PARABOLA

1. CONIC SECTIONS :

A conic section, or conic is the locus of a point which moves in a plane so that its distance from a fixed point is in a constant ratio to its perpendicular distance from a fixed straight line.

(a) The fixed point is called the FOCUS.

(b) The fixed straight line is called the DIRECTRIX.

(c) The constant ratio is called the ECCENTRICITY denoted by e.

(d) The line passing through the focus & perpendicular to the directrix is called the AXIS.

(e) A point of intersection of a conic with its axis is called a VERTEX.

2. GENERAL EQUATION OF A CONIC : FOCAL DIRECTRIX PROPERTY :

The general equation of a conic with focus (p, q) & directrix $lx + my + n = 0$ is :

$$(l^2 + m^2)\,[(x - p)^2 + (y - q)^2] = e^2\,(lx + my + n)^2$$
$$\equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

3. DISTINGUISHING BETWEEN THE CONIC :

The nature of the conic section depends upon the position of the focus S w.r.t. the directrix & also upon the value of the eccentricity e. Two different cases arise.

Case (i) When the focus lies on the directrix :

In this case $D \equiv abc + 2\,fgh - af^2 - bg^2 - ch^2 = 0$ & the general equation of a conic represents a pair of straight lines and if :

$e > 1$, $h^2 > ab$ the lines will be real & distinct intersecting at S.

$e = 1$, $h^2 = ab$ the lines will coincident.

$e < 1$, $h^2 < ab$ the lines will be imaginary.

ase (ii) When the focus does not lie on the directrix :

The conic represents :

a parabola	an ellipse	a hyperbola	a rectangular hyperbola
$e = 1; D \neq 0$ $h^2 = ab$	$0 < e < 1; D \neq 0$ $h^2 < ab$	$D \neq 0\ ; e > 1$ $h^2 > ab$	$e > 1\ ; D \neq 0$ $h^2 > ab\ ;\ a + b = 0$

4. PARABOLA :

A parabola is the locus of a point which moves in a plane, such that its distance from a fixed point (focus) is equal to its perpendicular distance from a fixed straight line (directrix).

Standard equation of a parabola is $y^2 = 4\,ax$. For this parabola :

(i) Vertex is (0, 0) **(ii)** Focus is (a, 0)

(iii) Axis is $y = 0$ **(iv)** Directrix is $x + a = 0$

(a) Focal distance :

The distance of a point on the parabola from the focus is called the FOCAL DISTANCE OF THE POINT.

(b) Focal chord :

A chord of the parabola, which passes through the focus is called a FOCAL CHORD.

(c) Double ordinate :

A chord of the parabola perpendicular to the axis of the symmetry is called a DOUBLE ORDINATE with respect to axis as diameter.

(d) Latus rectum :

A focal chord perpendicular to the axis of parabola is called the LATUS RECTUM. For $y^2 = 4ax$.

(i) Length of the latus rectum = 4a.

(ii) Length of the semi latus rectum = 2a.

(iii) Ends of the latus rectum are L(a, 2a) & L'(a, – 2a)

Note that :

(i) Perpendicular distance from focus on directrix = half the latus rectum.

(ii) Vertex is middle point of the focus & the point of intersection of directrix & axis.

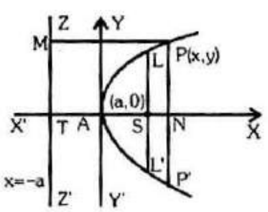
(iii) Two parabolas are said to be equal if they have latus rectum of same length.

5. PARAMETRIC REPRESENTATION :

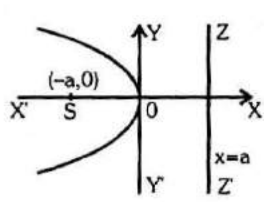
The simplest & the best form of representing the co-ordinates of a point on the parabola $y^2 = 4ax$ is $(at^2, 2at)$. The equation $x = at^2$ & $y = 2at$ together represents the parabola $y^2 = 4ax$, t being the parameter.

6. TYPE OF PARABOLA :

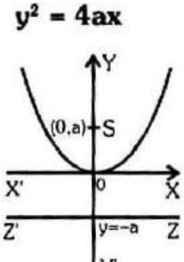
Four standard forms of the parabola are $y^2 = 4ax$; $y^2 = -4ax$; $x^2 = 4ay$; $x^2 = -4ay$



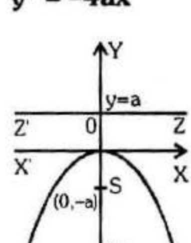
$y^2 = 4ax$



$y^2 = -4ax$



$x^2 = 4ay$



$x^2 = -4ay$

Parabola	Vertex	Fous	Axis	Directrix	Length of Latus rectum	Ends of Latus rectum	Parametric equation	Focal length
$y^2 = 4ax$	(0,0)	(a,0)	y=0	x=−a	4a	(a, ±2a)	$(at^2, 2at)$	x+a
$y^2 = -4ax$	(0,0)	(−a,0)	y=0	x=a	4a	(−a, ±2a)	$(-at^2, 2at)$	x−a
$x^2 = +4ay$	(0,0)	(0,a)	x=0	y=−a	4a	(±2a, a)	$(2at, at^2)$	y+a
$x^2 = -4ay$	(0,0)	(0,−a)	x=0	y=a	4a	(±2a, −a)	$(2at, -at^2)$	y−a
$(y-k)^2=4a(x-h)$	(h,k)	(h+a,k)	y=k	x+a−h=0	4a	(h+a, k±2a)	$(h+at^2, k+2at)$	x−h+a
$(x-p)^2=4b(y-q)$	(p,q)	(p, b+q)	x=p	y+b−q=0	4b	(p±2a,q+a)	$(p+2at, q+at^2)$	y−q+b

7. POSITION OF A POINT RELATIVE TO A PARABOLA :

The point (x_1, y_1) lies outside, on or inside the parabola $y^2 = 4ax$ according as the expression $y_1^2 - 4ax_1$ is positive, zero or negative.

8. CHORD JOINING TWO POINTS :

The equation of a chord of the parabola $y^2 = 4ax$ joining its two points $P(t_1)$ and $Q(t_2)$ is $y(t_1 + t_2) = 2x + 2at_1t_2$

Note :

(i) If PQ is focal chord then $t_1t_2 = -1$.

(ii) Extremities of focal chord can be taken as $(at^2, 2at)$ & $\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$

(iii) If $t_1t_2 = k$ then chord always passes a fixed point $(-ka, 0)$.

9. LINE & A PARABOLA :

(a) The line $y = mx + c$ meets the parabola $y^2 = 4ax$ in two points real, coincident or imaginary according as $a >=< cm$

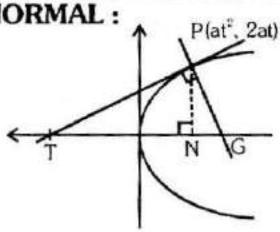
$\Rightarrow$ condition of tangency is, $c = \frac{a}{m}$.

Note : Line $y = mx + c$ will be tangent to parabola $x^2 = 4ay$ if $\mathbf{c = -am^2}$.

(b) Length of the chord intercepted by the parabola $y^2 = 4ax$ on the line $y = mx + c$ is : $\left(\frac{4}{m^2}\right)\sqrt{a(1+m^2)(a-mc)}$.

Note : length of the focal chord making an angle α with the x-axis is $4a\,\mathrm{cosec}^2\,\alpha$.

10. LENGTH OF SUBTANGENT & SUBNORMAL :



PT and PG are the tangent and normal respectively at the point P to the parabola $y^2 = 4ax$. Then

TN = length of subtangent = twice the abscisaa of the point P

(Subtangent is always bisected by the vertex)

NG = length of subnormal which is constant for all points on the parabola & equal to its semilatus rectum (2a).

11. TANGENT TO THE PARABOLA $y^2 = 4ax$:

(a) Point form :

Equation of tangent to the given parabola at its point (x_1, y_1) is
$yy_1 = 2a(x + x_1)$

(b) Slope form :

Equation of tangent to the given parabola whose slope is 'm', is

$$y = mx + \frac{a}{m}, (m \neq 0)$$

Point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

(c) Parametric form :

Equation of tangent to the given parabola at its point P(t), is -
$ty = x + at^2$

Note : Point of intersection of the tangents at the point t_1 & t_2 is $[at_1 t_2, a(t_1 + t_2)]$. (i.e. G.M. and A.M. of abscissae and ordinates of the points)

12. NORMAL TO THE PARABOLA $y^2 = 4ax$:

(a) Point form :

Equation of normal to the given parabola at its point (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

(b) Slope form :

Equation of normal to the given parabola whose slope is 'm', is
$y = mx - 2am - am^3$ foot of the normal is $(am^2, -2am)$

(c) Parametric form :

Equation of normal to the given parabola at its point P(t), is

$y + tx = 2at + at^3$

Note :

(i) Point of intersection of normals at t_1 & t_2 is $(a(t_1{}^2 + t_2{}^2 + t_1t_2 + 2), - at_1t_2 (t_1 + t_2))$.

(ii) If the normal to the parabola $y^2 = 4ax$ at the point t_1, meets the parabola again at the point t_2, then $t_2 = -\left(t_1 + \dfrac{2}{t_1}\right)$.

(iii) If the normals to the parabola $y^2 = 4ax$ at the points t_1 & t_2 intersect again on the parabola at the point 't_3' then $t_1t_2 = 2$; $t_3 = -(t_1 + t_2)$ and the line joining t_1 & t_2 passes through a fixed point $(-2a, 0)$.

13. PAIR OF TANGENTS :

The equation of the pair of tangents which can be drawn from any point $P(x_1, y_1)$ outside the parabola to the parabola $y^2 = 4ax$ is given by : $SS_1 = T^2$, where :

$S \equiv y^2 - 4ax$; $S_1 \equiv y_1{}^2 - 4ax_1$; $T \equiv yy_1 - 2a(x + x_1)$.

14. CHORD OF CONTACT :

Equation of the chord of contact of tangents drawn from a point $P(x_1, y_1)$ is $yy_1 = 2a(x + x_1)$

Remember that the area of the triangle formed by the tangents from the point (x_1, y_1) & the chord of contact is $\dfrac{\left(y_1^2 - 4ax_1\right)^{3/2}}{2a}$. Also note that the chord of contact exists only if the point P is not inside.

15. CHORD WITH A GIVEN MIDDLE POINT :

Equation of the chord of the parabola $y^2 = 4ax$ whose middle point is (x_1, y_1) is $y - y_1 = \dfrac{2a}{y_1}(x - x_1)$.

This reduced to $T = S_1$

where $T \equiv yy_1 - 2a(x + x_1)$ & $S_1 \equiv y_1^2 - 4ax_1$.

16. DIAMETER :

The locus of the middle points of a system of parallel chords of a Parabola is called a DIAMETER. Equation to the diameter of a parabola is $y = 2a/m$, where m = slope of parallel chords.

17. CONORMAL POINTS :

Foot of the normals of three concurrent normals are called conormals point.

(i) Algebraic sum of the slopes of three concurrent normals of parabola $y^2 = 4ax$ is zero.

(ii) Sum of ordinates of the three conormal points on the parabola $y^2 = 4ax$ is zero.

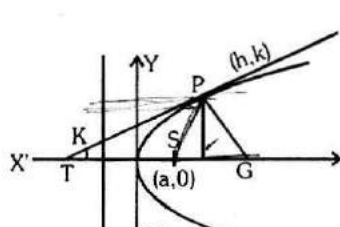
(iii) Centroid of the triangle formed by three co-normal points lies on the axis of parabola.

(iv) If $27ak^2 < 4(h - 2a)^3$ satisfied then three real and distinct normal are drawn from point (h, k) on parabola $y^2 = 4ax$.

(v) If three normals are drawn from point (h, 0) on parabola $y^2 = 4ax$, then $h > 2a$ and one of the normal is axis of the parabola and other two are equally inclined to the axis of the parabola.

18. IMPORTANT HIGHLIGHTS :

(a) If the tangent & normal at any point 'P' of the parabola intersect the axis at T & G then ST = SG = SP where 'S' is the focus. In other words the tangent and the normal at a point P on the parabola are the bisectors of the angle between



the focal radius SP & the perpendicular from P on the directrix. From this we conclude that all rays emanating from S will become parallel to the axis of the parabola after reflection.

(b) The portion of a tangent to a parabola cut off between the directrix & the curve subtends a right angle at the **focus**.

(c) The tangents at the extremities of a focal chord intersect at right angles on the **directrix,** and a circle on any focal chord as diameter touches the directrix. Also a circle on any focal radii of a point P $(at^2, 2at)$ as diameter touches the tangent at the vertex and intercepts a chord of length $a\sqrt{1+t^2}$ on a normal at the point P.

(d) Any tangent to a parabola & the perpendicular on it from the focus meet on the tangent at the vertex.

(e) Semi latus rectum of the parabola $y^2 = 4ax$, is the harmonic mean between segments of any focal chord

i.e. $2a = \frac{2bc}{b+c}$ or $\frac{1}{b}+\frac{1}{c}=\frac{1}{a}$.

(f) Image of the focus lies on diretrix with respect to any tangent of parabola $y^2 = 4ax$.