DAY SEVEN

System of Particles and Rigid Body

Learning & Revision for the Day

- Centre of Mass
- Momentum Conservation
- Rigid Bodies
- Moment of Inertia
- Radius of Gyration
- Theorems on Moment of Inertia
- Radius of Gyration
- Moment of Inertia for Simple Geometrical Objects

Centre of Mass

Centre of mass of a system (body) is a point that moves when external forces are applied on the body as though all the mass were concentrated at that point and all external forces were applied there.

Centre of Mass of Two Particle System

Centre of mass of a two particles system consisting of two particles of masses m_1, m_2 and respective position vectors $\mathbf{r}_1, \mathbf{r}_2$ is given by

$$\mathbf{r}_{CM} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$

If $m_1 = m_2 = m$ (say), then $\mathbf{r}_{CM} = \frac{\mathbf{r}_1 + \mathbf{r}_2}{2}$

Centre of Mass of *n*-Particle System

Centre of mass of n-particles system which consists n-particles of masses $m_1, m_2, \dots, m_n$ with $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n$ as their position vectors at a given instant of time.

The $\mathbf{r}_{CM}$ of the system at that instant is given by

$$\mathbf{r}_{CM} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \dots + m_n\mathbf{r}_n}{m_1 + m_2 + \dots + m_n} = \frac{\sum_{i=1}^{n} m_i\mathbf{r}_i}{M}$$

Cartesian Components of the Centre of Mass

The position vectors $\mathbf{r}_{CM}$ and $\mathbf{r}_i$ are related to their cartesian components by

$$\mathbf{r}_{CM} = x_{CM}\hat{\mathbf{i}} + y_{CM}\hat{\mathbf{j}} + z_{CM}\hat{\mathbf{k}} \quad \text{and} \quad \mathbf{r}_i = x_i\hat{\mathbf{i}} + y_i\hat{\mathbf{j}} + z_i\hat{\mathbf{k}}$$

The cartesian components of $\mathbf{r}_{CM}$ are given by

$$x_{CM} = \frac{\sum_{i=1}^{n} m_i x_i}{M}, \; y_{CM} = \frac{\sum_{i=1}^{n} m_i y_i}{M} \text{ and } z_{CM} = \frac{\sum_{i=1}^{n} m_i z_i}{M}$$

Motion of Centre of Mass

The position vector $\mathbf{r}_{CM}$ of the centre of mass of n particle system is defined by

$$\mathbf{r}_{CM} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3 + \ldots + m_n\mathbf{r}_n}{m_1 + m_2 + m_3 \ldots m_n}$$

$$= \frac{1}{M}(m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3 + \ldots + m_n\mathbf{r}_n)$$

$$\frac{d\mathbf{r}_{CM}}{dt} = \frac{1}{M}\left(m_1\frac{d\mathbf{r}_1}{dt} + m_2\frac{d\mathbf{r}_2}{dt} + \ldots + m_n\frac{d\mathbf{r}_n}{dt}\right)$$

Velocity of centre of mass $\mathbf{v}_{CM} = \frac{1}{M}(m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + \ldots m_n\mathbf{v}_n)$

$$\mathbf{v}_{CM} = \frac{\sum_{i=1}^{n} m_i\mathbf{v}_i}{M}$$

- Similarly, **acceleration of centre of mass** is given by

$$\mathbf{a}_{CM} = \frac{\sum_{i=1}^{n} m_i\mathbf{a}_i}{M}$$

- From Newton's second law of motion,

$$M\mathbf{a}_{CM} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots \mathbf{F}_n \Rightarrow M\mathbf{a}_{CM} = \mathbf{F}_{Ext}$$

For an isolated system, if external force on the body is zero.

$$\mathbf{F} = M\mathbf{a}_{CM} = M\frac{d}{dt}(\mathbf{v}_{CM}) = 0 \Rightarrow \mathbf{v}_{CM} = \text{constant}$$

i.e. Centre of mass of an isolated system moves with uniform velocity along a straight line path and momentum remain conserved.

NOTE
- If some mass or area is removed from a rigid body, then the position of centre of mass of the remaining portion is obtained from the following formula

$$\mathbf{r}_{CM} = \frac{m_1\mathbf{r}_1 - m_2\mathbf{r}_2}{m_1 - m_2} \quad \text{or} \quad \mathbf{r}_{CM} = \frac{A_1\mathbf{r}_1 - A_2\mathbf{r}_2}{A_1 - A_2}$$

Momentum Conservation

Let us consider a system of particles of masses $m_1, m_2 \ldots m_n$ are having respective velocities $\mathbf{v}_1, \mathbf{v}_2 \ldots \mathbf{v}_n$. The total linear momentum of the system would be the vector sum of the momentum of the individual particles.

i.e. $\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 \ldots \mathbf{p}_n = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + \ldots + m_n\mathbf{v}_n$

Velocity of centre of mass of a system

$$\mathbf{v}_{CM} = \frac{1}{M}(m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + \ldots\ldots + m_n\mathbf{v}_n)$$

where, M is the total mass of the system, therefore

$$\mathbf{p} = M\mathbf{v}_{CM}$$

Thus, total linear momentum of a system of particles is equal to the product of the total mass of the system and the velocity of its centre of mass.

Again $\quad \frac{d\mathbf{p}}{dt} = M\frac{d\mathbf{v}}{dt} = M\mathbf{a} = \mathbf{F}_{ext}$

If $\quad \mathbf{F}_{ext} = 0$, then $\frac{d\mathbf{p}}{dt} = 0$, i.e. $\mathbf{p} = \text{constant}$

If external force of a system is zero, then momentum of system of particle remain constant.

Rigid Bodies

A rigid body is defined as that body which does not undergo any change in shape (or) volume when external forces are applied on it. When a force is applied on a rigid body, the distance between any two particles of the body will remain unchanged, however, larger the forces may be.

Coordinates of centre of mass of a rigid body are

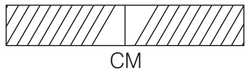
$$X_{CM} = \frac{1}{M}\sqrt{x}\,dm,$$

$$Y_{CM} = \frac{1}{M}\sqrt{y}\,dm$$

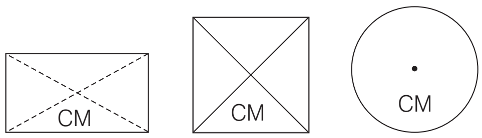
and $\quad Z_{CM} = \frac{1}{M}\sqrt{z}\,dm$

Centre of Mass of Some Rigid Bodies

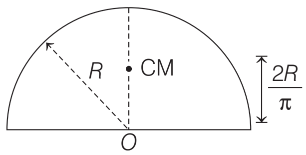
- The centre of mass of a uniform rod is located at its mid-point.



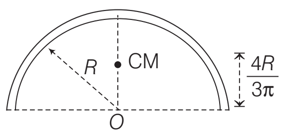
- Centre of mass of a uniform rectangular, square or circular plate lies at its centre.



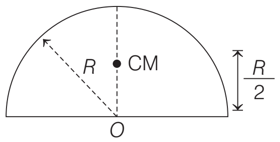
- Centre of mass of a uniform semi-circular ring lies at a distance of $h = \frac{2R}{\pi}$ from its centre, on the axis of symmetry, where R is the radius of the ring.



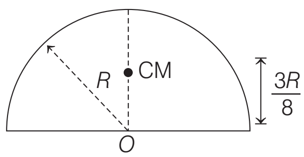
- Centre of mass of a uniform semi-circular disc of radius R lies at a distance of $h = \frac{4R}{3\pi}$ from the centre on the axis of symmetry as shown in figure.



- Centre of mass of a hemispherical shell of radius R lies at a distance of $h = \frac{R}{2}$ from its centre on the axis of symmetry as shown in figure.



- Centre of mass of a solid hemisphere of radius R lies at a distance of $h = \frac{3R}{8}$ from its centre on the axis of symmetry.



Moment of Inertia

Moment of inertia of a rotating body is its property to oppose any change in its state of uniform rotation.

If in a given rotational system particles of masses $m_1, m_2, m_3, \ldots$ be situated at normal distances $r_1, r_2, r_3, \ldots$ from the axis of rotation, then moment of inertia of the system about the axis of rotation is given by

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots = \Sigma\, mr^2$$

For a rigid body having continuous mass distribution

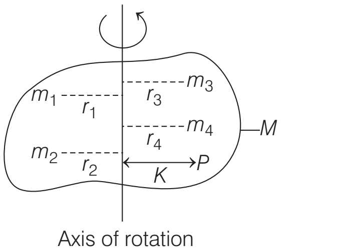
$$I = \int dm\, r^2$$

SI unit of moment of inertia is kg m^2. It is neither a scalar nor a vector i.e. it is a tensor.

Radius of Gyration

Radius of gyration of a given body about a given axis of rotation is the normal distance of a point from the axis, where if whole mass of the body is placed, then its moment of inertia will be exactly same as it has with its actual distribution of mass. Thus, radius of gyration

$$K = \sqrt{\frac{I}{M}} \text{ or } K = \left[\frac{r_1^2 + r_2^2 + r_3^2 + \ldots + r_n^2}{n}\right]^{1/2}$$



SI unit of radius of gyration is metre.

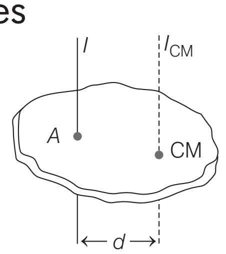
Radius of gyration depends upon shape and size of the body, position and configuration of the axis of rotation and also on distribution of mass of body w.r.t. axis of rotation.

Theorems on Moment of Inertia

There are two theorems based on moment of inertia are given below:

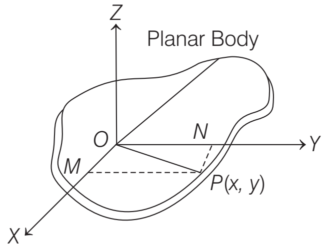
1. Theorem of Parallel Axes

Moment of inertia of a body about a given axis I is equal to the sum of moment of inertia of the body about a parallel axis passing through its centre of mass I_{CM} and the product of mass of body (M) and square of normal distance between the two axes. Mathematically, $I = I_{CM} + Md^2$



2. Theorem of Perpendicular Axes

The sum of moment of inertia of a plane laminar body about two mutually perpendicular axes lying in its plane is equal to its moment of inertia about an axis passing through the point of intersection of these two axes and perpendicular to the plane of laminar body.



If I_x and I_y be moment of inertia of the body about two perpendicular axes in its own plane and I_z be the moment of inertia about an axis passing through point O and perpendicular to the plane of lamina, then

$$I_z = I_x + I_y$$

In theorem of perpendicular axes, the point of intersection of the three axes (x, y and z) may be any point on the plane.

Moment of Inertia for Simple Geometrical Objects

- **Uniform Ring of Mass M and Radius R** About an axis passing through the centre and perpendicular to plane of ring, $I = MR^2$. About a diameter, $I = \frac{1}{2} MR^2$
- **Uniform Circular Disc of Mass M and Radius R** About an axis passing through the centre and perpendicular to plane of disc, $I = \frac{1}{2} MR^2$. About a diameter, $I = \frac{1}{4} MR^2$
- **Thin Uniform Rod of Mass M and Length l** About an axis passing through its centre and perpendicular to the rod, $I = \frac{1}{12} Ml^2$
- **Uniform Solid Cylinder of Mass M, Length l and Radius R** About its own axis, $I = \frac{1}{2} MR^2$. About an axis passing through its centre and perpendicular to its length,

$$I = M\left[\frac{l^2}{12} + \frac{R^2}{4}\right]$$

- **Uniform Solid Sphere** About its diameter, $I = \frac{2}{5} MR^2$.

  About its tangent, $I = \frac{7}{5} MR^2$

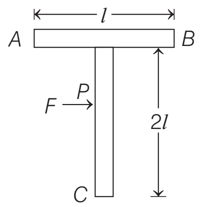
DAY PRACTICE SESSION 1

FOUNDATION QUESTIONS EXERCISE

1 In the HCl molecule, the separation between the nuclei of the two atoms is about 1.27 Å (1 Å $= 10^{-10}$ m). Given that, chlorine atom is about 35.5 times as massive as a hydrogen atom and all the mass of the atom is concentrated in its nucleus. The approximate location of centre of mass w.r.t. H atom is

(a) 1 Å (b) 1.24 Å
(c) 10^{-10} Å (d) 10^{-5} Å

2 A T shaped object with dimensions shown in the figure, is lying on a smooth floor. A force **F** is applied at the point P parallel to AB, such that the object has only the translational motion without rotation. Find the location of P with respect to C.



(a) $\frac{4}{3}l$ (b) l
(c) $\frac{2}{3}l$ (d) $\frac{3}{2}l$

3 If linear density of a rod of length 3m varies as $\lambda = 2 + x$, then the position of the centre of gravity of the rod is

(a) $\frac{7}{3}$m (b) $\frac{12}{7}$ m
(c) $\frac{10}{7}$ m (d) $\frac{9}{7}$ m

4 Two bodies of masres 1 kg and 3 kg have position vectors $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $-3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$, respectively. The centre of mass of this system has a position vector
➜ CBSE AIPMT 2009

(a) $-2\hat{\mathbf{i}} + 2\hat{\mathbf{k}}$ (b) $-2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$
(c) $2\hat{\mathbf{i}} - \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ (d) $-\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$

5 Three masses are placed on the X-axis : 300 g at origin, 500 g at $x = 40$ cm and 400 g at $x = 70$ cm. The distance of the centre of mass from the origin is

(a) 40 cm (b) 45 cm (c) 50 cm (d) 30 cm

6 Three masses of 2 kg, 4 kg and 4 kg are placed at three points (1, 0, 0), (1, 1, 0) and (0, 1, 0) respectively. The position vector of its centre of mass is

(a) $\frac{3}{5}\hat{\mathbf{i}} + \frac{4}{5}\hat{\mathbf{j}}$ (b) $3\hat{\mathbf{i}} + \hat{\mathbf{j}}$ (c) $\frac{2}{5}\hat{\mathbf{i}} + \frac{4}{5}\hat{\mathbf{j}}$ (d) $\frac{1}{5}\hat{\mathbf{i}} + \frac{4}{5}\hat{\mathbf{j}}$

7 Three bodies having masses 5 kg, 4 kg and 2 kg is moving at the speed of 5 m/s, 4 m/s and 2 m/s respectively along X-axis. The magnitude of velocity of CM is

(a) 1.0 m/s (b) 4 m/s (c) 0.9 m/s (d) 1.3 m/s

8. Two particles which are initially at rest, move towards each other under the action of their internal attraction. If their speeds are v and $2v$ at any instant, then the speed of centre of mass of the system will be ➜ CBSE AIPMT 2010

(a) $2v$ (b) 0 (c) $1.5v$ (d) v

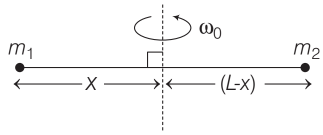
9 Two persons of masses 55 kg and 65 kg respectively, are at the opposite ends of a boat. The length of the boat is 3 m and weighs 100 kg. The 55 kg man walks upto the 65 kg man and sits with him. If the boat is in still water the centre of mass of the system shifts by
➜ CBSE AIPMT 2012

(a) 3 m (b) 2.3 m
(c) zero (d) 0.75 m

10 Two spherical bodies of mass M and $5M$ and radii R and $2R$ respectively are released in free space with initial separation between their centre equal to $12R$. If they attract each other due to gravitational force only, then the distance covered by the smaller body just before collision is

(a) $1.5R$ (b) $2.5R$
(c) $4.5R$ (d) $7.5R$

11 Point masses m_1 and m_2 are placed at the opposite ends of a rigid rod of length L and negligible mass. The rod is to be set rotating about an axis perpendicular to it. The position of point P on this rod through which the axis should pass, so that the work required to set the rod rotating with angular velocity ω_0 is minimum is given by



(a) $x = \frac{m_1 L}{m_1 + m_2}$ (b) $x = \frac{m_1}{m_2}L$
(c) $x = \frac{m_2}{m_1}L$ (d) $x = \frac{m_2 L}{m_1 + m_2}$

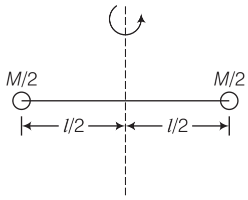
12 From a disc of radius R and mass M, a circular hole of diameter R, whose rim pases through the centre is cut. What is the moment of inertia of the remaining part of the disc about a perpendicular axis passing through the centre? ➜ NEET 2016

(a) $\frac{13MR^2}{32}$ (b) $\frac{11MR^2}{32}$ (c) $\frac{9MR^2}{32}$ (d) $\frac{15MR^2}{32}$

13 Point masses 1 kg, 2 kg, 3 kg and 4 kg are lying at the point (0, 0, 0), (2, 0, 0), (0, 3, 0) and (−2, −2, 0) respectively. The moment of inertia of this system about X axis will be

(a) 43 kg-m^2 (b) 34 kg-m^2
(c) 27 kg-m^2 (d) 72 kg-m^2

14 Two masses are joined with a light rod and the system is rotating about the fixed axis as shown in the figure. The MI of the system about the axis is



(a) $Ml^2/2$ (b) $Ml^2/4$
(c) Ml^2 (d) $Ml^2/6$

15 Four identical thin rods each of mass M and length l, form a square frame. Moment of inertia of this frame about an axis through the centre of the square and perpendicular to its plane is ➔ CBSE AIPMT 2009

(a) $\frac{4}{3}Ml^2$ (b) $\frac{2}{3}Ml^2$ (c) $\frac{13}{3}Ml^2$ (d) $\frac{1}{3}Ml^2$

16 The moment of inertia of a thin uniform circular disc about one of its diameter is I. The moment of inertia about an axis perpendicular to the circular surface and passing through its centre is

(a) $2I$ (b) $I/\sqrt{2}$
(c) $\sqrt{2}I$ (d) $I/2$

17 The moment of inertia of a uniform circular disc of radius R and mass M about an axis passing from the edge of the disc and normal to the disc is

(a) $\frac{1}{2}MR^2$ (b) MR^2
(c) $\frac{7}{2}MR^2$ (d) $\frac{3}{2}MR^2$

18 A thin wire to length l and mass m is bent in the form of a semi-circle. Its moment of inertia about an axis joining its free ends will be

(a) zero (b) ml^2
(c) $\frac{ml^2}{\pi^2}$ (d) $\frac{ml^2}{2\pi^2}$

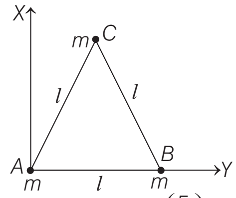
19 Two disc of the same material and thickness have radii 0.2 m and 0.6 m. Their moment of inertia about their axes will be in the ratio

(a) 1 : 81 (b) 1 : 27
(c) 1 : 91 (d) 1 : 3

20 A solid cylinder has mass M, length L and radius R. The momentum of inertia of this cylinder about a generator is

(a) $M\left(\frac{L^2}{12}+\frac{R^2}{4}\right)$ (b) $\frac{ML^2}{4}$ (c) $\frac{1}{2}MR^2$ (d) $\frac{3}{2}MR^2$

21 Three particles each of mass m gram situated at the vertices of an equilateral triangle ABC of side l cm (as shown in the figure). The moment of inertia of the system about a line AX perpendicular to AB and in the plane of ABC in gcm^2 units will be



(a) $\left(\frac{3}{4}\right)ml^2$ (b) $2ml^2$ (c) $\left(\frac{5}{4}\right)ml^2$ (d) $\left(\frac{3}{2}\right)ml^2$

22 Radius of gyration of uniform thin rod of length L about an axis passing normally through its centre of mass is

(a) $\frac{L}{\sqrt{12}}$ (b) $\frac{L}{12}$ (c) $\sqrt{12}L$ (d) $12L$

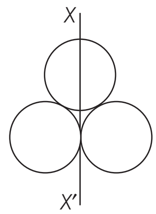
23 The radius of gyration of a body about an axis at a distance 6 cm from its centre of mass is 10 cm. Then, its radius of gyration about a parallel axis through its centre of mass will be

(a) 800 cm (b) 8 cm (c) 0.8 cm (d) 80 cm

24 The moment of inertia of a thin uniform rod of mass M and length L about an axis passing through its mid-point and perpendicular to its length is I_0. Its moment of inertia about an axis passing through one of its ends and perpendicular to its length is ➔ CBSE AIPMT 2011

(a) $I_0 + \frac{ML^2}{4}$ (b) $I_0 + 2ML^2$ (c) $I_0 + ML^2$ (d) $I_0 + \frac{ML^2}{2}$

25 Three identical spherical shells, each of mass m and radius r are placed as shown in figure. Consider an axis XX' which is touching to two shells and passing through diameter of third shell. Moment of inertia of the system consisting of these three spherical shells about XX' axis is ➔ CBSE AIPMT 2015



(a) $\frac{11}{5}mr^2$ (b) $3\,mr^2$ (c) $\frac{16}{5}mr^2$ (d) $4\,mr^2$

26 The moment of inertia of a thin rod of mass M and length L, about an axis perpendicular to the rod and distance $\frac{L}{4}$ from one end is

(a) $\frac{ML^2}{6}$ (b) $\frac{ML^2}{12}$ (c) $\frac{7ML^2}{24}$ (d) $\frac{7ML^2}{48}$

DAY PRACTICE SESSION 2

PROGRESSIVE QUESTIONS EXERCISE

1 Four particle of masses, m, $2m$, $3m$ and $4m$ are arranged at the corners of a parallelogram with each side equal to a and one of the angle between two adjacent sides is 60°. The parallelogram lies in the xy plane with mass m at the origin and $4m$ on the X-axis. The centre of mass of the arrangement will be located at

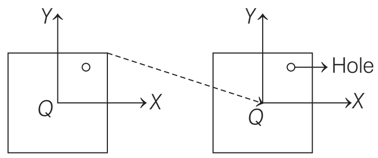
(a) $\left(\frac{\sqrt{3}}{2}a, 0.95a\right)$ (b) $\left(0.95a, \frac{\sqrt{3}}{4}a\right)$

(c) $\left(\frac{3a}{4}, \frac{a}{2}\right)$ (d) $\left(\frac{a}{2}, \frac{3a}{4}\right)$

2 A point P on the rim of wheel is initially at rest and in contact with the ground. Find the displacement of the point P, if the radius of the wheel is 5 m and the wheel rolls forward through half a revolution.

(a) 5 m (b) 10 m (c) 2.5 m (d) $5\sqrt{(\pi^2+4)}$ m

3 A uniform square plate has a small piece Q of an irregular shape removed and glued to the centre of the plate leaving a hole behind. The moment of inertia about the Z-axis is the



(a) increased (b) decreased
(c) the same
(d) changed in unpredicated manner

4 For spheres each of mass M and radius R are placed with their centres on the four corners A, B, C and D of a square of side b. The spheres A and B are hollow and C and D are solids. The moment of inertia of the system about side AD of square is

(a) $\frac{8}{3}MR^2 + 2Mb^2$ (b) $\frac{8}{5}MR^2 + 2Mb^2$

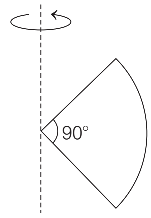
(c) $\frac{32}{15}MR^2 + 2Mb^2$ (d) $32MR^2 + 4Mb^2$

5 The surface density of a circular disc of radius a depends on the distance as $\rho(r) = A + Br$. The moment of inertia about the line perpendicular to the plane of the disc is

(a) $\pi a^4\left(\frac{A}{2} + \frac{2a}{5}B\right)$ (b) $\pi a^4\left(\frac{A}{2} + \frac{2B}{5}\right)$

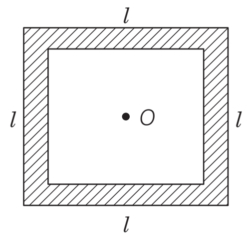
(c) $2\pi a^3\left(\frac{A}{2} + \frac{Ba}{5}\right)$ (d) None of these

6 One quarter sector is cut from a uniform circular disc of radius R. This sector has mass M. It is made to rotate about a line perpendicular to its plane and passing through the centre of the original disc. Its moment of inertia about the axis of rotation is



(a) $\frac{1}{2}mR^2$ (b) $\frac{1}{4}mR^2$ (c) $\frac{1}{8}mR^2$ (d) $\sqrt{2}MR^2$

7 Four thin rods of same mass M and same length l, form a square as shown in figure. Moment of inertia of this system about an axis through centre a and perpendicular to its plane is



(a) $\frac{4}{3}Ml^2$ (b) $\frac{Ml^2}{3}$ (c) $\frac{Ml^2}{6}$ (d) $\frac{2}{3}Ml^2$

8 Moment of inertia of circular loop of radius R about the axis of rotation parallel to horizontal diameter at a distance $R/2$ from it is

(a) MR^2 (b) $\frac{1}{2}MR^2$

(c) $2MR^2$ (d) $\frac{3}{4}MR^2$

9 What is the moment of inertia of a ring about a tangent to the periphery of the ring?

(a) $\frac{1}{2}MR^2$ (b) $\frac{3}{2}MR^2$

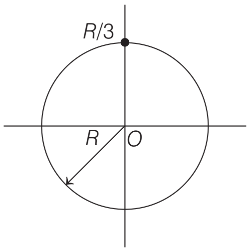
(c) MR^2 (d) $MR^2/9$

10 A uniform cylinder has a radius R and length L. If the moment of inertia of this cylinder about an axis passing through its centre and normal to its circular face is equal to the moment of inertia of the same cylinder about an axis passing through its centre and perpendicular to its length, then

(a) $L = R$ (b) $L = \sqrt{3}R$

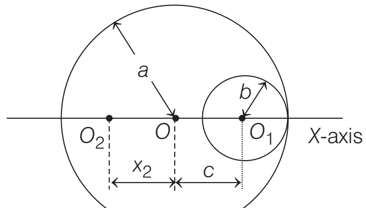
(c) $L = \frac{R}{\sqrt{3}}$ (d) $L = \sqrt{\frac{3}{2}}R$

11 From a circular disc of radius R and mass $9M$, a small disc of radius $\frac{R}{3}$ is removed from the disc, the moment of inertia of the remaining disc about an axis perpendicular to the plane of the disc and passing through O is



(a) $4MR^2$ (b) $\frac{40}{9}MR^2$ (c) $10MR^2$ (d) $\frac{37}{9}MR^2$

12 A uniform circular disc of radius a is taken. A circular portion of radius b has been removed from it as shown in the figure. If the centre of holes is at a distance c from the centre of the disc, the distance x_2 of the centre of mass of the remaining part from the initial centre of mass O is given by



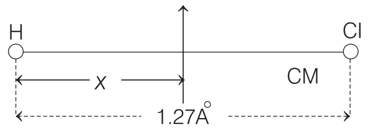
(a) $\frac{\pi b^2}{(a^2 - c^2)}$ (b) $\frac{cb^2}{(a^2 - b^2)}$ (c) $\frac{\pi c^2}{(a^2 - b^2)}$ (d) $\frac{ca^2}{(c^2 - b^2)}$

ANSWERS

SESSION 1	**1** (b)	**2** (a)	**3** (b)	**4** (b)	**5** (a)	**6** (a)	**7** (b)	**8** (b)	**9** (c)	**10** (d)
	11 (d)	**12** (a)	**13** (a)	**14** (b)	**15** (a)	**16** (a)	**17** (d)	**18** (d)	**19** (a)	**20** (d)
	21 (c)	**22** (a)	**23** (b)	**24** (a)	**25** (d)	**26** (d)				
SESSION 2	**1** (b)	**2** (d)	**3** (b)	**4** (c)	**5** (a)	**6** (a)	**7** (a)	**8** (d)	**9** (b)	**10** (b)
	11 (a)	**12** (b)								

Hints and Explanations

1 Let centre of mass be at a distance of x cm from H atom.



If x_1 and x_2 be the position vectors of H and Cl atoms w.r.t. the centre of mass as origin, then

$$x_{CM} = 0 = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$\Rightarrow \quad m_1 x_1 + m_2 x_2 = 0$$

$$\Rightarrow \quad m(-x) + 35.5m(1.27 - x) = 0$$

$$\Rightarrow \quad mx = 35.5m(1.27 - x)$$

$$\Rightarrow \quad x + 35.5x = 35.5 \times 1.27$$

$$\Rightarrow \quad x(1 + 35.5) = 45.08$$

$$36.5x = 45.08$$

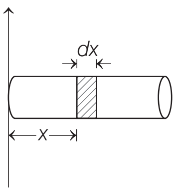
$$\Rightarrow \quad x = \frac{45.08}{36.5} = 1.24\,\text{Å}$$

2 Position vector of centre of mass of the system

$$r_{cm} = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2} = \frac{m2l + 2ml}{m + 2m}$$

$$= \frac{4ml}{3m} = \frac{4}{3}l$$

3 Linear density of the rod varies with distance



$$\frac{dm}{dx} = \lambda \text{ [given]}$$

$$\therefore \quad dm = \lambda dx$$

Position of centre of mass

$$x_{CM} = \frac{\int dm \times x}{\int dm} = \frac{\int_0^3 (\lambda dx) \times x}{\int_0^3 \lambda dx}$$

$$= \frac{\int_0^3 (2 + x) \times x dx}{\int_0^3 (2 + x) dx} = \frac{\left[x^2 + \frac{x^3}{3}\right]_0^3}{\left[2x + \frac{x^2}{2}\right]_0^3}$$

$$= \frac{9 + 9}{6 + \frac{9}{2}} = \frac{36}{21} = \frac{12}{7}\text{ m}$$

4 The position vector of centre of mass

$$\mathbf{r} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}$$

$$= \frac{1(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}) + 3(-3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}})}{1 + 3}$$

$$= \frac{1}{4}(-8\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 4\hat{\mathbf{k}})$$

$$= -2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$$

5
$$x_{CM} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3}$$

$$= \frac{300 \times (0) + 500(40) + 400(70)}{300 + 500 + 400}$$

$$= \frac{500 \times 40 + 400 \times 70}{1200}$$

$$x_{CM} = 40\text{ cm}$$

6 For centre of mass

$$x_{CM} = \frac{2 \times 1 + 4 \times 1 + 4 \times 0}{2 + 4 + 4} = \frac{6}{10} = \frac{3}{5}$$

$$y_{CM} = \frac{2 \times 0 + 4 \times 1 + 4 \times 1}{2 + 4 + 4} = \frac{8}{10} = \frac{4}{5}$$

Position vector of its centre of mass $= \frac{3}{5}\hat{\mathbf{i}} + \frac{4}{5}\hat{\mathbf{j}}$.

7 $V_{CM} = \dfrac{m_1v_1 + m_2v_2 + m_3v_3}{m_1 + m_2 + m_3}$

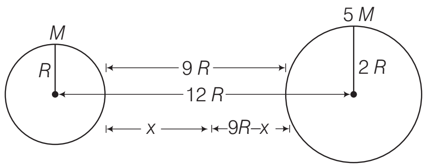
$= \dfrac{5\times 5 + 4\times 4 + 2\times 2}{5+4+2} = \dfrac{45}{11}$

$= 4.09 = 4$ m/s

8 As initially both the particles were at rest therefore velocity of centre of mass was zero and there is no external force on the system so speed of centre of mass remains constant i.e it should be equal to zero.

9 Here, on the entire system net external force on the system is zero hence centre of mass remains unchanged.

10 As the spherical bodies have their own size, so the distance covered by both the body $12R - 3R = 9R$, but individual distance covered by each body depends upon their masses.

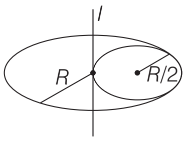


We know that bodies are moving under the effect of mutual attraction only, so their position of centre of mass remains unaffected.
Let smaller body cover distance x just before collision from $m_1r_1 = m_2r_2$, we get, $Mx = 5M(9R - x) \Rightarrow x = 7.5R$

11 The position of point P on rod through which the axis should pass, so that the work required to set the rod rotating with minimum angular velocity ω_0 is their centre of mass, we have

$m_1x = m_2(L - x) \Rightarrow x = \dfrac{m_2L}{m_1 + m_2}$

12 Considering the information given in the question, let us draw the figure



If the above figure is considered, then moment of inertia of disc will be given as

$I = I_{remain} + I_{(R/2)} \Rightarrow I_{remain} = I - I_{(R/2)}$

Putting the values, we get

$= \dfrac{MR^2}{2} - \left[\dfrac{\frac{M}{4}\left(\frac{R}{2}\right)^2}{2} + \dfrac{M}{4}\left(\dfrac{R}{2}\right)^2\right]$

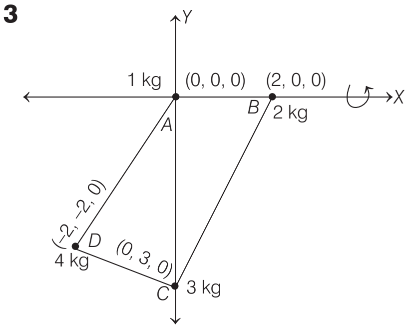
$= \dfrac{MR^2}{2} - \left[\dfrac{MR^2}{32} + \dfrac{MR^2}{16}\right]$

$= \dfrac{MR^2}{2} - \left[\dfrac{MR^2 + 2MR^2}{32}\right]$

$= \dfrac{MR^2}{2} - \dfrac{3MR^2}{32} = \dfrac{16MR^2 - 3MR^2}{32}$

$I_{remain} = \dfrac{13MR^2}{32}$

13



$I_{AB} = I_X = \Sigma m_i r_i^2$

$= m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + m_4r_4^2$

$= (1)(0)^2 + 2(0)^2 + 3(3)^2 + 4(-2)^2$

$= 27 + 16 = 43$ kg-m^2

14 Total moment of inertia is

$I = \dfrac{M}{2}\left(\dfrac{l}{2}\right)^2 + \dfrac{M}{2}\left(\dfrac{l}{2}\right)^2$

$= \dfrac{Ml^2}{8} + \dfrac{Ml^2}{8} = \dfrac{Ml^2}{4}$

15 Moment of inertia of rod about an axis through its centre of mass and perpendicular to rod = (mass of rod) × (perpendicular distance between two axes)

$= \dfrac{Ml^2}{12} + M\left(\dfrac{l}{2}\right)^2 = \dfrac{Ml^2}{3}$

Moment of inertia of the system

$= \dfrac{Ml^2}{3} \times 4 = \dfrac{4}{3}Ml^2$

16 $I = I_d = \dfrac{MR^2}{4}$ and $I_p = \dfrac{MR^2}{2}$

Hence, $I_p = 2I$

17 Moment of inertia of disc passing through its centre of gravity and perpendicular to its plane is

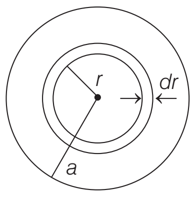
$I_{AB} = \dfrac{1}{2}MR^2$

Using theorem of parallel axes, we have

$I_{CD} = I_{AB} + MR^2$

$= \dfrac{1}{2}MR^2 + MR^2 = \dfrac{3}{2}MR^2$

18 For semi-circle,



$\pi r = l$

$\therefore \quad r = \dfrac{l}{\pi}$

Moment of inertia

$I = \dfrac{mr^2}{2}$

$= \dfrac{m}{2}\left(\dfrac{l}{\pi}\right)^2 = \dfrac{ml^2}{2\pi^2}$

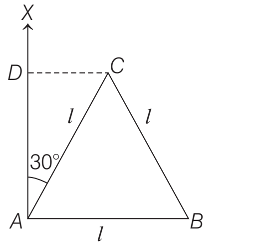
19 $I = \dfrac{1}{2}MR^2 = \dfrac{1}{2}(\pi R^2 t \times \rho) \times R^2 \Rightarrow I = R^2$

$\therefore \dfrac{I_1}{I_2} = \left(\dfrac{R_1}{R_2}\right)^4 = \left(\dfrac{0.2}{0.6}\right)^4 = \dfrac{1}{81}$

20 Generator axis of a cylinder in a line lying on it's surface and parallel to axis of cylinder. By parallel axis theorem,

$I = \dfrac{MR^2}{2} + MR^2 = \dfrac{3}{2}MR^2$

21 Moment of inertia of the system about AX is given by



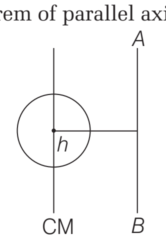
$= m_Ar_A^2 + m_Br_B^2 + m_Cr_C^2$

$= m(0)^2 + m(l)^2 + m(l\sin 30°)^2$

$= ml^2 + \dfrac{ml^2}{4} = \dfrac{5}{4}ml^2$

22 $\dfrac{ML^2}{12} = Mk^2 \Rightarrow k = \dfrac{L}{\sqrt{12}}$

23 From theorem of parallel axis,



$I = I_G + Mh^2 \Rightarrow I_{AB} = I_{CM} + Mh^2$

$MK_{AB}^2 = MK_{CM}^2 + Mh^2$

$\Rightarrow \quad K_{AB}^2 = K_{CM}^2 + h^2$

Given, $K_{AB} = 10$ cm, $h = 6$ cm

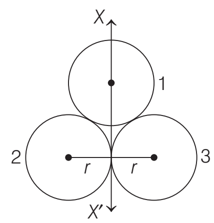
$\therefore \quad K_{CM} = 8$ cm

24 The theorem of parallel axis for moment of inertia.

$I = I_{CM} + Mh^2$,

$I = I_0 + M\left(\dfrac{L}{2}\right)^2 = I_0 + \dfrac{ML^2}{4}$

25

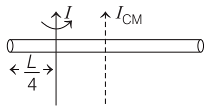


$I_{xx'} = I_1 + I_2 + I_3$

$$= \frac{2}{3}mr^2 + \left(\frac{2}{3}mr^2 + mr^2\right) + \left(\frac{2}{3}mr^2 + mr^2\right)$$

$$I_{xx'} = 2mr^2 + 2mr^2 = 4mr^2$$

26



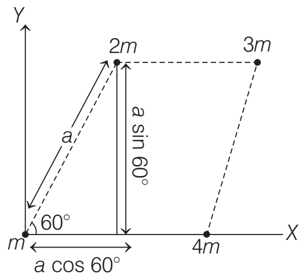
According to theorem of parallel axes,

$$I = I_{CM} + Mx^2 = \frac{ML^2}{12} + M\left(\frac{L}{4}\right)^2$$

$$= \frac{ML^2}{12} + \frac{ML^2}{16} = \frac{7ML^2}{48}$$

SESSION 2

1 Let $m_1 = m$, $m_2 = 2m$, $m_3 = 3m$, $m_4 = 4m$



$\mathbf{r}_1 = 0\hat{\mathbf{i}} + 0\hat{\mathbf{j}}$

$\mathbf{r}_2 = a\cos 60\hat{\mathbf{i}} + a\sin 60\hat{\mathbf{j}} = \frac{a}{2}\hat{\mathbf{i}} + \frac{a\sqrt{3}}{2}\hat{\mathbf{j}}$

$\mathbf{r}_3 = (a + a\cos 60)\hat{\mathbf{i}} + a\sin 60\hat{\mathbf{j}}$

$= \frac{3}{2}a\hat{\mathbf{i}} + \frac{a\sqrt{3}}{2}\hat{\mathbf{j}}$

$\mathbf{r}_4 = a\hat{\mathbf{i}} + 0\hat{\mathbf{j}}$

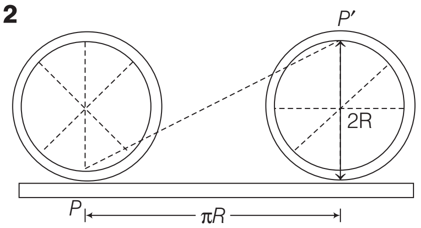
On substituting above value in the following formula

$$\mathbf{r} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3 + m_4\mathbf{r}_4}{m_1 + m_2 + m_3 + m_4}$$

$$= 0.95a\hat{\mathbf{i}} + \frac{\sqrt{3}}{4}a\hat{\mathbf{j}}$$

So, the location of centre of mass $\left[0.95a, \frac{\sqrt{3}}{4}a\right]$.

2



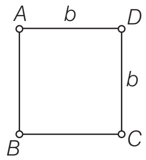
Displacement of point P after half revolution,

$$PP' = \sqrt{(\pi R)^2 + (2R)^2}$$

$$= R\sqrt{\pi^2 + 4} = 5\sqrt{\pi^2 + 4} \text{ m}$$

3 According to the theorem of perpendicular axes, $I_Z = I_X + I_Y$ with the hole, I_x and I_y both decrease and gluing the removed piece at the centre of square plate does not affect I_z. Hence, I_z decreases, overall.

4 Moment of inertia of a hollow sphere of radius R about the diameter passing through D is



$$I_A = \frac{2}{3}MR^2$$

Moment of inertia of solid sphere about diameter,

$$I_D = \frac{2}{5}MR^2$$

Moment of inertia of whole system about side AD is $= I_A + I_D + I_B + I_C$

$$= \frac{2}{3}MR^2 + \frac{2}{5}MR^2 + \left(Mb^2 + \frac{2}{3}MR^2\right) + \left(Mb^2 + \frac{2}{5}MR^2\right)$$

$$= \frac{32}{15}MR^2 + 2Mb^2$$

5 $dM = (2\pi r)dr(\rho) = (A + Br)(2\pi r dr)$

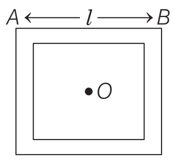
$$I = \int_0^a dMr^2 = \frac{\pi A a^4}{2} + \frac{2\pi B a^5}{5}$$

$$= \pi a^4\left(\frac{A}{2} + \frac{2Ba}{5}\right)$$

6 Mass of the entire disc would be $4M$ and its moment of inertia about the given axis would be $\frac{1}{2}(4M)R^2$.

For the given section the moment of inertia about the same axis would be one quarter of this, i.e. $\frac{1}{2}MR^2$.

7 Moment of inerita of rod AB about point P and perpendicular to the plane $= \frac{Ml^2}{12}$



M.I. of rod AB about point $O' = \frac{Ml^2}{12} + M\left(\frac{l}{2}\right)^2 = \frac{Ml^2}{3}$

But the system consists of four rods of similar type so by the symmetry

$$I_{\text{system}} = 4\left(\frac{Ml^2}{3}\right) = \frac{4}{3}Ml^2$$

8 Applying theorem of parallel axis

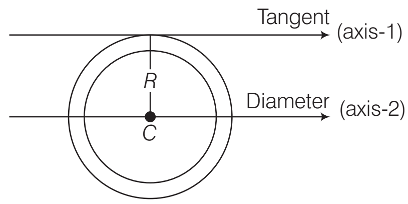
$$I = I_{CM} + M\left(\frac{R}{2}\right)^2$$

$$= \frac{1}{2}MR^2 + \frac{MR^2}{4} = \frac{3}{4}MR^2$$

9 The tangent to the ring in the plane of the ring is parallel to one of the diameters of the ring.

The distance between these two parallel axes (in figure) is R, the radius of the ring.

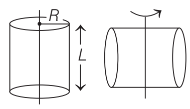
Using the parallel axes theorem,



$$I_{\text{tangent}} = I_{\text{dia}} + MR^2 = \frac{MR^2}{2} + MR^2$$

$$= \frac{3}{2}MR^2$$

10 Moment of inertia of a cylinder about its centre and parallel to its length $= \frac{MR^2}{2}$



Moment of inertia about its centre and perpendicular to its length

$$= M\left(\frac{L^2}{12} + \frac{R^2}{4}\right)$$

$$\therefore \quad \frac{ML^2}{12} + \frac{MR^2}{4} = \frac{MR^2}{2} \Rightarrow L = \sqrt{3}\,R$$

11 As the mass is uniformly distributed on the disc, so mass density (per unit area) $= \frac{9M}{\pi R^2}$

Mass of removed portion

$$= \frac{9M}{\pi R^2} \times \pi\left(\frac{R}{3}\right)^2 = M$$

So, moment of inertia of the removed portion about the stated axis by theorem of parallel axis.

$$I_1 = \frac{1}{2}M\left(\frac{R}{3}\right)^2 + M\left(\frac{2R}{3}\right)^2$$

If the disc would not have been removed, then the moment of inertia of complete disc about the stated axis.

$$I_2 = \frac{1}{2}9M(R)^2$$

So, the moment of inertia of the disc about required axis. $I = I_2 - I_1$

$$= \frac{1}{2}9M(R)^2 - \left[\frac{1}{2}M\left(\frac{R}{3}\right)^2 + M\left(\frac{2R}{3}\right)^2\right]$$

$$I = 4MR^2$$

12 Centre of mass of whole system was at point O.

Hence, x_2 (area of remaining portion) $= c$ (area of removed disc)

$$x_2(\pi a^2 - \pi b^2) = c(\pi b^2) \Rightarrow x_2 = \frac{cb^2}{a^2 - b^2}$$