C_r (7)^{\frac{6561-r}{3}} (11^{1/9})^r$

Here r should be multiple of 9

r = 0, 9,18...............6561

Number of terms = 730

Hindi $T_{r+1} = {}^{6561}C_r (7)^{\frac{6561-r}{3}} (11^{1/9})^r$

यहाँ r, 9 का गुणज होना चाहिए

r = 0, 9,18...............6561

पदों की संख्या = 730

A-7. The value of m, for which the coefficients of the $(2m + 1)^{th}$ and $(4m + 5)^{th}$ terms in the expansion of $(1 + x)^{10}$ are equal, is

$(1 + x)^{10}$ के प्रसार में (2m + 1)वें एवं (4m + 5)वें पदों के गुणांक समान हैं, तो m का मान है–

(A) 3 (B*) 1 (C) 5 (D) 8

Sol. $\left.\begin{matrix} T_{2m+1} \Rightarrow {}^{10}C_{2m} \\ T_{4m+5} \Rightarrow {}^{10}C_{4m+4} \end{matrix}\right]$ equal बराबर है।

$2m + 4m + 4 = 10 \Rightarrow 6m + 4 = 10$

m = 1

A-8. The co-efficient of x in the expansion of $(1 - 2x^3 + 3x^5)\left(1 + \frac{1}{x}\right)^8$ is :

$(1 - 2x^3 + 3x^5)\left(1 + \frac{1}{x}\right)^8$ के प्रसार में x का गुणांक है–

(A) 56 (B) 65 (C*) 154 (D) 62

Sol. $(1 - 2x^3 + 3x^5)\left(1 + \frac{1}{x}\right)^8$

Co-efficient of x (x का गुणांक) $= -2.\,{}^8C_2 + 3.{}^8C_4 = 154$

A-9. Given that the term of the expansion $(x^{1/3} - x^{-1/2})^{15}$ which does not contain x is 5 m, where $m \in N$, then m=

(A) 1100 (B) 1010 (C*) 1001 (D) 1002

यदि $(x^{1/3} - x^{-1/2})^{15}$ के प्रसार में x से स्वतंत्र पद 5 m के बराबर है, जहाँ $m \in N$, तो m =

(A) 1100 (B) 1010 (C) 1001 (D) 1002

Sol. $(x^{1/3} - x^{-1/2})^{15}$

$T_{r+1} = {}^{15}C_r\ x^{\left(\frac{15-r}{3}\right)} (-x^{-1/2})^r$

For constant term $\frac{15-r}{3} - \frac{r}{2} = 0 \Rightarrow r = 6$

Co-efficient of $x^0 = {}^{15}C_6 = 5\text{ m} \Rightarrow m = 1001$

Hindi. $(x^{1/3} - x^{-1/2})^{15}$

$T_{r+1} = {}^{15}C_r\ x^{\left(\frac{15-r}{3}\right)} (-x^{-1/2})^r$

अचर पद के लिये $\frac{15-r}{3} - \frac{r}{2} = 0 \Rightarrow r = 6$

x^0 का गुणांक $= {}^{15}C_6 = 5\text{ m} \Rightarrow m = 1001$

A-10. The term independent of x in the expansion of $\left(x - \frac{1}{x}\right)^4 \left(x + \frac{1}{x}\right)^3$ is:

$\left(x - \frac{1}{x}\right)^4 \left(x + \frac{1}{x}\right)^3$ के प्रसार में x से स्वतंत्र पद हैं–

(A) – 3 (B*) 0 (C) 1 (D) 3

Sol. $\left(x-\frac{1}{x}\right)\left(x^2-\frac{1}{x^2}\right)^3 = \left(x-\frac{1}{x}\right)({}^3C_0\,x^6 - {}^3C_1\,x^2 + {}^3C_2\,x^{-2} - {}^3C_3\,x^{-6})$

There is no term independent of x यहाँ कोई भी पद x से स्वतंत्र नहीं है।

Section (B) : Middle term, Remainder & Numerically/Algebrically Greatest terms
खण्ड (B) : मध्य पद, शेषफल और संख्यात्मक/बीजगणितीय महत्तम पद

B-1. If $k \in R^+$ and the middle term of $\left(\frac{k}{2}+2\right)^8$ is 1120, then value of k is:

(A) 3 (B*) 2 (C) 1 (D) 4

यदि $k \in R^+$ और $\left(\frac{k}{2}+2\right)^8$ का मध्य पद 1120 है, तो k का मान होगा :

(A) 3 (B*) 2 (C) 1 (D) 4

Sol. middle term = T_5

$T_5 = T_{4+1} = {}^8C_4 \,.\, k^4 = 1120 \quad \Rightarrow \quad k = 2$

Hindi. मध्य पद = T_5

$T_5 = T_{4+1} = {}^8C_4 \,.\, k^4 = 1120 \quad \Rightarrow \quad k = 2$

B-2. The remainder when 2^{2003} is divided by 17 is :
यदि 2^{2003} को 17 से विभाजित किया जाता है, तो शेषफल होगा –

(A) 1 (B) 2 (C*) 8 (D) 7

Sol. $2^{2003} = 8.(16)^{500}$

$= 8\,(17–1)^{500}$ $\therefore$ Remainder शेषफल = 8

B-3. The last two digits of the number 3^{400} are:
संख्या 3^{400} के अन्तिम दो अंक है :

(A) 81 (B) 43 (C) 29 (D*) 01

Sol. $(81)^{100} = (80 + 1)^{100} = {}^{100}C_0\,(80)^{100} + ... + {}^{100}C_{99}\,(80)^1 + 1$

Last two digits अन्तिम दो अंक = 01

B-4. The last three digits in 10 ! are :
10 ! के मान में अन्तिम तीन अंक है –

(A*) 800 (B) 700 (C) 500 (D) 600

Sol. Last two digits in 10! are 00 and third digit = 8

Hindi 10! के मान में अन्तिम दो अंक 00 है तथा तीसरा अंक = 8

B-5. The value of $\sum_{r=1}^{10} r \,.\, \frac{{}^nC_r}{{}^nC_{r-1}}$ is equal to

$\sum_{r=1}^{10} r \,.\, \frac{{}^nC_r}{{}^nC_{r-1}}$ का मान बराबर है–

(A*) 5 (2n – 9) (B) 10 n (C) 9 (n – 4) (D) n –2

Sol. $\sum_{r=1}^{10} r.\ \frac{^nC_r}{^nC_{r-1}} = \sum_{r=1}^{10} n-r+1 = (n + 1) \times 10 - \frac{10\times 11}{2} = 10n - 45$

B-6. $\sum_{r=0}^{n-1} \frac{^nC_r}{^nC_r + {}^nC_{r+1}} =$

(A*) $\frac{n}{2}$ (B) $\frac{n+1}{2}$ (C) $(n+1)\ \frac{n}{2}$ (D) $\frac{n\ (n-1)}{2\ (n+1)}$

Sol. $\sum_{r=0}^{n-1} \frac{^nC_r}{^nC_r + {}^nC_{r+1}} = \sum_{r=0}^{n-1} \frac{r+1}{n+1} = \frac{1}{n+1}\ [1 + 2 + ...+n] = \frac{1}{n+1} \times \frac{n(n+1)}{2} = \frac{n}{2}$

B-7. Find numerically greatest term in the expansion of $(2 + 3x)^9$, when x = 3/2.

$(2 + 3x)^9$ के प्रसार में x = 3/2 के लिए महत्तम संख्यात्मक मान वाला पद है –

(A*) $^9C_6.\ 2^9.\ (3/2)^{12}$ (B) $^9C_3.\ 2^9.\ (3/2)^6$ (C) $^9C_5.\ 2^9.\ (3/2)^{10}$ (D) $^9C_4.\ 2^9.\ (3/2)^8$

Sol. For numerically greatest term $r = \left[\frac{n+1}{1+\left|\frac{x}{a}\right|}\right] = \left[\frac{9+1}{1+\left|\frac{4}{9}\right|}\right]$ $\Rightarrow r = 6$

Numerically greatest term $T_{r+1} = {}^9C_6 (2)^3 \left(\frac{9}{2}\right)^6$

Hindi. महत्तम संख्यात्मक मान वाले पद के लिये $r = \left[\frac{n+1}{1+\left|\frac{x}{a}\right|}\right] = \left[\frac{9+1}{1+\left|\frac{4}{9}\right|}\right] \Rightarrow r = 6$

महत्तम संख्यात्मक पद $T_{r+1} = {}^9C_6 (2)^3 \left(\frac{9}{2}\right)^6$

B-8. The greatest integer less than or equal to $(\sqrt{2} + 1)^6$ is

$(\sqrt{2} + 1)^6$ से कम या बराबर महत्तम पूर्णांक है–

(A) 196 (B*) 197 (C) 198 (D) 199

Sol. T_{22} is the numerically greatest term. T_{22} संख्यात्मक महत्तम पद है।

$(\sqrt{2} + 1)^6 = I + f$

$(\sqrt{2} - 1)^6 = f'$

$2[^6C_0 + {}^6C_2 . 2 + {}^6C_4\ (2)^2 + ...] = I + f + f'$

$f + f' = 1$ or $f' = 1 - f$

$I = 2\ [^6C_0 + {}^6C_2.2 + {}^6C_4.4 + {}^6C_6.8] - 1$

$I = 2\ [1 + 30 + 60 + 8] - 1 = 197$

Section (C) : Summation of series, Variable upper index & Product of binomial coefficients
खण्ड (C) : श्रेणी का योग, चर ऊपरी सूचकांक एवं द्विपद गुणांको का गुणन

C-1. $\frac{^{11}C_0}{1} + \frac{^{11}C_1}{2} + \frac{^{11}C_2}{3} + \ ...... \ + \frac{^{11}C_{10}}{11} =$

(A) $\frac{2^{11}-1}{11}$ (B*) $\frac{2^{11}-1}{6}$ (C) $\frac{3^{11}-1}{11}$ (D) $\frac{3^{11}-1}{6}$

Sol. $\frac{^{11}C_0}{1} + \frac{^{11}C_1}{2} + \frac{^{11}C_2}{3} + ... + \frac{^{11}C_{10}}{11}$

$= \left[\frac{12}{1} \cdot {}^{11}C_0 + \frac{12}{2} \cdot {}^{11}C_1 + \frac{12}{3} \cdot {}^{11}C_2 + ... + \frac{12}{11} \cdot {}^{11}C_{10}\right] = \frac{1}{12}\left[{}^{12}C_1 + {}^{12}C_2 + {}^{12}C_3 + ... + {}^{12}C_{11}\right]$

$= \frac{1}{12}(2^{12} - 2) = \frac{2^{11}-1}{6}$

C-2. The value of $\frac{C_0}{1.3} - \frac{C_1}{2.3} + \frac{C_2}{3.3} - \frac{C_3}{4.3} + ... + (-1)^n \frac{C_n}{(n+1)\ .\ 3}$ is :

(A) $\frac{3}{n+1}$ (B) $\frac{n+1}{3}$ (C*) $\frac{1}{3(n+1)}$ (D) none of these

$\frac{C_0}{1.3} - \frac{C_1}{2.3} + \frac{C_2}{3.3} - \frac{C_3}{4.3} + ... + (-1)^n \frac{C_n}{(n+1)\ .\ 3}$ का मान होगा–

(A) $\frac{3}{n+1}$ (B) $\frac{n+1}{3}$ (C*) $\frac{1}{3(n+1)}$ (D) इनमें से कोई नहीं

Sol. $\int_0^1 (1-x)^n dx = \int_0^1 \left(C_0 - C_1x + C_2x^2 - C_3x^3 + ... + (-1)^n C_nx^n\right)dx$

$\Rightarrow \frac{1}{n+1} = \left[C_0 - \frac{C_1}{2} + \frac{C_2}{3} - .... + (-1)^n \frac{C_n}{n+1}\right]$

$\Rightarrow \frac{1}{3}\left(\frac{1}{n+1}\right) = \frac{1}{3}\left[C_0 - \frac{C_1}{2} + \frac{C_2}{3} - .... + (-1)^n \frac{C_n}{n+1}\right]$

C-3. The value of the expression ${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3$ is equal to :

व्यंजक ${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3$ का मान बराबर है–

(A) ${}^{47}C_5$ (B) ${}^{52}C_5$ (C*) ${}^{52}C_4$ (D) ${}^{49}C_4$

Sol. ${}^{47}C_4 + {}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + {}^{48}C_3 + {}^{47}C_3 = {}^{52}C_4$

C-4. The value of $\binom{50}{0}\binom{50}{1} + \binom{50}{1}\binom{50}{2} + ... + \binom{50}{49}\binom{50}{50}$ is, where ${}^nC_r = \binom{n}{r}$

$\binom{50}{0}\binom{50}{1} + \binom{50}{1}\binom{50}{2} + ... + \binom{50}{49}\binom{50}{50}$ का मान होगा, जहाँ ${}^nC_r =$

(A) $\binom{100}{50}$ (B*) $\binom{100}{51}$ (C) $\binom{50}{25}$ (D) $\binom{50}{25}^2$

Sol. ${}^{50}C_0 \times {}^{50}C_1 + {}^{50}C_1 \times {}^{50}C_2 + ... + {}^{50}C_{49} \times {}^{50}C_{50}$

$= {}^{50}C_0 \times {}^{50}C_{49} + {}^{50}C_1 \times {}^{50}C_{48} + \text{..........} + {}^{50}C_{49} \times {}^{50}C_0$

= co-eff. of x^{49} in $(1 + x)^{100}$ $= {}^{100}C_{49}$

Hindi ${}^{50}C_0 \times {}^{50}C_1 + {}^{50}C_1 \times {}^{50}C_2 + \text{..........} + {}^{50}C_{49} \times {}^{50}C_{50}$

$= {}^{50}C_0 \times {}^{50}C_{49} + {}^{50}C_1 \times {}^{50}C_{48} + \text{..........} + {}^{50}C_{49} \times {}^{50}C_0$

$= (1 + x)^{100}$ में x^{49} का गुणांक $= {}^{100}C_{49}$

Section (D) : Negative & fractional index, Multinomial theorem
खण्ड (D) : ऋणात्मक व भिन्न घातांक, बहुपदीय प्रमेय

D-1. If $|x| < 1$, then the co-efficient of x^n in the expansion of $(1 + x + x^2 + x^3 + \text{.......})^2$ is

यदि $|x| < 1$, तो $(1 + x + x^2 + x^3 + \text{.......})^2$ के प्रसार में x^n का गुणांक है –

(A) n (B) n – 1 (C) n + 2 (D*) n + 1

Sol. Co-efficient of x^n in $(1 - x)^{-2} = {}^{2+n-1}C_1 = n + 1$

Hindi. $(1 - x)^{-2}$ के प्रसार में x^n का गुणांक $= {}^{2+n-1}C_1 = n + 1$

D-2. The co–efficient of x^4 in the expansion of $(1 - x + 2x^2)^{12}$ is:

$(1 - x + 2x^2)^{12}$ के प्रसार में x^4 का गुणांक है –

(A) ${}^{12}C_3$ (B) ${}^{13}C_3$ (C) ${}^{14}C_4$ (D*) ${}^{12}C_3 + 3\ {}^{13}C_3 + {}^{14}C_4$

Sol. $(1 - x + 2x^2)^{12}$

General term $= \frac{12\,!}{r_1\,!\ r_2!\ r_3!}(1)^{r_1}(-x)^{r_2}\left(2x^2\right)^{r_3}$

$r_2 + 2r_3 = 4 \Rightarrow$ $r_3 = 0, r_2 = 4, r_1 = 8$

$r_3 = 1, r_2 = 2, r_1 = 9$

$r_3 = 2, r_2 = 0, r_1 = 10$

Co-efficient of $x^4 = \frac{12\,!}{4\,!\ 8\,!} + \frac{12\,!}{2\,!\ 10\,!}(2)^2 + \frac{12\,!}{2\,!\ 9\,!} \times (2)$

$= {}^{12}C_8 + 4.\ {}^{12}C_{10} + 6\ .\ {}^{12}C_9$

$= {}^{12}C_3 + 3\ .\ {}^{13}C_3 + {}^{14}C_4$ (after solving)

Hindi. $(1 - x + 2x^2)^{12}$

व्यापक पद $= \frac{12\,!}{r_1\,!\ r_2!\ r_3!}(1)^{r_1}(-x)^{r_2}\left(2x^2\right)^{r_3}$

$r_2 + 2r_3 = 4 \Rightarrow$ $r_3 = 0, r_2 = 4, r_1 = 8$

$r_3 = 1, r_2 = 2, r_1 = 9$

$r_3 = 2, r_2 = 0, r_1 = 10$

x^4 का गुणांक $= \frac{12\,!}{4\,!\ 8\,!} + \frac{12\,!}{2\,!\ 10\,!}(2)^2 + \frac{12\,!}{2\,!\ 9\,!} \times (2)$

$= {}^{12}C_8 + 4.\ {}^{12}C_{10} + 6\ .\ {}^{12}C_9 = {}^{12}C_3 + 3\ .\ {}^{13}C_3 + {}^{14}C_4$ हल करने पर

D-3. If $(1 + x)^{10} = a_0 + a_1x + a_2x^2 + \text{......} + a_{10}x^{10}$, then value of

$(a_0 - a_2 + a_4 - a_6 + a_8 - a_{10})^2 + (a_1 - a_3 + a_5 - a_7 + a_9)^2$ is

(A*) 2^{10} (B) 2 (C) 2^{20} (D) None of these

यदि $(1 + x)^{10} = a_0 + a_1x + a_2x^2 + \text{......} + a_{10}x^{10}$ हो, तो $(a_0 - a_2 + a_4 - a_6 + a_8 - a_{10})^2 + (a_1 - a_3 + a_5 - a_7 + a_9)^2$ का मान है–

(A) 2^{10} (B) 2 (C) 2^{20} (D) इनमें से कोई नहीं

Sol. $(1 + x)^{10} = a_0 + a_1 + a_2x^2 + ... + a_{10}x^{10}$

Put $x = i$,

$(1 + i)^{10} = a_0 - a_2 + a_4 + ...+ a_{10} + i\,(a_1 - a_3 + ...+ a_9)$

$a_0 - a_2 + a_4 + ...+ a_{10}$ = real part of $(1 + i)^{10} = 2^5\cos 10\pi/4$

$a_1 - a_3 + ...$ = imaginary part of $(1 + i)^{10} = 2^5 \sin 10\pi/4$ ....(2)

$(1)^2 + (2)^2 = 2^{10}$

Hindi. $(1 + x)^{10} = a_0 + a_1 + a_2x^2 + ... + a_{10}x^{10}$

$x = i$ रखने पर

$(1 + i)^{10} = a_0 - a_2 + a_4 + ...+ a_{10} + i\,(a_1 - a_3 + ...+ a_9)$

$a_0 - a_2 + a_4 + ...+ a_{10} = (1 + i)^{10}$ का वास्तविक भाग $= 2^5\cos 10\pi/4$

$a_1 - a_3 + ... = (1 + i)^{10}$ का काल्पनिक भाग $= 2^5 \sin 10\pi/4$ ....(2)

$(1)^2 + (2)^2 = 2^{10}$

PART - III : MATCH THE COLUMN

भाग - III : कॉलम को सुमेलित कीजिए (MATCH THE COLUMN)

1.

	Column – I		**Column – II**
(A)	If $(r + 1)^{th}$ term is the first negative term in the expansion of $(1 + x)^{7/2}$, then the value of r (where $0 < x < 1$) is	(p)	divisible by 2
(B)	If the sum of the co-efficients in the expansion of $(1 + 2x)^n$ is 6561, and T_r is the greatest term in the expansion for $x = 1/2$ then r is	(q)	divisible by 5
(C)	${}^nC_r$ is divisible by n, $(1 < r < n)$ if n is	(r)	divisible by 10
(D)	The coefficient of x^4 in the expression $(1 + 2x + 3x^2 + 4x^3 + ...\text{up to } \infty)^{1/2}$ is c, $(c \in N)$, then c + 1 (where $\lvert x \rvert < 1$) is	(s)	a prime number

Ans. (A) → (q, s), (B) → (q,s), (C) → (s), (D) → (p, s)

	स्तम्भ – I		**स्तम्भ – II**
(A)	यदि $(1 + x)^{7/2}$ के प्रसार में $(r + 1)$वाँ पद प्रथम ऋणात्मक पद है, तो r का मान है – (जहाँ $0 < x < 1$)	(p)	2 से विभाजित है
(B)	यदि $(1 + 2x)^n$ के प्रसार में गुणांको का योगफल 6561, और $x = 1/2$ के लिए T_r अधिकतम पद है तब r है।	(q)	5 से विभाजित है
(C)	${}^nC_r$, $(1 < r < n)$, n से विभाजित होगा यदि n है,	(r)	10 से विभाजित है
(D)	व्यंजक $(1 + 2x + 3x^2 + 4x^3 + ...$ अनन्त पदों तक$)^{1/2}$ में x^4 का गुणांक c, $(c \in N)$ है, तो c + 1 है – (जहाँ $\lvert x \rvert < 1$)	(s)	एक अभाज्य संख्या

Sol. (A) We have, $T_{r+1} = \dfrac{\frac{7}{2}\left(\frac{7}{2}-1\right)\left(\frac{7}{2}-2\right).....\left(\frac{7}{2}-r+1\right)x^r}{r\,!}$

This will be the first negative term when $\frac{7}{2} - r + 1 < 0$ i.e. $r > \frac{9}{2}$

Hence r = 5.

(B) $3^n = 6561$ (put x = 1) $\Rightarrow$ n = 8

$\dfrac{T_{r+1}}{T_r} = \dfrac{8-r+1}{r} \geq 1 \Rightarrow 8 - r + 1 \geq r \Rightarrow r \leq \dfrac{9}{2} \Rightarrow r = 4$ (5th term is greatest)

(C) Obviously a prime number.

(D) We have : $(1 + 2x + 3x^2 + 4x^3 + ...)^{1/2}$

$= [(1-x)^{-2}]^{1/2} = (1-x)^{-1} = 1 + x + x^2 + ... + x^n + ...$

Hence, coefficient of $x^4 = 1$ $\therefore$ c = 1, hence c + 1 = 2

Hindi (A) चूँकि $T_{r+1} = \dfrac{\frac{7}{2}\left(\frac{7}{2}-1\right)\left(\frac{7}{2}-2\right).....\left(\frac{7}{2}-r+1\right)x^r}{r\,!}$

यह प्रथम ऋणात्मक पद होगा यदि $\frac{7}{2} - r + 1 < 0$ i.e. $r > \frac{9}{2}$

अतः r = 5.

(B) $3^n = 6561$ (x = 1 रखने पर) $\Rightarrow$ n = 8

$\dfrac{T_{r+1}}{T_r} = \dfrac{8-r+1}{r} \geq 1 \Rightarrow 8 - r + 1 \geq r \Rightarrow r \leq \Rightarrow r = 4$ (5th पद महत्तम है।)

(C) स्पष्टतयाः एक अभाज्य संख्या।

(D) $(1 + 2x + 3x^2 + 4x^3 + ...)^{1/2}$

$= [(1-x)^{-2}]^{1/2} = (1-x)^{-1} = 1 + x + x^2 + ... + x^n + ...$

अतः x^4 का गुणांक = 1 $\therefore$ c = 1, अतः c + 1 = 2

Exercise-2

✍ **Marked questions are recommended for Revision.**
✍ **चिन्हित प्रश्न दोहराने योग्य प्रश्न है।**

PART - I : ONLY ONE OPTION CORRECT TYPE

भाग-I : केवल एक सही विकल्प प्रकार (ONLY ONE OPTION CORRECT TYPE)

1. In the expansion of

$\left(3\sqrt{\frac{a}{b}}+3\sqrt{\frac{b}{\sqrt{a}}}\right)^{21}$, the term containing same powers of a & b is

(A) 11^{th} term (B*) 13^{th} term (C) 12^{th} term (D) 6^{th} term

$\left(3\sqrt{\frac{a}{b}}+3\sqrt{\frac{b}{\sqrt{a}}}\right)^{21}$, के विस्तार में a और b की समान घातों का पद है –

(A) 11^{th} वां पद (B*) 13^{th} वां पद (C) 12^{th} वां पद (D) 6^{th} वां पद

Sol. $T_{r+1} = {}^{21}C_r\left(\frac{a}{b}\right)^{\frac{21-r}{3}}\left(\frac{b}{\sqrt{a}}\right)^{\frac{r}{3}}$

$= {}^{21}C_r .\ a^{\frac{21-r}{3}-\frac{r}{6}}\ \ b^{\frac{r}{3}-\frac{21-r}{3}}$

$= {}^{21}C_r .\ a^{\frac{42-3r}{6}} .\ b^{\frac{2r-21}{3}}$

$= {}^{21}C_r .\ a^{\frac{14-r}{2}} .\ b^{\frac{2r-21}{3}}$

$\frac{14-r}{2} = \frac{2r-21}{3}$

$42 - 3r = 4r - 42$

$2r = 84$

$r = 12 \quad \Rightarrow \quad T_{13}$ term (T_{13}वां पद)

2. Consider the following statements :

S_1 : Number of dissimilar terms in the expansion of $(1 + x + x^2 + x^3)^n$ is $3n + 1$

S_2 : $(1 + x)(1 + x + x^2)(1 + x + x^2 + x^3)......(1 + x + x^2 +...... + x^{100})$ when written in the ascending power of x then the highest exponent of x is 5000.

S_3 : $\sum_{k=1}^{n-r} {}^{n-k}C_r = {}^nC_{r+1}$

S_4 : If $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n}x^{2n}$, then $a_0 + a_2 + a_4 + ... + a_{2n} = \frac{3^n - 1}{2}$

State, in order, whether S_1, S_2, S_3, S_4 are true or false

माना कि निम्न कथन है –

S_1 : $(1 + x + x^2 + x^3)^n$ के प्रसार में असमान पदों की संख्या $3n + 1$ है।

S_2 : $(1 + x)(1 + x + x^2)(1 + x + x^2 + x^3)......(1 + x + x^2 +...... + x^{100})$ को यदि x की बढ़ती हुई घातों के क्रम में लिखा जाता है, तो x की अधिकतम घात 5000 होगी –

S_3 : $\sum_{k=1}^{n-r} {}^{n-k}C_r = {}^nC_{r+1}$

S_4 : यदि $(1 + x + x^2)^n = a_0 + a_1x + a_2 x^2 + ... + a_{2n} x^{2n}$, तो $a_0 + a_2 + a_4 + ... + a_{2n} = \frac{3^n - 1}{2}$

S_1, S_2, S_3, S_4 के सत्य (T) या असत्य (F) होने का सही क्रम है –

(A*) TFTF (B) TTTT (C) FFFF (D) FTFT

Sol. S1 : Number of dissimilar terms will be same as in $(1 + x)^{3n}$ i.e. 3n + 1

S_2 : $(1 + x)(1 + x + x^2)...... + (1 + x + ... + x^{100})$
Highest exponent of x = 1 + 2 + ...+ 100
= 5050

Hindi. $(1 + x)(1 + x + x^2)...... + (1 + x + ... + x^{100})$
x की अधिकतम घात = 1 + 2 + ...+ 100
= 5050

S_3 : $\sum_{k=1}^{n-r} {}^{n-k}C_r = {}^xC_y \quad \Rightarrow {}^xC_y = {}^{n-1}C_r + {}^{n-2}C_r + ... {}^rC_r$

$\Rightarrow {}^xC_y$ = co-efficient of x^r in $((1+x)^r +...+(1+x)^{n-1})$

= co-efficient of x^r in $(1+x)^r \left[\frac{(1+x)^{n-r} - 1}{x}\right]$ = co-efficient of x^{r+1} in $(1+x)^n = {}^nC_{r+1}$

Hindi $\sum_{k=1}^{n-r} {}^{n-k}C_r = {}^xC_y \quad \Rightarrow {}^xC_y = {}^{n-1}C_r + {}^{n-2}C_r + ... {}^rC_r$

$\Rightarrow {}^xC_y = ((1+x)^r +...+(1+x)^{n-1})$ में x^r का गुणांक

$= (1+x)^r \left[\frac{(1+x)^{n-r} - 1}{x}\right]$ में x^r का गुणांक = $(1+x)^n$ में x^{r+1} का गुणांक = ${}^nC_{r+1}$

S_4 : $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + ...+ a_{2n}x^{2n}$
put x = 1 रखने पर
$3^n = a_0 + a_1 + a_2 +.......+ a_{2n}$ ....(i)
x = – 1
$1 = a_0 - a_1 + a_2 + ...+ a_{2n}$ ....(ii)
adding (i) & (ii) (i) तथा (ii) को जोड़ने पर

$\frac{3^n + 1}{2} = a_0 + a_2 + ...+ a_{2n.}$

3. .If $\frac{{}^nC_r + 4\ {}^nC_{r+1} + 6\ {}^nC_{r+2} + 4\ {}^nC_{r+3} + {}^nC_{r+4}}{{}^nC_r + 3\,{}^nC_{r+1} + 3\ {}^nC_{r+2} + {}^nC_{r+3}} = \frac{n+k}{r+k}$ then the value of k is :

यदि $\frac{{}^nC_r + 4\ {}^nC_{r+1} + 6\ {}^nC_{r+2} + 4\ {}^nC_{r+3} + {}^nC_{r+4}}{{}^nC_r + 3\,{}^nC_{r+1} + 3\ {}^nC_{r+2} + {}^nC_{r+3}} = \frac{n+k}{r+k}$ हो तो k का मान ज्ञात कीजिए।

(A) 1 (B) 2 (C*) 4 (D) 5

Sol. Numerator $= {}^nC_r + {}^nC_{r+1} + 3({}^nC_{r+1} + {}^nC_{r+2}) + 3({}^nC_{r+2} + {}^nC_{r+3}) + {}^nC_{r+3} + {}^nC_{r+4}$
$= {}^{n+1}C_{r+1} + 3\ {}^{n+1}C_{r+2} + 3\ {}^{n+1}C_{r+3} + {}^{n+1}C_{r+4}$
$= {}^{n+1}C_{r+1} + {}^{n+1}C_{r+2} + 2({}^{n+1}C_{r+2} + {}^{n+1}C_{r+3}) + {}^{n+1}C_{r+3} + {}^{n+1}C_{r+4}$
$= {}^{n+2}C_{r+2} + 2\ {}^{n+2}C_{r+3} + {}^{n+2}C_{r+4}$

$= {}^{n+2}C_{r+2} + {}^{n+2}C_{r+3} + {}^{n+2}C_{r+4} + {}^{n+2}C_{r+3}$
$= {}^{n+3}C_{r+3} + {}^{n+3}C_{r+4} = {}^{n+4}C_{r+4}$
Denominator $= {}^{n}C_{r} + {}^{n}C_{r+1} + 2\,({}^{n}C_{r+1} + {}^{n}C_{r+2}) + ({}^{n}C_{r+2} + {}^{n}C_{r+3})$
$= {}^{n+1}C_{r+1} + 2\,{}^{n+1}C_{r+2} + {}^{n+1}C_{r+3}$
$= {}^{n+1}C_{r+1} + {}^{n+1}C_{r+2} + {}^{n+1}C_{r+2} + {}^{n+1}C_{r+3}$
$= {}^{n+2}C_{r+2} + {}^{n+2}C_{r+3} = {}^{n+3}C_{r+3}$

$\therefore$ The expression is equal to : $\dfrac{{}^{n+4}C_{r+4}}{{}^{n+3}C_{r+3}} = \dfrac{(n+4)!(r+3)!(n-r)!}{(r+4)!(n-r)!(n+3)!} = \dfrac{n+4}{r+4}$

$\therefore$ k = 4

4. The co-efficient of x^5 in the expansion of $(1 + x)^{21} + (1 + x)^{22} + ... + (1 + x)^{30}$ is :
$(1 + x)^{21} + (1 + x)^{22} + ... + (1 + x)^{30}$ के प्रसार में x^5 का गुणांक है :
(A) ${}^{51}C_5$ (B) ${}^{9}C_5$ (C*) ${}^{31}C_6 - {}^{21}C_6$ (D) ${}^{30}C_5 + {}^{20}C_5$

Sol. $(1 + x)^{21} [1 + (1 + x) + ... (1 + x)^9] = (1 + x)^{21} \left[\dfrac{(1+x)^{10}-1}{x}\right] = \dfrac{(1+x)^{31}-(1+x)^{21}}{x}$

Coefficient of $x^5 = {}^{31}C_6 - {}^{21}C_6$
x^5 का गुणांक $= {}^{31}C_6 - {}^{21}C_6$

5. The coefficient of x^{52} in the expansion $\sum\limits_{m=0}^{100} {}^{100}C_m\,(x-3)^{100-m}.\,2^m$ is :

$\sum\limits_{m=0}^{100} {}^{100}C_m\,(x-3)^{100-m}.\,2^m$ के प्रसार में x^{52} का गुणांक है –
(A) ${}^{100}C_{47}$ (B*) ${}^{100}C_{48}$ (C) $-{}^{100}C_{52}$ (D) $-{}^{100}C_{100}$

Sol. $S = \sum\limits_{m=0}^{100} {}^{100}C_m (x-3)^{100-m} 2^m$
$S = {}^{100}C_0\,(x-3)^{100} + {}^{100}C_1\,(x-3)^{99}\,.\,2 + ... + {}^{100}C_{100}\,.\,2^{100}$
$S = (2 + (x-3))^{100} = (x-1)^{100}$
Co-efficient of $x^{52} = {}^{100}C_{52} = {}^{100}C_{48}$
x^{52} का गुणांक $= {}^{100}C_{52} = {}^{100}C_{48}$

6. The sum of the coefficients of all the integral powers of x in the expansion of $\left(1+2\sqrt{x}\right)^{40}$ is :

$\left(1+2\sqrt{x}\right)^{40}$ के प्रसार में x की सभी पूर्णांक घातों के गुणांकों का योगफल है –

(A) $3^{40} + 1$ (B) $3^{40} - 1$ (C) $\dfrac{1}{2}(3^{40} - 1)$ (D*) $\dfrac{1}{2}(3^{40} + 1)$

Sol. $\left(1+2\sqrt{x}\right)^{40} = {}^{40}C_0 + {}^{40}C_1\, 2\sqrt{x} + ... + {}^{40}C_{40}\,(2\sqrt{x})^{40}$
$(1-2\sqrt{x})^{40} = {}^{40}C_0 - {}^{40}C_1\, 2\sqrt{x} + ... + {}^{40}C_{40}\,(2\sqrt{x})^{40}$
$(1+2\sqrt{x})^{40} + (1-2\sqrt{x})^{40} = 2\,[{}^{40}C_0 + {}^{40}C_2\,(2\sqrt{x})^2 + ... + {}^{40}C_{40}\,(2\sqrt{x})^{40}]$
Putting x = 1 रखने पर

${}^{40}C_0 + {}^{40}C_2(2)^2 + ... + {}^{40}C_{40}\,(2)^{40} = \dfrac{3^{40}+1}{2}$

7. $\sum\limits_{r=0}^{n} (-1)^r\ {}^{n}C_r.\dfrac{(1+r\ell n10)}{(1+\ell n10^n)^r} =$

$\sum_{r=0}^{n}(-1)^r\ {}^nC_r.\frac{(1+r\ell n10)}{(1+\ell n10^n)^r}$ का मान है–

(A*) 0 (B) 1/2 (C) 1 (D) None of these (इनमें से कोई)

Sol. $\sum_{r=0}^{n}(-1)^r\ {}^nC_r.\frac{1}{(1+n\ell n10)^r}+\sum_{r=0}^{n}(-1)^r r.{}^nC_r\ \frac{\ell n10}{(1+n\ell n10)^r}$

$= \left(1-\frac{1}{1+n\ell n10}\right)^n + n\ell n10\sum_{r=1}^{n}(-1)^r\ {}^{n-1}C_{r-1}\frac{1}{(1+n\ell n10)^r}$

$= \left(\frac{n\ell 10}{1+n\ell 10}\right)^n - \frac{n\ell 10}{1+\ell 10}\sum_{r=1}^{n}(-1)^{r-1}\ {}^{n-1}C_{r-1}\left(\frac{1}{n\ell n10}\right)^{r-1}$

$= \left(\frac{n\ell 10}{1+n\ell 10}\right)^n - \frac{n\ell 10}{1+n\ell 10}\times\left(1-\frac{1}{1+n\ell 10}\right)^{n-1}$

$= \left(\frac{n\ell 10}{1+\ell n10}\right)^n - \frac{n\ell 10}{1+\ell n\ell 10}\times\frac{(n\ell 10)^{n-1}}{(1+n\ell 10)^{n-1}} = 0$

8. The coefficient of the term independent of x in the expansion of $\left(\frac{x+1}{x^{\frac{2}{3}}-x^{\frac{1}{3}}+1}-\frac{x-1}{x-x^{\frac{1}{2}}}\right)^{10}$ is :

$\left(\frac{x+1}{x^{\frac{2}{3}}-x^{\frac{1}{3}}+1}-\frac{x-1}{x-x^{\frac{1}{2}}}\right)^{10}$ के प्रसार में x से स्वतंत्र पद का गुणांक है –

(A) 70 (B) 112 (C) 105 (D*) 210

Sol. $\left(\frac{x+1}{x^{\frac{2}{3}}-x^{\frac{1}{3}}+1}-\frac{x-1}{x-x^{\frac{1}{2}}}\right)^{10} = \left(x^{1/3}+1-1-\frac{1}{\sqrt{x}}\right)^{10}$

$T_{r+1} = {}^{10}C_r(x^{1/3})^{10-r}\left(-\frac{1}{\sqrt{x}}\right)^r$

For independent term $\frac{10-r}{3}-\frac{r}{2}=0\ \Rightarrow r = 4$

Coefficient of the term independent of x = ${}^{10}C_4$

Hindi. $\left(\frac{x+1}{x^{\frac{2}{3}}-x^{\frac{1}{3}}+1}-\frac{x-1}{x-x^{\frac{1}{2}}}\right)^{10} = \left(x^{1/3}+1-1-\frac{1}{\sqrt{x}}\right)^{10}$

$T_{r+1} = {}^{10}C_r(x^{1/3})^{10-r}\left(-\frac{1}{\sqrt{x}}\right)^r$

स्वतन्त्र पद हेतु $\frac{10-r}{3}-\frac{r}{2}=0\ \Rightarrow r = 4$

x से स्वतन्त्र पद का गुणांक = ${}^{10}C_4$

9.✍ Coefficient of x^{n-1} in the expansion of, $(x + 3)^n + (x + 3)^{n-1} (x + 2) + (x + 3)^{n-2} (x + 2)^2 + ... + (x + 2)^n$ is :

$(x + 3)^n + (x + 3)^{n-1} (x + 2) + (x + 3)^{n-2} (x + 2)^2 + ... + (x + 2)^n$ के प्रसार में x^{n-1} का गुणांक है –

(A) ${}^{n+1}C_2(3)$ (B) ${}^{n-1}C_2(5)$ (C*) ${}^{n+1}C_2(5)$ (D) ${}^{n}C_2(5)$

Sol. $(x+3)^n + (x+3)^{n-1}(x+2) + \ldots\ldots + (x+2)^n = (x+3)^n \left(\dfrac{1-\left(\dfrac{x+2}{x+3}\right)^{n+1}}{1-\dfrac{x+2}{x+3}} \right) = [(x+3)^{n+1} - (x+2)^{n+1}]$

Coefficient of x^{n-1} का गुणांक $= {}^{n+1}C_{n-1}(3)^2 - {}^{n+1}C_{n-1} \times 4$

10. Let $f(n) = 10^n + 3.4^{n+2} + 5$, $n \in N$. The greatest value of the integer which divides f(n) for all n is :

(A) 27 (B*) 9 (C) 3 (D) None of these

माना $f(n) = 10^n + 3.4^{n+2} + 5$, $n \in N$ तब वह अधिकतम पूर्णांक जो f(n) को n के प्रत्येक मान के लिए विभाजित करता है–

(A) 27 (B) 9 (C) 3 (D) इनमें से कोई नहीं

Sol. $f(n) = 10^n + 3.4^{n+2} + 5$

put n = 1

f(1) = 10 + 192 + 5 = 207 this is divisible by 3 and 9

f(1) = 10 + 192 + 5 = 207 यह 3 तथा 9 दोनों से भाज्य है।

11. If $(1+x)^n = \sum_{r=0}^{n} a_r x^r$ and $b_r = 1 + \dfrac{a_r}{a_{r-1}}$ and $\prod_{r=1}^{n} b_r = \dfrac{(101)^{100}}{100\,!}$, then n equals to :

(A) 99 (B*) 100 (C) 101 (D) 102

यदि $(1+x)^n = \sum_{r=0}^{n} a_r x^r$ और $b_r = 1 + \dfrac{a_r}{a_{r-1}}$ और $\prod_{r=1}^{n} b_r = \dfrac{(101)^{100}}{100\,!}$, तो n बराबर है :

(A) 99 (B) 100 (C) 101 (D) 102

Sol. $(1+x)^n = \sum_{r=0}^{n} a_r x^r = a_0 + a_1x + \ldots\ldots. + a_nx^n$

$b_r = 1 + \dfrac{a_r}{a_{r-1}} = 1 + \dfrac{n-r+1}{r} = \dfrac{n+1}{r}$

$\prod_{n=1}^{n} b_r = b_1b_2\ldots\ldots..b_n = \dfrac{(n+1)^n}{1.\ \ 2.3\ldots\ldots n} = \dfrac{(101)^{100}}{100!} \quad \Rightarrow n = 100$

12. Number of rational terms in the expansion of $\left(1+\sqrt{2}+\sqrt{5}\right)^6$ is :

$\left(1+\sqrt{2}+\sqrt{5}\right)^6$ के विस्तार में परिमेंय पदों की संख्या है –

(A) 7 (B*) 10 (C) 6 (D) 8

Sol. General term is = व्यापक पद

$= \dfrac{6!}{r_1!\ r_2!\ r_3!}(1)^{r_1}(\sqrt{2})^{r_2}(\sqrt{5})^{r_3} = \dfrac{6!}{r_1!\ r_2!\ r_3!}(2)^{\frac{r_2}{2}}(2)^{\frac{r_3}{2}}$ where जहाँ $r_1 + r_2 + r_3 = 6$

r_1	r_2	r_3
2	4	0
4	2	0
0	4	2
2	2	2
4	0	2
0	2	4
2	0	4
0	0	6
0	6	0
6	0	0

10 terms are possible

r_1	r_2	r_3
2	4	0
4	2	0
0	4	2
2	2	2
4	0	2
0	2	4
2	0	4
0	0	6
0	6	0
6	0	0

10 पद संभव है।

13. If $S = {}^{404}C_4 - {}^4C_1 . {}^{303}C_4 + {}^4C_2 . {}^{202}C_4 - {}^4C_3 . {}^{101}C_4 = (101)^k$ then k equals to :

यदि $S = {}^{404}C_4 - {}^4C_1 . {}^{303}C_4 + {}^4C_2 . {}^{202}C_4 - {}^4C_3 . {}^{101}C_4 = (101)^k$ तब k का मान है–

(A) 1 (B) 2 (C*) 4 (D) 6

Sol. S = coeff. of x^4 in S = x^4 में का गुणांक

$= \left[{}^4C_0 . \left((1+x)^{101}\right)^4 - {}^4C_1\left((1+x)^{101}\right)^3 + {}^4C_2\left((1+x)^{101}\right)^2 - {}^4C_3\left((1+x)^{101}\right)^1 + {}^4C_4 \right] - {}^4C_4$

$= \left((1+x)^{101} - 1\right)^4 - 1$

$= (1 + {}^{101}C_1\, x + {}^{101}C_2\, x^2 + {}^{101}C_3\, x^3 + {}^{101}C_4\, x^4 + \ldots\ldots\ldots - 1)^4 - 1$

$= x^4\left({}^{101}C_1 + {}^{101}C_2 x + {}^{101}C_3 x^2 + \ldots.\right)^4 - 1$

$= x^4\left(101 + ({}^{101}C_2 x + {}^{101}C_3 x^2 + \ldots.)\right)^4 - 1$

$= 101^4$

k = 4

14. ${}^{10}C_0^{\,2} - {}^{10}C_1^{\,2} + {}^{10}C_2^{\,2} - \ldots. - \left({}^{10}C_9\right)^2 + \left({}^{10}C_{10}\right)^2 =$

(A) 0 (B) $\left({}^{10}C_5\right)^2$ (C*) $-{}^{10}C_5$ (D) $2\,{}^9C_5$

Sol. $(1+x)^{10} = {}^{10}C_0 + {}^{10}C_1 x + {}^{10}C_2 x^2 + \ldots + {}^{10}C_9 x^9 + {}^{10}C_{10} x^{10}$

$(x-1)^{10} = {}^{10}C_0 x^{10} - {}^{10}C_1 x^9 + {}^{10}C_2 x^8 - \ldots - {}^{10}C_9 . x + {}^{10}C_{10}$

S = coff. of x^{10} in $\left(x^2 - 1\right)^{10}$

$= -{}^{10}C_5$

15. The sum $\sum_{r=0}^{n} (r+1)\, C_r^2$ is equal to :

योगफल $\sum_{r=0}^{n} (r+1)\, C_r^2$ बराबर है –

(A*) $\frac{(n+2)\,(2n-1)\,!}{n\,!\,(n-1)\,!}$ (B) $\frac{(n+2)\,(2n+1)\,!}{n\,!\,(n-1)\,!}$ (C) $\frac{(n+2)\,(2n+1)\,!}{n\,!\,(n+1)\,!}$ (D) $\frac{(n+2)\,(2n-1)\,!}{n\,!\,(n+1)\,!}$

Sol. $\because$ $(1+x)^n = C_0 + C_1 x + \ldots\ldots\ldots\ldots + C_n x^n$

Multiply by x & then differentiate

$(1+x)^n + x.\, n(1+x)^{n-1} = C_0 + 2C_1 x + \ldots\ldots\ldots\ldots + (n+1)C_n x^n$ ....(i)

and $(x+1)^n = C_0 x^n + C_1 x^{n-1} + \ldots\ldots\ldots\ldots + C_n$ ....(ii)

Multiply (i) & (ii) & equate the coefficient of x^n on both side

$C_0^2 + 2C_1^2 + \text{........} + (n+1)\,C_n^2 = {}^{2n}C_n + n.\,{}^{2n-1}C_{n-1} = \frac{(2n)\,!}{(n!)^2} + n\frac{(2n-1)\,!}{n!(n-1)!} = (n+2)\frac{(2n-1)\,!}{n!(n-1)!}$

Hindi $\because$ $(1+x)^n = C_0 + C_1x + \text{............} + C_nx^n$

x से गुणा करके अवकलन करने पर

$(1+x)^n + x.\,n(1+x)^{n-1} = C_0 + 2C_1x + \text{............} + (n+1)C_nx^n$ ....(i)

तथा $(x+1)^n = C_0x^n + C_1x^{n-1} + \text{............} + C_n$ ....(ii)

(i) और (ii) का गुणा करके दोनों पक्षों में x^n के गुणांकों की तुलना करने पर

$C_0^2 + 2C_1^2 + \text{........} + (n+1)\,C_n^2 = {}^{2n}C_n + n.\,{}^{2n-1}C_{n-1} = \frac{(2n)\,!}{(n!)^2} + n\frac{(2n-1)\,!}{n!(n-1)!} = (n+2)\frac{(2n-1)\,!}{n!(n-1)!}$

16. If $(1 + x + x^2 + x^3)^5 = a_0 + a_1x + a_2x^2 + \text{.......................} + a_{15}x^{15}$, then a_{10} equals to :

यदि $(1 + x + x^2 + x^3)^5 = a_0 + a_1x + a_2x^2 + \text{.......................} + a_{15}x^{15}$ हो, तो $a_{10} =$

(A) 99 (B*) 101 (C) 100 (D) 110

Sol. $(x^4 - 1)^5\,(x-1)^{-5} = {}^5C_0\,(x-1)^{-5} - {}^5C_1\,x^4\,(x-1)^{-5} + {}^5C_2\,x^8\,(x-1)^{-5} = {}^5C_0 \times {}^{14}C_4 - {}^5C_1 \times {}^{10}C_6 + {}^5C_2 \times {}^6C_2 = 101$

17. If $a_n = \sum_{r=0}^{n} \frac{1}{{}^nC_r}$, the value of $\sum_{r=0}^{n} \frac{n-2r}{{}^nC_r}$ is :

यदि $a_n = \sum_{r=0}^{n} \frac{1}{{}^nC_r}$ हो, तो $\sum_{r=0}^{n} \frac{n-2r}{{}^nC_r}$ का मान होगा –

(A) $\frac{n}{2}a_n$ (B) $\frac{1}{4}a_n$ (C) na_n (D*) 0

Sol. $a_n = \sum_{r=0}^{n} \frac{1}{{}^nC_r}$; $S = \sum_{r=0}^{n} \frac{n-2r}{{}^nC_r}$; $S = \sum_{r=0}^{n} \frac{n-2(n-r)}{{}^nC_r}$; $2S = 0 \Rightarrow S = 0$

18. The sum of: $3.{}^nC_0 - 8.{}^nC_1 + 13.{}^nC_2 - 18.{}^nC_3 + \text{....}$ upto (n+1) terms is $(n \geq 2)$:

(A*) zero (B) 1 (C) 2 (D) none of these

$3.{}^nC_0 - 8.{}^nC_1 + 13.{}^nC_2 - 18.{}^nC_3 + \text{....}$ के (n+1) पदों का योगफल है $(n \geq 2)$:

(A) शून्य (B) 1 (C) 2 (D) इनमें से कोई नहीं

Sol. $3\,.\,{}^nC_0 - 8\,.\,{}^nC_1 + 13\,.\,{}^nC_2 - 18\,{}^nC_3 + \text{....}$ up to (n + 1) terms

$(1 + x^5)^n = C_0 + C_1x^5 + C_2x^{10} + \text{......} + C_nx^{5n}$

Multiplying by x^3 and differentiating w.r.t. x

$x^3\,.\,n(1 + x^5)^{n-1}\,.\,5x^4 + 3x^2\,(1 + x^5)^n = 3C_0\,x^2 + 8C_1\,x^7 + 13C_2\,x^{12} + \text{......} + (5n+3)\,C_n\,x^{5n+2}$

Now put $x = -1$

$3C_0 - 8C_1 + 13\,C_2 + \text{..........} + (n+1)$ terms = 0

Hindi. $3\,.\,{}^nC_0 - 8\,.\,{}^nC_1 + 13\,.\,{}^nC_2 - 18\,{}^nC_3 + \text{....}$ (n + 1) पदों तक

$(1 + x^5)^n = C_0 + C_1x^5 + C_2x^{10} + \text{......} + C_nx^{5n}$

x^3 से गुणा करके x के सापेक्ष गुणा करने पर

$x^3\,.\,n(1 + x^5)^{n-1}\,.\,5x^4 + 3x^2\,(1 + x^5)^n = 3C_0\,x^2 + 8C_1\,x^7 + 13C_2\,x^{12} + \text{......} + (5n+3)\,C_n\,x^{5n+2}$

अब $x = -1$ रखने पर

$3C_0 - 8C_1 + 13\,C_2 + \text{..........} + (n+1)$ पदों तक = 0

19. If यदि $\sum_{r=0}^{n-1}\left(\frac{{}^nC_r}{{}^nC_r + {}^nC_{r+1}}\right)^3 = \frac{4}{5}$ then तब n =

(A*) 4 (B) 6 (C) 8 (D) None of these इनमें से कोइ नहीं

Sol. $\sum_{r=0}^{n-1}\left(\frac{{}^nC_r}{{}^nC_r + {}^nC_{r+1}}\right)^3 = \frac{4}{5}$ $\Rightarrow \sum_{r=0}^{n-1}\left(\frac{\frac{n!}{r!(n-r!)}}{\frac{(n+1)!}{(r+1)!(n-r)!}}\right)^3 = \frac{4}{5}$ $\Rightarrow \sum_{r=0}^{n-1}\left(\frac{r+1}{n+1}\right)^3 = \frac{4}{5}$

$\Rightarrow$ $\frac{1}{(n+1)^3}$ $(1^3 + 2^3 + 3^3 + \ldots\ldots\ldots + n^3) = 4/5$

$\Rightarrow$ $\frac{1}{(n+1)^3} \times \frac{n^2(n+1)^2}{4} = \frac{4}{5} \Rightarrow$ $5n^2 - 16n - 16 = 0$ $\Rightarrow$ $n = 4$

20. The number of terms in the expansion of $\left(x^2 + 1 + \frac{1}{x^2}\right)^n$, $n \in N$, is :

$\left(x^2 + 1 + \frac{1}{x^2}\right)^n$, $n \in N$ के प्रसार में पदों की संख्या है –

(A) 2n (B) 3n (C*) 2n + 1 (D) 3n + 1

Sol. $\left(\left(x + \frac{1}{x}\right)^2 - 1\right)^n = {}^nC_0 \left(x + \frac{1}{x}\right)^{2n} - {}^nC_1 \left(x + \frac{1}{x}\right)^{2n-2} + \ldots. + {}^nC_n (-1)^n$

Total number of terms = 2n + 1 कुल पदों की संख्या = 2n + 1

21. Suppose

$$\det\begin{bmatrix} \sum_{k=0}^{n} k & \sum_{k=0}^{n} {}^nC_k k^2 \\ \sum_{k=0}^{n} {}^nC_k k & \sum_{k=0}^{n} {}^nC_k 3^k \end{bmatrix} = 0$$

{[BT-BC]-M-305}

holds for some positive integer n. Then $\sum_{k=0}^{n} \frac{{}^nC_k}{k+1}$ equals.

मानाकि

किसी धनात्मक पूर्णांक n के लिए

$$\det\begin{bmatrix} \sum_{k=0}^{n} k & \sum_{k=0}^{n} {}^nC_k k^2 \\ \sum_{k=0}^{n} {}^nC_k k & \sum_{k=0}^{n} {}^nC_k 3^k \end{bmatrix} = 0, \text{ तो } \sum_{k=0}^{n} \frac{{}^nC_k}{k+1} \text{ का मान है–}$$

[JEE(Advanced) 2019, Paper-2 ,(4, –1)/62]

Ans. (6.20) **[Binomial Theorem_M]**

Sol. $\begin{vmatrix} \frac{n(n+1)}{2} & n2^{n-1}+n(n-1)2^{n-2} \\ n2^{n-1} & 4^n \end{vmatrix} = 0$

$\frac{n(n+1)}{2} - \frac{n^2}{4} - \frac{n^2(n-1)}{8} = 0$

n = 0 or $4(n + 1) - 2n - n(n - 1) = 0$

$4n + 4 - 2n - n^2 + n = 0$

$3n - n^2 + 4 = 0 \Rightarrow n^2 - 3n - 4 = 0$

$(n - 4)(n + 1) = 0$

$n = 4$

$\sum_{r=0}^{4} \frac{{}^4C_r}{r+1} = \sum_{r=0}^{4} \frac{{}^5C_{r+1}}{5} = \frac{2^5-1}{5} = \frac{31}{5} = 6.20$

PART-II: NUMERICAL VALUE QUESTIONS

भाग-II : संख्यात्मक प्रश्न (NUMERICAL VALUE QUESTIONS)

INSTRUCTION :

- The answer to each question is **NUMERICAL VALUE** with two digit integer and decimal upto two digit.
- If the numerical value has more than two decimal places **truncate/round-off** the value to **TWO** decimal placed.

निर्देश :

- इस खण्ड में प्रत्येक प्रश्न का उत्तर **संख्यात्मक मान** के रूप में है जिसमें दो पूर्णांक अंक तथा दो अंक दशमलव के बाद में है।
- यदि संख्यात्मक मान में दो से अधिक दशमलव स्थान है, तो संख्यात्मक मान को दशमलव के **दो** स्थानों तक **ट्रंकेट/राउंड ऑफ (truncate/round-off)** करें।

1. If $\frac{1}{1!10!}+\frac{1}{2!9!}+\frac{1}{3!8!}+......+\frac{1}{10!1!}=\frac{(2^{10}-1)}{k\ 10!}$ then find the value of k.

यदि $\frac{1}{1!10!}+\frac{1}{2!9!}+\frac{1}{3!8!}+......+\frac{1}{10!1!}=\frac{(2^{10}-1)}{k\ 10!}$ तब k का मान ज्ञात कीजिए।

Ans. 05.50

Sol. $\frac{1}{11!}\left[\frac{11!}{1!10!}+\frac{11!}{2!9!}+\frac{11!}{3!10!}+......+\frac{11!}{1!10!}\right] = \frac{1}{11!}\ [{}^{11}C_1 + {}^{11}C_2 + ...+ {}^{11}C_{10}]$

$= \frac{1}{11!}\,[2^{11}-2] = \frac{2}{11!}\,[2^{10}-1] \Rightarrow k = 11$

2. If the 6^{th} term in the expansion of $\left[\frac{1}{x^{8/3}} + x^2 \log_{10} x\right]^8$ is 5600, then x =

यदि $\left[\frac{1}{x^{8/3}} + x^2 \log_{10} x\right]^8$ के प्रसार में छठवां पद 5600 है, तो x =

Ans. **10.00**

Sol. $T_6 = {}^8C_5\left(\frac{1}{x^{8/3}}\right)^3 (x^2 \log_{10} x)^5 = 5600 \quad \Rightarrow \frac{1}{x^8} x^{10} (\log_{10} x)^5 = 100 \quad \Rightarrow x = 10$

3. The number of values of ' x ' for which the fourth term in the expansion,

$$\left(5^{\frac{2}{5}\log_5\sqrt{4^x+44}} + \frac{1}{5^{\log_5\sqrt[3]{2^{x-1}+7}}}\right)^8$$ is 336, is :

' x ' के मानों की संख्या होगी जिसके लिए व्यंजक $\left(5^{\frac{2}{5}\log_5\sqrt{4^x+44}} + \frac{1}{5^{\log_5\sqrt[3]{2^{x-1}+7}}}\right)^8$ में चौथा पद 336 है –

Ans. **02.00**

Sol. $T_4 = {}^8C_3\left(5^{\frac{1}{5}\log_5(4^x+44)}\right)^5\left(\frac{1}{5^{\frac{1}{3}\log_5(2^{x-1}+7)}}\right)^3 \quad\Rightarrow\quad {}^8C_3(4^x+44)\left(\frac{1}{2^{x-1}+7}\right) = 336$

$\Rightarrow \quad \frac{4^x+44}{2^{x-1}+7} = 6 \quad\Rightarrow\quad 4^x + 44 = 3.2^x + 42 \quad\Rightarrow\quad (2^x)^2 - 3.2^x + 2 = 0$

$\Rightarrow \quad (2^x - 1)(2^x - 2) = 0 \quad\Rightarrow\quad x = 0 \ \& \ 1$

4. If second, third and fourth terms in the expansion of $(x + a)^n$ are 240, 720 and 1080 respectively, then ratio of last term and first term is.
यदि $(x + a)^n$ के विस्तार में दुसरा, तीसरा और चौथा पद क्रमशः 240, 720 तथा 1080 है तब अन्तिम पद तथा प्रथम पद का अनुपात होगा –

Ans. **07.59**

Sol. $T_2 = {}^nC_1\,(x)^{n-1}\,.\,a = 240$ ....(i)
$T_3 = {}^nC_2\,(x)^{n-2}\,a^2 = 720$ ....(ii)
$T_4 = {}^nC_3\,(x)^{n-3}\,a^3 = 1080$ ....(iii)

From (i) and (ii) (i) तथा (ii) से

Here यहाँ $\frac{{}^nC_1(x)^{n-1}a}{{}^nC_2x^{n-2}a^2} = \frac{2x}{(n-1)a} = \frac{240}{720} = \frac{1}{3} \quad\Rightarrow\quad 6x = (n-1)a$

From (ii) and (iii) (ii) तथा (iii) से

$9x = 2(n-2)\,a$

On dividing भाग देने पर $\frac{3}{2} = \frac{2(n-2)}{(n-1)} \Rightarrow 3n - 3 = 4n - 8 \Rightarrow n = 5$

hence $\frac{T_6}{T_1} = \frac{{}^5C_5(x)^0(a)^5}{{}^5C_0x^5a^0} = \left(\frac{a}{x}\right)^5 = \left(\frac{3}{2}\right)^5 = \mathbf{07.59}$

5. Let the co-efficients of x^n in $(1 + x)^{2n}$ & $(1 + x)^{2n-1}$ be P & Q respectively, then $\left(\frac{P + Q}{P}\right)^4 =$

मानाकि $(1 + x)^{2n}$ एवं $(1 + x)^{2n-1}$ के प्रसार में x^n के गुणांक क्रमशः P एवं Q है, तो $\left(\frac{P + Q}{P}\right)^4 =$

Ans. **05.06**

Sol. $P = {}^{2n}C_n$ and तथा $Q = {}^{2n-1}C_n \quad\Rightarrow\quad \frac{P}{Q} = 2 \ ; \ \left(1+\frac{Q}{P}\right)^4 = (1 + \frac{1}{2})^4 = \frac{81}{16}$

6. In the expansion of $\left(3^{\frac{-x}{4}}+3^{\frac{5x}{4}}\right)^n$, the sum of the binomial coefficients is 256 and four times the term with greatest binomial coefficient exceeds the square of the third term by 21n, then find x.

$\left(3^{\frac{-x}{4}}+3^{\frac{5x}{4}}\right)^n$ के विस्तार में द्विपद गुणाकों का योग 256 है और अधिकतम द्विपद गुणांक का चार गुना, तीसरे पद के वर्ग से 21n, अधिक है तब x का मान ज्ञात कीजिए।

Ans. **00.50**

Sol. $2^n = 256 = 2^8$

$n = 8$

$\left(3^{\frac{-x}{4}}+3^{\frac{5x}{4}}\right)^8$

$4\,T_5 = T_3^2 + 21n$

$4 \times {}^8C_4 \times 3^{\frac{-x}{4}\times 4} \times 3^{\frac{5x}{4}\times 4} = \left({}^8C_2 \times 3^{\frac{-x}{4}\times 6} \times 3^{\frac{5x}{4}\times 2}\right)^2 + 21n$

$1120 \times 3^{x/4} = (28 \times 3^x)^2 + 21n$

$1120 \times 3^{x/4} = 28^2 \times 3^{2x} + 21 \times 8$

$\Rightarrow \quad x = \frac{1}{2}$

7. If $\sum_{k=1}^{19}\frac{(-2)^k}{k!(19-k)!} = \frac{1}{k\ 18!}$ then find k.

यदि $\sum_{k=1}^{19}\frac{(-2)^k}{k!(19-k)!} = \frac{1}{k\ 18!}$ हो तो k का मान ज्ञात कीजिए।

Ans. **09.50**

Sol. $\frac{1}{19!}\sum_{k=1}^{19}(-2)^k.{}^{19}C_k$

$= \frac{1}{19!}\sum_{k=1}^{19}(-1)^k.2^k.{}^{19}C_k$

$= \frac{1}{19!}\left[-{}^{19}C_1.2 + {}^{19}C_2.2^2 - {}^{19}C_3.2^3 + \ldots - 2^{19}.{}^{19}C_{19}\right]$

$= \frac{1}{19!}\left[{}^{19}C_0 - {}^{19}C_1 + {}^{19}C_2.2^2 - \ldots 2^{19}.{}^{19}C_{19} - 1\right]$

$= \frac{1}{19!}\left((1-2)^{19} - 1\right) = \frac{-2}{19!}$

8. The value of p, for which coefficient of x^{50} in the expression $(1 + x)^{1000} + 2x\,(1 + x)^{999} + 3x^2\,(1 + x)^{998} + \ldots.. + 1001\,x^{1000}$ is equal to ${}^{1002}C_p$, is :

व्यंजक $(1 + x)^{1000} + 2x\,(1 + x)^{999} + 3x^2\,(1 + x)^{998} + \ldots.. + 1001\,x^{1000}$ में x^{50} का गुणांक ${}^{1002}C_p$ है, तो p का मान है

Ans. **50.00**

Sol. Co-efficient of x^{50} (x^{50} का गुणांक)

$S = (1 + x)^{1000} + 2x(1 + x)^{999} + 3x^2(1 + x)^{998} + \ldots.. + 1001x^{1000}$ ....(i)

$$\frac{xS}{1+x} = x(1+x)^{999} + 2x^2(1+x)^{998} .... + 1000x^{1000} + \frac{1001\ \ x^{1001}}{(1+x)} \qquad ....(ii)$$

$$\frac{S}{1+x} = (1+x)^{1000} + x(1+x)^{999} + ... + x^{1000} - \frac{1001\ \ x^{1001}}{1+x}$$

$$\Rightarrow \qquad \frac{S}{1+x} = (1+x)^{1000}\left[\frac{1-\left(\frac{x}{1+x}\right)^{1001}}{1-\frac{x}{1+x}}\right] - \frac{1001\ \ x^{1001}}{(1+x)}$$

$$\Rightarrow \qquad S = (1+x)^{1002} - x^{1001}(1+x) - 1001\ x^{1001}$$

Co-efficient of $x^{50} = {}^{1002}c_{50}$ (x^{50} का गुणांक $= {}^{1002}c_{50}$)

9. If { x } denotes the fractional part of ' x ', and $\left\{\frac{3^{1001}}{82}\right\} = \frac{1}{\lambda}$ then value of λ is

यदि { x } , ' x ' के भिन्नात्मक भाग को प्रदर्शित करता है, तथा $\left\{\frac{3^{1001}}{82}\right\} = \frac{1}{\lambda}$ हो तो λ का मान होगा –

Ans. **27.33**

Sol. $$\left\{\frac{3^{1001}}{82}\right\} = \left\{\frac{3.(82-1)^{250}}{82}\right\} = \left\{\frac{3.[{}^{250}C_0(82)^{250} + {}^{250}C_1(82)^{249}(-1) + ... + {}^{250}C_{250}]}{82}\right\} = \frac{3}{82}$$

10. The index ' n ' of the binomial $\left(\frac{x}{5} + \frac{2}{5}\right)^n$ if the only 9th term of the expansion has numerically the greatest coefficient ($n \in N$) then find $\frac{T_9}{T_8}$ (where T_r denote coefficient of r^{th} term from beginning in the expansion)

यदि $\left(\frac{x}{5} + \frac{2}{5}\right)^n$, ($n \in N$) के प्रसार में केवल 9वाँ पद संख्यात्मक रूप से महत्तम गुणांक वाला पद हो, तो $\frac{T_9}{T_8}$ का मान ज्ञात कीजिए (जहाँ T_r प्रसार में प्रारम्भ से rवें पद के गुणांक को प्रदर्शित करता है)

Ans. **01.25**

Sol. For T_9 to be the numerically greatest term, $r = \left[\frac{n+1}{1+\left|\frac{x}{a}\right|}\right] = \left[\frac{n+1}{1+\left|\frac{1}{2}\right|}\right] = 8$

$$\Rightarrow \qquad 8 < \frac{2(n+1)}{3} < 9 \Rightarrow 11 < n < 12.5 \Rightarrow n = 12$$

then $$\frac{T_9}{T_8} = \frac{{}^{12}C_8\left(\frac{1}{5}\right)^4\left(\frac{2}{5}\right)^8}{{}^{12}C_7\left(\frac{1}{5}\right)^5\left(\frac{2}{5}\right)^7} = \frac{5}{4}$$

Hindi. चूँकि T_9 संख्यात्मक रूप से महत्तम पद है अतः $r = \left[\frac{n+1}{1+\left|\frac{x}{a}\right|}\right] = \left[\frac{n+1}{1+\left|\frac{1}{2}\right|}\right] = 8 \qquad \Rightarrow \qquad 8 < \frac{2(n+1)}{3} < 9$

$\Rightarrow 11 < n < 12.5 \Rightarrow n = 12$

तब $\frac{T_9}{T_8} = \frac{{}^{12}C_8\left(\frac{1}{5}\right)^4\left(\frac{2}{5}\right)^8}{{}^{12}C_7\left(\frac{1}{5}\right)^5\left(\frac{2}{5}\right)^7} = \frac{5}{4}$

11. Sum of square of all possible values of 'r' satisfying the equation,

$${}^{39}C_{3r-1} - {}^{39}C_{r^2} = {}^{39}C_{r^2-1} - {}^{39}C_{3r} \text{ is :}$$

समीकरण ${}^{39}C_{3r-1} - {}^{39}C_{r^2} = {}^{39}C_{r^2-1} - {}^{39}C_{3r}$ को संतुष्ट करने वाले 'r' के सभी संभावित मानों के वर्गों का योग होगा –

Ans. **34.00**

Sol. ${}^{39}C_{3r-1} - {}^{39}C_{r^2} = {}^{39}C_{r^2-1} - {}^{39}C_{3r}$

$\Rightarrow {}^{39}C_{3r-1} + {}^{39}C_{3r} = {}^{39}C_{r^2-1} + {}^{39}C_{r^2}$

$\Rightarrow {}^{40}C_{3r} = {}^{40}C_{r^2} \Rightarrow$ (i) $r^2 = 3r \Rightarrow r = 0, 3$

(ii) $r^2 + 3r = 40 \Rightarrow r = 5, -8$

possible values are 3,5

12. Find the value of

${}^6C_0 . {}^{12}C_6 - {}^6C_1\, {}^{11}C_6 + {}^6C_2\, {}^{10}C_6 - {}^6C_3 . {}^9C_6 + {}^6C_4 . {}^8C_6 - {}^6C_5 . {}^7C_6 + {}^6C_6 . {}^6C_6$ का मान ज्ञात कीजिए

Ans. **01.00**

Sol. Coeff of x^6 : ${}^6C_0.(1+x)^{12} - {}^6C_1(1+x)^{11} + {}^6C_2(1+x)^{10} - {}^6C_3(1+x)^9$

$+ {}^6C_4(1+x)^8 - {}^6C_5(1+x)^7 + {}^6C_6(1+x)^6$

$$= (1+x)^{12}\left[{}^6C_0 - {}^6C_1\left(\frac{1}{1+x}\right) + {}^6C_2\left(\frac{1}{1+x}\right)^2 - {}^6C_3\left(\frac{1}{1+x}\right)^3 + {}^6C_4\left(\frac{1}{1+x}\right)^4 - {}^6C_5\left(\frac{1}{1+x}\right)^5 + {}^6C_6\left(\frac{1}{1+x}\right)^6\right]$$

$= (1+x)^{12}.(1 - \frac{1}{1+x})^6$

$= (1+x)^6 . x^6$

$= 1 \times$ coeff of $x^6 = 1$

13. If n is a positive integer & $C_k = {}^nC_k$, find the value of $\left(\sum_{k=1}^{n}\frac{k^3}{n(n+1)^2.(n+2)}\left(\frac{C_k}{C_{k-1}}\right)^2\right)$ is :

यदि n एक धनात्मक पूर्णांक है तथा $C_k = {}^nC_k$ तब $\left(\sum_{k=1}^{n}\frac{k^3}{n(n+1)^2.(n+2)}\left(\frac{C_k}{C_{k-1}}\right)^2\right)$ का मान है–

Ans. **00.08**

Sol $\sum_{k=1}^{n} k^3\left(\frac{n-k+1}{k}\right)^2 = \sum_{k=1}^{n} k(n-k+1)^2 = \sum_{k=1}^{n}(n^2k + k^3 + k - 2nk^2 + 2nk - 2k^2)$

$= \frac{(n+1)^2.n(n+1)}{2} + \left[\frac{n(n+1)}{2}\right]^2 - \frac{2(n+1)n(n+1)(2n+1)}{6} = \frac{n(n+1)^2(n+2)}{12}$

14. The value of the expression $\left(\sum_{r=0}^{10} {}^{10}C_r\right)\left(\sum_{K=0}^{10}(-1)^K \frac{{}^{10}C_K}{2^K}\right)$ is :

व्यंजक $\left(\sum_{r=0}^{10} {}^{10}C_r\right)\left(\sum_{K=0}^{10}(-1)^K \frac{{}^{10}C_K}{2^K}\right)$ का मान है–

Ans. **01.00**

Sol. $\left(\sum_{r=0}^{10} {}^{10}C_r\right)\left(\sum_{k=0}^{10}(-1)^k \frac{{}^{10}C_k}{2^k}\right) = ({}^{10}C_0 + ... + {}^{10}C_{10})\left({}^{10}C_0 - \frac{{}^{10}C_1}{2} + \frac{{}^{10}C_2}{2^2} .... + \frac{{}^{10}C_{10}}{2^{10}}\right) = 2^{10} \times \left(1 - \frac{1}{2}\right)^{10} = 1$

15. The value of λ if $\sum_{m=97}^{100} {}^{100}C_m \,.\, {}^{m}C_{97} = \lambda\, {}^{99}C_{96}$ is :

λ का मान होगा यदि $\sum_{m=97}^{100} {}^{100}C_m \,.\, {}^{m}C_{97} = \lambda\, {}^{99}C_{96}$ है।

Ans. **08.24 or 08.25**

Sol. $\sum_{m=p}^{n} {}^{n}C_m . {}^{m}C_p = \sum_{m=p}^{n} \frac{n\,!}{m\,!(n-m)\,!} \times \frac{m\,!}{p\,!(m-p)\,!} = \sum_{m=p}^{n} {}^{n}C_p . {}^{n-p}C_{m-p}$

$= {}^{n}C_p\,[{}^{n-p}C_0 + {}^{n-p}C_1 + ... + {}^{n-p}C_{n-p}] = {}^{n}C_p\, 2^{n-p}$; where n = 100 and p = 97.

जहाँ n = 100 तथा p = 97

16. If $(1 + x + x^2 + ... + x^p)^n = a_0 + a_1x + a_2x^2 + ... + a_{np}\, x^{np}$, then the value of :

$\frac{1}{p(p+1)^7}\,[a_1 + 2a_2 + 3a_3 + ... + 7p\, a_{7p}]$ is :

यदि $(1 + x + x^2 + ... + x^p)^n = a_0 + a_1x + a_2x^2 + ... + a_{np}\, x^{np}$, हो, तो $\frac{1}{p(p+1)^7}\,[a_1 + 2a_2 + 3a_3 + ... + 7p\, a_{7p}]$ का मान है–

Ans. **03.50**

Sol $(1 + x + x^2 + ... + x^p)^n = a_0 + a_1x + ... + a_{np}\, x^{np}$

Differentiating both side w.r.t. x x के सापेक्ष अवकलन करने पर

$n(1 + x + x^2 + ... + x^p)^{n-1}(1 + 2x + ... + px^{p-1}) = a_1 + 2a_2x + ... + np\, a_{np}x^{np-1}$

Now put x = 1 रखने पर

$a_1 + 2a_2 + ... + np\, a_{np} = n(p + 1)^{n-1}(1 + 2 + ... + p) = \frac{n(p+1)^n.\ p}{2}$, where $n = \frac{7}{2}$

17. If $({}^{2n}C_1)^2 + 2.\,({}^{2n}C_2)^2 + 3.\,({}^{2n}C_3)^2 + ... + 2n.\,({}^{2n}C_{2n})^2 = 18\,.\,{}^{4n-1}C_{2n-1}$, then n is :

यदि $({}^{2n}C_1)^2 + 2.\,({}^{2n}C_2)^2 + 3.\,({}^{2n}C_3)^2 + ... + 2n.\,({}^{2n}C_{2n})^2 = 18\,.\,{}^{4n-1}C_{2n-1}$, तब n है –

Ans. **09.00**

Sol. $\because \quad (1+x)^{2n} = {}^{2n}C_0 + {}^{2n}C_1x + ... + {}^{2n}C_{2n}x^{2n}$

differentiating it

$2n(1+x)^{2n-1} = {}^{2n}C_1 + 2.\,{}^{2n}C_2x + ... + 2n\,{}^{2n}C_{2n}x^{2n-1}$

Again $(x+1)^{2n} = {}^{2n}C_0x^{2n} + {}^{2n}C_1x^{2n-1} + {}^{2n}C_2x^{2n-2} + ... + {}^{2n}C_{2n}$

Required expression = coefficient of x^{2n-1} in $2n\,(1+x)^{4n-1}$

$= 2n.\,{}^{4n-1}C_{2n-1}$

Hindi $\because \quad (1+x)^{2n} = {}^{2n}C_0 + {}^{2n}C_1x + ... + {}^{2n}C_{2n}x^{2n}$

अवकलन करने पर

$2n(1+x)^{2n-1} = {}^{2n}C_1 + 2.\,{}^{2n}C_2x + ...+ 2n{}^{2n}C_{2n}x^{2n-1}$

पुनः $(x+1)^{2n} = {}^{2n}C_0x^{2n} + {}^{2n}C_1x^{2n-1} + {}^{2n}C_2x^{2n-2} + ...+ {}^{2n}C_{2n}$

अभीष्ट व्यंजक $= 2n\,(1+x)^{4n-1}$ में x^{2n-1} का गुणांक

$= 2n.\,{}^{4n-1}C_{2n-1}$

18. If $\sum_{r=0}^{n}\frac{2r+3}{r+1}.{}^{n}C_r = \frac{(2n+3k)2^n-1}{n+1}$ then 'k' is

यदि $\sum_{r=0}^{n}\frac{2r+3}{r+1}.{}^{n}C_r = \frac{(2n+3k)2^n-1}{n+1}$ है तब 'k' का मान है

Ans. **01.33**

Sol. $\sum_{r=0}^{n}\frac{2r+3}{r+1}.{}^{n}C_r = \sum_{r=0}^{n}2.{}^{n}C_r + \sum_{r=0}^{n}\frac{1}{r+1}.{}^{n}C_r = 2\,.2^2 + \frac{1}{n+1}\,.\,\sum_{r=0}^{n}.{}^{n+1}C_{r+1}$

$= 2^{n+1} + \frac{1}{n+1}.(2^{n+1}-1) = \frac{(n+2).2^{n+1}-1}{n+1}$

19. If $\sum_{r=0}^{n}\frac{(-1)^r.C_r}{(r+1)(r+2)(r+3)} = \frac{a}{(n+b)}$, then a + b is

यदि $\sum_{r=0}^{n}\frac{(-1)^r.C_r}{(r+1)(r+2)(r+3)} = \frac{a}{(n+b)}$ है, तब a + b का मान है

Ans. **03.50**

Sol. $\sum_{r=0}^{n}\frac{(-1)^r.C_r}{(r+1)(r+2)(r+3)} = \frac{-1}{(n+1)(n+2)(n+3)}$

$\frac{-1}{(n+1)(n+2)(n+3)}\left[(1-1)^{n+3} - \left\{{}^{n+3}C_0(-1)^0 + {}^{n+3}C_1(-1)^1 + {}^{n+3}C_2(-1)^2\right\}\right]$

$\frac{1}{(n+1)(n+2)(n+3)}\;(n+2)\times\frac{(n+1)}{2} = \frac{1}{2(n+3)}$

20. $\sum_{k=1}^{3n}{}^{6n}C_{2k-1}(-3)^k$ is equal to :

$\sum_{k=1}^{3n}{}^{6n}C_{2k-1}(-3)^k$ बराबर है

Ans. **00.00**

Sol. $S = \sum_{k=1}^{3n}{}^{6n}C_{2k-1}(-3)^k \quad\Rightarrow\quad S = {}^{6n}C_1(-3) + {}^{6n}C_3(-3)^2 + ...+ {}^{6n}C_{6n-1}(-3)^{3n}$

$\Rightarrow S = (\sqrt{3}\;i)\sum_{k=1}^{3n}{}^{6n}C_{2k-1}(\sqrt{3}\;i)^{2k-1} \qquad \Rightarrow S = (\sqrt{3}\;i)\left[\frac{(1+\sqrt{3}\;i)^{6n}-(1-\sqrt{3}\;i)^{6n}}{2}\right] = 0$

21. If x is very large as compare to y, then the value of k in $\sqrt{\frac{x}{x+y}}\sqrt{\frac{x}{x-y}} = 1+\frac{ky^2}{x^2}$

यदि x, y की तुलना में बहुत बड़ा हो, तो $\sqrt{\frac{x}{x+y}}\sqrt{\frac{x}{x-y}} = 1+\frac{ky^2}{x^2}$ में k का मान है

Ans. **00.50**

Sol. $\sqrt{\frac{x}{x+y}}\sqrt{\frac{x}{x-y}} = \left(\frac{1}{1+\frac{y}{x}}\right)^{1/2}\left(\frac{1}{1-\frac{y}{x}}\right)^{1/2} = \left(1-\frac{y^2}{x^2}\right)^{-1/2} = 1+\frac{1}{2}\cdot\frac{y^2}{x^2}$ $\Rightarrow$ k = 2

PART - III : ONE OR MORE THAN ONE OPTIONS CORRECT TYPE
भाग - III : एक या एक से अधिक सही विकल्प प्रकार

1. In the expansion of $\left(\sqrt[3]{4}+\frac{1}{\sqrt[4]{6}}\right)^{20}$

(A*) the number of irrational terms is 19 (B*) middle term is irrational
(C*) the number of rational terms is 2 (D*) 9th term is rational

$\left(\sqrt[3]{4}+\frac{1}{\sqrt[4]{6}}\right)^{20}$ के प्रसार में

(A) अपरिमेय पदों की संख्या 19 है। (B) मध्य पद अपरिमेय है।
(C) परिमेय पदों की संख्या 2 है। (D) 9वाँ पद परिमेय है।

Sol. $\left(\sqrt[3]{4}+\frac{1}{\sqrt[4]{6}}\right)^{20}$

$T_{r+1} = {}^{20}C_r(4^{1/3})^{20-r}(6^{-1/4})^r$

For rational terms

$20 - r = 3k$ & $r = 4p$, where $k, p \in I$ $\Rightarrow$ r = 20 & r = 8

$\therefore$ no. of rational terms = 2 $\therefore$ no. of irrational terms = 19

Hindi $\left(\sqrt[3]{4}+\frac{1}{\sqrt[4]{6}}\right)^{20}$

$T_{r+1} = {}^{20}C_r(4^{1/3})^{20-r}(6^{-1/4})^r$

परिमेय पदों हेतु

$20 - r = 3k$ तथा $r = 4p$, जहाँ $k, p \in I$

$\Rightarrow$ r = 20 तथा r = 8 $\therefore$ परिमेय पदों की संख्या = 2 $\therefore$ अपरिमेय पदों की संख्या = 19

2. The coefficient of x^4 in $\left(\frac{1+x}{1-x}\right)^2$, $|x| < 1$, is

$\left(\frac{1+x}{1-x}\right)^2$, $|x| < 1$ में x^4 का गुणांक है–

(A) 4 (B) –4 (C*) $10 + {}^4C_2$ (D*) 16

Sol. $(1 + x)^2 (1 - x)^{-2} = (1 + x^2 + 2x)(1 - x)^{-2}$

Co-efficient of x^4 का गुणांक $= {}^5C_4 + {}^3C_2 + 2\,{}^4C_3 = 16$

3. $7^9 + 9^7$ is divisible by :

(A*) 16 (B) 24 (C*) 64 (D) 72

$7^9 + 9^7$ विभाजित हैं –

(A) 16 से (B) 24 से (C) 64 से (D) 72 से

Sol. $7^9 + 9^7 = (8-1)^9 + (8+1)^7 = {}^9C_0(8)^9 - {}^9C_1(8)^8 + {}^9C_2(8)^7 \,.....+ {}^9C_8(8) - {}^9C_9 + {}^7C_0(8)^7 + ...+ {}^7C_6(8) + {}^7C_7$

This is divisible by 64 & 16 यह 64 और 16 से भाजित है

4. The sum of the series $\sum_{r=1}^{n}(-1)^{r-1}.\ {}^nC_r(a-r)$ is equal to :

(A*) 5 if a = 5 (B) –5 if a = 5 (C*) –5 if a = –5 (D) 5 if a = –5

श्रेणी $\sum_{r=1}^{n}(-1)^{r-1}.\ {}^nC_r(a-r)$ का योगफल बराबर है

(A*) 5 यदि a = 5 (B) –5 यदि a = 5 (C*) –5 यदि a = –5 (D) 5 यदि a = –5

Sol. $= a\sum_{r=1}^{n}(-1)^{r-1}.\ {}^nC_r - \sum_{r=1}^{n} r.\ {}^nC_r(-1)^{r-1} = a[{}^nC_1 - {}^nC_2 + {}^nC_3 \,.....+ (-1)^{n-1}.{}^nC_n] - n\sum_{r=1}^{n}(-1)^{r-1}\ {}^{n-1}C_{r-1}$

$= a(1) - n[{}^{n-1}C_0 - {}^{n-1}C_1 + ...+ (-1)^{(n-1)}\,{}^{n-1}C_{n-1}] = a - n(0) = a$

5. Let $a_n = \frac{1000^n}{n!}$ for $n \in N$, then a_n is greatest, when

मानाकि $n \in N$ के लिए $a_n = \frac{1000^n}{n!}$ हो, तो a_n महत्तम होगा, यदि

(A) n = 997 (B) n = 998 (C*) n = 999 (D*) n = 1000

Sol. $a_n = \frac{(1000)(1000)............(1000)}{1.2.........n}$

$a_{999} = a_{1000}$

a_n is maximum for n = 999 and n = 1000

Hindi $a_n = \frac{(1000)(1000)............(1000)}{1.2.........n}$

$a_{999} = a_{1000}$

a_n , n = 999 तथा n = 1000 के लिए अधिकतम है।

6. ${}^nC_0 - 2.3\ {}^nC_1 + 3.3^2\ {}^nC_2 - 4.3^3\ {}^nC_3 + ...+ (-1)^n (n+1)\ {}^nC_n\ 3^n$ is equal to

(A*) $2^n\left(\frac{3n}{2}+1\right)$ if n is even (B) $2^n\left(n+\frac{3}{2}\right)$ if n is even

(C*) $-\ 2^n\left(\frac{3n}{2}+1\right)$ if n is odd (D) $2^n\left(n+\frac{3}{2}\right)$ if n is odd

${}^nC_0 - 2.3\ {}^nC_1 + 3.3^2\ {}^nC_2 - 4.3^3\ {}^nC_3 + ...+ (-1)^n (n+1)\ {}^nC_n\ 3^n$ का मान बराबर है

(A*) $2^n\left(\frac{3n}{2}+1\right)$ यदि n सम है। (B) $2^n\left(n+\frac{3}{2}\right)$ यदि n सम है।

(C*) $-\ 2^n\left(\frac{3n}{2}+1\right)$ यदि n विषम है। (D) $2^n\left(n+\frac{3}{2}\right)$ यदि n विषम है।

Sol. $(1+x)^n = {}^nC_0 + {}^nC_1\,x + {}^nC_2\,x^2 + ...+ {}^nC_n\,x^n$

Multiply it by x

$x(1+x)^n = {}^nC_0\,x + {}^nC_1\,x^2 + {}^nC_2x^3 + ... {}^nC_nx^{n+1}$

Differentiate w.r. to x and put x = – 3

$n\,x\,(1+x)^{n-1} + (1+x)^n = {}^nC_0 + 2{}^nC_1\,x + 3\,{}^nC_2\,x^2 + 4\,{}^nC_3\,x^3 + ...+ (n+1)\,{}^nC_n\,x^n$

So answer, $-\,3n\,(-2)^{n-1} + (-2)^n$

$$= (-2)^n\left(1+\frac{3n}{2}\right) = (-1)^n 2^n\left(\frac{3n}{2}+1\right)$$

Hindi $(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + ... + {}^nC_n x^n$

x से गुणा करने पर

$x(1 + x)^n = {}^nC_0 x + {}^nC_1 x^2 + {}^nC_2 x^3 + ... {}^nC_n x^{n+1}$

x के सापेक्ष अवकलन करके $x = -3$ रखने पर

$n x (1 + x)^{n-1} + (1 + x)^n = {}^nC_0 + 2{}^nC_1 x + 3\,{}^nC_2 x^2 + 4\,{}^nC_3 x^3 + ... + (n + 1)\,{}^nC_n x^n$

अतः उत्तर $-3n(-2)^{n-1} + (-2)^n = (-2)^n\left(1+\frac{3n}{2}\right) = (-1)^n 2^n\left(\frac{3n}{2}+1\right)$

7. Element in set of values of r for which, ${}^{18}C_{r-2} + 2.\,{}^{18}C_{r-1} + {}^{18}C_r \ge {}^{20}C_{13}$ is :

r के मानों की संख्या होगी जिसके लिए ${}^{18}C_{r-2} + 2.\,{}^{18}C_{r-1} + {}^{18}C_r \ge {}^{20}C_{13}$ है

(A*) 9 (B) 5 (C*) 7 (D*) 10

Sol. ${}^{18}C_{r-2} + 2.{}^{18}C_{r-1} + {}^{18}C_r \ge {}^{20}C_{13}$

or या ${}^{19}C_{r-1} + {}^{19}C_r \ge {}^{20}C_{13}$

or या ${}^{20}C_r \ge {}^{20}C_{13}$

r = 7,8,9,10,11,12,13

8. The expansion of $(3x + 2)^{-1/2}$ is valid in ascending powers of x, if x lies in the interval.

$(3x + 2)^{-1/2}$ का प्रसार x की बढ़ती हुई घातों के लिए वैध है यदि x अन्तराल में स्थित होगा।

(A*) (0, 2/3) (B) (–3/2, 3/2)

(C*) (–2/3, 2/3) (D) (–∞, –3/2) (3/2, ∞)

Sol. $(3x + 2)^{-1/2}$ has infinite expansion when $\left|\frac{3x}{2}\right| < 1 \Rightarrow x \in \left(-\frac{2}{3}, \frac{2}{3}\right)$

Hindi. $(3x + 2)^{-1/2}$ का प्रसार अनन्त पदों तक होगा यदि $\left|\frac{3x}{2}\right| < 1 \Rightarrow x \in \left(-\frac{2}{3}, \frac{2}{3}\right)$

9. If $(1 + 2x + 3x^2)^{10} = a_0 + a_1x + a_2x^2 + ... + a_{20}x^{20}$, then :

यदि $(1 + 2x + 3x^2)^{10} = a_0 + a_1x + a_2x^2 + ... + a_{20}x^{20}$ हो, तो

(A*) $a_1 = 20$ (B*) $a_2 = 210$

(C*) $a_4 = 8085$ (D) $a_{20} = 2^2.\,3^7.\,7$

Sol. General term व्यापक पद $= \frac{10!}{r_1!\ r_2!\ r_3!}(1)^{r_1}(2x)^{r_2}(3x^2)^{r_3}$

a_1 = Coeff. of x (x का गुणांक)

$r_2 + 2r_3 = 1 \Rightarrow r_2 = 1, r_1 = 9, r_3 = 0$ $\therefore$ $a_1 = \frac{10!}{1!\,9!}(2)^1 = 20$

a_2 = Coeff. of x^2 (x^2 का गुणांक)

$r_2 + 2r_3 = 2 \Rightarrow r_2 = 2, r_1 = 8, r_3 = 0$

$r_2 = 0, r_1 = 9, r_3 = 1$

$a_2\ \frac{10!}{2!\ 8!} = (2)^2 + \frac{10!}{9!\ 1!}(3) = 210$

a_4 = coeff. of x^4 (x^4 का गुणांक)

$r_2 + 2r_3 = 4 \Rightarrow r_2 = 4, r_1 = 6, r_3 = 0$

$r_2 = 2, r_1 = 7, r_3 = 1$

$r_2 = 0, r_1 = 8, r_3 = 2$

$a_4 = \frac{10!}{4!\,6!}\ (2)^4 + \frac{10!}{2!7!1!}\ (2)^2\,(3) + \frac{10!}{8!2!}\ (3)^2 = 8085$

$a_{20} = 3^{10}$

10. In the expansion of $(x + y + z)^{25}$

(A*) every term is of the form $^{25}C_r.\ ^rC_k.\ x^{25-r}.\ y^{r-k}.\ z^k$
(B*) the coefficient of $x^8\,y^9\,z^9$ is 0
(C) the number of terms is 325
(D) none of these

$(x + y + z)^{25}$ के प्रसार में

(A) प्रत्येक पद $^{25}C_r.\ ^rC_k.\ x^{25-r}.\ y^{r-k}.\ z^k$ रूप में होगा।
(B) $x^8\,y^9\,z^9$ का गुणांक 0 है।
(C) पदों की संख्या 325 है।
(D) इनमें से कोई नहीं

Sol. $(x + y + z)^{25}$

General term व्यापक पद $= \frac{25!}{r_1!\ r_2!\ r_3!}\ x^{r_1}y^{r_2}z^{r_3}$

Putting $r_3 = k$, $r_2 = r - k$ and तथा $r_1 = 25 - r$ रखने पर

$= \frac{25!}{(25-r)!(r-k)!(k)!} \times \frac{r!}{r!} \times x^{25-r}\,y^{r-k}\,z^k = {}^{25}C_r\ .\ {}^rC_k\ .\ x^{25-r}\,y^{r-k}\,z^k$

$r_1 + r_2 + r_3 = 25$

$\therefore$ coefficient of $x^8\,y^9\,z^9$ is 0 $\qquad \therefore$ $x^8\,y^9\,z^9$ का गुणांक $= 0$

11. If $(1 + x + 2x^2)^{20} = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_{40}x^{40}$, then $a_0 + a_2 + a_4 \ldots\ldots + a_{38}$ is equal to :

यदि $(1 + x + 2x^2)^{20} = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_{40}x^{40}$ हो, तो $a_0 + a_2 + a_4 \ldots\ldots + a_{38}$ बराबर है –

(A) $2^{19}\,(2^{30} + 1)$ (B*) $2^{19}(2^{20} - 1)$ (C*) $2^{39} - 2^{19}$ (D) $2^{39} + 2^{19}$

Sol. $(1 + x + 2x^2)^{20} = a_0 + a_1x + \ldots\ldots + a_{40}x^{40}$

$x = 1$, then $a_0 + a_1 + \ldots\ldots + a_{40} = 4^{20}$

$x = -1$, then $a_0 - a_1 + a_2 - \ldots\ldots + a_{40} = 2^{20}$

$2^{20} + 2^{40} = 2[a_0 + a_2 + \ldots\ldots + a_{38} + a_{40}]$

$\Rightarrow a_0 + a_2 + \ldots\ldots + a_{38} = 2^{19} + 2^{39} - 2^{20} = 2^{19}\,(2^{20}-1) \quad \therefore a_{40} = a^{20}$

Hindi. $(1 + x + 2x^2)^{20} = a_0 + a_1x + \ldots\ldots + a_{40}x^{40}$

$x = 1$, तो $a_0 + a_1 + \ldots\ldots + a_{40} = 4^{20}$

$x = -1$, तो $a_0 - a_1 + a_2 - \ldots\ldots + a_{40} = 2^{20}$

$2^{20} + 2^{40} = 2[a_0 + a_2 + \ldots\ldots + a_{38} + a_{40}]$

$\Rightarrow a_0 + a_2 + \ldots\ldots + a_{38} = 2^{19} + 2^{39} - 2^{20} = 2^{19}\,(2^{20}-1) \quad \therefore a_{40} = a^{20}$

12. $n^n\left(\frac{n+1}{2}\right)^{2n}$ is $(n \in N)$

(A) Less than $\left(\frac{n+1}{2}\right)^3$
(B*) Greater than or equal to $\left(\frac{n+1}{2}\right)^3$
(C) Less than $(n!)^3$
(D*) Greater than or equal to $(n!)^3$.

$n^n\left(\frac{n+1}{2}\right)^{2n}$ है– $(n \in N)$

(A) $\left(\frac{n+1}{2}\right)^3$ से छोटा
(B*) $\left(\frac{n+1}{2}\right)^3$ से बड़ा या बराबर
(C) $(n!)^3$ से छोटा
(D*) $(n!)^3$ से बड़ा या बराबर

Sol. $n^n\left(\frac{n+1}{2}\right)^{2n} = \left(\frac{\left(\frac{n(n+1)}{2}\right)^2}{n}\right)^n = \left(\frac{1^3+2^3+...+n^3}{n}\right)^n$; $\frac{1^3+2^3+....+n^3}{n} \geq \sqrt[n]{(n!)^3}$

13. If recursion polynomials $P_k(x)$ are defined as $P_1(x) = (x-2)^2$, $P_2(x) = ((x-2)^2-2)^2$
$P_3(x) = ((x-2)^2 - 2)^2 - 2)^2$ .... (In general $P_k(x) = (P_{k-1}(x) - 2)^2$, then the constant term in $P_k(x)$ is
(A*) 4 (B) 2 (C) 16 (D*) a perfect square
यदि व्यंजक $P_k(x)$ इस तरह से परिभाषित है कि $P_1(x) = (x-2)^2$, $P_2(x) = ((x-2)^2-2)^2$
$P_3(x) = ((x-2)^2 - 2)^2 - 2)^2$ (व्यापक रूप से $P_k(x) = (P_{k-1}(x) - 2)^2$), तो $P_k(x)$ में अचर पद है–
(A*) 4 (B) 2 (C) 16 (D*) एक पूर्ण वर्ग

Sol. Constant term in $P_1(x)$ is 4
If the constant term in $P_k(x)$ is also 4, then
$P_k(x) = 4 + a_1x + a_2x^2 + ...$
and $P_{k+1}(x) = (P_k(x) - 2)^2 = (a_1x + a_2x^2 + ...+ 2)^2$

Hindi $P_1(x)$ में नियत पद 4 है।
$P_k(x)$ में भी नियत पद 4 है, तो
$P_k(x) = 4 + a_1x + a_2x^2 + ...$
और $P_{k+1}(x) = (P_k(x) - 2)^2 = (a_1x + a_2x^2 + ...+ 2)^2$

PART - IV : COMPREHENSION

भाग - IV : अनुच्छेद (COMPREHENSION)

Comprehension # 1 (Q. No. 1 to 3)

Consider, sum of the series $\sum\sum_{0 \le i < j \le n} f(i)\, f(j)$

In the given summation, i and j are not independent.

In the sum of series $\sum_{i=1}^{n}\sum_{j=1}^{n} f(i) \quad f(j) = \sum_{i=1}^{n}\left(f(i) \left(\sum_{j=1}^{n} f(j)\right)\right)$ i and j are independent. In this summation, three types of terms occur, those when i < j, i > j and i = j.

Also, sum of terms when i < j is equal to the sum of the terms when i > j if f(i) and f(j) are symmetrical. So, in that case

$$\sum_{i=1}^{n}\sum_{j=1}^{n} f(i)f(j) = \sum\sum_{0\le i<j\le n} f(i)f(j) + \sum\sum_{0\le i<j\le n} f(i)f(j) + \sum\sum_{i=j} f(i)f(j)$$

$$= 2\sum\sum_{0\le i<j\le n} f(i)f(j) + \sum\sum_{i=j} f(i)f(j)$$

$$\Rightarrow \quad \sum\sum_{0\le i<j\le n} f(i)f(j) = \frac{\sum_{i=0}^{n}\sum_{j=0}^{n} f(i)f(j) - \sum\sum_{i=j} f(i)f(j)}{2}$$

When f(i) and f(j) are not symmetrical, we find the sum by listing all the terms.

अनुच्छेद # 1 (Q. No. 1 to 3)

माना कि श्रेणियों का योगफल $\sum\sum_{0 \le i < j \le n}$ f(i) f(j) सूत्र से दिया जाता है जहाँ i तथा j स्वतंत्र नहीं है।

श्रेणियों के योगफल $\sum_{i=1}^{n}\sum_{j=1}^{n} f(i)\ \ f(j) = \sum_{i=1}^{n}\left(f(i)\ \left(\sum_{j=1}^{n} f(j)\right)\right)$ में i व j स्वतंत्र है। इस योगफल में तीन प्रकार के पद होते है। जिनमें i < j, i > j तथा i = j तथा जब i < j के लिए पदो का योगफल, i > j के लिए पदों के योगफल के बराबर है। यदि f(i) तथा f(j) सममित है। इस स्थिति में

$$\sum_{i=1}^{n}\sum_{j=1}^{n} f(i)f(j) = \sum\sum_{0\le i<j\le n} f(i)f(j)$$

$$+ \sum\sum_{0\le i<j\le n} f(i)f(j) + \sum\sum_{i=j} f(i)f(j)$$

$$= 2\sum\sum_{0\le i<j\le n} f(i)f(j) + \sum\sum_{i=j} f(i)f(j)$$

$$\Rightarrow \qquad \sum\sum_{0\le i<j\le n} f(i)f(j) = \frac{\sum_{i=0}^{n}\sum_{j=0}^{n} f(i)f(j) - \sum\sum_{i=j} f(i)f(j)}{2}$$

जब f(i) तथा f(j) सममित नहीं है। तब हम सभी पदों का योगफल ज्ञात करते है।

1. $\sum\sum_{0\le i<j\le n} {}^nC_i\ \ {}^nC_j$ is equal to

$\sum\sum_{0\le i<j\le n} {}^nC_i\ \ {}^nC_j$ का मान बराबर है–

(A*) $\frac{2^{2n} - {}^{2n}C_n}{2}$ (B) $\frac{2^{2n} + {}^{2n}C_n}{2}$ (C) $\frac{2^{2n} - {}^{n}C_n}{2}$ (D) $\frac{2^{2n} + {}^{n}C_n}{2}$

Sol. $\sum\sum_{0\le i<j\le n} {}^nC_i\ \ {}^nC_j$

$$= \frac{\left(\sum_{i=0}^{n}\sum_{j=0}^{n} {}^nC_i\ \ {}^nC_j\right) - \sum_{i=0}^{n} ({}^nC_i)^2}{2} = \frac{\left(\sum_{i=0}^{n} {}^nC_i\ \ 2^n\right) - \sum_{i=0}^{n} ({}^nC_i)^2}{2} = \frac{2^n2^n - \sum_{i=0}^{n} ({}^nC_i)^2}{2} = \frac{2^{2n} - {}^{2n}C_n}{2}$$

2. Let ${}^0C_0 = 1$, then $\sum_{m=0}^{n}\ \sum_{p=0}^{m} {}^nC_m\ .\ {}^mC_p$ is equal to

माना ${}^0C_0 = 1$, तब $\sum_{m=0}^{n}\ \sum_{p=0}^{m} {}^nC_m\ .\ {}^mC_p$ का मान बराबर है–

(A) $2^n–1$ (B*) 3^n (C) $3^n–1$ (D) 2^n

Sol. $\sum_{m=0}^{n}\ \sum_{p=0}^{m} {}^nC_m\ .\ {}^mC_p = \sum_{m=0}^{n} {}^nC_m\ \left(\sum_{p=0}^{m} {}^mC_p\right) = \sum_{m=0}^{n} {}^nC_m\ \left(2^m\right) = 3^n$

3. $\sum\sum_{0\le i\le j\le n} \left({}^nC_i + {}^nC_j\right)$

(A*) $(n + 2)2^n$ (B) $(n + 1)2^n$ (C) $(n – 1)2^n$ (D) $(n + 1)2^n–1$

Sol. $\sum\sum_{0\le i\le j\le n}\left({}^nC_i + {}^nC_j\right) = \dfrac{\left(\sum_{i=0}^{n}\sum_{j=0}^{n}\left({}^nC_i + {}^nC_j\right)\right) - \sum_{i=0}^{n} 2\ {}^nC_i}{2} = \dfrac{\left(\sum_{i=0}^{n}\left(\sum_{j=0}^{n}{}^nC_i + \sum_{j=0}^{n}{}^nC_j\right)\right) - 2\times 2^n}{2}$

$$= \frac{\left(\sum_{i=0}^{n}\left({}^nC_i \sum_{j=0}^{n} 1 + 2^n\right)\right) - 2^{n+1}}{2} = \frac{\left(\sum_{i=0}^{n}\left({}^nC_i(n+1) + 2^n\right)\right) - 2^{n+1}}{2}$$

$$= \frac{(n+1)\sum_{i=1}^{n}{}^nC_i + 2^n\sum_{i=0}^{n} 1 - 2^{n+1}}{2} = \frac{(n+1)2^n + 2^n(n+1) - 2^{n+1}}{2}$$

$= (n + 1)2^n - 2^n = n2^n$

Comprehension # 2 (Q. No. 4 to 6)
अनुच्छेद # 2 (Q. No. 4 to 6) **[Revision Planner]**

Let P be a product given by $P = (x + a_1)(x + a_2) \ldots\ldots\ldots (x + a_n)$

and Let $S_1 = a_1 + a_2 + \ldots\ldots. + a_n = \sum_{i=1}^{n} a_i$, $S_2 = \sum\sum_{i<j} a_i.a_j$, $S_3 = \sum\sum_{i<j<k}\sum a_i.a_j.a_k$ and so on,

then it can be shown that

$P = x^n + S_1 x^{n-1} + S_2 x^{n-2} + \ldots\ldots\ldots + S_n$.

यदि एक गुणनफल P इस प्रकार है, $P = (x + a_1)(x + a_2) \ldots\ldots\ldots (x + a_n)$

तथा माना ० $S_1 = a_1 + a_2 + \ldots\ldots. + a_n = \sum_{i=1}^{n} a_i$, $S_2 = \sum\sum_{i<j} a_i.a_j$, $S_3 = \sum\sum_{i<j<k}\sum a_i.a_j.a_k$ इसी प्रकार आगे,

तो

निम्न को सिद्ध किया जा सकता है –

$P = x^n + S_1 x^{n-1} + S_2 x^{n-2} + \ldots\ldots\ldots + S_n$.

4. The coefficient of x^8 in the expression $(2 + x)^2 (3 + x)^3 (4 + x)^4$ must be
व्यंजक $(2 + x)^2 (3 + x)^3 (4 + x)^4$ में x^8 का गुणांक है–
(A) 26 (B) 27 (C) 28 (D*) 29

Sol. The expression $(2 + x)^2 (3 + x)^3 (4 + x)^4 = (x + 2)(x + 2)(x + 3)(x + 3)(x + 3)(x + 4)(x + 4)(x + 4)(x + 4)$
$= x^9 + (2 + 2 + 3 + 3 + 3 + 4 + 4 + 4 + 4)\, x^8 + \ldots\ldots\ldots\ldots.$
$\Rightarrow$ Co-efficient of $x^8 = 29$

Hindi. व्यंजक $(2 + x)^2 (3 + x)^3 (4 + x)^4 = (x + 2)(x + 2)(x + 3)(x + 3)(x + 3)(x + 4)(x + 4)(x + 4)(x + 4)$
$= x^9 + (2 + 2 + 3 + 3 + 3 + 4 + 4 + 4 + 4)\, x^8 + \ldots\ldots\ldots\ldots.$
$\Rightarrow$ x^8 का गुणांक $= 29$

5. The coefficient of x^{203} in the expression $(x - 1)(x^2 - 2)(x^3 - 3) \ldots\ldots\ldots. (x^{20} - 20)$ must be
व्यंजक $(x - 1)(x^2 - 2)(x^3 - 3) \ldots\ldots\ldots. (x^{20} - 20)$ में x^{203} का गुणांक है–
(A) 11 (B) 12 (C*) 13 (D) 15

Sol. Expression $= x. x^2. x^3 \ldots\ldots\ldots.x^{20} \left(1-\frac{1}{x}\right)\left(1-\frac{2}{x^2}\right)\left(1-\frac{3}{x^3}\right)\ldots\ldots.\left(1-\frac{20}{x^{20}}\right)$

Let $E = \left(1-\frac{1}{x}\right)\left(1-\frac{2}{x^2}\right)\left(1-\frac{3}{x^3}\right)\ldots\ldots.\left(1-\frac{20}{x^{20}}\right)$

Now Co-efficient of x^{203} in original expression

$\Rightarrow$ Co-efficient of x^{-7} in E.
But

$$E = 1 - \left(\frac{1}{x}+\frac{2}{x^2}+\frac{3}{x^3}+......\right) + \left(\frac{1}{x}.\frac{6}{x^6}+\frac{2}{x^2}.\frac{5}{x^5}+\frac{3}{x^3}.\frac{4}{x^4}+......\right) - \left(\frac{1}{x}.\frac{2}{x^2}.\frac{4}{x^4}+......\right)$$

= Co-efficient of $x^{-7} = -7 + 6 + 10 + 12 - 8 = 13$

Hindi. व्यंजक $= x.\ x^2.\ x^3\ ....x^{20} \left(1-\frac{1}{x}\right)\left(1-\frac{2}{x^2}\right)\left(1-\frac{3}{x^3}\right).......\left(1-\frac{20}{x^{20}}\right)$

माना $E = \left(1-\frac{1}{x}\right)\left(1-\frac{2}{x^2}\right)\left(1-\frac{3}{x^3}\right).......\left(1-\frac{20}{x^{20}}\right)$

अब मूल व्यंजक में x^{203} का गुणांक = E में x^{-7} का गुणांक
परन्तु

$$E = 1 - \left(\frac{1}{x}+\frac{2}{x^2}+\frac{3}{x^3}+......\right) + \left(\frac{1}{x}.\frac{6}{x^6}+\frac{2}{x^2}.\frac{5}{x^5}+\frac{3}{x^3}.\frac{4}{x^4}+......\right) - \left(\frac{1}{x}.\frac{2}{x^2}.\frac{4}{x^4}+......\right)$$

$\Rightarrow$ x^{-7} का गुणांक $= -7 + 6 + 10 + 12 - 8 = 13$

6. The coefficient of x^{98} in the expression of $(x - 1)(x - 2)$ .... $(x - 100)$ must be
(A) $1^2 + 2^2 + 3^2 + ... + 100^2$
(B) $(1 + 2 + 3 + ... + 100)^2 - (1^2 + 2^2 + 3^2 + ... + 100^2)$
(C*) $\frac{1}{2}$ $[(1 + 2 + 3 + ... + 100)^2 - (1^2 + 2^2 + 3^2 + ... + 100^2)]$
(D) None of these
व्यंजक $(x - 1)(x - 2)$ .... $(x - 100)$ में x^{98} का गुणांक है–
(A) $1^2 + 2^2 + 3^2 + ... + 100^2$
(B) $(1 + 2 + 3 + ... + 100)^2 - (1^2 + 2^2 + 3^2 + ... + 100^2)$
(C*) $\frac{1}{2}[(1 + 2 + 3 + ... + 100)^2 - (1^2 + 2^2 + 3^2 + ... + 100^2)]$
(D) इनमें से कोई नहीं

Sol. The Co-efficient of $x^{98} = (1.2 + 2.3 + ...99.100)$
= Sum of product of first 100 natural numbers taken two at a time
$= \frac{1}{2}$ $[(1 + 2 + 3 + ...+ 100^2)^2 - (1^2 + 2^2 + 3^2 +..........+ 100^2)]$

Hindi. x^{98} का गुणांक $= (1.2 + 2.3 + ...99.100)$
= दो –दो एक साथ लेने पर प्रथम 100 प्राकृत संख्याओं के गुणनफलों का योगफल
$= \frac{1}{2}$ $[(1 + 2 + 3 + ...+ 100^2)^2 - (1^2 + 2^2 + 3^2 +..........+ 100^2)]$

Comprehension # 3 (Q.No. 7 to 9)

Let $(7 + 4\sqrt{3})^n = I + f = {}^nC_0.7^n + {}^nC_1.7^{n-1}.(4\sqrt{3})^1 + ...$ ....(i)
where I & f are its integral and fractional parts respectively.
It means $0 < f < 1$
Now, $0 < 7 - 4\sqrt{3} < 1$ $\Rightarrow$ $0 < (7 - 4\sqrt{3})^n < 1$
Let $(7 - 4\sqrt{3})^n = f' = {}^nC_0.7^n - {}^nC_1.7^{n-1}.(4\sqrt{3})^1 + ...$ ....(ii)
$\Rightarrow$ $0 < f' < 1$
Adding (i) and (ii) (so that irrational terms cancelled out)
$I + f + f' = (7 + 4\sqrt{3})^n + (7 - 4\sqrt{3})^n$
$= 2\,[{}^nC_0\,7^n + {}^nC_2\,7^{n-2}\,(4\sqrt{3})^2 + ...]$
$I + f + f'$ = even integer $\Rightarrow$ ($f + f'$ must be an integer)
$0 < f + f' < 2$ $\Rightarrow$ $f + f' = 1$

with help of above analysis answer the following questions

अनुच्छेद # 3 (Q.No. 7 to 9)

माना $(7 + 4\sqrt{3})^n = I + f = {}^nC_0.7^n + {}^nC_1.7^{n-1}.(4\sqrt{3})^1 + ...$ ....(i)

जहाँ I तथा f इसके पूर्णांक व भिन्नात्मक भाग है

अर्थात् $0 < f < 1$

अब, $0 < 7 - 4\sqrt{3} < 1 \Rightarrow 0 < (7 - 4\sqrt{3})^n < 1$

मानाकि $(7 - 4\sqrt{3})^n = f' = {}^nC_0.7^n - {}^nC_1.7^{n-1}.(4\sqrt{3})^1 + ...$ ....(ii)

$\Rightarrow \quad 0 < f' < 1$

(i) व (ii) का योग करने पर इससे अपरिमेय पद निरस्त हो जायेगें

$I + f + f' = (7 + 4\sqrt{3})^n + (7 - 4\sqrt{3})^n$

$= 2\,[{}^nC_0\, 7^n + {}^nC_2\, 7^{n-2}\, (4\sqrt{3})^2 + ...]$

$I + f + f'$ = समपूर्णांक $\Rightarrow$ ($f + f'$ एक पूर्णांक होना चाहिए)

$0 < f + f' < 2 \quad \Rightarrow \quad f + f' = 1$

उपरोक्त विश्लेषण के आधार पर निम्न प्रश्नों के उत्तर दीजिए।

7. If $\left(3\sqrt{3} + 5\right)^n = p + f$, where p is an integer and f is a proper fraction, then find the value of $\left(3\sqrt{3} - 5\right)^n$, $n \in N$, is

(A*) $1 - f$, if n is even (B) f, if n is even (C) $1 - f$, if n is odd (D*) f, if n is odd

यदि $\left(3\sqrt{3} + 5\right)^n = p + f$, जहाँ p पूर्णांक है और f भिन्नात्मक भाग है, तो $\left(3\sqrt{3} - 5\right)^n$, $n \in N$ का मान है।

(A*) $1 - f$, यदि n सम है। (B) f, यदि n सम है।

(C) $1 - f$, यदि n विषम है। (D*) f, यदि n विषम है।

Sol. $p + f = (3\sqrt{3} + 5)^n = {}^nC_0(3\sqrt{3})^n 5^0 + {}^nC_1(3\sqrt{3})^{n-1} 5^1 + ...$

$f' = (3\sqrt{3} - 5)^n = {}^nC_0\,(3\sqrt{3})^n 5^0 - {}^nC_1(3\sqrt{3})^{n-1} 5^1 + ...$

$p + f + f' = 2\,[{}^nC_0(3\sqrt{3})^n + {}^nC_2\,(3\sqrt{3})^{n-2}5^2 + ...]$

$\Rightarrow \quad p + f + f'$ = even integer सम पूर्णांक (if n is even) (यदि n सम है)

$\Rightarrow \quad f + f' = 1 \Rightarrow f' = 1 - f$

$p + f - f' = 2\,[{}^nC_1(3\sqrt{3})^{n-1}(5) + {}^nC_3(3\sqrt{3})^{n-3}5^3 + ...]$ (if n is odd) (यदि n विषम है)

$\Rightarrow \quad f - f' = 0 \quad \Rightarrow \quad f' = f$

8. If $\left(9 + \sqrt{80}\right)^n = I + f$, where I, n are integers and $0 < f < 1$, then :

(A*) I is an odd integer (B) I is an even integer

(C*) $(I + f)(1 - f) = 1$ (D*) $1 - f = \left(9 - \sqrt{80}\right)^n$

यदि $\left(9 + \sqrt{80}\right)^n = I + f$ जहाँ I, n पूर्णांक हैं और $0 < f < 1$, तो –

(A) I एक विषम पूर्णांक हैं। (B) I एक सम पूर्णांक है।

(C) $(I + f)(1 - f) = 1$ (D) $1 - f = \left(9 - \sqrt{80}\right)^n$

Sol. $\left(9 + \sqrt{80}\right)^n = I + f$

$\left(9 - \sqrt{80}\right)^n = f'$

$2[{}^nC_0\,(9)^n + {}^nC_2\,(9)^{n-2}\,(\sqrt{80})^2 + ...] = I + f + f'$

$\therefore$ I = 2(integer पूर्णांक) – 1 $(\because f + f' = 1)$

$\therefore$ $(I + f)(1 - f) = 1$

9. The integer just above $(\sqrt{3} + 1)^{2n}$ is, for all $n \in N$.
(A*) divisible by 2^n (B*) divisible by 2^{n+1}
(C*) divisible by 8 (D) divisible by 16

$(\sqrt{3} + 1)^{2n}$ से ठीक 1 अधिक पूर्णांक हो, समीकरण $n \in N$ के लिए
(A*) 2^n का भाग देने पर (B*) 2^{n+1} का भाग देने पर
(C*) 8 का भाग देने पर (D) 16 का भाग देने पर

Sol. Let (माना) $(\sqrt{3} + 1)^{2n} = (4 + 2\sqrt{3})^n = 2^n (2 + \sqrt{3})^n = I + f$ ....(i)
where I and f are its integral & fractional parts respectively
जहाँ I तथा f इसके पूर्णांक इसके क्रमशः पूर्णांक व भिन्नात्मक भाग है
$0 < f < 1$.
Now अब $0 < \sqrt{3} - 1 < 1$
$0 < (\sqrt{3} - 1)^{2n} < 1$
Let माना कि $(\sqrt{3} - 1)^{2n} = (4 - 2\sqrt{3})^n = 2^n (2 - \sqrt{3})^n = f'$. ....(ii)
$0 < f' < 1$
adding (i) and (ii) (i) और (ii) को जोड़ने पर
$I + f + f' = (\sqrt{3} + 1)^{2n} + (\sqrt{3} - 1)^{2n}$
$= 2^n [(2 + \sqrt{3})^n + (2\sqrt{3} -)^n] = 2.2^n [{}^nC_0 2^n + {}^nC_2 2^{n-2} (\sqrt{3})^2 + ...]$
$I + f + f' = 2^{n+1} k$ (where k is a positive integer)
$I + f + f' = 2^{n+1} k$ (जहाँ k धनात्मक पूर्णांक है)
$0 < f + f' < 2 \Rightarrow f + f' = 1$
$I + 1 = 2^{n+1} k$.
I + 1 is the integer just above $(\sqrt{3} + 1)^{2n}$ and which is divisible by 2^{n+1}.
I + 1 ठीक अधिक पूर्णांक है $(\sqrt{3} + 1)^{2n}$ तथा यह 2^{n+1} से विभाजित है।
for से n = 1, $(\sqrt{3} + 1)^{2n} = (\sqrt{3} + 1)^2 \Rightarrow I + 1 = 8$
so it is divisible by 8 but not by 16.
यह 8 से विभाजित हो परन्तु 16 से हो।

Exercise-3

Marked questions are recommended for Revision.
चिन्हित प्रश्न दोहराने योग्य प्रश्न है।

PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

भाग - I : JEE (ADVANCED) / IIT-JEE (पिछले वर्षो) के प्रश्न

*** Marked Questions may have more than one correct option.**
*** चिन्हित प्रश्न एक से अधिक सही विकल्प वाले प्रश्न है -**

1. Coefficient of t^{24} in $(1 + t^2)^{12} (1 + t^{12}) (1 + t^{24})$ is: **[IIT-JEE-2003, Scr, (3, – 1), 84]**
$(1 + t^2)^{12} (1 + t^{12}) (1 + t^{24})$ में t^{24} का गुणांक है :
(A) ${}^{12}C_6 + 3$ (B) ${}^{12}C_6 + 1$ (C) ${}^{12}C_6$ (D*) ${}^{12}C_6 + 2$

Sol. $(1 + t^2)^{12} (1 + t^{12} + t^{24} + t^{36}) = (1 + t^{12} + t^{24}) (1 + t^2)^{12}$

coefficient of $t^{24} = {}^{12}C_{12} + {}^{12}C_6 + {}^{12}C_0 = {}^{12}C_6 + 2$

Hindi $(1 + t^2)^{12} (1 + t^{12} + t^{24} + t^{36}) = (1 + t^{12} + t^{24}) (1 + t^2)^{12}$

t^{24} का गुणांक $= {}^{12}C_{12} + {}^{12}C_6 + {}^{12}C_0 = {}^{12}C_6 + 2$

2. Prove that $2^k \binom{n}{0}\binom{n}{k} - 2^{k-1}\binom{n}{1}\binom{n-1}{k-1} + 2^{k-2}\binom{n}{2}\binom{n-2}{k-2} - \ldots\ldots + (-1)^k \binom{n}{k}\binom{n-k}{0} = \binom{n}{k}$.

[IIT-JEE-2003, Main, (2, 0), 60]

सिद्ध कीजिए कि $2^k \binom{n}{0}\binom{n}{k} - 2^{k-1}\binom{n}{1}\binom{n-1}{k-1} + 2^{k-2}\binom{n}{2}\binom{n-2}{k-2} - \ldots\ldots + (-1)^k \binom{n}{k}\binom{n-k}{0} = \binom{n}{k}$.

Sol. $S = 2^k\, {}^nC_0 . {}^nC_k - 2^{k-1}\, {}^nC_1\, {}^{n-1}C_{k-1} + 2^{k-2}\, {}^nC_2\, {}^{n-2}C_{k-2} + \ldots\ldots\ldots\ldots$

$\Rightarrow S = \sum_{r=0}^{k} (-1)^r\ {}^nC_r\ {}^{n-r}C_{k-r} \cdot 2^{k-r}$ $\qquad \Rightarrow S = \sum_{r=0}^{k} (-1)^r \frac{n!}{r!(n-r)!} \times \frac{(n-r)!.2^{k-r}}{(n-k)!\ (k-r)!}$

$= \sum_{r=0}^{k} (-1)^r\ 2^{k-r} \frac{n!}{k!\ (n-k)!} \times \frac{k!}{r!\ (k-r)!} = 2^k\ {}^nC_k \left(1 - \frac{1}{2}\right)^k = {}^nC_k$

3. If ${}^{(n-1)}C_r = (k^2 - 3)\, {}^nC_{r+1,}$ then an interval in which k lies is **[IIT-JEE-2004, Scr, (3, – 1), 84]**

यदि ${}^{(n-1)}C_r = (k^2 - 3)\, {}^nC_{r+1}$ हो, तो k के मान का अन्तराल है – **[Scr, (3, – 1), 84]**

(A) $(2, \infty)$ (B) $(-\infty, -2)$ (C) $\left[-\sqrt{3}, \sqrt{3}\right]$ (D*) $\left(\sqrt{3}, 2\right]$

Sol. ${}^{(n-1)}C_r = (k^2 - 3)\, {}^nC_{r+1}$

or या ${}^{(n-1)}C_{n-(r+1)} = (k^2 - 3)\, {}^nC_{n-(r+1)}$

$1 \ge k^2 - 3 > 0 \quad \Rightarrow k \in [-2, -\sqrt{3}) \ \cup\ (\sqrt{3},\ 2]$

4. The value of **[IIT-JEE-2005, Scr, (3, – 1), 84]**

$\binom{30}{0}\binom{30}{10} - \binom{30}{1}\binom{30}{11} + \binom{30}{2}\binom{30}{12} - \ldots\ldots\ldots + \binom{30}{20}\binom{30}{30}$ is :

(A) $\binom{60}{20}$ (B*) $\binom{30}{10}$ (C) $\binom{30}{15}$ (D) None of these

$\binom{30}{0}\binom{30}{10} - \binom{30}{1}\binom{30}{11} + \binom{30}{2}\binom{30}{12} - \ldots\ldots\ldots + \binom{30}{20}\binom{30}{30}$ का मान है– **[Scr, (3, – 1), 84]**

(A) $\binom{60}{20}$ (B) $\binom{30}{10}$ (C) $\binom{30}{15}$ (D) इनमें से कोई नहीं

Sol. $S = {}^{30}C_0\ {}^{30}C_{20} - {}^{30}C_1\ {}^{30}C_{19} + {}^{30}C_2\ {}^{30}C_{18} \ldots\ldots\ldots\ldots\ldots\ldots$

S = Co-efficient of x^{20} in $(1-x)^{30} (1 + x)^{30}$

S = Co-efficient of x^{20} in $(1-x^2)^{30} = {}^{30}C_{10}$

Hindi. $S = {}^{30}C_0\ {}^{30}C_{20} - {}^{30}C_1\ {}^{30}C_{19} + {}^{30}C_2\ {}^{30}C_{18} \ldots\ldots\ldots\ldots\ldots\ldots$

$S = (1-x)^{30} (1 + x)^{30}$ में x^{20} का गुणांक

$S = (1-x^2)^{30}$ में x^{20} का गुणांक $= {}^{30}C_{10}$

5. For r = 0, 1, ..., 10, let A_r, B_r and C_r denote, respectively, the coefficient of x^r in the expansions of $(1 + x)^{10}$, $(1 + x)^{20}$ and $(1 + x)^{30}$. Then $\sum_{r=1}^{10} A_r(B_{10}B_r - C_{10}A_r)$ is equal to

मानाकि r = 0, 1, ..., 10 के लिए A_r, B_r तथा C_r क्रमशः $(1 + x)^{10}$, $(1 + x)^{20}$ तथा $(1 + x)^{30}$ के प्रसार में x^r के गुणांक हैं।

तो $\sum_{r=1}^{10} A_r(B_{10}B_r - C_{10}A_r)$ का मान निम्न है **[IIT-JEE 2010, Paper-2, (5, –2)/79]**

(A) $B_{10} - C_{10}$ (B) $A_{10}(B^2_{10} - C_{10}A_{10})$ (C) 0 (D*) $C_{10} - B_{10}$

Sol. $B_{10}\sum_{r=1}^{10} A_rB_r - C_{10}\sum_{r=1}^{10}(A_r)^2 = {}^{20}B_{10}({}^{30}C_{20} - 1) - {}^{30}C_{10}({}^{20}C_{10} - 1) = {}^{30}C_{10} - {}^{20}C_{10} = C_{10} - B_{10}$

6. The coefficients of three consecutive terms of $(1 + x)^{n+5}$ are in the ratio 5 : 10 : 14. Then n =

$(1 + x)^{n+5}$ के तीन क्रमागत पदों के गुणांक 5 : 10 : 14 के अनुपात में है, तब n =

Ans. 6

[JEE (Advanced) 2013, Paper-1, (4, – 1)/60]

Sol. ${}^{n+5}C_{r-1} : {}^{n+5}C_r : {}^{n+5}C_{r+1} = 5 : 10 : 14$

$\Rightarrow \quad \frac{{}^{n+5}C_r}{{}^{n+5}C_{r-1}} = \frac{10}{5} \quad \& \quad \frac{{}^{n+5}C_{r+1}}{{}^{n+5}C_r} = \frac{14}{10}$

$\Rightarrow \quad \frac{(n+5) - r + 1}{r} = 2 \quad \& \quad \frac{(n+5) - (r+1) + 1}{r+1} = \frac{7}{5}$

$\Rightarrow \quad \frac{n+6}{r} = 3 \quad \& \quad \frac{n+6}{r+1} = \frac{12}{5} \quad \Rightarrow \quad 3r = \frac{12}{5}(r + 1) \quad \Rightarrow \quad r = 4$

$\therefore \quad n + 6 = 12 \quad \Rightarrow \quad n = 6$

7. Coefficient of x^{11} in the expansion of $(1 + x^2)^4 (1 + x^3)^7 (1 + x^4)^{12}$ is

[JEE (Advanced) 2014, Paper-2, (3, –1)/60]

$(1 + x^2)^4 (1 + x^3)^7 (1 + x^4)^{12}$ विस्तार में (expansion) x^{11} का गुणांक (coefficient) है–

[JEE (Advanced) 2014, Paper-2, (3, –1)/60]

(A) 1051 (B) 1106 (C) 1113 (D) 1120

Ans. **(C)**

Sol. Coefficent of $x^{11} \equiv \frac{(1+x^2)^4(1+x^3)^7(1+x^4)^{12}(1-x^2)^4}{(1-x^2)^4}$

Coefficent of $x^{11} \equiv (1 - x^8)^4 (1 + x^4)^8 (1 + x^3)^7 (1 - x^2)^{-4}$

$= (1 - 4x^8)(1 + x^4)^8 (7x^3 + 35x^9)(1 - x^2)^{-4}$

$= (7x^3 + 35x^9 - 28x^{11})(1 + x^4)^8 (1 - x^2)^{-4}$

Coefficent of $x^8 = (7x + 35x^6 - 28x^8)(1 + 8x^4 + 28x^8)(1 - x^2)^{-4}$

$= (7 + 35x^6 - 28x^8 + 56x^4 + 196x^8)(1 - x^2)^{-4}$

Coefficent of $t^4 \equiv (7 + 56t^2 + 35t^3 + 168t^4)(1 - t)^{-4}$

$= 7 \cdot {}^7C_3 + 56 \cdot {}^5C_3 + 35 \cdot {}^4C_3 + 168$

$= 245 + 700 + 168 = 1113.$

Hindi $\frac{(1+x^2)^4(1+x^3)^7(1+x^4)^{12}(1-x^2)^4}{(1-x^2)^4}$ में x^{11} का गुणांक

$1 - x^8)^4 (1 + x^4)^8 (1 + x^3)^7 (1 - x^2)^{-4}$ में x^{11} का गुणांक

$= (1 - 4x^8)(1 + x^4)^8 (7x^3 + 35x^9)(1 - x^2)^{-4}$

$= (7x^3 + 35x^9 - 28x^{11})(1 + x^4)^8 (1 - x^2)^{-4}$

$(7x + 35x^6 - 28x^8)(1 + 8x^4 + 28x^8)(1 - x^2)^{-4}$ में x^8 का गुणांक

$= (7 + 35x^6 - 28x^8 + 56x^4 + 196x^8)(1 - x^2)^{-4}$

$(7 + 56t^2 + 35t^3 + 168t^4)(1 - t)^{-4}$ में t^4 का गुणांक

$= 7 \cdot {}^7C_3 + 56 \cdot {}^5C_3 + 35 \cdot {}^4C_3 + 168$

$= 245 + 700 + 168 = 1113.$

Alterantive : वैकल्पिक हल

$2x + 3y + 4z = 11$

$(x, y, z) = (0, 1, 2)$ ${}^4C_0 \times {}^7C_1 \times {}^{12}C_2$

$(1, 3, 0)$ ${}^4C_1 \times {}^7C_3$

$(2, 1, 1)$ ${}^4C_2 \times {}^7C_1 \times {}^{12}C_1$

$(4, 1, 0)$ ${}^7C_1$

coefficient of x^{11} का गुणांक $= 66 \times 7 + 35 \times 4 + 42 \times 12 + 7$

$= 1113$. **Ans.**

8. The coefficient of x^9 in the expansion of $(1 + x)(1 + x^2)(1 + x^3)$ ......$(1 + x^{100})$ is **(Moderate)**

$(1 + x)(1 + x^2)(1 + x^3)$ ......$(1 + x^{100})$ के विस्तार में x^9 के गुणांक का मान है

[JEE (Advanced) 2015, P-2 (4, 0) / 80]

Ans. 8

Sol. 9 = (0, 9) (1, 8), (2, 7), (3, 6), (4, 5) # 5 cases

9 = (1,2,6), (1,3,5), (2, 3, 4) # 3 cases

total = 8

9. Let m be the smallest positive integer such that the coefficient of x^2 in the expansion of $(1 + x)^2 + (1 + x)^3 + ... + (1 + x)^{49} + (1 + mx)^{50}$ is $(3n + 1)\ {}^{51}C_3$ for some positive integer n. Then the value of n is

[JEE (Advanced) 2016, Paper-1, (3, 0)/62]

माना कि m ऐसा न्यूनतम धनात्मक पूर्णांक (smallest positive integer) है कि $(1 + x)^2 + (1 + x)^3 + ... + (1 + x)^{49} + (1 + mx)^{50}$ के विस्तार में x^2 का गुणांक $(3n + 1)\ {}^{51}C_3$ किसी धनात्मक पूर्णांक n के लिए है। तब n का मान है–

Ans. 5

Sol. Coeff. x^2 का गुणांक

${}^2C_2 + {}^3C_2 + {}^4C_2 + ... + {}^{49}C_2 + {}^{50}C_2\ m^2 = (3n + 1)\ {}^{51}C_3$

${}^3C_3 + {}^3C_2 + {}^4C_2 + ... + {}^{49}C_2 + {}^{50}C_2\ m^2 = (3n + 1)\ {}^{51}C_3$

${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r \quad \Rightarrow \quad {}^{50}C_3 + {}^{50}C_2 . m^2 = (3n + 1)\ {}^{51}C_3$

${}^{50}C_3 + {}^{50}C_2 + (m^2 - 1)\ {}^{50}C_2 = 3n. \dfrac{51}{3}. {}^{50}C_2 + {}^{51}C_3 \quad \Rightarrow \quad {}^{51}C_3 + (m^2 - 1)\ {}^{50}C_2 = 51n . {}^{50}C_2 + {}^{51}C_3$

$m^2 - 1 = 51n \quad \Rightarrow \quad m^2 = 51n + 1$

min value of m^2 for $51n + 1$ is integer for $n = 5$ ($51n + 1$ के पूर्णांक होने के लिए m^2 का न्यूनतम मान $n = 5$)

10. Let $X = ({}^{10}C_1)^2 + 2({}^{10}C_2)^2 + 3({}^{10}C_3)^2 + ... + 10({}^{10}C_{10})^2$ where ${}^{10}C_r$, $r \in \{1, 2, ..., 10\}$ denote binomial coefficients. Then the value of $\dfrac{1}{1430}$ X is _______ . **[JEE (Advanced) 2018, Paper-1, (3, 0)/60]**

माना कि $X = ({}^{10}C_1)^2 + 2({}^{10}C_2)^2 + 3({}^{10}C_3)^2 + ... + 10({}^{10}C_{10})^2$, जहॉं ${}^{10}C_r$, $r \in \{1, 2, ..., 10\}$, द्विपद गुणांकों (binomial coefficients) को दर्शाते हैं। तब $\dfrac{1}{1430}$ X का मान है _______ ।

Ans. (646)

Sol. $X = \sum_{r=1}^{10} r.\ {}^{10}C_r . {}^{10}C_r = 10. \sum_{r=1}^{10} {}^9C_{r-1} . {}^{10}C_{10-r} = 10 . {}^{19}C_9$

Now अब $\dfrac{X}{1430} = \dfrac{10.{}^{19}C_9}{1430} = \dfrac{{}^{19}C_9}{143} = \dfrac{{}^{19}C_9}{11 \times 13} = \dfrac{19.17.16}{8} = 19 \times 34 = 646$

PART - II : JEE (MAIN) / AIEEE PROBLEMS (PREVIOUS YEARS)

भाग - II : JEE (MAIN) / AIEEE (पिछले वर्षो) के प्रश्न

1. Let $S_1 = \sum_{j=1}^{10} j\ (j-1)\ {}^{10}C_j$, $S_2 = \sum_{j=1}^{10} j\ {}^{10}C_j$ and $S_3 = \sum_{j=1}^{10} j^2\ {}^{10}C_j$. **[AIEEE 2009, (4, –1), 144]**

Statement -1 : $S_3 = 55 \times 2^9$.

Statement -2 : $S_1 = 90 \times 2^8$ and $S_2 = 10 \times 2^8$.

(1) Statement -1 is true, Statement-2 is true ; Statement -2 is not a correct explanation for Statement -1.

(2*) Statement-1 is true, Statement-2 is false.

(3) Statement -1 is false, Statement -2 is true.

(4) Statement -1 is true, Statement -2 is true; Statement-2 is a correct explanation for Statement-1.

माना $S_1 = \sum_{j=1}^{10} j\ (j-1)\ {}^{10}C_j$, $S_2 = \sum_{j=1}^{10} j\ {}^{10}C_j$ तथा $S_3 = \sum_{j=1}^{10} j^2\ {}^{10}C_j$.

प्रकथन -1 : $S_3 = 55 \times 2^9$.

प्रकथन -2 : $S_1 = 90 \times 2^8$ तथा $S_2 = 10 \times 2^8$.

(1) प्रकथन-1 सत्य है, प्रकथन-2 सत्य है ; प्रकथन-2, प्रकथन-1 की सही व्याख्या नहीं है।

(2*) प्रकथन-1 सत्य है, प्रकथन-2 मिथ्या है।

(3) प्रकथन-1 मिथ्या है, प्रकथन-2 सत्य है।

(4) प्रकथन-1 सत्य है, प्रकथन-2 सत्य है ; प्रकथन-2, प्रकथन-1 की सही व्याख्या है।

Sol. $S_1 = \sum_{j=1}^{10} j(j-1)\ .\ \frac{10(10-1)}{j\ (j-1)}\ {}^{8}C_{j-2}$

$\Rightarrow \quad S_1 = 9 \times 10 \sum_{j=2}^{10} {}^{8}C_{j-2} \quad \Rightarrow \quad S_1 = 90\ .\ 2^8$

$S_2 = \sum_{j=1}^{10} j\ .\ \frac{10}{j}\ {}^{9}C_{j-1} = 10.2^9$

$S_3 = \sum_{j=1}^{10} (j(j-1) + j)\ {}^{10}C_j = \sum_{j=1}^{10} j(j-1)\ {}^{10}C_j + \sum_{j=1}^{10} j\ {}^{10}C_j = 90 \sum_{j=2}^{10} {}^{8}C_{j-2} + 10 \sum_{j=1}^{10} {}^{9}C_{j-1}$

$= 90 \times 2^8 + 10 \times \sum_{j=2}^{10} {}^{8}C_{j-2}\ \ 2^9 = (45 + 10)\ .\ 2^9 = (45 + 10)\ .\ 2^9 = 55.2^9$

so statement-1 is true and statement 2 is false.

इसलिए कथन-1 सत्य है तथा कथन - 2 असत्य है।

Hence correct option is (2)

अतः सही विकल्प (2) है।

2. The coefficient of x^7 in the expansion of $(1 - x - x^2 + x^3)^6$ is : **[AIEEE 2011, (4, –1), 120]**

$(1 - x - x^2 + x^3)^6$ के प्रसार में x^7 का गुणांक है :

(1) 144 (2) – 132 (3*) – 144 (4) 132

Sol. **(3)**

$(1 - x - x^2 + x^3)^6$

$(1 - x)^6 (1 - x^2)^6$

$({}^6C_0 - {}^6C_1 x^1 + {}^6C_2 x^2 - {}^6C_3 x^3 + {}^6C_4 x^4 - {}^6C_5 x^5 + {}^6C_6 x^6)\ ({}^6C_0 - {}^6C_1 x^2 + {}^6C_2 x^4 - {}^6C_3 x^6 + {}^6C_4 x^8 + \ldots\ldots\ldots\ldots + {}^6C_6 x^{12})$

Now coefficient of $x^7 = {}^6C_1{}^6C_3 - {}^6C_3{}^6C_2 + {}^6C_5{}^6C_1$

$= 6 \times 20 - 20 \times 15 + 36$

$= 120 - 300 + 36$

$= 156 - 300 = -144$ **Ans.**

Hindi $(1 - x - x^2 + x^3)^6$

$(1 - x)^6 (1 - x^2)^6$

$({}^6C_0 - {}^6C_1 x^1 + {}^6C_2 x^2 - {}^6C_3 x^3 + {}^6C_4 x^4 - {}^6C_5 x^5 + {}^6C_6 x^6)\ ({}^6C_0 - {}^6C_1 x^2 + {}^6C_2 x^4 - {}^6C_3 x^6 + {}^6C_4 x^8 + \ldots\ldots\ldots\ldots + {}^6C_6 x^{12})$

x^7 का गुणांक $= {}^6C_1{}^6C_3 - {}^6C_3{}^6C_2 + {}^6C_5{}^6C_1$

$= 6 \times 20 - 20 \times 15 + 36$

$= 120 - 300 + 36$
$= 156 - 300$
$= - 144$ **Ans.**

3. If n is a positive integer, then $\left(\sqrt{3}+1\right)^{2n} - \left(\sqrt{3}-1\right)^{2n}$ is : **[AIEEE 2012, (4, –1), 120]**

(1*) an irrational number
(2) an odd positive integer
(3) an even positive integer
(4) a rational number other than positive integers

यदि n एक धनपूर्णांक है, तो $\left(\sqrt{3}+1\right)^{2n} - \left(\sqrt{3}-1\right)^{2n}$ – :

(1*) एक अपरिमेय संख्या है।
(2) एक विषम धनपूर्णांक है।
(3) एक सम धनपूर्णांक है।
(4) धनपूर्णांकों को छोड़ कर एक परिमेय संख्या है।

Sol. **Ans. (1)**
$(\sqrt{3} + 1)^{2n} - (\sqrt{3} - 1)^{2n}$
$= 2[{}^{2n}C_1(\sqrt{3})^{2n-1} + {}^{2n}C_3(\sqrt{3})^{2n-3} + {}^{2n}C_5(\sqrt{3})^{2n-5} + ...]$
= which is an irrational number

Hindi. $(\sqrt{3} + 1)^{2n} - (\sqrt{3} - 1)^{2n}$
$= 2[{}^{2n}C_1(\sqrt{3})^{2n-1} + {}^{2n}C_3(\sqrt{3})^{2n-3} + {}^{2n}C_5(\sqrt{3})^{2n-5} + ...]$
= जो कि एक अपरिमेय संख्या है।

4. The term independent of x in expansion of $\left(\frac{x+1}{x^{2/3}-x^{1/3}+1} - \frac{x-1}{x-x^{1/2}}\right)^{10}$ is :**[AIEEE - 2013, (4, –1),120]**

$\left(\frac{x+1}{x^{2/3}-x^{1/3}+1} - \frac{x-1}{x-x^{1/2}}\right)^{10}$ के प्रसार में x से स्वतंत्र पद है : **[AIEEE - 2013, (4, –1),120]**

(1) 4 (2) 120 (3*) 210 (4) 310

Sol. (3)

$$\left((x^{1/3}+1) - \left(\frac{\sqrt{x}+1}{\sqrt{x}}\right)\right)^{10}$$

$(x^{1/3} - x^{-1/2})^{10}$

$T_{r+1} = {}^{10}c_r\,(x^{1/3})^{10-r}\,(-x^{-1/2})^r$

$$\frac{10-r}{3} - \frac{r}{2} = 0 \quad \Rightarrow \quad 20 - 2r - 3r = 0$$

$\Rightarrow \quad r = 4$

$$T_5 = {}^{10}C_4 = \frac{10\times9\times8\times7}{4\times3\times2\times1} = 210$$

5. If the coefficients of x^3 and x^4 in the expansion of $(1 + ax + bx^2)(1 - 2x)^{18}$ in powers of x are both zero, then (a, b) is equal to **[Binomial Theorem]** **[JEE(Main) 2014, (4, – 1), 120]**

यदि $(1 + ax + bx^2)(1 - 2x)^{18}$ के x की घातों में प्रसार में x^3 तथा x^4 दोनों के गुणांक शून्य हैं, तो (a, b) बराबर है–
[Binomial Theorem] **[JEE(Main) 2014, (4, – 1), 120]**

(1) $\left(14, \frac{272}{3}\right)$ (2*) $\left(16, \frac{272}{3}\right)$ (3) $\left(16, \frac{251}{3}\right)$ (4) $\left(14, \frac{251}{3}\right)$

Sol. **Ans. (2)**
$(1 + ax + bx^2)(1 - 2x)^{18}$
coeff of x^3 का गुणांक $= {}^{18}C_3(-2)^3 + a.(-2)^2 \,.\, {}^{18}C_2 + b.(-2).\, {}^{18}C_1 = 0$
coeff of x^4 का गुणांक $\equiv {}^{18}C_4(-2)^4 + a(-2)^3 \,.\, {}^{18}C_3 + b(-2)^2 \,.\, {}^{18}C_2 = 0$

$\Rightarrow \quad 51a - 3b = 544$ and और $32a - 3b = 240$

Subtracting we get घटाने पर प्राप्त होता है $a = 16$

$\Rightarrow \quad b = \frac{272}{3}$

6.✍ The sum of coefficients of integral powers of x in the binomial expansion of $(1 - 2\sqrt{x})^{50}$ is

$(1 - 2\sqrt{x})^{50}$ के द्विपद प्रसार में x की पूर्णांकीय घातो के गुणांकों का योग है :**[JEE(Main) 2015, (4, – 1), 120]**

(1) $\frac{1}{2}(3^{50} + 1)$ (2) $\frac{1}{2}(3^{50})$ (3) $\frac{1}{2}(3^{50} - 1)$ (4) $\frac{1}{2}(2^{50} + 1)$

Ans. **(1)**

Sol. $\left(1-2\sqrt{x}\right)^{50} = C_0 - C_1\ 2\sqrt{x} + C_2\left(2\sqrt{x}\right)^2 + \ldots\ldots + C_{50}\left(2\sqrt{x}\right)^{50}$

$\left(1+2\sqrt{x}\right)^{50} = C_0 + C_1\left(2\sqrt{x}\right) + C_2\left(2\sqrt{x}\right)^2 + \ldots\ldots + C_{50}\left(2\sqrt{x}\right)^{50}$

Put x = 1 रखने पर

$\therefore \quad \frac{1+3^{50}}{2} = C_0 + C_2(2)^2 + \ldots$

7. If the number of terms in the expansion of $\left(1-\frac{2}{x}+\frac{4}{x^2}\right)^n$, $x \neq 0$, is 28, then the sum of the coefficients of all the terms in this expansion, is **[JEE(Main) 2016, (4, – 1), 120]**

यदि $\left(1-\frac{2}{x}+\frac{4}{x^2}\right)^n$, $x \neq 0$ के प्रसार में पदों की संख्या 28 है, तो इस प्रसार में आने वाले सभी पदों के गुणांकों का योग है–

(1) 2187 (2) 243 (3) 729 (4) 64

Ans. **(3) or Bonus**

Sol. Theortically the number of terms are 2N + 1 (i.e. odd) But As the number of terms being odd hence considering that number clubbing of terms is done hence the solutions follwos :

Number of terms = ${}^{n+2}C_2 = 28 \quad \therefore \quad n = 6$

sum of cofficient = $3^n = 3^6 = 729$

put x = 1

सिद्धान्तः पदों की संख्या 2N + 1 है (अर्थात् विाम) परन्तु जैसा कि पदों की संख्या विषम है अतः पदों के मिश्रण के अनुसार हल लिखने पर।

पदो की संख्या = ${}^{n+2}C_2 = 28 \quad \therefore \quad n = 6$

गुणांको का योग = $3^n = 3^6 = 729$

x = 1 रखने पर

8.✍ The value of $({}^{21}C_1 - {}^{10}C_1) + ({}^{21}C_2 - {}^{10}C_2) + ({}^{21}C_3 - {}^{10}C_3) + ({}^{21}C_4 - {}^{10}C_4) + \ldots\ldots + ({}^{21}C_{10} - {}^{10}C_{10})$ is

$({}^{21}C_1 - {}^{10}C_1) + ({}^{21}C_2 - {}^{10}C_2) + ({}^{21}C_3 - {}^{10}C_3) + ({}^{21}C_4 - {}^{10}C_4) + \ldots\ldots + ({}^{21}C_{10} - {}^{10}C_{10})$ का मान है–

[JEE(Main) 2017, (4, – 1), 120]

(1) $2^{21} - 2^{11}$ (2) $2^{21} - 2^{10}$ (3) $2^{20} - 2^9$ (4) $2^{20} - 2^{10}$

Ans. **(4)**

Sol. $({}^{21}C_1 + {}^{21}C_2 + {}^{21}C_3 + \ldots\ldots {}^{21}C_{10}) - ({}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 + \ldots\ldots {}^{10}C_{10}) = S_1 - S_2$

$S_1 = {}^{21}C_1 + {}^{21}C_2 + {}^{21}C_3 + \ldots\ldots {}^{21}C_{10}$

$S_1 = \frac{1}{2}({}^{21}C_1 + {}^{21}C_2 + \ldots\ldots + {}^{21}C_{20}) = \frac{1}{2}({}^{21}C_0 + {}^{21}C_1 + {}^{21}C_2 + \ldots\ldots + {}^{21}C_{20} + {}^{21}C_{21} - 2)$

$S_1 = 2^{20} - 1$

$S_2 = ({}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 + \ldots\ldots {}^{10}C_{10}) = 2^{10} - 1$

Therefore इसलिए, $S_1 - S_2 = 2^{20} - 2^{10}$

9. The sum of the co-efficients of all odd degree terms in the expansion of $\left(x+\sqrt{x^3-1}\right)^5+\left(x-\sqrt{x^3-1}\right)^5$, $(x > 1)$ is : **[JEE(Main) 2018, (4, – 1), 120]**

$\left(x+\sqrt{x^3-1}\right)^5+\left(x-\sqrt{x^3-1}\right)^5$, $(x > 1)$ के प्रसार में सभी विषम घातों वाले पदों के गुणांको का योग है :

(1) 1 (2) 2 (3) –1 (4) 0

Sol. **(2)**

$\left(x+\sqrt{x^3-1}\right)^5+\left(x-\sqrt{x^3-1}\right)^5$

$= (T_1 + T_2 + T_3 + T_4 + T_5 + T_6) + (T_1 - T_2 + T_3 - T_4 + T_5 - T_6)$

$= 2(T_1 + T_3 + T_5)$

$= 2({}^5C_0(x)^5 + {}^5C_2(x)^3 \left(\sqrt{x^3-1}\right)^2 + {}^5C_4(x)^1 \left(\sqrt{x^3-1}\right)^4$

$= 2(x^5 + 10x^3 (x^3 -1) + 5x (x^6 + 1 -2x^3)]$

$= 2(x^5 + 10x^6 - 10x^3 + 5x^7 + 5x - 10x^4)$

$= 2(5x^7 + 10x^6 + x^5 - 10x^4 - 10x^3 + 5x)$

sum of odd degree terms विषम घात के पदों का योगफल $= 10 + 2 -20 + 10 = 2$

10. If the fractional part of the number $\frac{2^{403}}{15}$ is $\frac{k}{15}$, then k is equal to :

[JEE(Main) 2019, Online (09-01-19),P-1 (4, – 1), 120]

यदि संख्या $\frac{2^{403}}{15}$ का भिन्नात्मक भाग (fractional part) $\frac{k}{15}$ है तो k बराबर है–

(1) 14 (2) 8 (3) 6 (4) 4

Ans. **(2)**

Sol. $\left\{\frac{2^{203}}{15}\right\}$

$8.2^{200} \Rightarrow 8.16^{50} = 8(1 + 15)^{50} = 8(1 + 15)$

hence remainder is 8. (अतः शेषफल 8 है I)

11. If $\sum_{i=1}^{20}\left(\frac{{}^{20}C_{i-1}}{{}^{20}C_i+{}^{20}C_{i-1}}\right)^3=\frac{k}{21}$, then k equals : **[JEE(Main) 2019, Online (10-01-19),P-1 (4, – 1), 120]**

यदि $\sum_{i=1}^{20}\left(\frac{{}^{20}C_{i-1}}{{}^{20}C_i+{}^{20}C_{i-1}}\right)^3=\frac{k}{21}$, तो k बराबर है:

(1) 50 (2) 400 (3) 200 (4) 100

Ans. **(4)**

Sol. $\sum_{i=1}^{20}\left(\frac{{}^{20}C_{i-1}}{{}^{20}C_i+{}^{20}C_{i-1}}\right)^3$

Now अब $\frac{{}^{20}C_{i-1}}{{}^{20}C_i+{}^{20}C_{i-1}}=\frac{{}^{20}C_{i-1}}{{}^{21}C_i}=\frac{i}{21}$

Let given sum be S, so माना दिया गया योग S है, तब

$S = \sum_{i=1}^{20}\frac{(i)^3}{21^3}=\frac{1}{(21)^3}\left(\frac{20.21}{2}\right)^2 = \frac{100}{21}$

Given दिया गया है कि $S = \frac{k}{21} \Rightarrow k = 100$

12. If $\sum_{r=0}^{25}\left\{{}^{50}C_r.{}^{50-r}C_{25-r}\right\}=K\left({}^{50}C_{25}\right)$, then K is equal to :

यदि $\sum_{r=0}^{25}\left\{{}^{50}C_r.{}^{50-r}C_{25-r}\right\} = K\left({}^{50}C_{25}\right)$ है, तो K बराबर है :**[JEE(Main) 2019, Online (10-01-19),P-2 (4, – 1), 120]**

(1) 2^{25} (2) $2^{25} - 1$ (3) $(25)^2$ (4) 2^{24}

Ans. **(1)**

Sol. $\sum_{r=0}^{25} {}^{50}C_r \; {}^{50-r}C_{25-r}$

$= \sum_{r=0}^{25} \frac{50!}{r!(50-r)!}\frac{(50-r)!}{(25-r)!(25)!}$

$= \sum_{r=0}^{25} \frac{50! \; 25!}{r!(25-r)!(25)!25!}$

$= {}^{50}c_{25} \sum_{r=0}^{25} {}^{25}c_r = {}^{50}c_{25} \times 2^{25} = k\left({}^{50}c_{25}\right)$

$\Rightarrow k = 2^{25}$

13. Let $S_n = 1 + q + q^2 + \ldots + q^n$ and $T_n = 1 + \left(\frac{q+1}{2}\right) + \left(\frac{q+1}{2}\right)^2 + \ldots + \left(\frac{q+1}{2}\right)^n$.

where q is a real number and $q \neq 1$. If ${}^{101}C_1 + {}^{101}C_2 . S_1 + \ldots + {}^{101}C_{101} . S_{100} = \alpha T_{100}$ then α is equal to

माना $S_n = 1 + q + q^2 + \ldots + q^n$ तथा $T_n = 1 + \left(\frac{q+1}{2}\right) + \left(\frac{q+1}{2}\right)^2 + \ldots + \left(\frac{q+1}{2}\right)^n$. जहाँ q एक वास्तविक संख्या है तथा $q \neq 1$ यदि ${}^{101}C_1 + {}^{101}C_2 . S_1 + \ldots + {}^{101}C_{101} . S_{100} = \alpha T_{100}$ तो α बराबर है –

[JEE(Main) 2019, Online (11-01-19),P-2 (4, – 1), 120]

(1) 200 (2) 2^{99} (3) 2^{100} (4) 202

Ans. **(3)**

Sol. $\sum_{r=1}^{101} {}^{101}C_r S_{r-1} = \sum_{r=1}^{101} {}^{101}C_r \frac{q^r - 1}{q-1} = \frac{1}{q-1}\left(\sum_{r=1}^{101} {}^{101}C_r q^r - \sum_{r=1}^{101} {}^{101}C_r\right) = \frac{1}{q-1}\left((1+q)^{101} - 1 - 2^{101} + 1\right)$

$\Rightarrow \quad \alpha \frac{\left(\frac{q+1}{2}\right)^{101} - 1}{\frac{q+1}{2} - 1} = \frac{1}{q-1}\left((1+q)^{101} - 2^{101}\right)$

$\Rightarrow \quad \frac{\alpha}{2^{100}}\left(\frac{(1+q)^{101} - 2^{101}}{q-1}\right) = \frac{1}{q-1}\left((1+q)^{101} - 2^{101}\right)$

$\Rightarrow$ Hence अतः $\alpha = 2^{100}$

High Level Problems (HLP)

SUBJECTIVE QUESTIONS

विषयात्मक प्रश्न (SUBJECTIVE QUESTIONS)

1. Find the coefficient of x^{49} in

$$\left(x+\frac{C_1}{C_0}\right)\left(x+2^2\frac{C_2}{C_1}\right)\left(x+3^2\frac{C_3}{C_2}\right)........\left(x+50^2\frac{C_{50}}{C_{49}}\right)$$ where $C_r = {}^{50}C_r$

Ans. 22100

$$\left(x+\frac{C_1}{C_0}\right)\left(x+2^2\frac{C_2}{C_1}\right)\left(x+3^2\frac{C_3}{C_2}\right)........\left(x+50^2\frac{C_{50}}{C_{49}}\right)$$ में x^{49} का गुणांक ज्ञात कीजिए। (जहाँ $C_r = {}^{50}C_r$)

Sol. Coeff. of $x^{49} = \left(\frac{C_1}{C_0}+2^2\frac{C_2}{C_1}+3^2\frac{C_3}{C_2}+.............+50^2\frac{C_{50}}{C_{49}}\right) = \sum_{r=1}^{50} r^2\frac{{}^{50}C_r}{{}^{50}C_{r-1}} = \sum_{r=1}^{50} r^2\left(\frac{50-r+1}{r}\right)$

$= \sum_{r=1}^{50} r(51-r) = \frac{51\times 50\times 51}{2}-\frac{50\times 51\times 101}{6} = 22100$

2. The expression, $\left(\sqrt{2x^2+1}+\sqrt{2x^2-1}\right)^6+\left(\frac{2}{\sqrt{2x^2+1}+\sqrt{2x^2-1}}\right)^6$ is a polynomial of degree

व्यंजक $\left(\sqrt{2x^2+1}+\sqrt{2x^2-1}\right)^6+\left(\frac{2}{\sqrt{2x^2+1}+\sqrt{2x^2-1}}\right)^6$ किस घात का एक बहुपद है–

Ans. 6

Sol. $\left(\sqrt{2x^2+1} + \sqrt{2x^2-1}\right)^6+\left(\frac{2\left(\sqrt{2x^2+1} - \sqrt{2x^2-1}\right)}{(\ 2x^2+1 - 2x^2+1\ )}\right)^6$

$= \left(\sqrt{2x^2+1} + \sqrt{2x^2-1}\right)^6+\left(\sqrt{2x^2+1} - \sqrt{2x^2-1}\right)^6$

$= 2\left({}^6C_0(2x^2+1)^3 + {}^6C_2(2x^2+1)^2(2x^2-1) + {}^6C_4 (2x^2+1)(2x^2-1)^2+{}^6C_6 (2x^2-1)^3\right)$

clearly '6'

Hindi $\left(\sqrt{2x^2+1} + \sqrt{2x^2-1}\right)^6+\left(\frac{2\left(\sqrt{2x^2+1} - \sqrt{2x^2-1}\right)}{(\ 2x^2+1 - 2x^2+1\ )}\right)^6$

$= \left(\sqrt{2x^2+1} + \sqrt{2x^2-1}\right)^6+\left(\sqrt{2x^2+1} - \sqrt{2x^2-1}\right)^6$

$= 2\left({}^6C_0(2x^2+1)^3 + {}^6C_2(2x^2+1)^2(2x^2-1) + {}^6C_4 (2x^2+1)(2x^2-1)^2+{}^6C_6 (2x^2-1)^3\right)$

स्पष्टया '6'

3. Find the co-efficient of x^5 in the expansion of $(1 + x^2)^5 (1 + x)^4$.
$(1 + x^2)^5 (1 + x)^4$ के विस्तार में x^5 का गुणांक ज्ञात कीजिए।

Ans. 60

Sol. Co-efficient of x^5 in $(1 + x^2)^5 (1+x)^4 = {}^4C_1 \,.\, {}^5C_2 + {}^4C_3 \,.\, {}^5C_1 = 40 + 20 = 60$

Hindi $(1 + x^2)^5 (1+x)^4$ में x^5 का गुणांक $= {}^4C_1 \,.\, {}^5C_2 + {}^4C_3 \,.\, {}^5C_1 = 40 + 20 = 60$

4. Prove that the co-efficient of x^{15} in $(1 + x + x^3 + x^4)^n$ is $\sum_{r=0}^{5} {}^nC_{15-3r}\,{}^nC_r$.

सिद्ध कीजिए कि $(1 + x + x^3 + x^4)^n$ के विस्तार में x^{15} का गुणांक $\sum_{r=0}^{5} {}^nC_{15-3r}\,{}^nC_r$ है

Sol. $(1 + x + x^3 + x^4)^n = [(1 + x)(1 + x^3)]^n = (1 + x)^n (1 + x^3)^n$

power of x in $(1 + x^3)^n$ expansion is multiple of 3

so possible cases to get x^{15} are :-

$(9n)\ (1 + x)^n$ $\Downarrow$	$9n(1 + x^3)^n$ $\Downarrow$		
${}^nC_{15}\,x^{15}$	${}^nC_0\,(x^3)^0$	=	${}^nC_{15}\,.\,{}^nC_0\,x^{15}$
${}^nC_{12}\,x^{12}$	${}^nC_1\,(x^3)^1$	=	${}^nC_{12}\,.\,{}^nC_1\,x^{15}$
${}^nC_{12}\,x^9$	${}^nC_2\,(x^3)^2$	=	${}^nC_9\,.\,{}^nC_2\,x^{15}$
${}^nC_6\,x^6$	${}^nC_3\,(x^3)^3$	=	${}^nC_6\,.\,{}^nC_3\,x^{15}$
${}^nC_3\,x^3$	${}^nC_4\,(x^3)^4$	=	${}^nC_3\,.\,{}^nC_4\,x^{15}$
${}^nC_0\,x^0$	${}^nC_5\,(x^3)^5$	=	${}^nC_0\,.\,{}^nC_5\,x^{15}$

$\therefore$ coefficient of x^{15} is

$= {}^nC_{15}\,.\,{}^nC_0 + {}^nC_{12}\,.\,{}^nC_1 + {}^nC_9\,.\,{}^nC_2 + {}^nC_6\,.\,{}^nC_3 + {}^nC_3\,.\,{}^nC_4 + {}^nC_0\,.\,{}^nC_5$

$= \sum_{r_2=0}^{5} {}^nC_{15-3r}\,.{}^nC_r$ hence prooved

Hindi $(1 + x + x^3 + x^4)^n = [(1 + x)(1 + x^3)]^n = (1 + x)^n (1 + x^3)^n$

$(1 + x^3)^n$ में x की घात, 3 का गुणज है।

x^{15} की संभाविय स्थिति

$(9n)\ (1 + x)^n$ $\Downarrow$	$9n(1 + x^3)^n$ $\Downarrow$		
${}^nC_{15}\,x^{15}$	${}^nC_0\,(x^3)^0$	=	${}^nC_{15}\,.\,{}^nC_0\,x^{15}$
${}^nC_{12}\,x^{12}$	${}^nC_1\,(x^3)^1$	=	${}^nC_{12}\,.\,{}^nC_1\,x^{15}$
${}^nC_{12}\,x^9$	${}^nC_2\,(x^3)^2$	=	${}^nC_9\,.\,{}^nC_2\,x^{15}$
${}^nC_6\,x^6$	${}^nC_3\,(x^3)^3$	=	${}^nC_6\,.\,{}^nC_3\,x^{15}$
${}^nC_3\,x^3$	${}^nC_4\,(x^3)^4$	=	${}^nC_3\,.\,{}^nC_4\,x^{15}$
${}^nC_0\,x^0$	${}^nC_5\,(x^3)^5$	=	${}^nC_0\,.\,{}^nC_5\,x^{15}$

$\therefore$ x^{15} का गुणांक है।

$= {}^nC_{15}\,.\,{}^nC_0 + {}^nC_{12}\,.\,{}^nC_1 + {}^nC_9\,.\,{}^nC_2 + {}^nC_6\,.\,{}^nC_3 + {}^nC_3\,.\,{}^nC_4 + {}^nC_0\,.\,{}^nC_5$

$= \sum_{r_2=0}^{5} {}^nC_{15-3r}\,.{}^nC_r$ अतः सिद्व हुआ

5. If n is even natural and coefficient of x^r in the expansion of $\frac{(1+x)^n}{1-x}$ is 2^n, $(|x| < 1)$, then prove that $r \geq n$

यदि n सम प्राकृत संख्या है तथा $\frac{(1+x)^n}{1-x}$ के विस्तार में x^r का गुणांक 2^n $(|x| < 1)$ है तब सिद्ध कीजिए $r \geq n$

Sol. $y = (1 - x)^{-1} (1 + x)^n$

$y = (1 + x + x^2 + \ldots \infty)(1 + x)^n$

$y = (1 + x)^n + x(1 + x)^n + x^2 (1 + x)^n + \ldots.$

Co-efficient of x^r =

${}^nC_r + {}^nC_{r-1} + \ldots.\ {}^nC_0 = 2^n$

$r \geq n$ (As ${}^nC_{n+1}) = 0$

Hindi $y = (1-x)^{-1}(1+x)^n$

$y = (1 + x + x^2 + ... \infty)(1+x)^n$

$y = (1+x)^n + x(1+x)^n + x^2(1+x)^n + ...$

x^r का गुणांक =

${}^nC_r + {}^nC_{r-1} + ... {}^nC_0 = 2^n$

$r \geq n$ (As ${}^nC_{n+1}) = 0$

6. Find the coefficient of x^n in polynomial $(x + {}^{2n+1}C_0)(x + {}^{2n+1}C_1)........(x + {}^{2n+1}C_n)$.

बहुपद $(x + {}^{2n+1}C_0)(x + {}^{2n+1}C_1)........(x + {}^{2n+1}C_n)$ में x^n का गुणांक ज्ञात कीजिए।

Ans. 2^{2n}

Sol. Co-efficient of $x^n = {}^{2n+1}C_0 + {}^{2n+1}C_1 + ... + {}^{2n+1}C_n = 2^{2n}$

Hindi x^n का गुणांक $= {}^{2n+1}C_0 + {}^{2n+1}C_1 + ... + {}^{2n+1}C_n = 2^{2n}$

7. Find the value of $\sum_{r=1}^{n}\left(\sum_{p=0}^{r-1} {}^nC_r\,{}^rC_p 2^p\right)$.

$\sum_{r=1}^{n}\left(\sum_{p=0}^{r-1} {}^nC_r\,{}^rC_p 2^p\right)$ का मान ज्ञात कीजिए।

Ans. $4^n - 3^n$

Sol. $\sum_{r=1}^{n}\left(\sum_{p=0}^{r-1} {}^nC_r\;{}^rC_p 2^p\right) = \sum_{r=1}^{n} {}^nC_r \sum_{p=0}^{r-1} {}^rC_p.2^p = \sum_{r=1}^{n} {}^nC_r\;[{}^rC_0 + {}^rC_1 . 2 + ... + {}^rC_{r-1} 2^{r-1}]$

$= \sum_{r=1}^{n} {}^nC_r\;(3^r - 2^r) = 4^n - 3^n$

Comprehension (Q-8 to Q.10)
अनुच्छेद

For $k, n \in N$, we define

$B(k, n) = 1.2.3......... k + 2.3.4........(k+1) + ...+ n(n+1).......(n+k-1)$, $S_0(n) = n$ and $S_k(n) = 1^k + 2^k + ... + n^k$.

To obtain value B(k, n), we rewrite B(k, n) as follows

$$B(k,n) = k!\left[{}^kC_k + {}^{k+1}C_k + {}^{k+2}C_k + ... + {}^{n+k-1}C_k\right] = k!\left({}^{n+k}C_{k+1}\right)$$

$$= \frac{n(n+1)........(n+k)}{k+1}$$

where ${}^nC_k = \dfrac{n!}{k!\,(n-k)!}$

$k, n \in N$ के लिए परिभाषित किया जाता है कि

$B(k, n) = 1.2.3......... k + 2.3.4........(k+1) + ...+ n(n+1).......(n+k-1)$, $S_0(n) = n$ एवं $S_k(n) = 1^k + 2^k + ... + n^k$.

B(k, n) का मान ज्ञात करने के लिए B(k, n) को निम्न प्रकार पुनः लिखने पर

$$B(k,n) = k!\left[{}^kC_k + {}^{k+1}C_k + {}^{k+2}C_k + ... + {}^{n+k-1}C_k\right] = k!\left({}^{n+k}C_{k+1}\right)$$

$= \dfrac{n(n+1)........(n+k)}{k+1}$ जहाँ ${}^nC_k = \dfrac{n!}{k!\,(n-k)!}$

8. Prove that $S_2(n) + S_1(n) = B(2, n)$

सिद्ध कीजिए $S_2(n) + S_1(n) = B(2, n)$

Sol. $S_2(n) + S_1(n) = \sum n^2 + \sum n$

$= \sum n(n+1)$

$= 1.2 + 2.3 + 3.4 + ... + n(n + 1)$

$= B(2, n)$

9. Prove that सिद्ध कीजिए $S_3(n) + 3S_2(n) = B(3, n) - 2B(1, n)$

Sol. $S_3(n) + 3S_2(n) + 2S_1(n) - 2\,S_1(n)$

$= \sum n^3 + 3\sum n^2 + 2\sum n - 2\sum n$

$= \sum n(n+1)(n+2) - 2\sum n$

$= B(3, n) - 2\,B(1, n)$

10. If $(1 + x)^p = 1 + {}^pC_1\,x + {}^pC_2x^2 + ... + {}^pC_p\,x^p,\ p \in N$, then show that ${}^{k+1}C_1\,S_k(n) + {}^{k+1}C_2\,S_{k-1}(n) + ... + {}^{k+1}C_k\,S_1(n) + {}^{k+1}C_{k+1}\,S_0(n) = (n + 1)^{k+1} - 1$

यदि $(1 + x)^p = 1 + {}^pC_1\,x + {}^pC_2x^2 + ... + {}^pC_p\,x^p,\ p \in N$ तब दर्शाइये कि ${}^{k+1}C_1\,S_k(n) + {}^{k+1}C_2\,S_{k-1}(n) + ... + {}^{k+1}C_k\,S_1(n) + {}^{k+1}C_{k+1}\,S_0(n) = (n + 1)^{k+1} - 1$

Sol. $(1 + x)^{k+1} = {}^{k+1}C_{k+1} + {}^{k+1}C_k\,x + {}^{k+1}C_{k-1}\,x^2 + {}^{k+1}C_{k-2}\,x^3 + ... + {}^{k+1}C_0\,x^{k+1}$

Put $x = 1, 2, ..., n$ रखने पर

$2^{k+1} = {}^{k+1}C_{k+1} + {}^{k+1}C_k\,.\,1 + {}^{k+1}C_{k-1}.1^2 + {}^{k+1}C_{k-2}\,.\,1^3 + ... + {}^{k+1}C_0\,.\,1^{k+1}$

$3^{k+1} = {}^{k+1}C_{k+1} + {}^{k+1}C_k\,.\,2 + {}^{k+1}C_{k-1}\,.2^2 + ... + {}^{k+1}C_0.\,2^{k+1}$

$\vdots$

$(1 + n)^{k+1} = {}^{k+1}C_{k+1} + {}^{k+1}C_k\,.\,n + {}^{k+1}C_{k-1},\,n^2 + ... + {}^{k+1}C_0\,n^{k+1}$

$2^{k+1} + 3^{k+1} + ... + (1 + n)^{k+1} = {}^{k+1}C_{k+1}\,S_0(n) + {}^{k+1}C_k\,S_1(n) + {}^{k+1}C_{k-1}\,S_2(n) + ... + {}^{k+1}C_0\,S_{(k+1)}\,(n)$

$2^{k+1} + 3^{k+1} + ... + (n+1)^{k+1} = {}^{k+1}C_{k+1}\,S_0(n) + {}^{k+1}C_k\,S_1(n) + ... + {}^{k+1}C_1\,S_k(n) + 1^{k+1} + 2^{k+1} + 3^{k+1} + ... + n^{k+1}$

So $(n + 1)^{k+1} - 1$

Hindi $(1 + x)^{k+1} = {}^{k+1}C_{k+1} + {}^{k+1}C_k\,x + {}^{k+1}C_{k-1}\,x^2 + {}^{k+1}C_{k-2}\,x^3 + ... + {}^{k+1}C_0\,x^{k+1}$

$x = 1, 2, ..., n$ रखने पर

$2^{k+1} = {}^{k+1}C_{k+1} + {}^{k+1}C_k\,.\,1 + {}^{k+1}C_{k-1}.1^2 + {}^{k+1}C_{k-2}\,.\,1^3 + ... + {}^{k+1}C_0\,.\,1^{k+1}$

$3^{k+1} = {}^{k+1}C_{k+1} + {}^{k+1}C_k\,.\,2 + {}^{k+1}C_{k-1}\,.2^2 + ... + {}^{k+1}C_0.\,2^{k+1}$

$\vdots$

$(1 + n)^{k+1} = {}^{k+1}C_{k+1} + {}^{k+1}C_k\,.\,n + {}^{k+1}C_{k-1},\,n^2 + ... + {}^{k+1}C_0\,n^{k+1}$

$2^{k+1} + 3^{k+1} + ... + (1 + n)^{k+1} = {}^{k+1}C_{k+1}\,S_0(n) + {}^{k+1}C_k\,S_1(n) + {}^{k+1}C_{k-1}\,S_2(n) + ... + {}^{k+1}C_0\,S_{(k+1)}\,(n)$

$2^{k+1} + 3^{k+1} + ... + (n+1)^{k+1} = {}^{k+1}C_{k+1}\,S_0(n) + {}^{k+1}C_k\,S_1(n) + ... + {}^{k+1}C_1\,S_k(n) + 1^{k+1} + 2^{k+1} + 3^{k+1} + ... + n^{k+1}$

अतः $(n + 1)^{k+1} - 1$

11. Show that $25^n - 20^n - 8^n + 3^n,\ n \in I^+$ is divisible by 85.

प्रदर्शित कीजिए कि $25^n - 20^n - 8^n + 3^n,\ n \in I^+$, 85 से भाज्य है।

Sol. $85 = 17 \times 5$ (Both are prime number)

$25^n = (20 + 5)^n$

and $8^n = (5 + 3)^n$

So clearly $(20 + 5)^n - 20^n - (5 + 3)^n + 3^n$ is divisible by 5

Also $(17 + 8)^n - 8^n - (17 + 3)^n + 3^n$ is divisible by 17

So expression is divisible by 85

Hindi $85 = 17 \times 5$ (दोनो अभाज्य संख्याएँ है।)

$25^n = (20 + 5)^n$ तथा $8^n = (5 + 3)^n$

अतः स्पष्टतया $(20 + 5)^n - 20^n - (5 + 3)^n + 3^n$, 5 से भाज्य है

पुनः $(17 + 8)^n - 8^n - (17 + 3)^n + 3^n$, 17 से भाज्य है अतः व्यंजक 85 से भाज्य है।

12. Prove that ${}^nC_1\ ({}^nC_2)^2\ ({}^nC_3)^3........({}^nC_n)^n \le \left(\dfrac{2^n}{n+1}\right)^{{}^{n+1}C_2}$.

सिद्ध कीजिए कि ${}^nC_1\ ({}^nC_2)^2\ ({}^nC_3)^3........({}^nC_n)^n \le \left(\dfrac{2^n}{n+1}\right)^{{}^{n+1}C_2}$.

Sol. A.M. $\ge$ G.M

$$\frac{{}^nC_1 + 2.{}^nC_2 + 3.{}^nC_3 + ... + n.{}^nC_n}{1+2+3+........... + n} \ge \sqrt[\frac{n(n+1)}{2}]{{}^nC_1\left({}^nC_2\right)^2........\left({}^nC_n\right)^n}$$

$$= \frac{n.2^{n-1}\cdot 2}{n(n+1)} \ge \sqrt[\frac{n(n+1)}{2}]{{}^nC_1.\left({}^nC_2\right)^2.........\left({}^nC_n\right)^n}$$

$${}^nC_1\left({}^nC_2\right)^2.......\left({}^nC_n\right)^n \le \left(\frac{2^n}{n+1}\right)^{\frac{n(n+1)}{2}} \qquad \left(\text{Also}\frac{n(n+1)}{2} = {}^{n+1}C_2\right)$$

13. If p is nearly equal to q and n > 1, show that $\dfrac{(n+1)\ \ p+(n-1)q}{(n-1)p+(n+1)q} = \left(\dfrac{p}{q}\right)^{1/n}$. Hence find the approximate value of $\left(\dfrac{99}{101}\right)^{1/6}$.

यदि p, q के लगभग बराबर है तथा n > 1, प्रदर्शित कीजिए कि $\dfrac{(n+1)\ \ p+(n-1)q}{(n-1)p+(n+1)q} = \left(\dfrac{p}{q}\right)^{1/n}$. इसकी सहायता से $\left(\dfrac{99}{101}\right)^{1/6}$ का निकटतम मान ज्ञात कीजिए।

Ans. $\dfrac{1198}{1202}$

Sol. Let p = q + h (say), where h is small that its square and higher powers may be neglected. then -

$$\frac{(n+1)\ \ p\ +\ (n-1)\ \ q}{(n-1)p\ +\ (n+1)\ \ q} = \frac{(n+1)\ \ (q+h)\ +\ (n-1)\ \ q}{(n-1)\ \ (q+h)\ +\ (n+1)\ \ q} \quad (\because\ p = q + h)$$

$$= \frac{2nq+(n+1)\ \ h}{2nq\ +\ (n-1)\ \ h} = \left(1+\left(\frac{n+1}{2nq}\right)\ h\right)\left(1+\left(\frac{n-1}{2nq}\right)\ h\right)^{-1} = 1 + \frac{h}{nq} = \left(1+\frac{h}{q}\right)^{1/n} = \left(\frac{p}{q}\right)^{1/n}$$

put p = 99 , q = 101 and n = 6

$$\frac{(6+1)\ \times 99+(6-1)\times 101}{(6-1)\ \times 99+(6+1)\ \times\ 101} = \left(\frac{99}{101}\right)^{1/6} = \frac{1198}{1202}$$

Hindi मानाकि p = q + h (माना), जहाँ h इतना छोटा है कि इसके वर्ग तथा अन्य बड़ी घातों को नगण्य मान सकते है, तो

$$\frac{(n+1)\ \ p\ +\ (n-1)\ \ q}{(n-1)p\ +\ (n+1)\ \ q} = \frac{(n+1)\ \ (q+h)\ +\ (n-1)\ \ q}{(n-1)\ \ (q+h)\ +\ (n+1)\ \ q} \quad (\because p = q + h)$$

$$= \frac{2nq+(n+1)\ \ h}{2nq\ +\ (n-1)\ \ h} = \left(1+\left(\frac{n+1}{2nq}\right)\ h\right)\left(1+\left(\frac{n-1}{2nq}\right)\ h\right)^{-1} = 1 + \frac{h}{nq} = \left(1+\frac{h}{q}\right)^{1/n} = \left(\frac{p}{q}\right)^{1/n}$$

p = 99 , q = 101 तथा n = 6 रखने पर

$$\frac{(6+1)\ \times 99+(6-1)\times 101}{(6-1)\ \times 99+(6+1)\ \times\ 101} = \left(\frac{99}{101}\right)^{1/6} = \frac{1198}{1202}$$

14. If $(18x^2 + 12x + 4)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n}\ x^{2n}$, then prove that

$a_r = 2^n\ 3^r\ \left({}^{2n}C_r + {}^nC_1\ {}^{2n-2}C_r\ +\ {}^nC_2\ {}^{2n-4}C_r\ +\ ...\right)$

यदि $(18x^2 + 12x + 4)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n}\ x^{2n}$, तब सिद्ध कीजिए

$a_r = 2^n\ 3^r\ \left({}^{2n}C_r + {}^nC_1\ {}^{2n-2}C_r\ +\ {}^nC_2\ {}^{2n-4}C_r\ +\ ...\right)$

Sol. a_r is the coefficient of x^r in R.H.S.

$(18x^2 + 12x + 4)^n = 2^n\ (1+(1+3x)^2)^n$

$= 2^n\left({}^nC_0(1+3x)^{2n} + {}^nC_1\ (1+3x)^{2n-2} + {}^nC_2\ (1+3x)^{2n-4} + ...\right)$

$= 2^n\left({}^nC_0 3^r\ {}^{2n}C_r + {}^nC_1 3^r\ {}^{2n-2}C_r + {}^nC_2 3^r\ {}^{2n-4}C_r + ...\right)$

Hindi. a_r , R.H.S. में x^r का गुणांक है।

$(18x^2 + 12x + 4)^n = 2^n\ (1+(1+3x)^2)^n$

$= 2^n\left({}^nC_0(1+3x)^{2n} + {}^nC_1\ (1+3x)^{2n-2} + {}^nC_2\ (1+3x)^{2n-4} + ...\right)$

$= 2^n\ \left({}^nC_0 3^r\ {}^{2n}C_r + {}^nC_1 3^r\ {}^{2n-2}C_r + {}^nC_2 3^r\ {}^{2n-4}C_r + ...\right)$

15. Prove that $1^2. C_0 + 2^2. C_1 + 3^2. C_2 + 4^2. C_3 +.... + (n+1)^2 C_n = 2^{n-2} (n+1) (n+4)$.

Sol. $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$

multiply by x and then differentiate

$(1 + x)^n + x . n (1 + x)^{n-1} = C_0 + 2.C_1x + 3.C_2 x^2 + ... + (n + 1) . C_n x^n$

again multiply by x and then differentiate

$(1 + x)^n + nx (1 + x)^{n-1} + 2nx (1 + x)^{n-1} + n (n - 1) x^2 (1 + x)^{n-2} = C_0 + 2^2 C_1x + 3^2 C_2x^2 +...+ (n+1)^2 C_nx^n$

put x = 1

then $S = 2^n + n. 2^{n-1} + 2n. 2^{n-1} + n (n - 1) 2^{n-2}$

$= 2^{n-2} [4 + 2n + 4n + n^2 - n]$

$= 2^{n-2} (n + 1) (n + 4)$

Hindi. $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$

x से गुणा करने के बाद अवकलन करने पर

$(1 + x)^n + x . n (1 + x)^{n-1} = C_0 + 2.C_1x + 3.C_2 x^2 + ... + (n + 1) . C_n x^n$

दुबारा x से गुणा करके अवकलन करने पर

$(1 + x)^n + nx (1 + x)^{n-1} + 2nx (1 + x)^{n-1} + n (n - 1) x^2 (1 + x)^{n-2} = C_0 + 2^2 C_1x + 3^2 C_2x^2 +...+ (n+1)^2 C_nx^n$

x = 1 रखने पर

तब $S = 2^n + n. 2^{n-1} + 2n. 2^{n-1} + n (n - 1) 2^{n-2}$

$= 2^{n-2} [4 + 2n + 4n + n^2 - n]$

$= 2^{n-2} (n + 1) (n + 4)$.

16. If $(1 - x)^{-n} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 +........$, find the value of, $a_0 + a_1 + a_2 +....... + a_n$.

$(1 - x)^{-n} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 +........$, $a_0 + a_1 + a_2 +....... + a_n$ का मान ज्ञात करो–

Ans. $\frac{(2n)!}{(n!)^2}$

Sol. $(1 - x)^{-n} = a_0 + a_1x + a_2x^2 + ... a_nx^n = \sum_{r=0}^{n} {}^{n+r-1}C_r x^r$

$a_0 + a_1 + ... + a_n = {}^{n-1}C_0 + {}^nC_1 + {}^{n+1}C_2 + ... + {}^{2n-1}C_n$
$= {}^{n-1}C_{n-1} + {}^nC_{n-1} + {}^{n+1}C_{n-1} + ... + {}^{2n-1}C_{n-1}$
$= {}^nC_n + {}^nC_{n-1} + {}^{n+1}C_{n-1} + ... + {}^{2n-1}C_{n-1} = {}^{2n}C_n$ $\{{}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r\}$

17. Find the remainder when $32^{32^{32}}$ is divided by 7.

$32^{32^{32}}$ को 7 से भाग देने पर शेषफल ज्ञात करो।

Ans. 4

Sol. $32^{32} = (2^5)^{32} = 2^{160}$

$(3-1)^{160} = 3\lambda + 1$

$32^{32^{32}} = (2^5)^{(3\lambda+1)} = 2^{(15\lambda+3)+2} = 4 . (2^3)^{(5\lambda+1)} = 4(7+1)^{\beta} = 4(7\mu + 1)$

$\therefore$ remainder is 4

Hindi. $32^{32} = (2^5)^{32} = 2^{160}$

$(3-1)^{160} = 3\lambda + 1$

$32^{32^{32}} = (2^5)^{(3\lambda+1)} = 2^{(15\lambda+3)+2} = 4 . (2^3)^{(5\lambda+1)} = 4(7+1)^{\beta} = 4(7\mu + 1)$

$\therefore$ शेषफल 4 है।

18. If n is an integer greater than 1, show that : $a - {}^nC_1(a-1) + {}^nC_2(a-2) - ..... + (-1)^n (a-n) = 0$.

यदि n (> 1) एक पूर्णांक है तब प्रदर्शित कीजिए : $a - {}^nC_1(a-1) + {}^nC_2(a-2) - ..... + (-1)^n (a-n) = 0$.

Sol. $S = a\,[{}^nC_0 - {}^nC_1 + {}^nC_2 ....+ (-1)^n\, {}^nC_n] + {}^nC_1 - 2.\, {}^nC_2 + ... + (-1)^{n+1} n\, {}^nC_n$

$(1-x)^n = {}^nC_0 - {}^nC_1x + {}^nC_2x^2 .... + (-1)^n.\, {}^nC_n\, x^n$

$n(1-x)^{n-1} = -\,{}^nC_1 + 2.\, {}^nC_2x + ... + (-1)^n\, n\, {}^nC_n\, x^{n-1}$

put x = 1

then S = 0

Hindi. $S = a\,[{}^nC_0 - {}^nC_1 + {}^nC_2 ....+ (-1)^n\, {}^nC_n] + {}^nC_1 - 2.\, {}^nC_2 + ... + (-1)^{n+1} n\, {}^nC_n$

$(1-x)^n = {}^nC_0 - {}^nC_1x + {}^nC_2x^2 .... + (-1)^n.\, {}^nC_n\, x^n$

$n(1-x)^{n-1} = -\,{}^nC_1 + 2.\, {}^nC_2x + ... + (-1)^n\, n\, {}^nC_n\, x^{n-1}$

x = 1 रखने पर

तब S = 0

19. If $(1+x)^n = p_0 + p_1 x + p_2 x^2 + p_3 x^3 +.......$, then prove that :

यदि $(1+x)^n = p_0 + p_1 x + p_2 x^2 + p_3 x^3 +.......$, तब सिद्ध करो कि

(a) $p_0 - p_2 + p_4 -....... = 2^{n/2} \cos \frac{n\pi}{4}$ (b) $p_1 - p_3 + p_5 -....... = 2^{n/2} \sin \frac{n\pi}{4}$

Sol. $(1+x)^n = p_0 + p_1x + p_2x^2 ....$

$(1-x)^n = p_0 - p_1x + p_2x^2 ....$

$(1+x)^n + (1-x)^n = 2[p_0 + p_2 x^2 + p_4x^4 + ...]$

Put x = i

then $p_0 - p_2 + p_4 ....= \frac{(1+i)^n + (1-i)^n}{2} = 2^{n/2} \cos \frac{n\pi}{4}$

and $(1+x)^n - (1-x)^n = 2\,[p_1x + p_3x^3 + ...]$

or $p_1 - p_3 + p_5 .... = \frac{(1+i)^n - (1-i)^n}{2i} = 2^{n/2} \sin \frac{n\pi}{4}$

Hindi. $(1+x)^n = p_0 + p_1x + p_2x^2 ....$

$(1-x)^n = p_0 - p_1x + p_2x^2 ....$

$(1+x)^n + (1-x)^n = 2[p_0 + p_2 x^2 + p_4x^4 + ...]$

x = i रखने पर

तब $p_0 - p_2 + p_4$ ....= $\dfrac{(1+i)^n + (1-i)^n}{2} = 2^{n/2} \cos \dfrac{n\pi}{4}$

और $(1 + x)^n - (1 - x)^n = 2 [p_1x + p_3x^3 + ...]$

या $p_1 - p_3 + p_5$ .... = $\dfrac{(1+i)^n - (1-i)^n}{2i} = 2^{n/2} \sin \dfrac{n\pi}{4}$

20. Show that if the greatest term in the expansion of $(1 + x)^{2n}$ has also the greatest co-efficient, then ' x ' lies between, $\dfrac{n}{n+1}$ & $\dfrac{n+1}{n}$.

प्रदर्शित करो कि यदि $(1 + x)^{2n}$ के प्रसार में अधिकतम पद का गुणांक भी अधिकतम है, तो ' x ' का मान $\dfrac{n}{n+1}$ और $\dfrac{n+1}{n}$ के बीच में है।

Sol. Middle term has greatest co-efficient in this case so r = n

मध्य पद का गुणांक महत्तम होता है। अतः इस स्थिति में r = n है

$r = \left[\dfrac{2n+1}{1+|x|}\right] \Rightarrow \dfrac{2n+1}{1+|x|} - 1 < n < \dfrac{2n+1}{1+|x|} \qquad \Rightarrow \qquad \dfrac{2n+1}{n+1} - 1 < |x|$ and तथा $|x| < 1 + \dfrac{1}{n}$

$\Rightarrow \qquad \dfrac{n}{n+1} < x < \dfrac{n+1}{n}$

21. Prove that if ' p ' is a prime number greater than 2, then $\left[(2+\sqrt{5})^p\right] - 2^{p+1}$ is divisible by p, where [.] denotes greatest integer function.

सिद्ध करो कि यदि ' p ', 2 से बड़ी एक अभाज्य संख्या है, तो $\left[(2+\sqrt{5})^p\right] - 2^{p+1}$, p से विभाजित होगा, जहाँ [.] महत्तम पूर्णांक फलन है।

Sol. $\left[(2+\sqrt{5})^p\right] - 2^{p+1}$

Let $(\sqrt{5}+2)^p = I + f$ so $[(\sqrt{5}+2)^p] = I$, where I is an integer and $f \in (0, 1)$

$(\sqrt{5}-2)^p = f' \in (0, 1)$

$2[{}^pC_0 2^p + {}^pC_2 2^{p-2} (\sqrt{5})^2 + ...] = I + f - f'$

$\Rightarrow f' - f = 0 \qquad \Rightarrow f = f' \quad \Rightarrow [(2+\sqrt{5})^p] - 2^{p+1} = \quad 2[{}^pC_0 2^p + {}^pC_2 2^{p-2}.5 + ...] - 2^{p+1}$

$= \quad {}^pC_2.2^{p-1}.5 + {}^pC_4 2^{p-3}.5^2 + \;........$

This is always divisible by p because for a prime number p, ${}^pC_r(1 < r < p)$ is always divisible by p.

Hindi $\left[(2+\sqrt{5})^p\right] - 2^{p+1}$

माना $(\sqrt{5}+2)^p = I + f$ इसलिए $[(\sqrt{5}+2)^p] = I$, जहाँ I एक पूर्णांक है तथा $f \in (0, 1)$

$(\sqrt{5}-2)^p = f' \in (0, 1)$

$2[{}^pC_0 2^p + {}^pC_2 2^{p-2} (\sqrt{5})^2 + ...] = I + f - f'$

$\Rightarrow f' - f = 0 \qquad \Rightarrow \; f = f' \;\Rightarrow\; [(2+\sqrt{5})^p] - 2^{p+1}$

$= \quad 2[{}^pC_0 2^p + {}^pC_2 2^{p-2}.5 + ...] - 2^{p+1}$

$= \quad {}^pC_2.2^{p-1}.5 + {}^pC_4 2^{p-3}.5^2 + \;........$

यह हमेशा p से भाजित है क्योंकि एक अभाज्य संख्या p हेतु ${}^pC_r(1 < r < p)$ सदैव p से भाजित है।

22. If $\sum_{r=0}^{n} (-1)^r.\ {}^nC_r \left[\frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + ... \text{ to m terms}\right] = k\left(1 - \frac{1}{2^{mn}}\right)$, then find the value of k.

यदि $\sum_{r=0}^{n} (-1)^r.\ {}^nC_r \left[\frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + ... \text{ m पदों तक}\right] = k\left(1 - \frac{1}{2^{mn}}\right)$ हो, तो k का मान ज्ञात कीजिए।

Ans. $\frac{1}{2^n - 1}$

Sol. $\sum_{r=0}^{n} (-1)^r.\ {}^nC_r \left[\frac{1}{2^r} + \left(\frac{3}{4}\right)^r + \left(\frac{7}{8}\right)^r + ...\text{m terms}\right]$

$= \sum_{r=0}^{n} \left[(-1)^r\ {}^nC_r \frac{1}{2^r} + (-1)^r\ {}^nC_r \left(\frac{3}{4}\right)^r + ...\text{m terms}\right]$

$= [\,{}^nC_0 - {}^nC_1 \left(\frac{1}{2}\right) + {}^nC_2 \left(\frac{1}{2}\right)^2 + ...] +$

$[\,{}^nC_0 - {}^nC_1 + {}^nC_2 + ...] + ...\text{m terms}$

$= \left(1 - \frac{1}{2}\right)^n + \left(1 - \frac{3}{4}\right)^n + \left(1 - \frac{7}{8}\right)^n + ...\text{m terms} = \frac{1}{2^n}\left(\frac{1 - \frac{1}{2^{mn}}}{1 - \frac{1}{2^n}}\right) = \frac{1}{2^n - 1}\left(1 - \frac{1}{2^{mn}}\right)$

Hindi. $\sum_{r=0}^{n} (-1)^r.\ {}^nC_r = \left[\frac{1}{2^r} + \left(\frac{3}{4}\right)^r + \left(\frac{7}{8}\right)^r + ...\text{m terms}\right]$

$= \left[(-1)^r\ {}^nC_r \frac{1}{2^r} + (-1)^r\ {}^nC_r \left(\frac{3}{4}\right)^r + ...\text{m terms}\right]$

$= [\,{}^nC_0 - {}^nC_1 \left(\frac{1}{2}\right) + {}^nC_2 \left(\frac{1}{2}\right)^2 + ...] +$

$[\,{}^nC_0 - {}^nC_1 + {}^nC_2 + ...] + ...$m पद

$= \left(1 - \frac{1}{2}\right)^n + \left(1 - \frac{3}{4}\right)^n + \left(1 - \frac{7}{8}\right)^n + ...$m पद

$\frac{1}{2^n}\left(\frac{1 - \frac{1}{2^{mn}}}{1 - \frac{1}{2^n}}\right) = \frac{1}{2^n - 1}\left(1 - \frac{1}{2^{mn}}\right)$

23. Given $s_n = 1 + q + q^2 + ... + q^n$ & $S_n = 1 + \frac{q+1}{2} + \left(\frac{q+1}{2}\right)^2 + ... + \left(\frac{q+1}{2}\right)^n$, $q \neq 1$,

prove that ${}^{n+1}C_1 + {}^{n+1}C_2.s_1 + {}^{n+1}C_3.s_2 + ... + {}^{n+1}C_{n+1}.s_n = 2^n.\ S_n.$

यदि $s_n = 1 + q + q^2 + ... + q^n$ तथा $S_n = 1 + \frac{q+1}{2} + \left(\frac{q+1}{2}\right)^2 + ... + \left(\frac{q+1}{2}\right)^n$, $q \neq 1$ हो, तो

सिद्ध करो कि ${}^{n+1}C_1 + {}^{n+1}C_2.s_1 + {}^{n+1}C_3.s_2 + ... + {}^{n+1}C_{n+1}.s_n = 2^n.\ S_n.$

Sol. $s_n = \frac{q^{n+1} - 1}{q - 1}$ and और $S_n = \frac{\left(\frac{q+1}{2}\right)^{n+1} - 1}{\frac{q+1}{2} - 1}$

$^{n+1}C_1 + {}^{n+1}C_2 s_1 + ...+ {}^{n+1}C_{n+1}\, s_n$

$= {}^{n+1}C_1 + {}^{n+1}C_2\left(\frac{q^2-1}{q-1}\right) + ...+ {}^{n+1}C_{n+1}\ \frac{q^{n+1}-1}{q-1}$

$= \frac{1}{q-1}\ [{}^{n+1}C_1 q + {}^{n+1}C_2 q^2 + ...+ {}^{n+1}C_{n+1} q^{n+1} - {}^{n+1}C_1 - {}^{n+1}C_2 - .... - {}^{n+1}C_{n+1}]$

$= \frac{1}{q-1}\ [(1+q)^{n+1} - 1 - 2^{n+1} + 1] = \frac{1}{q-1}\ [(1+q)^{n+1} - 2^{n+1}] = \left(\frac{\left(\frac{q+1}{2}\right)^{n+1} - 1}{\frac{q-1}{2}}\right) . 2^n = 2^n S_n$

24. If $(1+x)^{15} = C_0 + C_1.\, x + C_2.\, x^2 + ... + C_{15}.\, x^{15}$, then find the value of : $C_2 + 2C_3 + 3C_4 + ... + 14C_{15}$

यदि $(1+x)^{15} = C_0 + C_1.\, x + C_2.\, x^2 + ... + C_{15}.\, x^{15}$ हो, तो $C_2 + 2C_3 + 3C_4 + ... + 14C_{15}$ का मान ज्ञात करो।

Ans. 212993

Sol. $(1 + x)^{15} = C_0 + C_1x + ...+ C_{15}x^{15}$

Divide by x & then differentiating both side

$\frac{(1+x)^{15}}{x} = \frac{C_0}{x} + C_1 + C_2x + C_3x^2 + ...+ C_{15}x^{14}$

$.\ \frac{1}{x} 15(1 + x)^{14} - \frac{(1+x)^{15}}{x^2} = -\frac{C_0}{x^2} + C_2 + ...+ 14\, C_{15}x^{13}$

Put x = 1 then $C_2 + 2C_3 + ...+ 14C_{15} = 15.2^{14} - 2^{15} + 1$

Hindi. $(1 + x)^{15} = C_0 + C_1x + ...+ C_{15}x^{15}$

x से भाग देकर दोनों पक्षों का अवकलन करने पर

$\frac{(1+x)^{15}}{x} = \frac{C_0}{x} + C_1 + C_2x + C_3x^2 + ...+ C_{15}x^{14}$

$.\ \frac{1}{x} 15(1 + x)^{14} - \frac{(1+x)^{15}}{x^2} = -\frac{C_0}{x^2} + C_2 + ...+ 14\, C_{15}x^{13}$

x = 1 रखने पर $C_2 + 2C_3 + ...+ 14C_{15} = 15.2^{14} - 2^{15} + 1$

25. Prove that, $\frac{1}{2}\ {}^nC_1 - \frac{2}{3}\,{}^nC_2 + \frac{3}{4}\,{}^nC_3 - \frac{4}{5}\,{}^nC_4 + ... + \frac{(-1)^{n+1}\, n}{n+1} . {}^nC_n = \frac{1}{n+1}$

सिद्ध कीजिए कि : $\frac{1}{2}\ {}^nC_1 - \frac{2}{3}\,{}^nC_2 + \frac{3}{4}\,{}^nC_3 - \frac{4}{5}\,{}^nC_4 + ... + \frac{(-1)^{n+1}\, n}{n+1} . {}^nC_n = \frac{1}{n+1}$

Sol. $(1+x)^n = C_0 + C_1x + C_2x^2 + ...+ C_nx^n$

$\Rightarrow n(1 + x)^{n-1} = C_1 + 2.C_2x + 3.C_3x^2 + ...+ n.C_nx^{n-1}$

Now multiply by x & integrate from 0 to x

$n(1 + x)^{n-1}.\ x = C_1x + 2C_2x^2 + ...+ nC_nx^n$

$\Rightarrow \frac{C_1}{2}x^2 + \frac{2C_2x^3}{3} + \frac{3C_3x^4}{4} + ...+ \frac{n}{n+1}C_nx^{n+1} = x(1+x)^n - \frac{(1+x)^{n+1}-1}{n+1}$

Putting x = –1

$\frac{C_1}{2} - \frac{2}{3}C_2 + \frac{3}{4}C_3 + ... = \frac{1}{n+1}$

Hindi. $(1+x)^n = C_0 + C_1x + C_2x^2 + ...+ C_nx^n$

$\Rightarrow n(1 + x)^{n-1} = C_1 + 2.C_2x + 3.C_3x^2 + ...+ n.C_nx^{n-1}$

x से गुणा करके 0 से x सीमाओं में समाकलन करने पर

$n(1 + x)^{n-1}.\ x = C_1x + 2\, C_2x^2 + ...+ nC_nx^n$

$\Rightarrow \frac{C_1}{2}x^2 + \frac{2C_2x^3}{3} + \frac{3C_3x^4}{4}$ +.........+ $\frac{n}{n+1}C_nx^{n+1}$ = $x(1+x)^n - \frac{(1+x)^{n+1}-1}{n+1}$

x = –1 रखने पर

$\frac{C_1}{2} - \frac{2}{3}C_2 + \frac{3}{4}C_3 + ... = \frac{1}{n+1}$

26. Prove that $\sum_{r=0}^{n} r^2 \; {}^nC_r \, p^r q^{n-r} = npq + n^2p^2$, if p + q = 1.

सिद्ध करो कि $\sum_{r=0}^{n} r^2 \, {}^nC_r \, p^r q^{n-r} = npq + n^2p^2$ होगा, जबकि p + q = 1 हो।

Sol. $\sum_{r=0}^{n} r^2 \, {}^nC_r \; p^r q^{n-r} = n \, . \, \sum_{r=0}^{n} r \, . \, {}^{n-1}C_{r-1} p^r q^{n-r}$

$= n. \left[\sum_{r=1}^{n} (r-1) \; . \; {}^{n-1}C_{r-1} p^r q^{n-r} + \sum_{r=1}^{n} {}^{n-1}C_{r-1}. \;\; p^r q^{n-r} \right]$

$= n \left[(n-1)p^2 \sum_{r=2}^{n} {}^{n-2}C_{r-2} p^{r-2} q^{n-r} + p \sum_{r=1}^{n} {}^{n-1}C_{r-1}. \;\; p^{r-1} q^{n-r} \right]$

$= n[(n-1)p^2(p+q)^{n-2} + p(p+q)^{n-1}]$

$= n[np^2 - p^2 + p] = n^2p^2 - np^2 + pn = n^2p^2 + npq$

27. Prove that : $(n-1)^2. C_1 + (n-3)^2. C_3 + (n-5)^2. C_5 + ... = n(n+1)2^{n-3}$

सिद्ध कीजिए कि : $(n-1)^2. C_1 + (n-3)^2. C_3 + (n-5)^2. C_5 + ... = n(n+1)2^{n-3}$

Sol. $(n-1)^2 \, {}^nC_1 + (n-3)^2 \, {}^nC_3 + (n-5)^2 \; {}^nC_5 + ...$

$= n^2({}^nC_1 + {}^nC_3 + {}^nC_5 + ...) - 2n({}^nC_1 + 3{}^nC_3 + 5{}^nC_5 + ...) + ({}^nC_1 + 9{}^nC_3 + 25{}^nC_5 + ...)$

$= n^2 . 2^{n-1} - 2n^2({}^{n-1}C_0 + {}^{n-1}C_2 + {}^{n-1}C_4 + ...) + n({}^{n-1}C_0 + 3{}^{n-1}C_2 + 5{}^{n-1}C_4 + ...)$

$= n^2 . 2^{n-1} - 2n^2 . (2^{n-2}) + n({}^{n-1}C_0 + {}^{n-1}C_2 + {}^{n-1}C_4 + ...) + n(2{}^{n-1}C_2 + 4\,{}^{n-1}C_4 + 6\,{}^{n-1}C_6 + ...)$

$= n^2 . 2^{n-1} - n^2 . 2^{n-1} + n . 2^{n-2} + n(n-1)({}^{n-2}C_1 + {}^{n-2}C_3 + {}^{n-2}C_5 + ...)$

$= n . 2^{n-2} + n(n-1)\,2^{n-3}$

$= n(n+1)\,2^{n-3}$.

28. Prove that ${}^nC_r + 2\;{}^{n+1}C_r + 3\;{}^{n+2}C_r + ... + (n+1)\;{}^{2n}C_r = {}^nC_{r+2} + (n+1)\;{}^{2n+1}C_{r+1} - {}^{2n+1}C_{r+2}$

सिद्ध करो ${}^nC_r + 2\;{}^{n+1}C_r + 3\;{}^{n+2}C_r + ... + (n+1)\;{}^{2n}C_r = {}^nC_{r+2} + (n+1)\;{}^{2n+1}C_{r+1} - {}^{2n+1}C_{r+2}$

Sol. Let ${}^nC_r + 2.\;{}^{n+1}C_r + 3.\;{}^{n+2}C_r + ... = S$

S = co-efficient of x^r in $(1+x)^n + 2.(1+x)^{n+1} + 3.(1+x)^{n+2}$ + ...

Let $S' = (1+x)^n + 2(1+x)^{n+1} + 3(1+x)^{n+2} + ...+ (n+1)(1+x)^{2n}$ .... (1)

$(1+x)\,S' = (1+x)^{n+1} + 2(1+x)^{n+2} + ...+ (n+1)(1+x)^{2n+1}$ .... (2)

(1) – (2) :

$-x\,S' = (1+x)^n + (1+x)^{n+1} + ...+ (1+x)^{2n} - (n+1)\;(1+x)^{2n+1}$

$\Rightarrow -x\,S' = (1+x)^n \left[\frac{(1+x)^{n+1}-1}{x} \right] - (n+1)(1+x)^{2n+1} \quad \Rightarrow S' = \frac{(1+x)^{2n+1} - (1+x)^n}{-x^2} + \frac{(n+1)(1+x)^{2n+1}}{x}$

Now S = co-efficient of x^r in S'

$= -\,{}^{2n+1}C_{r+2} + {}^nC_{r+2} + (n+1)\;{}^{2n+1}C_{r+1}$

Hindi. माना ${}^nC_r + 2.\;{}^{n+1}C_r + 3.\;{}^{n+2}C_r + ... = S$

$S = (1+x)^n + 2.(1+x)^{n+1} + 3.(1+x)^{n+2}$ + ... में x^r का गुणांक

माना $S' = (1+x)^n + 2(1+x)^{n+1} + 3(1+x)^{n+2} + ...+ (n+1)(1+x)^{2n}$ .... (1)

$(1+x)\, S' = (1+x)^{n+1} + 2(1+x)^{n+2} + ...+ (n+1)(1+x)^{2n+1}$ .... (2)

(1) – (2) :

$-x\, S' = (1+x)^n + (1+x)^{n+1} + ...+ (1+x)^{2n} - (n+1) \quad (1+x)^{2n+1}$

$\Rightarrow -x\, S' = (1+x)^n \left[\dfrac{(1+x)^{n+1}-1}{x}\right] - (n+1)(1+x)^{2n+1} \quad \Rightarrow S' = \dfrac{(1+x)^{2n+1}-(1+x)^n}{-x^2} + \dfrac{(n+1)(1+x)^{2n+1}}{x}$

अब $\quad S = S'$ में x^r का गुणांक

$= -\,{}^{2n+1}C_{r+2} + {}^{n}C_{r+2} + (n+1)\ {}^{2n+1}C_{r+1}$

29. Show that, $\sqrt{3} = 1 + \dfrac{1}{3} + \dfrac{1}{3}.\dfrac{3}{6} + \dfrac{1}{3}.\dfrac{3}{6}.\dfrac{5}{9} + \dfrac{1}{3}.\dfrac{3}{6}.\dfrac{5}{9}.\dfrac{7}{12} + ...$

प्रदर्शित कीजिए $\sqrt{3} = 1 + \dfrac{1}{3} + \dfrac{1}{3}.\dfrac{3}{6} + \dfrac{1}{3}.\dfrac{3}{6}.\dfrac{5}{9} + \dfrac{1}{3}.\dfrac{3}{6}.\dfrac{5}{9}.\dfrac{7}{12} + ...$

Sol. $\sqrt{3} = \left(\dfrac{1}{3}\right)^{-1/2} = \left(1-\dfrac{2}{3}\right)^{-1/2}$

$= 1 + \dfrac{1}{2}\cdot\dfrac{2}{3} + \dfrac{1}{2}\cdot\left(\dfrac{3}{2}\right)\cdot\left(\dfrac{2}{3}\right)^2\cdot\dfrac{1}{2} + \dfrac{1}{2}\cdot\dfrac{3}{2}\cdot\dfrac{5}{2}\cdot\left(\dfrac{2}{3}\right)^3\cdot\dfrac{1}{3!} + ...$

$= 1 + \dfrac{1}{3} + \dfrac{1}{3}\left(\dfrac{3}{6}\right) + \dfrac{1}{3}\left(\dfrac{3}{6}\right)\left(\dfrac{5}{9}\right) + ...$

30. If $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$, show that for $m \geq 2$

$C_0 - C_1 + C_2 - .... + (-1)^{m-1}C_{m-1} = (-1)^{m-1}\ {}^{n-1}C_{m-1}$.

यदि $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$ हो, तो प्रदर्शित कीजिए कि $m \geq 2$ के लिए

$C_0 - C_1 + C_2 - .... + (-1)^{m-1}C_{m-1} = (-1)^{m-1}\ {}^{n-1}C_{m-1}$.

Sol. $(x-1)^n = C_0x^n - C_1x^{n-1} + C_2x^{n-2} - C_3x^{n-3} + ...+ (-1)^{m-1}C_{m-1}x^{n-m+1} + ...$

$\Rightarrow \quad \dfrac{1-x^m}{1-x} = 1 + x + x^2 + ...+ x^{m-1}$

$C_0 - C_1 + C_2 - C_3 + ...+(-1)^{m-1}C_{m-1}$

$=$ Co-efficient of x^n in $(x-1)^n \left(\dfrac{1-x^m}{1-x}\right)$

$=$ Co-efficient of x^n in $(x^m-1)(x-1)^{n-1}$

$=$ Co-efficient of x^{n-m} in $(x-1)^{n-1}$

$= (-1)^{m-1}\ {}^{n-1}C_{m-1}$

Hindi $(x-1)^n = C_0x^n - C_1x^{n-1} + C_2x^{n-2} - C_3x^{n-3} + ...+ (-1)^{m-1}C_{m-1}x^{n-m+1} + ...$

$\Rightarrow \quad \dfrac{1-x^m}{1-x} = 1 + x + x^2 + ...+ x^{m-1}$

$C_0 - C_1 + C_2 - C_3 + ...+(-1)^{m-1}C_{m-1}$

$= (x-1)^n\left(\dfrac{1-x^m}{1-x}\right)$ में x^n का गुणांक

$= (x^m-1)(x-1)^{n-1}$ में x^n का गुणांक $= (x-1)^{n-1}$ में x^{n-m} का गुणांक

$= (-1)^{m-1}\ {}^{n-1}C_{m-1}$

31. If $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_n x^n$, then show that the sum of the products of the C_i's taken two at a time, represented by $\underset{0 \le i < j \le n}{\sum\sum} C_i C_j$ is equal to $2^{2n-1} - \frac{2n!}{2(n!)^2}$.

यदि $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_n x^n$, तब दर्शाओं कि दो C_i's को एक साथ लेने पर उनके गुणनफलनों का योग जोकि $\underset{0 \le i < j \le n}{\sum\sum} C_i C_j$ द्वारा प्रदर्शित होता है $2^{2n-1} - \frac{2n!}{2(n!)^2}$ के बराबर है।

Sol. $(1+x)^n = C_0 + C_1x + ... + C_n x^n$

$S = \sum\sum C_iC_j = \frac{1}{2}\left[(C_0 + C_1 + C_2 + ...+ C_n)^2 - (C_0^2 + C_1^2 + C_2^2 + ...+ C_n^2)\right]$

$\Rightarrow 2S = 2^{2n} - {}^{2n}C_n \qquad \Rightarrow S = 2^{2n-1} - \frac{{}^{2n}C_n}{2}$

32. If $a_0, a_1, a_2,.....$ be the coefficients in the expansion of $(1 + x + x^2)^n$ in ascending powers of x, then prove that :

(i) $a_0 a_1 - a_1 a_2 + a_2 a_3 - .... = 0$

(ii) $a_0a_2 - a_1a_3 + a_2a_4 - ..... + a_{2n-2}\, a_{2n} = a_{n+1}$

(iii) $E_1 = E_2 = E_3 = 3^{n-1}$; where $E_1 = a_0 + a_3 + a_6 + ...$; $E_2 = a_1 + a_4 + a_7 + ...$ & $E_3 = a_2 + a_5 + a_8 + ...$

यदि $a_0, a_1, a_2,.....$, $(1 + x + x^2)^n$ के प्रसार में x की बढ़ती हुई घातों के गुणांक हैं, तो सिद्ध करो कि :

(i) $a_0 a_1 - a_1 a_2 + a_2 a_3 - .... = 0$

(ii) $a_0a_2 - a_1a_3 + a_2a_4 - ..... + a_{2n-2}\, a_{2n} = a_{n+1}$

(iii) $E_1 = E_2 = E_3 = 3^{n-1}$; जहाँ $E_1 = a_0 + a_3 + a_6 + ...$; $E_2 = a_1 + a_4 + a_7 + ...$ एवं $E_3 = a_2 + a_5 + a_8 + ...$

Sol. $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + a_3x^3 + ...+ a_{2n}x^{2n}$

taking $-\frac{1}{x}$ in place of x.

$$\left(1 - \frac{1}{x} + \frac{1}{x^2}\right)^n = a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} - \frac{a_3}{x^3} + ...$$

(i) $\therefore \quad a_0a_1 - a_1a_2 + a_2a_3 ....$ = coefficient of x in $(1 + x + x^2)^n \left(\frac{x^2 - x + 1}{x^2}\right)^n$

= coefficient of x in $\frac{1}{x^{2n}}(x^4 + x^2 + 1)^n$

$= 0$

(ii) $a_0 a_2 - a_1a_3 + a_2a_4 ....$ = coeff. of x^2 in $\frac{1}{x^{2n}}(x^4 + x^2 + 1)^n$

$= a_{n+1}$

(iii) putting $x = 1, \omega$ & ω^2 respectively we get

$3^n = a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + ...$ (1)

$0 = a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3 + a_4\omega^4 + a_5\omega^5 + a_6\omega^6 + ...$ (2)

$0 = a_0 + a_1\omega^2 + a_2\omega^4 + a_3\omega^6 + a_4\omega^8 + a_5\,\omega^{10} + a_6\omega^{12} + ...$ (3)

on adding

$3^n = 3(a_0 + a_3 + a_6 + ...)$

$\Rightarrow \quad E_1 = 3^{n-1}$

(1) + ω^2 (2) + ω(3) gives

$3^n = 3(a_1 + a_4 + a_7 + ...)$

$\Rightarrow \quad E_2 = 3^{n-1}$

Similarly

$(1) + \omega(2) + \omega^2(3)$ gives

$E_3 = 3^{n-1}$.

Hindi. $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + a_3x^3 + ...+ a_{2n}x^{2n}$

x के स्थान पर $-\dfrac{1}{x}$ रखने पर

$$\left(1-\frac{1}{x}+\frac{1}{x^2}\right)^n = a_0 - \frac{a_1}{x}+\frac{a_2}{x^2}-\frac{a_3}{x^3}+......$$

(i) $\therefore \quad a_0a_1 - a_1a_2 + a_2a_3 ....= (1 + x + x^2)^n \left(\dfrac{x^2-x+1}{x^2}\right)^n$ में x का गुणांक

$= \dfrac{1}{x^{2n}} (x^4 + x^2 + 1)^n$ में x का गुणांक

$= 0$

(ii) $a_0 a_2 - a_1a_3 + a_2a_4 .... = \dfrac{1}{x^{2n}} (x^4 + x^2 + 1)^n$ में x^2 का गुणांक

$= a_{n+1}$

(iii) x = 1, ω, और ω^2 क्रमशः रखने पर

$3^n = a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + ...$ (1)

$0 = a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3 + a_4\omega^4 + a_5\omega^5 + a_6\omega^6 + ...$ (2)

$0 = a_0 + a_1\omega^2 + a_2\omega^4 + a_3\omega^6 + a_4\omega^8 + a_5 \omega^{10} + a_6\omega^{12} + ...$ (3)

(1), (2) व (3) को जोड़ने पर

$3^n = 3(a_0 + a_3 + a_6 + ...)$

$\Rightarrow \quad E_1 = 3^{n-1}$

$(1) + \omega^2 (2) + \omega(3)$ से प्राप्त होता है–

$3^n = 3(a_1 + a_4 + a_7 + ...)$

$\Rightarrow \quad E_2 = 3^{n-1}$

इसी प्रकार

$(1) + \omega(2) + \omega^2(3)$ से प्राप्त होता है–

$E_3 = 3^{n-1}$.