4 Binomial Theorem

KEY FACTS

1. Expansion of a Binomial

If $\boldsymbol{x}$ and $\boldsymbol{a}$ are real numbers, then for all $n \in N$,

$$(x+a)^n = {}^nC_0\, x^n a^0 + {}^nC_1\, x^{n-1} a^1 + {}^nC_2\, x^{n-2} a^2 + \ldots\ldots + {}^nC_r\, x^{n-r} a^r + \ldots\ldots + {}^nC_{n-1}\, x^1 a^{n-1} + {}^nC_n\, x^0 a^n$$

$$\Rightarrow \quad (x+a)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} a^r.$$

2. Properties of Binomial Expansion

(*a*) The **total number of terms** in the expansion of $(x+a)^n$ is $\boldsymbol{n+1}$.

(*b*) The sum of the indices of $\boldsymbol{x}$ and $\boldsymbol{a}$ in each term is $\boldsymbol{n}$.
Thus, the $(r+1)$th term $= {}^nC_r\, x^{n-r} a^r$ and sum of indices $= (n-r+r) = \boldsymbol{n}$.

(*c*) The coefficients of terms equidistant from the beginning and end are equal. These coefficients are known as binomial coefficients.

$${}^nC_r = {}^nC_{n-r}\ \forall\ r = 0, 1, 2, \ldots.., n \quad \Rightarrow \quad {}^nC_0 = {}^nC_n,\ {}^nC_1 = {}^nC_{n-1},\ {}^nC_2 = {}^nC_{n-2} \text{ and so on.}$$

(*d*) Replacing $\boldsymbol{a}$ by $\boldsymbol{-a}$ in the expansion of $(x+a)^n$, we have

$$(x-a)^n = {}^nC_0\, x^n a^0 - {}^nC_1\, x^{n-1} a^1 + {}^nC_2\, x^{n-2} a^2 - {}^nC_3\, x^{n-3} a^3 + \ldots\ldots. + (-1)^r\, {}^nC_r\, x^{n-r} a^r \ldots\ldots + (-1)^r\, {}^nC_n\, x^0 a^n.$$

$$\Rightarrow \quad (x-a)^n = \sum_{r=0}^{n} (-1)^r\ {}^nC_r \,.\, x^{n-r} a^r$$

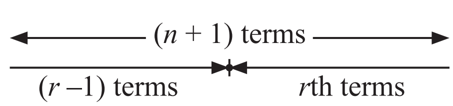
(*e*) $(x+a)^n + (x-a)^n = 2({}^nC_0\, x^n a^0 + {}^nC_2\, x^{n-2} a^2 + \ldots\ldots\ldots.)$
$(x+a)^n - (x-a)^n = 2({}^nC_1\, x^{n-1} a^1 + {}^nC_3\, x^{n-3} a^3 + \ldots\ldots\ldots.)$

(*f*) The **$(r+1)$th term** called the general term in the **expansion of $(x+a)^n$** is given by $\boldsymbol{T_{r+1} = {}^nC_r\, x^{n-r} a^r}$.
In the **expansion of $(x-a)^n$, $T_{r+1} = (-1)^r\, {}^nC_r\, x^{n-r} a^r$.**

(*g*) **Middle Term:** In the expansion of $(x+a)^n$, if

- ***n* is even natural number,** then middle term $= \left(\dfrac{n}{2}+1\right)$th term
- ***n* is odd natural number**, then $\left(\dfrac{n+1}{2}\right)$th term and $\left(\dfrac{n+3}{2}\right)$th term are the two middle terms.

(*h*) **Term from the end:** In the expansion of $(x+a)^n$, the rth term from the end is
$= [(n+1) - (r-1)]$th term from the beginning
$=$ **$(n-r+2)$th term** from the beginning.



(*i*) The sum of the binomial coefficients $= 2^n$, *i.e.*,

$$C_0 + C_1 + C_2 + \ldots\ldots + C_n = 2^n$$

(*j*) $C_1 + C_3 + C_5 + \ldots\ldots = C_0 + C_2 + C_4 + \ldots\ldots = 2^{n-1}$

General Term and Expansion of Binomial Theorem

SOLVED EXAMPLES

Ex. 1. Expand $\left(2x+\frac{y}{2}\right)^5$.

Sol. $\left(2x+\frac{y}{2}\right)^5 = {}^5C_0\,(2x)^5 + {}^5C_1\,(2x)^4\left(\frac{y}{2}\right) + {}^5C_2\,(2x)^3\left(\frac{y}{2}\right)^2 + {}^5C_3\,(2x)^2\left(\frac{y}{2}\right)^3 + {}^5C_4\,(2x)\left(\frac{y}{2}\right)^4 + {}^5C_5\left(\frac{y}{2}\right)^5$

$= 32x^5 + 5\times 16x^4\times\frac{y}{2} + \frac{5\times 4}{2\times 1}\times 8x^3\times\frac{y^2}{4} + \frac{5\times 4}{2\times 1}\times 4x^2\times\frac{y^3}{8} + 5\times 2x\times\frac{y^4}{16} + \frac{y^5}{32}$

$= \mathbf{32x^5 + 40x^4y + 20x^3y^2 + 5x^2y^3 + \frac{5}{8}xy^4 + \frac{y^5}{32}}$.

Ex. 2. Expand $\left(\frac{2}{x}-\frac{x}{2}\right)^6,\ x\neq 0$.

Sol. $\left(\frac{2}{x}-\frac{x}{2}\right)^6 = {}^6C_0\left(\frac{2}{x}\right)^6 + {}^6C_1\left(\frac{2}{x}\right)^5\left(-\frac{x}{2}\right) + {}^6C_2\left(\frac{2}{x}\right)^4\left(-\frac{x}{2}\right)^2 + {}^6C_3\left(\frac{2}{x}\right)^3\left(-\frac{x}{2}\right)^3 + {}^6C_4\left(\frac{2}{x}\right)^2\left(-\frac{x}{2}\right)^4$

$+ {}^6C_5\left(\frac{2}{x}\right)\left(-\frac{x}{2}\right)^5 + {}^6C_6\left(-\frac{x}{2}\right)^6$

$= \frac{64}{x^6} + 6\times\frac{32}{x^5}\times\left(-\frac{x}{2}\right) + \frac{6\times 5}{2\times 1}\times\frac{16}{x^4}\times\frac{x^2}{4} + \frac{6\times 5\times 4}{3\times 2\times 1}\times\frac{8}{x^3}\times\left(-\frac{x^3}{8}\right) + \frac{6\times 5}{2\times 1}\times\frac{4}{x^2}\times\frac{x^4}{16}$

$+ 6\times\frac{2}{x}\times\left(-\frac{x^5}{32}\right) + \frac{x^6}{64}$

$= \mathbf{\frac{64}{x^6} - \frac{96}{x^4} + \frac{60}{x^2} - 20 + \frac{15}{4}x^2 - \frac{3}{8}x^4 + \frac{x^6}{64}}$.

Ex. 3. Find the coefficient of x^5 in the expansion of $(1 + 2x)^6\,(1 - x)^7$.

Sol. $(1 + 2x)^6\,(1 - x)^7 = [1 + {}^6C_1\,(2x) + {}^6C_2\,(2x)^2 + {}^6C_3\,(2x)^3 + {}^6C_4\,(2x)^4 + {}^6C_5\,(2x)^5 + (2x)^6] \times [1 - {}^7C_1x + {}^7C_2x^2 - {}^7C_3x^3 + {}^7C_4x^4 - {}^7C_5x^5 + {}^7C_6x^6 - x^7]$

$= [1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6] \times [1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7]$

The terms containing x^5 in the product are obtained an multiplying constant term by the term containing x^5, term containing x by the term containing x^4 and so on.

These products are $(1 \times (-21x^5)) + (12x \times 35x^4) + (60x^2 \times (-35x^3)) + (160x^3 \times 21x^2) + (240x^4 \times (-7x)) + (192x^5 \times 1)$

$= -21x^5 + 420x^5 - 2100x^5 + 3360x^5 - 1680x^5 + 192x^5$

$\therefore$ Coefficient of $x^5 = (-21 + 420 - 2100 + 3360 - 1680 + 192) =$ **171.**

Ex. 4. Find the 6th term in the expansion of $\left(2x^2-\frac{1}{3x^2}\right)^{10}$. **(*Karnataka CET 2007*)**

Sol. The general term ($(n + 1)$th term) in the expansion of $\left(2x^2-\frac{1}{3x^2}\right)^{10}$ is

$$T_{r+1} = (-1)^r\,{}^{10}C_r\,(2x^2)^{10-r}\left(\frac{1}{3x^2}\right)^r = (-1)^r\,{}^{10}C_r\,2^{10-r}\,x^{20-2r}\,(3)^{-r}\,x^{-2r} = (-1)^r\,{}^{10}C_r\,2^{10-r}\,(3)^{-r}\,x^{20-4r}$$

$\therefore \quad T_6 = T_{5+1} = (-1)^5\, {}^{10}C_5\, 2^{10-5} (3)^{-5} x^{20-20} = (-1)\times\dfrac{10\times9\times8\times7\times6}{5\times4\times3\times2\times1}\times\dfrac{2^5}{3^5}\times x^0 = (-1)\times252\times\dfrac{32}{243} = -\dfrac{\mathbf{896}}{\mathbf{243}}.$

Ex. 5. Find the coefficient of x^{20} in the expansion of $(1 + 3x + 3x^2 + x^3)^{20}$ (*DCE 2007*)

Sol. $(1 + 3x + 3x^3 + x^3)^{20} = ((1 + x^3))^{20} = (1 + x)^{60}$

$\because \quad (1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \dots\dots\dots + {}^nC_nx^n,$

The coefficient of x^{20} in the expansion of $(1 + x)^{60} = {}^{60}C_{20}$ or ${}^{60}C_{40}$ $\quad (\because {}^nC_r = {}^nC_{n-r})$

Ex. 6. If the coefficients of $(m + 1)$th term and $(m + 3)$th term in the expansion of $(1 + x)^{20}$ are equal, then find the value of m? (*UPSEE 2007*)

Sol. $(1 + x)^{20} = {}^{20}C_0 + {}^{20}C_1x + {}^{20}C_2x^2 + \dots\dots\dots + {}^{20}C_rx^r + \dots.. + {}^{20}C_{20}x^{20}$

General term $T_{r+1} = {}^{20}C_rx^r \quad \therefore \Rightarrow \quad T_{m+1} = {}^{20}C_mx^m$

$T_{m+3} = {}^{20}C_{m+2}\, x^{m+2}$

Given, coeff. of $(m + 1)$th term = coeff. of $(m + 3)$th term

$\Rightarrow \quad {}^{20}C_m = {}^{20}C_{m+2} \quad \Rightarrow \quad {}^{20}C_{20-m} = {}^{20}C_{m+2}$ $\quad (\because {}^nC_r = {}^nC_{n-r})$

$\Rightarrow \quad 20 - m = m + 2 \quad \Rightarrow \quad 2m = 18 \quad \Rightarrow \quad m = \mathbf{9}.$

Ex. 7. If second, third and fourth terms in the expansion of $(x + a)^n$ are 240, 720 and 1080 respectively, then find the value of n.

Sol. $(x + a)^n = {}^nC_0x^n + {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 + {}^nC_3x^{n-3}a^3 + \dots\dots.. + {}^nC_na^n$

$\therefore \quad T_2 = {}^nC_1x^{n-1}a = 240$ ...(*i*)

$T_3 = {}^nC_2x^{n-2}a^2 = 720$ ...(*ii*)

$T_4 = {}^nC_3x^{n-3}a^3 = 1080$ ...(*iii*)

On dividing (*ii*) by (*i*), we get $\dfrac{{}^nC_2\,x^{n-2}\,a^2}{{}^nC_1\,x^{n-1}\,a} = \dfrac{720}{240} \Rightarrow \dfrac{\dfrac{n(n-1)}{2\times1}\times a}{n\times x} = 3 \Rightarrow \dfrac{(n-1)a}{2x} = 3$ ...(*iv*)

On dividing (*iii*) by (*ii*) we get $\dfrac{{}^nC_3\,x^{n-3}\,a^3}{{}^nC_2\,x^{n-2}\,a^2} = \dfrac{1080}{720} \Rightarrow \dfrac{\dfrac{n(n-1)(n-2)}{3\times2\times1}\times a}{\dfrac{n(n-1)}{2}\times x} = \dfrac{3}{2} \Rightarrow \dfrac{(n-2)a}{(3x)} = \dfrac{3}{2}$ ...(*v*)

Now dividing (*v*) by (*iv*) we get $\dfrac{\left(\dfrac{n-2}{3}\right)\dfrac{a}{x}}{\left(\dfrac{n-1}{2}\right)\dfrac{a}{x}} = \dfrac{3/2}{3} \Rightarrow \dfrac{2(n-2)}{3(n-1)} = \dfrac{1}{2} \Rightarrow 4n - 8 = 3n - 3 \Rightarrow n = \mathbf{5}.$

Ex. 8. What is the coefficient of x^4 in the expansion of $(1 + x)^{11} + (1 + x)^{12} + \dots.. + (1 + x)^{20}$?

Sol. $(1 + x)^{11} + (1 + x)^{12} + \dots.. + (1 + x)^{20}$

$= \dfrac{(1+x)^{11}\{(1+x)^{10} - 1\}}{(1+x) - 1}$ $\quad \left[\begin{array}{l}\because \text{This is the sum of a G.P. with 10 terms whose} \\ \text{first term} = (1+x)^{11} \text{ and common ratio} = (1+x)\end{array}\right]$

$= \dfrac{1}{x}\,[(1 + x)^{21} - (1 + x)^{11}].$

$\therefore$ Coeff. of x^4 in the expansion of $\dfrac{1}{x}\,[(1 + x)^{21} - (1 + x)^{11}]$

$=$ Coeff. of x^5 in $[(1 + x)^{21} - (1 + x)^{11}] = \mathbf{{}^{21}C_5 - {}^{11}C_5}.$

Ex. 9. If the coefficients of the second, third and fourth terms in the expansion of $(1 + x)^{2n}$ are in A.P., show that $2n^2 - 9n + 7 = 0$. ***(AMU, IIT)***

Sol. $(1+x)^{2n} = {}^{2n}C_0 + {}^{2n}C_1 x + {}^{2n}C_2 x^2 + ... + {}^{2n}C_n x^n + ... + {}^{2n}C_{2n} x^{2n}$

$T_2 = {}^{2n}C_1 x,\ T_3 = {}^{2n}C_2 x^2,\ T_4 = {}^{2n}C_3 x^3$

Given, coefficients of T_2, T_3 and T_4 are in A.P.

$\Rightarrow\quad 2\,.\,{}^{2n}C_2 = {}^{2n}C_1 + {}^{2n}C_3 \quad\Rightarrow\quad 2\,.\dfrac{(2n)!}{(2n-2)!\,2!} = \dfrac{(2n)!}{(2n-1)!} + \dfrac{(2n)!}{(2n-3)!\,3!}$

$\Rightarrow (2n)(2n-1) = 2n + \dfrac{2n(2n-1)(2n-2)}{6} \quad\Rightarrow\quad 6(2n-1) = 6 + (2n-1)(2n-2)$

$\Rightarrow\quad 12n - 6 = 6 + 4n^2 - 6n + 2 \quad\Rightarrow\quad 4n^2 - 18n + 14 = 0 \quad\Rightarrow\quad \mathbf{2n^2 - 9n + 7 = 0.}$

PRACTICE SHEET–1
(General Term and Expansion of Binomial Theorem)

1. The ninth term in the expansion $\left(3x - \dfrac{1}{2x}\right)^8$ is

(*a*) $\dfrac{1}{512\,x^9}$ (*b*) $-\dfrac{1}{512\,x^9}$

(*c*) $-\dfrac{1}{256\,x^8}$ (*d*) $\dfrac{1}{256\,x^8}$ ***(KCET 2007)***

2. If the fourth term in the expansion of $\left(ax + \dfrac{1}{x}\right)^n$ is $\dfrac{5}{2}$, then

(*a*) $a = \dfrac{1}{2}, n = 6$ (*b*) $a = \dfrac{1}{3}, n = 5$

(*c*) $a = 2, n = 3$ (*d*) $a = \dfrac{1}{4}, n = 1$

(J&K CET 2003, AMU 2013)

3. The two consecutive terms in the expansion of $(3 + 2x)^{74}$, whose coefficients are equal are

(*a*) 11, 12 (*b*) 7, 8 (*c*) 30, 31 (*d*) None of these

(Manipal Engg. 2009)

4. The coefficient of x^{-10} in $\left(x^2 - \dfrac{1}{x^3}\right)^{10}$ is

(*a*) – 252 (*b*) 210 (*c*) – (5 !) (*d*) – 120

(WB JEE 2009)

5. The coefficient of x^4 in the expansion of $\left(\dfrac{x}{2} - \dfrac{3}{x^2}\right)^{10}$ is

(*a*) $\dfrac{405}{256}$ (*b*) $\dfrac{450}{263}$ (*c*) $\dfrac{504}{259}$ (*d*) $\dfrac{540}{269}$

(RPET 2001)

6. If in the expansion of $(a - 2b)^n$, the sum of the 5th and 6th term is zero, then the value of $\dfrac{a}{b}$ is

(*a*) $\dfrac{n-4}{5}$ (*b*) $\dfrac{2(n-4)}{5}$ (*c*) $\dfrac{5}{n-4}$ (*d*) $\dfrac{5}{2(n-4)}$

(BCECE 2009)

7. If x^{-7} occurs in the rth term of $\left(ax - \dfrac{b}{x^2}\right)^{11}$, then the value of r is

(*a*) 6 (*b*) 7 (*c*) 8 (*d*) 9

(MP PET 2011)

8. In the expansion of $\left(x - \dfrac{1}{x}\right)^6$, the coefficient of x^0 is

(*a*) 20 (*b*) – 20 (*c*) 30 (*d*) – 30

(UPSEE 2009)

9. If the coefficients of 5th, 6th and 7th terms in the expansion of $(1 + x)^n$ are in A.P., then ***n*** equals

(*a*) 7 (*b*) 5 (*c*) 3 (*d*) 10

(UPSEAT 2000)

10. If rth and $(r + 1)$th terms in the expansion of $(p + q)^n$ are equal, then $\dfrac{(n+1)\,q}{r\,(p+q)}$ is

(*a*) 0 (*b*) 1 (*c*) $\dfrac{1}{4}$ (*d*) $\dfrac{1}{2}$

(KCET 2011)

11. The first three terms in the expansion of $(1 + ax)^n$ $(n \neq 0)$ are 1, $6x$ and $16x^2$. Then the value of ***a*** and ***n*** are respectively

(*a*) 2 and 9 (*b*) 3 and 2 (*c*) 2/3 and 9 (*d*) 3/2 and 6

12. In the expansion of $(2 - 3x^3)^{20}$, if the ratio of the 10th term to the 11th term is 45/22, then ***x*** is equal to

(*a*) $-\dfrac{2}{3}$ (*b*) $-\dfrac{3}{2}$ (*c*) $-\sqrt[3]{\dfrac{2}{3}}$ (*d*) $-\sqrt[3]{\dfrac{3}{2}}$

(Odisha JEE 2012)

13. If the second term in the expansion of $\left[\sqrt[13]{a} + \dfrac{a}{\sqrt{a^{-1}}}\right]^n$ is $14a^{5/2}$, then the value of $\dfrac{{}^nC_3}{{}^nC_2}$ is

(*a*) 4 (*b*) 3 (*c*) 12 (*d*) 6

(DCE 2006)

14. Let t_n denote the nth term in a binomial expansion. If $\frac{t_6}{t_5}$ in the expansion of $(a+b)^{n+4}$ and $\frac{t_5}{t_4}$ in the expansion of $(a+b)^n$ are equal, then n equals

(*a*) 9 (*b*) 11 (*c*) 13 (*d*) 15

(*Kerala PET 2013*)

15. If T_r denotes the rth term in the expansion of $\left(x+\frac{1}{x}\right)^{23}$, then

(*a*) $T_{12}=x^2\,T_{13}$ (*b*) $x^2-T_{13}=T_{12}$
(*c*) $T_{12}=T_{13}$ (*d*) $T_{12}+T_{13}=25$

16. Let the coefficient of x^n in the expansion of $(1+x)^{2n}$ be $\boldsymbol{p}$ and the coefficient of x^n in the expansion of $(1+x)^{2n-1}$ be $\boldsymbol{g}$, then

(*a*) $p=2g$ (*b*) $2p=3g$ (*c*) $2p=g$ (*d*) $3p=2g$

(*WBJEE 2011*)

17. If $0\le r\le n$, then the coefficient of x^r in the expansion of $P=1+(1+x)+(1+x)^2+(1+x)^3+.....+(1+x)^n$ is

(*a*) ${}^nC_r$ (*b*) ${}^nC_{r+1}$ (*c*) ${}^{n+1}C_r$ (*d*) ${}^{n+1}C_{r+1}$

(*AMU 2007*)

18. The coefficient of $\frac{1}{x}$ in the expansion of $\left(\frac{1}{x}+1\right)^n(1+x)^n$ is

(*a*) ${}^{2n}C_n$ (*b*) ${}^{2n}C_{n-1}$ (*c*) ${}^{2n}C_1$ (*d*) ${}^nC_{n-1}$

(*Odisha JEE 2004*)

19. If the magnitude of the coefficient of x^7 in the expansion of $\left(ax^2+\frac{1}{bx}\right)^8$, where a, b are positive numbers is equal in magnitude of the coefficient of x^{-7} in the expansion of $\left(ax-\frac{1}{bx^2}\right)^8$, then $\boldsymbol{a}$ and $\boldsymbol{b}$ are connected by the relation:

(*a*) $ab=1$ (*b*) $ab=2$ (*c*) $a^2b=1$ (*d*) $ab^2=2$

(*WBJEE 2008*)

20. In the expansion of $(1+x+x^2+x^3)^6$, the coefficient of x^{14} is

(*a*) 130 (*b*) 120 (*c*) 128 (*d*) 125

(*Kerala CEE 2007*)

ANSWERS

1. (*d*)	**2.** (*a*)	**3.** (*c*)	**4.** (*b*)	**5.** (*a*)	**6.** (*b*)	**7.** (*b*)	**8.** (*b*)	**9.** (*a*)	**10.** (*b*)
11. (*c*)	**12.** (*a*)	**13.** (*a*)	**14.** (*d*)	**15.** (*a*)	**16.** (*a*)	**17.** (*d*)	**18.** (*b*)	**19.** (*a*)	**20.** (*b*)

HINTS AND SOLUTIONS

1. Given, $\left(3x-\frac{1}{2x}\right)^8$

General term $=T_{r+1}={}^8C_r\,(3x)^{8-r}\left(-\frac{1}{2x}\right)^r$

$\therefore$ Ninth term $=T_9={}^8C_8\,(3x)^{8-8}\left(-\frac{1}{2x}\right)^8$

(Here $r+1=9\Rightarrow r=8$)

$$=\frac{\mathbf{1}}{\mathbf{256x^8}}.$$

2. Given, $\left(ax+\frac{1}{x}\right)^n$

General term $=T_{r+1}={}^nC_r\,(ax)^{n-r}\left(\frac{1}{x}\right)^r$

$$\therefore\quad T_4={}^nC_3\,(ax)^{n-3}\left(\frac{1}{x}\right)^3$$

$$={}^nC_3\,a^{n-3}\,.\,x^{n-3}\,.\,x^{-3}={}^nC_3\,a^{n-3}\,x^{n-6}$$

Given, $T_4=\frac{5}{2}\Rightarrow {}^nC_3\,a^{n-3}x^{n-6}=\frac{5}{2}$ ...(*i*)

Clearly, the fourth term does not contain x, so power of $x=0$.

$\therefore\quad n-6=0\Rightarrow n=6$

$\Rightarrow\quad {}^6C_3\,a^{6-3}=\frac{5}{2}$ (From (*i*))

$$\Rightarrow\quad \frac{\lfloor 6}{\lfloor 3\lfloor 3}a^3=\frac{5}{2}\Rightarrow\frac{6\times5\times4}{3\times2}a^3=\frac{5}{2}$$

$$\Rightarrow\quad a^3=\frac{5}{5\times4\times2}=\frac{1}{8}\Rightarrow a=\frac{\mathbf{1}}{\mathbf{2}}.$$

$$\therefore\quad \boldsymbol{a}=\frac{\mathbf{1}}{\mathbf{2}},\ \boldsymbol{n}=\mathbf{6}.$$

3. General term of $(3+2x)^{74}$ is $T_{r+1}={}^{74}C_r\,(3)^{74-r}\,(2x)^r$
$={}^{74}C_r\,(3)^{74-r}\,2^r\,.\,x^r$

Let the two consecutive terms be $(r+1)$th term and $(r+2)$th term.

Then, $T_{r+2}={}^{74}C_{r+1}\,(3)^{74-(r+1)}\,2^{r+1}\,x^{r+1}$

Given, Coefficient of T_{r+1} = Coefficient of T_{r+2}

$\Rightarrow\ {}^{74}C_r\,(3)^{74-r}\,.\,2^r={}^{74}C_{r+1}\,(3)^{73-r}\,.\,2^{r+1}$

$$\Rightarrow\ \frac{{}^{74}C_{r+1}}{{}^{74}C_r}=\frac{3^{74-r}}{3^{73-r}}\cdot\frac{2^r}{2^{r+1}}\Rightarrow\frac{\dfrac{\lfloor 74}{\lfloor 74-(r+1)\lfloor r+1}}{\dfrac{\lfloor 74}{\lfloor 74-r\lfloor r}}=\frac{3}{2}$$

$$\Rightarrow\ \frac{74-r}{r+1}=\frac{3}{2}\Rightarrow 148-2r=3r+3$$

$$\Rightarrow\ 5r=145\Rightarrow r=\mathbf{29}.$$

$\therefore$ The two consecutive terms are **30th** and **31st.**

4. Given, $\left(x^2 - \frac{1}{x^3}\right)^{10}$

General term $= T_{r+1} = {}^{10}C_r\,(x^2)^{10-r}\left(-\frac{1}{x^3}\right)^r$

$= {}^{10}C_r\,x^{20-2r}\,(-1)^r\,(x^{-3r})$

$= {}^{10}C_r\,x^{20-5r}\,(-1)^r.$

Since the term contains x^{-10}, $\therefore\ 20 - 5r = -10$

$\Rightarrow\quad 5r = 30 \quad\Rightarrow\quad r = 6$

$\therefore$ Coefficient of $x^{-10} = {}^{10}C_6\,(-1)^6 = \frac{\lfloor 10}{\lfloor 6\,\lfloor 4}$

$= \frac{10\times 9\times 8\times 7}{4\times 3\times 2} = \mathbf{210.}$

5. Given, $\left(\frac{x}{2} - \frac{3}{x^2}\right)^{10}$

General term $= T_{r+1} = {}^{10}C_r\left(\frac{x}{2}\right)^{10-r}\left(-\frac{3}{x^2}\right)^r$

$= {}^{10}C_r\left(\frac{1}{2}\right)^{10-r}(-3)^r\;x^{10-r}\,.\,x^{-2r}$

$= {}^{10}C_r\left(\frac{1}{2}\right)^{10-r}(-3)^r\;x^{10-3r}$

Since the term contains x^4, $10 - 3r = 4 \Rightarrow 3r = 6 \Rightarrow r = 2$

$\therefore$ Coefficient of $x^4 = {}^{10}C_2\left(\frac{1}{2}\right)^{10-2}(-3)^2$

$= \frac{\lfloor 10}{\lfloor 8\,\lfloor 2}\times\left(\frac{1}{2}\right)^8.(-3)^2$

$= \frac{10\times 9}{2}\times\frac{1}{256}\times 9 = \mathbf{\frac{405}{256}}.$

6. Given, $(a - 2b)^n$.

General term $= T_{r+1} = {}^nC_r\,(a)^{n-r}\,(-2b)^r$

$= {}^nC_r\,(a)^{n-r}\,(-2)^r\,b^r$

$T_5 = {}^nC_4\,a^{n-4}\,(-2)^4\,b^4 \qquad (\because r+1 = 5 \Rightarrow r = 4)$

$T_6 = {}^nC_5\,a^{n-5}\,(-2)^5\,b^5 \qquad (\because r+1 = 6 \Rightarrow r = 5)$

Given, $T_5 + T_6 = 0$

$\Rightarrow {}^nC_4\,a^{n-4}\,.\,16\,b^4 + {}^nC_5\,a^{n-5}\,(-32)\,b^5 = 0$

$\Rightarrow 16\,{}^nC_4\,a^{n-4}\,b^4 = 32\,.\,{}^nC_5\,a^{n-5}\,b^5$

$\Rightarrow \frac{{}^nC_5\,a^{n-5}\,b^5}{{}^nC_4\,a^{n-4}\,b^4} = \frac{16}{32} = \frac{1}{2} \quad\Rightarrow\quad \frac{b}{a} = \frac{1}{2}\times\frac{{}^nC_4}{{}^nC_5}$

$= \frac{1}{2}\times\frac{\frac{\lfloor n}{\lfloor 4\,\lfloor n-4}}{\frac{\lfloor n}{\lfloor 5\,\lfloor n-5}} = \frac{1}{2}\times\frac{5}{n-4} = \frac{5}{2\,(n-4)}$

$\Rightarrow\quad \frac{a}{b} = \mathbf{\frac{2\,(n-4)}{5}}.$

7. Given, $\left(ax - \frac{b}{x^2}\right)^{11}$

General Term $= T_{r+1} = {}^{11}C_r\,(ax)^{11-r}\left(-\frac{b}{x^2}\right)^r$

$= {}^{11}C_r\,a^{11-r}\,.\,x^{11-r}\,(-b)^r\,.\,x^{-2r}$

$= {}^{11}C_r\,a^{11-r}\,(-b)^r\,x^{11-3r}$

$\therefore\quad r$th term $= T_r = {}^{11}C_{r-1}\,a^{11-(r-1)}\,(-b)^{r-1}\,x^{11-3(r-1)}$

$= {}^{11}C_{r-1}\,a^{12-r}\,x^{14-3r}\,(-b)^{r-1}$

Since x^{-7} occurs in the rth term

$14 - 3r = -7 \quad\Rightarrow\quad 3r = 21 \quad\Rightarrow\quad r = 7.$

8. Let $(r+1)$th term be the coefficient of x^0 in the expansion of $\left(x - \frac{1}{x}\right)^6$.

$\therefore\quad T_{r+1} = {}^6C_r\,x^{6-r}\left(-\frac{1}{x}\right)^r = (-1)^r\;{}^6C_r\,x^{6-r}\;x^{-r}$

$= (-1)^r\,{}^6C_r\,x^{6-2r}$

As the power of x is 0, this means that the given term is a constant term.

$\therefore\quad 6 - 2r = 0 \quad\Rightarrow\quad r = 3$

$\therefore\quad T_4 = (-1)^3\,{}^6C_3 = \mathbf{-20.}$

9. $(1+x)^n = {}^nC_0 + {}^nC_1\,x + {}^nC_2\,x^2 + \ldots.. + {}^nC_{n-1}\,x^{n-1} + {}^nC_n\,x^n$

Given, coefficients of 5th, 6th and 7th terms are in A.P.

$\Rightarrow {}^nC_4, {}^nC_5, {}^nC_6$ are in A.P $\Rightarrow\ 2\,.\,{}^nC_5 = {}^nC_4 + {}^nC_6$

$\Rightarrow\ 2\,.\,\frac{n\,!}{5!\,(n-5)!} = \frac{n\,!}{4!\,(n-4)!} + \frac{n\,!}{6!\,(n-6)!}$

$\Rightarrow\ 2\times\frac{1}{5\times 4!\times(n-5)\times(n-6)!}$

$= \frac{1}{4!\times(n-4)\times(n-5)\times(n-6)!} + \frac{1}{6\times 5\times 4!\,(n-6)!}$

$\Rightarrow\ \frac{2}{5\,(n-5)} = \frac{1}{(n-4)\,(n-5)} + \frac{1}{30}$

$\Rightarrow\ 12\,(n-4) = 30 + (n-4)\,(n-5)$

$\Rightarrow\ n^2 - 9n + 50 = 12n - 48 \quad\Rightarrow\quad n^2 - 21n + 98 = 0$

$\Rightarrow\ (n-14)\,(n-7) = 0 \quad\Rightarrow\quad n = 14, 7.$

10. Given, $(p+q)^n$

$\therefore$ General term $= T_{r+1} = {}^nC_r\,p^{n-r}\,q^r$

$\therefore\quad r$th term $= T_r = {}^nC_{r-1}\,(p)^{n-(r-1)}\,q^{r-1}$

$= {}^nC_{r-1}\,p^{n-r+1}\,q^{r-1}$

$(r+1)$th term $= T_{r+1} = {}^nC_r\,p^{n-r}\,q^r$

Given, $T_r = T_{r+1}$

$\Rightarrow {}^nC_{r-1}\,p^{n-r+1}\,q^{r-1} = {}^nC_r\,p^{n-r}\,q^r$

$\Rightarrow\ \frac{\lfloor n}{\lfloor n-r+1\,\lfloor r-1}\,p^{n-r+1}\,q^{r-1} = \frac{\lfloor n}{\lfloor n-r\,\lfloor r}\,p^{n-r}q^r$

$\Rightarrow\ \frac{1}{(n-r+1)\,\lfloor n-r\,\lfloor r-1}\times\lfloor n-r\,.\,r.\,\lfloor r-1$

$= \frac{p^{n-r}\;q^r}{p^{n-r+1}\;q^{r-1}}$

$\Rightarrow\ \frac{r}{n-r+1} = \frac{q}{p} \quad\Rightarrow\quad pr = qn - qr + q$

$\Rightarrow\ (p+q)r = (n+1)\,q \quad\Rightarrow\quad \mathbf{\frac{(n+1)\,q}{r\,(p+q)} = 1.}$

11. $(1 + ax)^n = {}^nC_0 + {}^nC_1 (ax) + {}^nC_2 (ax)^2 + {}^nC_3 (ax)^3 + ... + {}^nC_n (ax)^n$

Given, $T_2 = 6x$ and $T_3 = 16x^2$

$\Rightarrow {}^nC_1\, ax = 6x$ and ${}^nC_2 (ax)^2 = 16x^2$

$\Rightarrow nax = 6x$ and $\dfrac{n(n-1)}{2} a^2 x^2 = 16x^2$

$\Rightarrow na = 6 \Rightarrow n^2a^2 = 36$ and $n(n-1)a^2 = 32$

$\therefore \dfrac{n(n-1)a^2}{n^2a^2} = \dfrac{32}{36} \Rightarrow \dfrac{n-1}{n} = \dfrac{8}{9}$

$\Rightarrow 9n - 9 = 8n \Rightarrow n = 9$

$\because na = 6 \quad \therefore \quad a = \dfrac{6}{n} = \dfrac{6}{9} = \mathbf{\dfrac{2}{3}}$.

12. General Term of $(2 - 3x^3)^{20}$ is $T_{r+1} = {}^{20}C_r (2)^{20-r} (-3x^3)^r$

$\therefore T_{10} = {}^{20}C_9\, 2^{11} (-3x^3)^9 = {}^{20}C_9 (-1)^9 . 2^{11} . 3^9 . x^{27}$

$T_{11} = {}^{20}C_{10}\, 2^{10} (-3x^3)^{10} = {}^{20}C_{10}\, 2^{10} . 3^{10} . x^{30}$

Given, $\dfrac{T_{10}}{T_{11}} = \dfrac{{}^{20}C_9 . (-1)^9\, 2^{11} . 3^9 . x^{27}}{{}^{20}C_{10} . 2^{10} . 3^{10} . x^{30}} = \dfrac{45}{22}$

$\Rightarrow -\dfrac{10}{11} \times \dfrac{2}{3x^3} = \dfrac{45}{22} \Rightarrow x^3 = -\dfrac{8}{27} \Rightarrow x = -\mathbf{\dfrac{2}{3}}$.

13. Given, $\left(\sqrt[13]{a} + \dfrac{a}{\sqrt{a^{-1}}}\right)^n$

$= \left(a^{1/13} + \dfrac{a^1}{a^{-1/2}}\right)^n = \left(a^{1/13} + a^{3/2}\right)^n$

$\Rightarrow T_2 = 14a^{5/2} \Rightarrow {}^nC_1 \left(a^{\frac{1}{13}}\right)^{n-1} \left(a^{\frac{3}{2}}\right) = 14\, a^{\frac{5}{2}}$

$\Rightarrow n\, a^{\frac{n-1}{13} + \frac{3}{2}} = 14a^{\frac{5}{2}} \Rightarrow n\, a^{\frac{2n-2+39}{26}} = 14a^{\frac{5}{2}}$

$\Rightarrow n\, a^{\frac{2n-37}{26}} = 14a^{\frac{5}{2}} \Rightarrow n = 14$

$\therefore \dfrac{{}^nC_3}{{}^nC_2} = \dfrac{{}^{14}C_3}{{}^{14}C_2} = \dfrac{\frac{14!}{11!\,3!}}{\frac{14!}{12!\,2!}} = \mathbf{4}.$

14. General term is expansion of $(a + b)^{n+4}$ is $T_{r+1} = {}^{n+4}C_r\, a^{n+4-r}\, b^r$

$\therefore \dfrac{t_6}{t_5} = \dfrac{{}^{n+4}C_5\, a^{n+4-5}\, b^5}{{}^{n+4}C_4\, a^{n+4-4}\, b^4} = \dfrac{{}^{n+4}C_5}{{}^{n+4}C_4} . \dfrac{b}{a}$ ...(*i*)

General term in the expansion of $(a+b)^n$ is $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

$\therefore \dfrac{t_5}{t_4} = \dfrac{{}^nC_4\, a^{n-4}\, b^4}{{}^nC_3\, a^{n-3}\, b^3} = \dfrac{{}^nC_4}{{}^nC_3} . \dfrac{b}{a}$ ...(*ii*)

According to the given condition, from (*i*) and (*ii*),

$\dfrac{{}^{n+4}C_5}{{}^{n+4}C_4} . \dfrac{b}{a} = \dfrac{{}^nC_4}{{}^nC_3} . \dfrac{b}{a}$

$\Rightarrow \dfrac{\frac{(n+4)!}{(n-1)!\,5!}}{\frac{(n+4)!}{n!\,4!}} = \dfrac{\frac{n!}{(n-4)!\,4!}}{\frac{n!}{(n-3)!\,3!}} \Rightarrow \dfrac{n}{5} = \dfrac{n-3}{4} \Rightarrow n = \mathbf{15}.$

15. The general term in the expansion of $\left(x + \dfrac{1}{x}\right)^{23}$ is

$T_{r+1} = {}^{23}C_r\, x^{23-r} \left(\dfrac{1}{x}\right)^r = {}^{23}C_r\, x^{23-2r}$

Now, $T_{12} = {}^{23}C_{11}\, x^{23-22} = {}^{23}C_{11}\, x$ ...(*ii*)

$(\because r + 1 = 12 \Rightarrow r = 11)$

$T_{13} = {}^{23}C_{12} \left(\dfrac{1}{x}\right)^{23-24} = {}^{23}C_{12} \left(\dfrac{1}{x}\right)$

$(\because r + 1 = 13 \Rightarrow r = 12)$

$= {}^{23}C_{11} \left(\dfrac{1}{x}\right)$ ... (*iii*)

$(\because {}^{23}C_{11} = {}^{23}C_{23-11} = {}^{23}C_{12})$

$\therefore \dfrac{T_{12}}{T_{13}} = \dfrac{x}{\frac{1}{x}} = x^2 \Rightarrow \mathbf{T_{12} = x^2 . T_{13}}.$

16. $(1 + x)^{2n} = {}^{2n}C_0 + {}^{2n}C_1\, x + {}^{2n}C_2\, x^2 + {}^{2n}C_3\, x^3 + ... + {}^{2n}C_n\, x^n + ... + {}^{2n}C_{2n}\, x^{2n}$

$(1 + x)^{2n-1} = {}^{2n-1}C_0 + {}^{2n-1}C_1\, x + {}^{2n-1}C_2\, x^2 + {}^{2n-1}C_3\, x^3 + ... + {}^{2n-1}C_n\, x^n + ... + {}^{2n-1}C_{2n-1}\, x^{2n-1}$

Given, p = coeff. of x^n in expansion of $(1 + x)^{2n} = {}^{2n}C_n$

g = coeff. of x^n in expansion of $(1 + x)^{2n-1} = {}^{2n-1}C_n$

Now $p = {}^{2n}C_n = \dfrac{\lfloor 2n}{\lfloor n \lfloor n} = \dfrac{2n \times \lfloor 2n-1}{n \lfloor (n-1) \lfloor n} = 2\left[\dfrac{\lfloor 2n-1}{\lfloor n-1 \lfloor n}\right]$ ...(*i*)

$g = {}^{2n-1}C_n = \dfrac{\lfloor 2n-1}{\lfloor 2n-1-n \lfloor n} = \dfrac{\lfloor 2n-1}{\lfloor n-1 \lfloor n}$ ...(*ii*)

$\therefore$ From (*i*) and (*ii*) $\mathbf{p = 2g}$.

17. Given, $P = 1 + (1 + x) + (1 + x)^2 + (1 + x)^3 + ... + (1 + x)^n$

P is the sum of a G.P of $(n + 1)$ terms with first term = 1, common ratio = $(1 + x)$.

$\therefore P = \dfrac{1\,((1+x)^{n+1} - 1)}{(1+x) - 1} \quad \left[\because S_n = \dfrac{a\,(r^n - 1)}{r - 1} \text{ when } n > 1\right]$

$= \dfrac{1}{x}\left\{(1+x)^{n+1} - 1)\right\}$

$= \dfrac{1}{x} \{(1 + {}^{n+1}C_1\, x + {}^{n+1}C_2\, x^2 + ... + {}^{n+1}C_r\, x^r + {}^{n+1}C_{r+1}\, x^{r+1} + ... x^{n+1}) - 1\}$

$= \{{}^{n+1}C_1 + {}^{n+1}C_2\, x + ... + {}^{n+1}C_r\, x^{r-1} + {}^{n+1}C_{r+1}\, x^r + ... x^n\}$

Thus, the coefficient of x^r in this expansion is $\mathbf{{}^{n+1}C_{r+1}}$ **or** $\mathbf{{}^{n+1}C_{n-r}}$.

18. $\left(\dfrac{1}{x} + 1\right)^n (1 + x)^n = \left(\dfrac{1+x}{x}\right)^n (1 + x)^n = \dfrac{1}{x^n}(1 + x)^{2n}$

$= \dfrac{1}{x^n}(1 + {}^{2n}C_1\, x + {}^{2n}C_2\, x^2 + ... + {}^{2n}C_{n-1}\, x^{n-1} + ... + {}^{2n}C_{2n}\, x^{2n})$

$= \left(\dfrac{1}{x^n} + {}^{2n}C_1 \dfrac{1}{x^{n-1}} + {}^{2n}C_2 \dfrac{1}{x^{n-2}} + ... + {}^{2n}C_{n-1} \dfrac{1}{x} + ... + {}^{2n}C_{2n} \dfrac{1}{x^n}\right).$

The coefficient of $\dfrac{1}{x}$ is $\mathbf{{}^{2n}C_{n-1}}$.

19. Let the term containing x^7 in the expansion of $\left(ax^2+\frac{1}{bx}\right)^8$ is T_{r+1}.

Then, $T_{r+1} = {}^8C_r\,(ax^2)^{8-r}\left(\frac{1}{bx}\right)^r$

$= {}^8C_r\,\frac{a^{8-r}}{b^r}\,.\,x^{16-2r}\,.\,x^{-r} = {}^8C_r\,\frac{a^{8-r}}{b^r}\,x^{16-3r}$

Since the term contains x^7, $16-3r=7 \Rightarrow 3r=9 \Rightarrow r=3$.

$\therefore$ Coefficient of x^7 in the expansion of $\left(ax^2+\frac{1}{bx}\right)^8$

$= {}^8C_3\,\frac{a^{8-3}}{b^3} = {}^8C_3\,\frac{a^5}{b^3}$

Now, let the term containing x^{-7} in expansion of $\left(ax-\frac{1}{bx^2}\right)^8$ is T_{R+1}

Then, $T_{R+1} = {}^8C_R\,(ax)^{8-R}\left(-\frac{1}{bx^2}\right)^R$

$= (-1)^R\,{}^8C_R\,\frac{a^{8-R}}{b^R}\,x^{8-R}\,.\,x^{-2R} = (-1)^R\,{}^8C_R\,\frac{a^{8-R}}{b^R}\,x^{8-3R}$

Since this term contains x^{-7}

$\therefore\ 8-3R=-7 \Rightarrow 3R=15 \Rightarrow R=5$

$\therefore$ Coefficient of x^{-7} in the expansion of $\left(ax-\frac{1}{bx^2}\right)^8$

$= (-1)^5\,{}^8C_5\,\frac{a^{8-5}}{b^5} = -\,{}^8C_5\,\frac{a^3}{b^5}$

Given, $\left|{}^8C_3\,\frac{a^5}{b^3}\right| = \left|{}^8C_5\,\frac{a^3}{b^5}\right|$

$\Rightarrow \frac{a^5}{b^3} = \frac{a^3}{b^5}$ $\quad (\because {}^8C_3 = {}^8C_{8-3} = {}^8C_5)$

$\Rightarrow a^2b^2 = 1 \Rightarrow$ $\boldsymbol{ab = 1.}$

20. $(1+x+x^2+x^3)^6 = [(1+x)+x^2(1+x)]^6$

$= [(1+x)(1+x^2)]^6 = (1+x)^6(1+x^2)^6$

$= ({}^6C_0 + {}^6C_1x + {}^6C_2x^2 + {}^6C_3x^3 + {}^6C_4x^4 + {}^6C_5x^5 + {}^6C_6x^6)$
$({}^6C_0 + {}^6C_1x^2 + {}^6C_2x^4 + {}^6C_3x^6 + {}^6C_4x^8 + {}^6C_5x^{10} + {}^6C_6x^{12})$

$\therefore$ Coefficient of $x^{14} = {}^6C_6\,.\,{}^6C_4 + {}^6C_4\,.\,{}^6C_5 + {}^6C_2\,.\,{}^6C_6$

$= \frac{6\times5}{2} + \frac{6\times5}{2}\times6 + \frac{6\times5}{4} = 30+90 = \mathbf{120}.$

(Middle Term, Term independent of x, Greatest term, p^{th} term from the end)

SOLVED EXAMPLES

Ex. 1. Write the middle term in the expansion of $\left(x-\frac{1}{2x}\right)^{10}$.

Sol. The general term in the expansion of $\left(x-\frac{1}{2x}\right)^6$ is

$$T_{r+1} = (-1)^r\,{}^6C_r x^{6-r}\left(\frac{1}{2x}\right)^r = (-1)^r\,.\,{}^6C_r x^{6-r}\,.\,2^{-r}x^{-r} = (-1)^r\,.\,{}^6C_r x^{6-2r}\,.\,2^{-r}$$

Now the power of binomial expansion being 6, (even), the middle term is $\left(\frac{6}{2}+1\right)$th term = 4th term.

$$\therefore\quad T_4 = T_{3+1} = (-1)^3\,{}^6C_3\,x^{6-6}\,2^{-3} = (-1)\times\frac{6\times5\times4}{3\times2\times1}\times x^0\times\frac{1}{8} = -\frac{5}{2}.$$

Ex. 2. Find the coefficient of the term independent of x in the expansion of $\left[\sqrt{\frac{x}{3}}+\frac{\sqrt{3}}{x}\right]^{10}$.

Sol. Given, $\left[\sqrt{\frac{x}{3}}+\frac{\sqrt{3}}{x^2}\right]^{10}$,

General term $= T_{r+1} = {}^{10}C_r\left(\sqrt{\frac{x}{3}}\right)^{10-r}\left(\frac{\sqrt{3}}{x^2}\right)^r = {}^{10}C_r\left(\frac{x}{3}\right)^{\frac{10-r}{2}}(\sqrt{3})^r\,x^{-2r} = {}^{10}C_r\left(\frac{1}{3}\right)^{\frac{10-r}{2}}(\sqrt{3})^r\,x^{\frac{10-r}{2}-2r}$

For the term independent of x,

$$\frac{10-r}{2}-2r = 0 \Rightarrow \frac{10-r}{2} = 2r \Rightarrow 10-r = 4r \Rightarrow 5r = 10 \Rightarrow r = 2$$

$\therefore\ T_{2+1}$, *i.e.*, T_3 is the term independent of x and

$$T_3 = {}^{10}C_2\left(\frac{1}{3}\right)^{\frac{10-2}{2}}(\sqrt{3})^2 = \frac{10\times9}{2}\times\left(\frac{1}{3}\right)^4\times3 = 45\times\frac{1}{27} = \frac{5}{3}.$$

Ex. 3. Show that the coefficient of the middle term in the expansion of $(1 + x)^{2n}$ is the sum of the coefficients of the two middle terms in the expansion of $(1 + x)^{2n-1}$.

Sol. The expansion of $(1 + x)^{2n}$ contains $(2n + 1)$ terms.

The middle term here is $\left(\frac{2n}{2}+1\right)$ th term, *i.e.*, $(n + 1)$th term.

$$\therefore \quad T_{n+1} = {}^{2n}C_n\, x^n = \frac{(2n)!}{n!\,n!}x^n = \frac{(2n)!}{(n!)^2}x^n \qquad ...(i)$$

In the expansion of $(1 + x)^{2n-1}$, $(2n - 1)$ being odd, there are two middle term $\left(\frac{2n-1+1}{2}\right)$th term and $\left(\frac{2n-1+3}{2}\right)$th term, *i.e.*, nth term and $(n + 1)$th term.

$\therefore$ In case of $(1 + x)^{2n-1}$

$$t_n = {}^{2n-1}C_{n-1}\, x^{n-1},\ t_{n+1} = {}^{2n-1}C_n\, x^n$$

$\therefore$ Sum of coefficients of t_n and $t_{n+1} = {}^{2n-1}C_{n-1} + {}^{2n-1}C_n = {}^{2n-1+1}C_n = {}^{2n}C_n \qquad (\because {}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1})$

$$= \frac{2n!}{(n!)^2} = \text{coefficient of middle term of } (1 + x)^{2n}. \qquad \text{(from } (i))$$

Ex. 4. Find the 5th term from the end in the expansion of $\left(\frac{2}{x}-\frac{x^3}{5}\right)^9$.

Sol. Using rth term from the end $= (m - r + 2)$th term, we have 5th term from the end in the expansion of $\left(\frac{2}{x}-\frac{x^3}{5}\right)^9$

$= (9 - 5 + 2)$th from the end $=$ 6th term from the beginning in the given expansion

$$= T_6 = T_{5+1} = {}^9C_5\left(\frac{2}{x}\right)^{9-5}\left(-\frac{x^3}{5}\right)^5 = \frac{-9\times8\times7\times6}{4\times3\times2\times1}\times\left(\frac{2}{x}\right)^4\times\left(\frac{x^3}{5}\right)^5$$

$$= -126\times\frac{16}{x^4}\times\frac{x^{15}}{3125} = \mathbf{\frac{-2016}{3125}x^{11}}.$$

$T_1\ T_2\ T_3\ T_4\ T_5\ T_6\ T_7\ T_8\ T_9\ T_{10}$

6th term from the beginning → T_6 ← 5th term from the end

PRACTICE SHEET–2
(Middle Term, Term independent of x, Greatest term, p^{th} term from the end)

1. The middle term in the expansion of $\left(\frac{2x^2}{3}+\frac{3}{2x^2}\right)^{10}$ is

(*a*) 252 (*b*) 260 (*c*) 274 (*d*) 450

(*MP PET 2013*)

2. The middle term in the expansion of $\left(x^2+\frac{1}{x^2}+2\right)^n$ is

(*a*) $\frac{n}{[(n/2)!]^2}$ (*b*) $\frac{(2n!)}{[(n/2)!]^2}$

(*c*) $\frac{1.3.5.....(2n+1)}{n!}2^n$ (*d*) $\frac{(2n)!}{(n!)^2}$

(*Manipal Engg. 2012*)

3. In the expansion of $(1 - 3x + 3x^2 - x^3)^{2n}$, the middle term is

(*a*) $(n + 1)$th term (*b*) $(2n + 1)$th term

(*c*) $(3n + 1)$th term (*d*) None of these

(*MP PET 2011*)

4. The coefficient of the middle term in the expansion of $(1 + x)^{2n}$ is

(*a*) $\frac{2^n\,(2n-1)\,(2n-3)\,.....\,3.1}{n\,(n-1)\,(n-2)\,.....\,3.2.1}$

(*b*) $n\,(n + 1)\,(n + 2)\,.....\,2n$

(*c*) $2^n\,(2n - 1)\,(2n - 3)\,.....\,3.\,1$

(*d*) $\frac{2^n\,(n)\,(n+1)\,(n+2)\,.....\,(2n)}{n\,(n-1)\,(n-2)\,.....\,3.2.1}$

(*Odisha JEE 2011*)

5. The coefficient of the middle term in the binomial expansion in powers of x of $(1 + \alpha x)^4$ and of $(1 - \alpha x)^6$ is the same if α equals

(*a*) $-\frac{3}{10}$ (*b*) $-\frac{5}{3}$ (*c*) $\frac{3}{5}$ (*d*) $\frac{10}{3}$

(*AIEEE 2004*)

6. The numerically greatest term in the expansion of $(3 + 2x)^{44}$, when $x = \frac{1}{5}$ is

(*a*) 4th term (*b*) 5th term (*c*) 6th term (*d*) 7th term

(*AMU 2009*)

7. The greatest term in the expansion of $(1 + 3x)^{54}$, where $x = \frac{1}{3}$ is

(*a*) T_{28} (*b*) T_{25} (*c*) T_{26} (*d*) T_{24}

(*DCE 2007*)

8. The term idependent of x in the expansion of $\left(\frac{2\sqrt{x}}{5} - \frac{1}{2x\sqrt{x}}\right)^{11}$ is

(*a*) 5th term (*b*) 6th term (*c*) 11th term (*d*) no term

(*J&K CET 2007*)

9. The term independent of x in $\left(\sqrt{x} - \frac{2}{x}\right)^{18}$ is

(*a*) $^{18}C_{12}\, 2^8$ (*b*) $^{18}C_6\, 2^{12}$ (*c*) $^{18}C_6\, 2^4$ (*d*) $^{18}C_{12}\, 2^6$

(*MP PET 2009*)

10. The term idependent of x in the expansion of $(1 + x)^n \left[1 + \left(\frac{1}{x}\right)\right]^n$ is

(*a*) $C_1 + C_2 + C_3 + ... + C_n$

(*b*) $C_1 + 2C_2 + 3C_3 + ... + {}^nC_n$

(*c*) $C_0^{\,2} + C_1^{\,2} + C_2^{\,2} + ... + C_n^{\,2}$

(*d*) $C_0^{\,2} + 2C_1^{\,2} + 3C_2^{\,2} + ... + (n + 1)\, C_n^{\,2}$

(*Kerala PET 2000*)

11. If in the expansion of $\left(\frac{3\sqrt{x}}{7} - \frac{5}{2x\sqrt{x}}\right)^{13n}$ contains a term independent of x, then n should be a multiple of

(*a*) 3 (*b*) 4 (*c*) 5 (*d*) 6

(*Kerala PET 2008*)

12. The 13th term in the expansion of $\left(x^2 + \frac{2}{x}\right)^n$ is independent of x, then the sum of the divisors of $\boldsymbol{n}$ is

(*a*) 36 (*b*) 37 (*c*) 38 (*d*) 39

(*Karnataka CET 2012*)

13. The term independent of x in the expansion $\left(x - \frac{1}{x}\right)^4 \left(x + \frac{1}{x}\right)^3$ is

(*a*) – 3 (*b*) 0 (*c*) 3 (*d*) 1

(*Gujarat CET 2007*)

14. What is the ratio of coefficient of x^{15} to the term independent of x in $\left(x^2 + \frac{2}{x}\right)^{15}$?

(*a*) $\frac{1}{64}$ (*b*) $\frac{1}{32}$ (*c*) $\frac{1}{16}$ (*d*) $\frac{1}{4}$

(*NDA/NA 2011*)

15. The term independent of x in the expansion of $\left(x + \frac{1}{x} + 2\right)^{11}$ is

(*a*) $\frac{11!}{6!\,6!}$ (*b*) $\frac{11!}{5!\,6!}$ (*c*) $^{22}C_{10}$ (*d*) $^{22}C_{11}$

16. 5th term from the end in the expansion of $\left(\frac{x^3}{2} - \frac{2}{x^2}\right)^{12}$ is

(*a*) $– 7920\, x^{–4}$ (*b*) $7920\, x^4$ (*c*) $7920\, x^{–4}$ (*d*) $– 7920\, x^4$

17. In the expansion of $\left(3x - \frac{2}{x^2}\right)^{15}$, if the pth term from the end does not depend on the value of x, then the value of $\boldsymbol{p}$ is

(*a*) 9 (*b*) 10 (*c*) 11 (*d*) 12

(*Rajasthan PET 2009*)

18. Find the sixth term of the expansion of $(y^{1/2} + x^{1/3})^n$, if the binomial coefficient of the third term from the end is 45.

(*a*) $240\, y^{3/2} x^{2/3}$ (*b*) $252\, y^{5/2} x^{5/3}$

(*c*) $252\, y^{3/2} x^{5/3}$ (*d*) $240\, y^{5/2} x^{5/3}$

ANSWERS

1. (*a*)	**2.** (*d*)	**3.** (*c*)	**4.** (*a*)	**5.** (*a*)	**6.** (*c*)	**7.** (*a*)	**8.** (*d*)	**9.** (*d*)	**10.** (*c*)	**11.** (*b*)
12. (*d*)	**13.** (*b*)	**14.** (*b*)	**15.** (*d*)	**16.** (*c*)	**17.** (*c*)	**18.** (*b*)				

HINTS AND SOLUTIONS

1. In the expansion of $\left(\frac{2x^2}{3} + \frac{3}{2x^2}\right)^{10}$, the middle term is the $\left(\frac{10}{2} + 1\right)$th term, *i.e.*, 6th term

Now $T_6 = T_{5+1} = {}^{10}C_5 \left(\frac{2x^2}{3}\right)^{10-5} \left(\frac{3}{2x^2}\right)^5$

$(\because\ T_{r+1} = {}^nC_r\, (a)^{n-r}\, b^r)$.

$= \frac{10!}{5!\,5!} \frac{2^5\, x^{10}}{3^5} \cdot \frac{3^5}{2^5\,.\,x^{10}} = \frac{10!}{5!\,5!} = \frac{10\times9\times8\times7\times6}{5\times4\times3\times2\times1} = \mathbf{252.}$

2. $\left(x^2 + \frac{1}{x^2} + 2\right)^n = \left(\left(x + \frac{1}{x}\right)^2\right)^n = \left(x + \frac{1}{x}\right)^{2n}$

$\therefore$ The middle term in the expansion of $\left(x + \frac{1}{x}\right)^{2n}$ $= \left(\frac{2n}{2} + 1\right)$th term $= (n + 1)$th term.

Now $T_{n+1} = {}^{2n}C_n\, (x)^{2n-n} \left(\frac{1}{x}\right)^n = \frac{(2n!)}{n!\,n!}\, x^n \cdot \frac{1}{x^n} = \frac{\mathbf{(2n!)}}{\mathbf{(n!)^2}}.$

3. $(1 – 3x + 3x^2 – x^3)^{2n} = ((1 – x)^3)^{2n} = (1 – x)^{6n}$

$\therefore$ Middle term $= \left(\frac{6n}{2} + 1\right)$th term $= (3n + 1)$th term.

4. The middle term in the expansion of $(1+x)^{2n}$ is $\left(\frac{2n}{2}+1\right)$th term, *i.e.*, $(n+1)$th term.

$\therefore$ Coefficient of $(n+1)$th term in the expansion of $(1+x)^{2n} = {}^{2n}C_n$

$$= \frac{(2n)!}{n!\,n!} = \frac{2n\,(2n-1)\,(2n-2)\,.....\,3.2.1}{\{n(n-1)\,(n-2)\,.....\,3.2.1\}^2}$$

$$= \frac{\{2n\,(2n-2)\,(2n-4)\,.....\,4.2.\}\,\{(2n-1)\,(2n-3)\,....\,3.1\}}{\{n\,(n-1)\,(n-2)\,....\,3.2.1\}^2}$$

$$= \frac{2^n\{n\,(n-1)\,(n-2)\,.....\,3.2.1\}\,\{(2n-1)\,(2n-3)\,....\,3.1\}}{\{n\,(n-1)\,(n-2)\,....\,3.2.1\}^2}$$

$$= \frac{2^n\{(2n-1)\,(2n-3)\,.....\,3.1.\}}{\{n\,(n-1)\,(n-2)\,....\,3.2.1\}}.$$

5. Middle term in the expansion of $(1+\alpha x)^4$ is the $\left(\frac{4}{2}+1\right)$th term, *i.e.*, 3rd term.

$\therefore \quad t_3 = t_{2+1} = {}^4C_2\,(\alpha x)^2$

Middle term in the expansion of $(1-\alpha x)^6$ is the $\left(\frac{6}{2}+1\right)$th term, *i.e.*, 4th term.

$\therefore \quad T_4 = T_{3+1} = {}^6C_3\,(-1)^3\,(\alpha x)^3$

$\therefore$ Coefficient of t_3 = Coefficient of T_4

$\Rightarrow \quad {}^4C_2\,\alpha^2 = {}^6C_3\,(-1)^3\,\alpha^3$

$$\Rightarrow \quad \frac{4\times 3}{2}\alpha^2 = (-1)\times\frac{6\times5\times4}{3\times2}\alpha^3 \quad \Rightarrow \quad 6\alpha^2 = -20\alpha^3$$

$$\Rightarrow \quad \alpha = -\frac{6}{20} = \mathbf{-\frac{3}{10}}.$$

6. The general term in the expansion of $(3+2x)^{44}$ is

$T_{r+1} = {}^{44}C_r\,(3)^{44-r}\,(2x)^r$

$\therefore \quad T_r = {}^{44}C_{r-1}\,(3)^{44-(r-1)}\,(2x)^{r-1} = {}^{44}C_{r-1}\,3^{45-r}\,(2x)^{r-1}$

For T_{r+1} to be the greatest term, $\left(\text{where } x = \frac{1}{5}\right)$

$$\frac{T_{r+1}}{T_r} > 1 \quad \Rightarrow \quad \frac{{}^{44}C_r\,3^{44-r}\,(2x)^r}{{}^{44}C_{r-1}\,3^{45-r}\,(2x)^{r-1}} > 1$$

$$\Rightarrow \quad \frac{\frac{(44)!}{(44-r)!\,r!}}{\frac{(44)!}{(44-r+1)!\,(r-1)!}} \times \frac{2x}{3} > 1 \quad \Rightarrow \quad \frac{(45-r)}{r}\times\frac{2x}{3} > 1$$

$$\Rightarrow \quad \frac{(45-r)}{r}\times\frac{2}{3}\times\frac{1}{5} > 1 \quad \Rightarrow \quad 90-2r > 15r$$

$$\Rightarrow \quad 90 > 17r \quad \text{or} \quad 17r < 90 \quad \Rightarrow \quad r < \frac{90}{17} = 5\frac{5}{7}$$

$$\Rightarrow \quad \frac{T_{r+1}}{T_r} > 1 \ \text{ for all value of } r \le 5$$

$\Rightarrow \quad T_6$, *i.e.*, the 6th term is numerically the greatest term.

7. Similar to Q. 6.

8. The general term in the expansion of $\left(\frac{2\sqrt{x}}{5} - \frac{1}{2x\sqrt{x}}\right)^{11}$ is

$$T_{r+1} = {}^{11}C_r\left(\frac{2\sqrt{x}}{5}\right)^{11-r}\left(-\frac{1}{2x\sqrt{x}}\right)^r$$

$$= {}^{11}C_r\,(-1)^r\,2^{11-r}\,(x^{1/2})^{11-r}\,(5^{-1})^{11-r}.\,2^{-r}\,.\,(x^{-3/2})^r$$

$$= {}^{11}C_r\,(-1)^r\,2^{11-2r}\,5^{r-11}\,x^{\frac{11}{2}-\frac{r}{2}-\frac{3r}{2}}$$

$$= {}^{11}C_r\,(-1)^r\,2^{11-2r}\,5^{r-11}\,x^{\frac{11}{2}-2r}$$

This term will be independent of x, if $\frac{11}{2} - 2r = 0 \Rightarrow 2r = \frac{11}{2}$

$\Rightarrow \ r = \frac{11}{4}$. Since $r = \frac{11}{4}$ is not an integral value, there is no term in the expansion of $\left(\frac{2\sqrt{x}}{5} - \frac{1}{2x\sqrt{x}}\right)^{11}$ which is independent of x.

9. The general term in the expansion of $\left[\sqrt{x} - \frac{2}{x}\right]^{18}$ is

$$T_{r+1} = {}^{18}C_r\,(\sqrt{x})^{18-r}\left(-\frac{2}{x}\right)^r = {}^{18}C_r\,x^{\frac{18-r}{2}}\,(-1)^r\,2^r\,x^{-r}$$

$$= {}^{18}C_r\,x^{\frac{18-3r}{2}}\,(-1)^r\,2^r$$

This term will be independent of x if $\frac{18-3r}{2} = 0 \Rightarrow 18-3r = 0 \Rightarrow r = 6$.

$\therefore$ Reqd. term $= T_{6+1} = {}^{18}C_6\,(-1)^6\,2^6 = {}^{18}C_6\,2^6 = {}^{18}C_{12}\,2^6$

$(\because {}^{18}C_6 = {}^{18}C_{12})$

10. $(1+x)^n\left(1+\frac{1}{x}\right)^n$

$= [{}^nC_0 + {}^nC_1\,x + {}^nC_2\,x^2 + ... + {}^nC_n\,x^n]$

$$\times\left[{}^nC_0 + {}^nC_1\left(\frac{1}{x}\right) + {}^nC_2\left(\frac{1}{x^2}\right) + ... + {}^nC_n\left(\frac{1}{x^n}\right)\right]$$

$\therefore$ The term independent of x in this expansion is

$[{}^nC_0\,.\,{}^nC_0 + {}^nC_1\,.\,{}^nC_1 + {}^nC_2\,.\,{}^nC_2 + ... + {}^nC_n\,.\,{}^nC_n]$

$= ({}^nC_0)^2 + ({}^nC_1)^2 + ({}^nC_2)^2 + ... + ({}^nC_n)^2$

$= \mathbf{C_0^2 + C_1^2 + C_2^2 + ... + C_n^2}$.

11. The general term in the expansion of $\left(\frac{3\sqrt{x}}{7} - \frac{5}{2x\sqrt{x}}\right)^{13n}$ is

$$T_{r+1} = {}^{13n}C_r\left(\frac{3\sqrt{x}}{7}\right)^{13n-r}\left(\frac{-5}{2x\sqrt{x}}\right)^r$$

$$= {}^{13n}C_r\,(-1)^r\,(3)^{13n-r}\,(7)^{r-13n}\,(5)^r\,(2)^{-r}\,x^{\frac{13n-r}{2}}\,.\,x^{-\frac{-3r}{2}}$$

$$= {}^{13n}C_r\,(-1)^r\,(3)^{13n-r}\,(7)^{r-13n}\,(5)^r\,(2)^{-r}\,x^{\frac{13n}{2}-2r}$$

This term is independent of x if $\frac{13n}{2} - 2r = 0$

$$\Rightarrow \quad \frac{13n}{2} = 2r \quad \Rightarrow \quad n = 4\left(\frac{r}{13}\right)$$

$\Rightarrow \quad n$ should be a **multiple of 4.**

12. 13th term in the expansion of $\left(x^2+\frac{2}{x}\right)^n$ is given by

$$T_{13} = {}^nC_{12}\,(x^2)^{n-12}\left(\frac{2}{x}\right)^{12} = {}^nC_{12}\,x^{2n-24}\,\frac{2^{12}}{x^{12}}$$
$$= {}^nC_{12}\,x^{2n-36}\,.\,2^{12}$$

If the 13th term is independent of x, then $2n - 36 = 0$

$\Rightarrow n = 18$.

The divisors of $n = 18$ are 1, 2, 3, 6, 9, 18 and their sum $= 1 + 2 + 3 + 6 + 9 + 18 =$ **39.**

13. $\left(x-\frac{1}{x}\right)^4\left(x+\frac{1}{x}\right)^3$

$$= \left({}^4C_0\,x^4 - {}^4C_1\,x^2 + {}^4C_2 - {}^4C_3\,\frac{1}{x^2} + {}^4C_4\,\frac{1}{x^4}\right)$$
$$\times\left({}^3C_0\,x^3 + {}^3C_1\,x + {}^3C_2\,\frac{1}{x} + {}^3C_3\,\frac{1}{x^3}\right)$$

As can be seen from the given product, there is no term free of x on RHS, therefore the term independent of x is **0.**

14. Given, binomial expression $= \left(x^2+\frac{2}{x}\right)^{15}$

General term $= T_{r+1} = {}^{15}C_r\,(x^2)^{15-r}\left(\frac{2}{x}\right)^r$

$= {}^{15}C_r\,x^{30-2r}\,2^r\,.\,x^{-r} = {}^{15}C_r\,x^{30-3r}\,.\,2^r$

For coefficient of x^{15}, put $30 - 3r = 15 \Rightarrow 3r = 15 \Rightarrow r = 5$

$\therefore \quad T_{5+1} = T_6 = {}^{15}C_5\,x^{15}\,.\,2^5$

For term independent of x, put $30 - 3r = 0 \Rightarrow 3r = 30$ $\Rightarrow r = 10$

$\therefore \quad T_{10+1} = T_{11} = {}^{15}C_{10}\,2^{10}$

$\therefore$ Required ratio $= \frac{T_6}{T_{11}} = \frac{{}^{15}C_5\,2^5}{{}^{15}C_{10}\,2^{10}} = \frac{{}^{15}C_{10}}{{}^{15}C_{10}}\times\frac{1}{2^5} = \frac{1}{32}$

($\because$ $^nC_r = {}^nC_{n-r}$)

15. $\left(x+\frac{1}{x}+2\right)^{11} = \left(\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2\right)^{11} = \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^{22}$

General term in the expansion of $\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^{22}$ is

$$T_{r+1} = {}^{22}C_r\,(\sqrt{x})^{22-r}\left(\frac{1}{\sqrt{x}}\right)^r = {}^{22}C_r\,(x^{1/2})^{22-2r}$$
$$= {}^{22}C_r\,x^{11-r}$$

This term is independent of x, if $11 - r = 0 \Rightarrow r = 11$

$\therefore$ Term independent of x is $T_{11+1} = T_{12} = {}^{22}C_{11}\,x^{22-22} =$ **$^{22}C_{11}$.**

16. 5th term from the end

$= (12 - 5 + 2)$th term from the beginning

$= 9$th term from the beginning in the expansion of $\left(\frac{x^3}{2}-\frac{2}{x^2}\right)^{12}$

$$= {}^{12}C_8\left(\frac{x^3}{2}\right)^{12-8}\left(-\frac{2}{x^2}\right)^8 = {}^{12}C_8\,x^{-4}\,.\,2^4$$
$$= \frac{12\times11\times10\times9}{4\times3\times2\times1}\times2^4\times x^{-4} = \mathbf{7920\,x^{-4}}.$$

17. The general term in the expansion $\left(3x-\frac{2}{x^2}\right)^{15}$ is

$$T_{r+1} = {}^{15}C_r\,(3x)^{15-r}\left(-\frac{2}{x^2}\right)^r$$
$$= {}^{15}C_r\,(3)^{15-r}\,(x)^{15-r}\,(-1)^r\,(2)^r\,(x)^{-2r}$$
$$= {}^{15}C_r\,(-1)^r\,(2)^r\,(3)^{15-r}\,(x)^{15-3r}$$

This term is independent of x if $15 - 3r = 0 \Rightarrow r = 5$

$\therefore\ T_{5+1} = T_6 =$ 6th term from the beginning is independent of x.

$\Rightarrow$ $(15 - 6 + 2)$th term is independent of x from the end

$\Rightarrow$ 11th term from the end is independent of x.

$\Rightarrow$ $p =$ **11.**

18. 3rd term from the end in the expansion of $(y^{1/2} + x^{1/3})^n$

$= (n - 3 + 2)$th term from the beginning in the given expansion

$= (n - 1)$th term from the beginning in the expansion of $(y^{1/2} + x^{1/3})^n$

$$= T_{n-1} = {}^nC_{n-2}\,(y^{1/2})^{n-(n-2)}\,(x^{1/3})^{n-2} = \frac{n\,(n-1)}{2}\times y\times x^{\frac{n-2}{3}}$$

Given, Binomial Coeff. of 3rd term, i.e., T_{2+1} from the end

$= 45$, *i.e.*, $^nC_2 = 45 \Rightarrow \frac{n\,(n-1)}{2} = 45$

$\Rightarrow\ n(n-1) = 90 \quad\Rightarrow\quad n = 10$

$\therefore\ T_6$ in the expansion of $(y^{1/2} + x^{1/3})^{10}$

$$= {}^{10}C_5\,(y^{1/2})^{10-5}\,(x^{1/3})^5 = \frac{10\times9\times8\times7\times6}{5\times4\times3\times2\times1}\times y^{5/2}\times x^{5/3}$$

$= \mathbf{252\,y^{5/2}\,x^{5/3}}.$

KEY FACTS (Contd.)

1. Binomial theorem for negative or fractional index

Let ***n*** be a negative integer or a fraction (+ve or –ve) and x be a real number such that $|x| < 1$, then

$$(1+x)^n = 1 + nx + \frac{n\,(n-1)}{2!}\,x^2 + \frac{n\,(n-1)\,(n-2)}{3!}\,x^3 + \ldots. + \frac{n\,(n-1)\,(n-2)\ldots(n-r+1)}{r!}\,x^r + \ldots\infty$$

Note 1. The expansion is valid only if $|x| < 1$.

2. The first term of the binomial is unity, *i.e.*, it is of the form $(1 + x)$ where $x \in R$ and $|x| < 1$. Also the expansion of $(1 + x)$ to negative or fractional index contains infinite terms.

$\therefore$ $$(x + a)^n = a^n\left(1+\frac{x}{a}\right)^n$$ (n being a negative integer or fraction (+ve or –ve))

$$= a^n\left(1+n.\frac{x}{a}+\frac{n(n-1)}{2!}\left(\frac{x}{a}\right)^2+.....\right)=a^n+na^{n-1}x+\frac{n(n-1)a^{n-2}}{2!}x^2+.....$$

This expression is valid only when $\left|\frac{x}{a}\right| < 1$ or $|x| < |a|$.

2. General term in the expansion of $(1 + x)^n$, when n is an integer or a fractional rational number.

$$T_{r+1} = \frac{n(n-1)(n-2).....(n-r+1)}{r!}.x^r$$

3. Some special expansions and their general terms.

a. $(1 - x)^{-1} = 1 + x + x^2 + ... + x^r + ...$

b. $(1 + x)^{-1} = 1 - x + x^2 - ..... + (-1)^r x^r + ...$

c. $(1 - x)^{-2} = 1 + 2x + 3x^2 + ... + (r + 1)x^r + ...$

d. $(1 + x)^{-2} = 1 - 2x + 3x^2 - ..... + (-1)^r (r + 1)x^r + ...$

e. $(1 - x)^{-3} = 1 + 3x + 6x^2 + ... + \dfrac{(r+1)(r+2)}{2}x^r + ...$

f. $(1 + x)^{-3} = 1 - 3x + 6x^2 - ..... + (-1)^r\dfrac{(r+1)(r+2)}{2}x^r + ...$

SOLVED EXAMPLES

Ex. 1. Write down and simplify the first four terms in the binomial expansion of $(1 - 2x)^{2/3}$.

Sol. $$(1-2x)^{2/3} = 1+\frac{2}{3}(-2x)+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{2.1}(-2x)^2+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)\left(\frac{2}{3}-2\right)}{3.2.1}(-2x)^3+.....$$

$$= 1-\frac{4}{3}x-\frac{4}{9}x^2-\frac{32}{81}x^3+.....$$

Ex. 2. Find the coefficient of x^4 in the expansion of $\left(\dfrac{1-x}{1+x}\right)^2$.

Sol. $\left(\dfrac{1-x}{1+x}\right)^2 = (1 - x)^2 (1 + x)^{-2} = (1 - 2x + x^2)(1 - 2x + 3x^2 - 4x^3 + 5x^4 .....)$

The coefficient of x^4 in this expansion or product is

$(1 \times 5) + (-2) \times (-4) + 1 \times 3 = 5 + 8 + 3 =$ **16.**

$(1 - 2x + x^2)(1 - 2x + 3x^2 - 4x^3 + 5x^4)$

Ex. 3. Find the coefficient of x in the expansion of $(1 + x + x^2 + x^3)^{-3}$.

Sol. $(1 + x + x^2 + x^3)^{-3} = \{(1 + x) + x^2(1 + x)\}^{-3} = \{(1 + x)(1 + x^2)\}^{-3} = (1 + x)^{-3}(1 + x^2)^{-3}$

$= \{1 - 3x + 6x^2 - 10x^3 + ...\}\{1 - 3x^2 + 6x^4 - 10x^6 + ...\}$

$\therefore$ The coefficient of x here $=$ **– 3.**

Ex. 4. Find the square root of $1 + 2x + 3x^2 + 4x^3 + ...$? **(*RPET 2007*)**

Sol. $1 + 2x + 3x^2 + 4x^3 + ... = (1 - x)^{-2}$

$\therefore$ Reqd. square root $= \left[(1-x)^{-2}\right]^{\frac{1}{2}} = \mathbf{(1-x)^{-1}}$

Ex. 5. Find the coefficient of x^n in the expansion of $(1 - 4x)^{-1/2}$.

Sol. $$(1-4x)^{-1/2} = 1 + \left(-\frac{1}{2}\right)(-4x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{1.2}(-4x)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{1.2.3}(-4x)^3 + ...$$
$$+ \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)......\left(-\frac{1}{2}-r+1\right)}{1.2.3......r}.(-4x)^r + ...$$

$$= 1 + \frac{1}{2}(4x) + \frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)}{2!}(4x)^2 + \frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)}{3!}(4x)^3 + ... + \frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)...\left(\frac{2r-1}{2}\right)}{r!}(4x)^r + ...$$

$$\therefore \quad T_{r+1} = \text{General term} = \frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)...\left(\frac{2r-1}{2}\right)}{r!}(4x)^r = \frac{1.3.5....(2r-1)}{2^r.r!}.2^{2r}.x^r = \frac{1.3....(2r-1)}{r!}2^r.x^r$$

$$= \frac{1.2.3.....(2r-1).2r}{2.4.6......2r}.\frac{2^r.x^r}{r!} = \frac{(2r)!}{2^r(1.2.3......r)}.\frac{2^r.x^r}{r!} = \frac{(2r)!}{r!.r!}.x^r$$

$$\therefore \quad \text{coefficient of } x^r = \frac{(2r)!}{r!r!} \Rightarrow \text{coefficient of } x^n = \mathbf{\frac{(2n)!}{(n!)^2}}.$$

PRACTICE SHEET–3
(Binomial Theorem for Negative or Fractional Index)

1. The coefficient of x^4 in the expansion of $(1 + x^{-2})$, where $|x| < 1$
(*a*) –5 (*b*) –3 (*c*) 4 (*d*) 5
(*J&K CET 2010*)

2. If $|x| < 1$, then the coefficient of x^6 in the expansion of $(1 + x + x^2)^{-3}$ is
(*a*) 3 (*b*) 6 (*c*) 9 (*d*) 12
(*Kerala PET 2009*)

3. Coefficient of $\boldsymbol{x}$ in the expansion of $\left(\frac{1+x}{1-x}\right)^n$ is
(*a*) $2n$ (*b*) $4n$ (*c*) n^2 (*d*) $\frac{n(n+1)}{2}$
(*AMU 2003*)

4. If $|x| < 1$, then the coefficient of x^n in $(1 + 2x + 3x^2 + 4x^3 + ...)^{1/2}$ is
(*a*) 1 (*b*) $-n$ (*c*) n (*d*) $n + 1$
(*WB JEE 2009*)

5. The coefficient of x^4 in the expansion of $\frac{(1-3x)^2}{(1-2x)}$ is
(*a*) 1 (*b*) 2 (*c*) 3 (*d*) 4
(*EAMCET 2001*)

6. For $|x| < 1$, the constant term in the expansion of $\frac{1}{(x-1)^2(x-2)}$ is
(*a*) $-\frac{1}{2}$ (*b*) 0 (*c*) 1 (*d*) 2

7. The coefficient of x^n in the expansion of $(1 - 2x)^{-1/2}$ is
(*a*) $\frac{(2n)!}{(n!)^2}$ (*b*) $\frac{(2n)!}{2^n\, n!}$ (*c*) $\frac{(2n)!}{2^n\,(n!)^2}$ (*d*) $\frac{(2n)!}{n!}$

8. If $|x| < \frac{1}{2}$, then the coefficient of x^r in the expansion of $\frac{1+2x}{(1-2x)^2}$ is
(*a*) $r\,2^r$ (*b*) $r\,.\,2^{2r+1}$ (*c*) $(2r-1)2^r$ (*d*) $(2r+1).2^r$
(*EAMCET 2005*)

ANSWERS

1. (*d*) **2.** (*a*) **3.** (*a*) **4.** (*a*) **5.** (*d*) **6.** (*a*) **7.** (*c*) **8.** (*d*)

HINTS AND SOLUTIONS

1. $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4$

$\Rightarrow$ Coefficient of x^4 in the expansion of $(1 + x)^{-2} =$ **5.**

2. $(1+x+x^2)^{-3} = \left(\dfrac{1-x^3}{1-x}\right)^{-3} = \left(\dfrac{1-x}{1-x^3}\right)^{3} = (1-x)^3\,(1-x^3)^{-3}$

$= ({}^3C_0 - {}^3C_1\,x + {}^3C_2\,x^2 - {}^3C_3\,x^3)$

$$\left[1 + (-3)\,.\,(-x^3) + \frac{(-3)\,(-3-1)}{1.2}(-x^3)^2 + \frac{(-3)\,(-3-1)\,(-3-2)}{1.2.3}(-x^3)^3 + \ldots\right]$$

$= (1 - 3x + 3x^2 - x^3)\,(1 + 3x^3 + 6x^6 + 10x^9 + \ldots.)$

$\Rightarrow$ Coefficient of x^6 in the given expansion

$= (1 \times 6 + (-1) \times 3) = 6 - 3 =$ **3.**

3. $\left(\dfrac{1+x}{1-x}\right)^n = (1 + x)^n\,(1 - x)^{-n}$

$$= \left(1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots\right)\left(1 + (-n)\,(-x) + \frac{-n\,(-n-1)}{1.2.}(-x)^2 + \ldots\right)$$

$\therefore$ Coefficient of x in the given expansion $= n + n =$ **2*n*.**

4. $(1 + 2x + 3x^2 + 4x^3 + \ldots.)^{1/2} = [(1 - x)^{-2}]^{1/2}$

$= (1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots..$

$\Rightarrow$ coefficient of each term = 1

$\Rightarrow$ coefficient of $x^n =$ **1.**

5. $= (1 - 3x)^2\,(1 - 2x)^{-1}$

$$= (1 - 6x + 9x^2)\,(1 + (-1))\,(-2x) + \frac{(-1)\,(-1-1)}{1.2}\,(-2x)^2 + \frac{(-1)\,(-1-1)\,(-1-2)}{1.2.3}\,.\,(-2x)^3 + \frac{(-1)\,(-1-1)\,(-1-2)\,(-1-3)}{1.2.3.4}(-2x)^4 + \ldots.$$

$= (1 - 6x + 9x^2)\,(1 + 2x + 4x^2 + 8x^3 + 16x^4 + \ldots..)$

$\Rightarrow$ coefficient of x^4 in the given product

$= (1 \times 16) + (-6 \times 8) + (9 \times 4) = -48 + 36 + 16 =$ **4.**

6. $\dfrac{1}{(x-1)^2\,(x-2)} = \dfrac{1}{-2\,(1-x)^2\,(1-x/2)}$

$$= -\frac{1}{2}\left[(1-x)^{-2}\,\left(1-\frac{x}{2}\right)^{-1}\right]$$

[**Note:** The steps of taking 2 out from $(x - 2)$]

$$= -\frac{1}{2}\left[(1 + 2x + 3x^2 + \ldots..)\left(1 + \frac{x}{2} + \ldots..\right)\right]$$

$\therefore$ Coefficient of constant term $= -\dfrac{\mathbf{1}}{\mathbf{2}}$.

We have done it because expansion of $(x + a)^n$ when n is a –ve integer or fraction is valid only when $|a| < 1$.

$\therefore$ We write $(x+a)^n = a^n\left(1+\dfrac{x}{a}\right)^n$, where $\left|\dfrac{x}{a}\right| < 1$.

7. General term in the expansion of $(1 - 2x)^{-1/2}$

$$T_{r+1} = \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\ldots.\left(-\frac{1}{2}-r+1\right)}{r!}(-2x)^r$$

$$= \frac{(-1)^r\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\ldots..\left(\frac{2r-1}{2}\right)}{r!}(-1)^r\,2^r\,.x^r$$

$$= (-1)^{2r}\,\frac{1\,.\,3\,.\,5\,.\,7\,\ldots..\,(2r-1)}{r!\;2^r}\,.\,2^r\,.\,x^r$$

$$= \frac{1\,.\,3\,.\,5\,.\,7\,\ldots...\,(2r-1)}{r!}\,x^r$$

For this term to contain x^n, $r = n$

$$\therefore\quad T_{n+1} = \frac{1\,.\,3\,.\,5\,.\,7.\ldots...\,(2n-1)}{n!}\,x^n$$

$$= \frac{1\,.\,2\,.\,3\,.\,4\,.\,5.\ldots..\,(2n-2)\,(2n-1)\,(2n)}{2\,.\,4\,.\,6.\ldots..\,(2n-2)\,(2n)\,n!}\,x^n$$

$$= \frac{(2n)!}{2^n\,(1\,.\,2\,.\,3.\ldots..\,(n-1)\,.\,n)\,n!}\,x^n = \frac{(2n)!}{2^n.\,n!\,n!}\,x^n$$

$\therefore$ Required coefficient $= \dfrac{\mathbf{2n!}}{\mathbf{2^n\,(n!)^2}}$.

8. $\dfrac{1+2x}{(1-2x)^2} = (1 + 2x)\,(1 - 2x)^{-2}$

$$= (1 + 2x)\,(1 + -2(-2x)) + \frac{-2\,(-2-1)}{2!}(-2x)^2 + \ldots..\frac{-2\,(-2-1)\,(-2-2)\,\ldots..\,(-2-\overline{r-1}+1)}{(r-1)!}(-2x)^{r-1} + \frac{-2\,(-2-1)\,\ldots...\,(-2-r+1)}{r!}(-2x)^r + \ldots.\}$$

$= (1 + 2x)\,(1 + 2(2x) + 3\,(2x)^2 + \ldots.\; r\,(2x)^{r-1} + (r + 1)\,(2x)^r + \ldots.)$

$\therefore$ Coefficient of x^r in the given expansion

$= (r + 1)2^r + 2\,.\,r\,.\,2^{r-1} = (r + 1)\,2^r + r\,.\,2^r =$ **(2*r* + 1) . 2*^r*.**

Properties of Binomial Coefficients

SOLVED EXAMPLES

Ex. 1. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + ... + C_n x^n$, then find the values of

(*i*) $C_1 + C_2 + C_3 + ... + C_n$

(*ii*) $C_1 - C_2 + C_3 - ..... + (-1)^n C_n$

(*iii*) $\frac{C_1}{C_o} + \frac{2C_2}{C_1} + \frac{3C_3}{C_2} + ... + \frac{nC_n}{C_{n-1}}$

Sol. (*i*) $(1+x)^n = C_0 + C_1 x + C_2 x^2 + ... + C_n x^n$

Sum of binomial coefficients = 2^n

$\therefore\ C_0 + C_1 + C_2 + ... + C_n = 2^n \Rightarrow C_1 + C_2 + ... + C_n = 2^n - C_0 = 2^n - 1$

(*ii*) $(1+x)^n = C_0 + C_1 x + C_2 x^2 + ... + C_n x^n$

Put $x = -1$, in the above equation, we have

$0 = C_0 - C_1 + C_2 - C_3 + ... + (-1)^n C_n \Rightarrow C_0 - C_1 + C_2 - C_3 + ... + (-1)^n C_n = 0$

$\Rightarrow C_1 - C_2 + C_3 + ... - (-1)^n C_n = C_0 = \mathbf{1}$

(*iii*) $\frac{C_1}{C_0} + \frac{2C_2}{C_1} + \frac{3C_3}{C_2} + ... + \frac{nC_n}{C_{n-1}} = \frac{n}{1} + \frac{2n(n-1)}{2.1.n} + \frac{3n(n-1)(n-2)\times 2}{3.2.1.n(n-1)} + ... + \frac{n\times 1}{n}$

$= n + (n-1) + (n-2) + ... + 1 = 1 + 2 + 3 + ... + (n-2) + (n-1) + n$

$= \mathbf{\frac{n(n+1)}{2}}$ (Sum of an A.P.)

Ex. 2. If $(1 + x - 3x^2)^{10} = 1 + a_1x + a_2x^2 + ... + a_{20}x^{20}$, then find $a_2 + a_4 + a_6 + ... + a_{20}$. (***Kerala PET 2007***)

Sol. $(1 + x - 3x^2)^{10} = 1 + a_1x + a_2x^2 + ... + a_{20}x^{20}$ ...(*i*)

Putting $x = 1$ in (*i*), we get:

$1 + a_1 + a_2 + ... + a_{20} = (-1)^{10} = 1$ ...(*ii*)

Putting $x = -1$ in (*i*), we get

$1 - a_1 + a_2 - ..... + a_{20} = (-3)^{10} = 3^{10}$ ...(*iii*)

Adding (*ii*) and (*iii*), we get

$2(1 + a_2 + a_4 + ... + a_{20}) = 3^{10} + 1$

$\Rightarrow 2(a_2 + a_4 + ... + a_{20}) = 3^{10} - 1 \Rightarrow a_2 + a_4 + ... + a_{20} = \mathbf{\frac{3^{10}-1}{2}}$.

Ex. 3. Given that $(1+x)^n = C_0 + C_1 x + C_2 x^2 + ... + C_n x^n$ and $C_0 + C_1 + C_2 + C_3 + ... + C_n = 1024$, find the value of n and hence find the term in the expansion of $\left(x^2 + \frac{1}{x}\right)^n$ which contains x^{11}.

Sol. Sum of coefficients in the expansion of $(1+x)^n$

$= C_0 + C_1 + C_2 + ... + C_n = 2^n$

Given, $2^n = 1024 \Rightarrow 2^n = 2^{10} \Rightarrow n = 10.$

Now, general term T_{r+1} in the expansion of $\left(x^2 + \frac{1}{x}\right)^{10}$

$= {}^{10}C_r\,(x^2)^{10-r}\left(\frac{1}{x}\right)^r = {}^{10}C_r\, x^{20-3r}$

$\therefore$ For term containing x^{11}, $20 - 3r = 11 \Rightarrow 3r = 9 \Rightarrow r = 3$

$\therefore\ T_4 = T_{3+1} = {}^{10}C_3\, x^{20-9} = \frac{10\times 9\times 8}{3\times 2}\times x^{11} = \mathbf{120\, x^{11}}.$

Ex. 4. If $C_0, C_1, C_2, ..., C_n$ be the coefficients in the expansion of $(1 + x)^n$, then find the value of $C_0^2 + C_1^2 + C_2^2 + ... + C_n^2$.

Sol. $(1 + x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + ... + C_n x^n$ ...(*i*)

Also $(1 + x)^n = C_n x^n + C_{n-1} x^{n-1} + ... + C_2 x^2 + C_1 x + C_0$ ...(*ii*)

Multiplying both the sides of eqn. (*i*) and (*ii*), we have

$$(1 + x)^{2n} = (C_0 + C_1 x + C_2 x^2 + ... + C_{n-1} x^{n-1} + C_n x^n) \times (C_n x^n + C_{n-1} x^{n-1} + ... + C_2 x^2 + C_1 x + C_0)$$

Now equating coefficients of x^n on both the sides, we have

$$^{2n}C_n = C_0 C_n + C_1 C_{n-1} + C_2 C_{n-2} + ... + C_{n-1} C_1 + C_n C_0$$

$$\Rightarrow \quad ^{2n}C_n = C_0^2 + C_1^2 + C_2^2 + ... + (C_{n-1})^2 + C_n^2 \qquad (\because C_0 = C_n, C_1 = C_{n-1}, ...)$$

$$\Rightarrow C_0^2 + C_1^2 + C_2^2 + ... + C_n^2 = {}^{2n}C_n = \frac{\underline{|2n}}{(\underline{|n})^2} = \frac{(2n)!}{(n!)^2}.$$

Ex. 5. If $(1 + x)^n = C_0 + C_1 x + C_2 x^2 + ... + C_n x^n$, then find the value of $C_0 + \frac{1}{2}C_1 + \frac{1}{3}C_2 + ... + \frac{1}{(n+1)}C_n$.

Sol. $C_0 + \frac{1}{2}C_1 + \frac{1}{3}C_2 + ... + \frac{1}{n+1}C_n = {}^nC_0 + \frac{1}{2}{}^nC_1 + \frac{1}{3}{}^nC_2 + ... + \frac{1}{n+1}{}^nC_n$

$$= 1 + \frac{n}{2} + \frac{n(n-1)}{3.2} + ... + \frac{1}{n+1} = \frac{1}{n+1}\left[(n+1) + \frac{(n+1)n}{2} + \frac{(n+1)n(n-1)}{3.2} + ... + 1\right]$$

$$= \frac{1}{n+1}\left[{}^{n+1}C_1 + {}^{n+1}C_2 + ... + {}^{n+1}C_{n+1}\right] = \frac{1}{n+1}\left[{}^{n+1}C_0 + {}^{n+1}C_1 + {}^{n+1}C_2 + ... + {}^{n+1}C_{n+1} - {}^{n+1}C_0\right]$$

$$= \frac{1}{n+1}(2^{n+1} - 1). \qquad (\because \text{Sum of coefficients of } (1 + x)^n = 2^n)$$

PRACTICE SHEET–4
(Properties of Binomial Coefficients)

1. The value of $^{15}C_8 + {}^{15}C_9 - {}^{15}C_6 - {}^{15}C_7$ is

(*a*) –1 (*b*) 0 (*c*) 1 (*d*) None of these

(*Odisha JEE 2009*)

2. The value of $(^7C_0 + {}^7C_1) + (^7C_1 + {}^7C_2) + ... + (^7C_6 + {}^7C_7)$ is

(*a*) $2^8 - 1$ (*b*) $2^8 + 1$ (*c*) $2^8 - 2$ (*d*) 2^8

(*Kerala CEE 2008*)

3. If $n = 5$, then

$(^nC_0)^2 + (^nC_1)^2 + (^nC_2)^2 + ... + (^nC_5)^2$ is equal to

(*a*) 250 (*b*) 254 (*c*) 252 (*d*) 245

4. The value of $^{50}C_4 + \sum_{r=1}^{6} {}^{56-r}C_3$ is

(*a*) $^{56}C_4$ (*b*) $^{56}C_3$ (*c*) $^{55}C_3$ (*d*) $^{55}C_4$

(*AIEEE 2005*)

5. $\frac{^8C_0}{6} - {}^8C_1 + {}^8C_2 . 6 - {}^8C_3 . 6^2 + ... + {}^8C_8 . 6^7$ is equal to

(*a*) 0 (*b*) 6^7 (*c*) 6^8 (*d*) $\frac{5^8}{6}$

(*J&K CET 2009*)

6. The sum of the last eight coefficients in the expansion of $(1 + x)^{15}$ is **(*DCE 2007*)**

(*a*) 2^{16} (*b*) 2^{15} (*c*) 2^{14} (*d*) None of these

7. $^{20}C_4 + 2 . {}^{20}C_3 + {}^{20}C_2 - {}^{22}C_{18}$ is equal to

(*a*) 0 (*b*) 1242 (*c*) 7315 (*d*) 6345

(*RPET 2005*)

8. The sum of the series

$^{20}C_0 - {}^{20}C_1 + {}^{20}C_2 - {}^{20}C_3 + ... + {}^{20}C_{10}$ is

(*a*) $^{20}C_{10}$ (*b*) $\frac{1}{2}\,{}^{20}C_{10}$ (*c*) 0 (*d*) $^{20}C_{10}$

(*AIEEE 2007*)

9. In the expansion of $(1 + x)^{30}$, the sum of the coefficients of the odd powers of x is

(*a*) 2^{30} (*b*) 2^{31} (*c*) 0 (*d*) 2^{29}

10. If $(1 + x + x^2)^n = a_0 + a_1 x + a_2 x^2 + ... + a_{2n} x^{2n}$, then $a_0 - a_1 + a_2 - a_3 + ... + a_{2n}$ is equal to

(*a*) 0 (*b*) 1 (*c*) $2n$ (*d*) $^{2n}C_n$

11. If $(1 - x + x^2)^n = a_0 + a_1 x + a_2 x^2 + ... + a_{2n} x^{2n}$, then $a_0 + a_2 + a_4 + ... + a_{2n}$ equals

(*a*) $\frac{3^n - 1}{2}$ (*b*) $\frac{1 - 3^n}{2}$ (*c*) $3^n + \frac{1}{2}$ (*d*) $\frac{3^n + 1}{2}$

(*WBJEE 2010*)

12. If n is an odd positive integer and $(1 + x + x^2 + x^3)^n = \sum_{r=0}^{3n} a_r \cdot x^r$, then $a_0 - a_1 + a_2 - a_3 + \ldots - a_{3n}$ equals

(*a*) –1 (*b*) 1 (*c*) 4^n (*d*) 0

(***Karnataka CET 2011***)

13. If $(1 + x - 2x^2)^6 = 1 + a_1x + a_2x^2 + \ldots + a_{12}x^{12}$, then the value of $a_1 + a_3 + \ldots + a_{11}$ is

(*a*) 32 (*b*) – 32 (*c*) 64 (*d*) – 64

14. Let $(1 + x)^n = 1 + a_1x + a_2x^2 + \ldots + a_nx^n$. If a_1, a_2 and a_3 are in A.P., then the value of ***n*** is

(*a*) 4 (*b*) 5 (*c*) 6 (*d*) 7

(***BITSAT 2010***)

15. If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$, then the sum $C_1 + 2 \cdot C_2 + 3 \cdot C_3 + \ldots + n \cdot C_n$ equals

(*a*) 2^{n-1} (*b*) $(n - 1)\, 2^{n-1}$ (*c*) $n \cdot 2^{n-1}$ (*d*) $(n - 1)\, 2^n$

16. If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$, then $C_0 + 3 \cdot C_1 + 5 \cdot C_2 + \ldots + (2n + 1) \cdot C_n$ equals

(*a*) 2^{2n-1} (*b*) $(n + 1)\, 2^n$ (*c*) $n \cdot 2^{n+1}$ (*d*) $(n - 1)\, 2^{n-1}$

17. $\left(1+\frac{C_1}{C_0}\right)\left(1+\frac{C_2}{C_1}\right)\left(1+\frac{C_3}{C_2}\right)\ldots\left(1+\frac{C_n}{C_{n-1}}\right)$ is equal to

(*a*) $\frac{n+1}{n!}$ (*b*) $\frac{(n+1)^n}{(n-1)!}$ (*c*) $\frac{(n-1)^n}{n!}$ (*d*) $\frac{(n+1)^n}{n!}$

(***Kerala CEE 2013***)

18. If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$, then $C_0C_1 + C_1C_2 + \ldots + C_{n-1}C_n$ is equal to

(*a*) $\frac{(2n)!}{(n-1)!\,(n+1)!}$ (*b*) $\frac{(2n-1)!}{(n-1)!\,(n+1)!}$

(*c*) $\frac{(2n)!}{(n+2)!\,(n+1)!}$ (*d*) None of these

(***Odisha JEE 2008***)

ANSWERS

1. (*b*)	**2.** (*c*)	**3.** (*c*)	**4.** (*a*)	**5.** (*d*)	**6.** (*c*)	**7.** (*a*)	**8.** (*b*)	**9.** (*d*)	**10.** (*b*)
11. (*d*)	**12.** (*d*)	**13.** (*b*)	**14.** (*d*)	**15.** (*c*)	**16.** (*b*)	**17.** (*d*)	**18.** (*a*)		

HINTS AND SOLUTIONS

1. ${}^{15}C_8 + {}^{15}C_9 - {}^{15}C_6 - {}^{15}C_7$

$= {}^{15}C_8 + {}^{15}C_9 - {}^{15}C_9 - {}^{15}C_8$ $(\because {}^nC_r = {}^nC_{n-r})$

$= 0$

2. $({}^7C_0 + {}^7C_1) + ({}^7C_1 + {}^7C_2) + \ldots + ({}^7C_6 + {}^7C_7)$

$= {}^8C_1 + {}^8C_2 + {}^8C_3 + \ldots + {}^8C_7$ $(\because {}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1})$

$= {}^8C_1 + {}^8C_2 + {}^8C_3 + \ldots + {}^8C_7 + ({}^8C_0 + {}^8C_8) - ({}^8C_0 + {}^8C_8)$

$= {}^8C_0 + {}^8C_1 + {}^8C_2 + {}^8C_3 + \ldots + {}^8C_7 + {}^8C_8 - (2)$

$(\because {}^nC_0 = {}^nC_n = 1)$

$= \mathbf{2^8 - 2.}$ $(\because {}^nC_0 + {}^nC_2 + \ldots + {}^nC_n = 2^n)$

3. $({}^nC_0)^2 + ({}^nC_1)^2 + ({}^nC_2)^2 + \ldots + ({}^nC_5)^2$

$= ({}^5C_0)^2 + ({}^5C_1)^2 + ({}^5C_2)^2 + \ldots + ({}^5C_5)^2$

$= 1^2 + 5^2 + 10^2 + 10^2 + 5^2 + 1^2$

$= 1 + 25 + 100 + 100 + 25 + 1 = \mathbf{252.}$

4. ${}^{50}C_4 + \sum_{r=1}^{6} {}^{56-r}C_3$

$= {}^{50}C_4 + {}^{55}C_3 + {}^{54}C_3 + {}^{53}C_3 + {}^{52}C_3 + {}^{51}C_3 + {}^{50}C_3$

$= {}^{50}C_3 + {}^{50}C_4 + {}^{51}C_3 + {}^{52}C_3 + {}^{53}C_3 + {}^{54}C_3 + {}^{55}C_3$

$= {}^{51}C_4 + {}^{51}C_3 + {}^{52}C_3 + {}^{53}C_3 + {}^{54}C_3 + {}^{55}C_3$

$(\because {}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1})$

$= {}^{52}C_4 + {}^{52}C_3 + {}^{53}C_3 + {}^{54}C_3 + {}^{55}C_3$

$= {}^{53}C_4 + {}^{53}C_3 + {}^{54}C_3 + {}^{55}C_3 = {}^{54}C_4 + {}^{54}C_3 + {}^{55}C_3$

$= {}^{55}C_4 + {}^{55}C_3 = \mathbf{{}^{56}C_4.}$

5. $\frac{{}^8C_0}{6} - {}^8C_1 + {}^8C_2 \cdot 6 - {}^8C_3 \cdot 6^2 + \ldots + {}^8C_8 \cdot 6^7$

$= \frac{1}{6}\,[{}^8C_0 - 6 \cdot {}^8C_1 + 6^2 \cdot {}^8C_2 - 6^3 \cdot {}^8C_3 + \ldots + 6^8 \cdot {}^8C_8]$

$= \frac{1}{6}\left[(1-6)^8\right] = \frac{5^8}{6}$ $[\because (1-x)^n = {}^nC_0 - x \cdot {}^nC_1 + x^2 \cdot {}^nC_2 - x^3 \cdot {}^nC_3 + \ldots + x^n \cdot {}^nC_n]$

6. $(1+x)^{15} = {}^{15}C_0 + {}^{15}C_1 \cdot x + {}^{15}C_2 \cdot x^2 + {}^{15}C_3 \cdot x^3 + \ldots + {}^{15}C_n \cdot x^n$

As, sum of binomial coefficients $= 2^n$

$\therefore$ ${}^{15}C_0 + {}^{15}C_1 + {}^{15}C_2 + \ldots + {}^{15}C_{14} + {}^{15}C_{15} = 2^{15}$

$\Rightarrow$ ${}^{15}C_{15} + {}^{15}C_{14} + {}^{15}C_{13} + \ldots + {}^{15}C_8 + {}^{15}C_8 + {}^{15}C_{14} + {}^{15}C_{15} = 2^{15}$ $(\because {}^nC_r = {}^nC_{n-r})$

$\Rightarrow$ $2\,({}^{15}C_8 + {}^{15}C_9 + \ldots + {}^{15}C_{14} + {}^{15}C_{15}) = 2^{15}$

$\Rightarrow$ $({}^{15}C_8 + {}^{15}C_9 + \ldots + {}^{15}C_{15}) = 2^{14}$

$\Rightarrow$ Required sum = Sum of last eight coefficients = $\mathbf{2^{14}}$.

7. ${}^{20}C_4 + 2 \cdot {}^{20}C_3 + {}^{20}C_2 - {}^{22}C_{18}$

$= {}^{20}C_4 + {}^{20}C_3 + {}^{20}C_3 + {}^{20}C_2 - {}^{22}C_{18}$

$= {}^{21}C_4 + {}^{21}C_3 - {}^{22}C_{18}$ $(\because {}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1})$

$= {}^{22}C_4 - {}^{22}C_{18} = {}^{22}C_{18} - {}^{22}C_{18} = \mathbf{0}$

$(\because {}^nC_r = {}^nC_{n-r} \Rightarrow {}^{22}C_4 = {}^{22}C_{22-4} = {}^{22}C_{18})$

8. $(1+x)^{20} = {}^{20}C_0 + {}^{20}C_1\,x + {}^{20}C_2\,x^2 + {}^{20}C_3\,x^3 + \ldots + {}^{20}C_{19}\,x^{19} + {}^{20}C_{20}\,x^{20}$

On putting $x = -1$, we get

$0 = {}^{20}C_0 - {}^{20}C_1 + {}^{20}C_2 - {}^{20}C_3 + \ldots - {}^{20}C_{19} + {}^{20}C_{20}$

$\Rightarrow$ $0 = {}^{20}C_0 - {}^{20}C_1 + \ldots - {}^{20}C_9 + {}^{20}C_{10} - {}^{20}C_9 + {}^{20}C_8 \ldots + {}^{20}C_0$ $(\because {}^nC_r = {}^nC_{n-r})$

$\Rightarrow\ 0 = 2\,({}^{20}C_0 - {}^{20}C_1 + ... - {}^{20}C_9) + {}^{20}C_{10}$

$\Rightarrow\ 2\,({}^{20}C_0 - {}^{20}C_1 + ... - {}^{20}C_9) = -\,{}^{20}C_{10}$

$\Rightarrow\ 2\,({}^{20}C_0 - {}^{20}C_1 + ... - {}^{20}C_9 + {}^{20}C_{10}) = {}^{20}C_{10}$.

(Adding 2. ${}^{20}C_{10}$ on both the sides)

$\Rightarrow\ {}^{20}C_0 - {}^{20}C_1 + ... - {}^{20}C_9 + {}^{20}C_{10} = \frac{1}{2}\,{}^{20}C_{10}$.

9. Sum of the coefficients of odd powers of x in $(1 + x)^{30}$

$= C_1 + C_3 + C_5 + ... + C_{29} = 2^{30-1} = \mathbf{2^{29}}$.

10. Given, $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n}\,x^2$

Putting $x = -1$ in the above equation, we have

$(1 - 1 + 1)^n = a_0 - a_1 + a_2 - a_3 + ... + a_{2n}$

$\Rightarrow\ a_0 - a_1 + a_2 - a_3 + ... + a_{2n} = 1^n = \mathbf{1}$.

11. Given, $(1 - x + x^2)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n}\,x^{2n}$ ...(*i*)

Putting $x = 1$ in the given equation (*i*), we have

$1 = a_0 + a_1 + a_2 + a_3 + ... + a_{2n}$ ...(*ii*)

Putting $x = -1$ in the above given equation (*i*), we have

$(1 - (-1) + 1)^n = a_0 - a_1 + a_2 - a_3 + ... + a_{2n}$

$\Rightarrow\ 3^n = a_0 - a_1 + a_2 - a_3 + ... + a_{2n}$ ...(*iii*)

Adding eqns (*ii*) and (*iii*), we have

$1 + 3^n = 2(a_0 + a_2 + a_4 + ... + a_{2n})$

$\Rightarrow\ a_0 + a_2 + a_4 + ... + a_{2n} = \mathbf{\frac{3^n + 1}{2}}$.

12. Given, $(1 + x + x^2 + x^3)^n = \sum_{r=0}^{3n} a_r\,x^r$

$\Rightarrow\ (1 + x + x^2 + x^3)^n = a_0 + a_1x + a_2x^2 + ... + a_{3n}\,x^{3n}$

Putting $x = -1$ in the above given equation, we have

$0 = a_0 - a_1 + a_2 - a_3 + ... - a_{3n}$

13. Given, $(1 + x - 2x^2)^6 = 1 + a_1x + a_2x^2 + ... + a_{12n}\,x^{12}$ ...(*i*)

Putting $x = 1$, in (*i*) we have

$(1 + 1 - 2 \times 1)^6 = 1 + a_1 + a_2 + ... + a_{12}$

$\Rightarrow\ 1 + a_1 + a_2 + ... + a_{12} = 0$ ...(*ii*)

Putting $x = -1$, in (*i*), we have

$(1 + (-1) - 2(-1)^2)^6 = 1 - a_1 + a_2 - ..... + a_{12}$

$\Rightarrow\ 1 - a_1 + a_2 - ..... + a_{12} = (-2)^6 = 64$ ...(*iii*)

Subtracting eqn. (*ii*) from eqn. (*i*), we have

$2\,(a_1 + a_3 + a_5 + ... a_{11}) = -\,64$

$\Rightarrow\ a_1 + a_3 + a_5 + ... a_{11} = \mathbf{-\,32}$.

14. $(1 + x)^n = {}^nC_0 + {}^nC_1\,x + {}^nC_2\,x^2 + {}^nC_3\,x^3 + ... + {}^nC_n\,x^n$

$= 1 + a_1x + a_2x^2 + a_3\,x^3 + ... + a_n\,x^n$

Given, a_1, a_2, a_3 are in A.P.

$\Rightarrow\ {}^nC_1, {}^nC_2, {}^nC_3$, are in A.P $\Rightarrow$ $2.\,{}^nC_2 = {}^nC_1 + {}^nC_3$

$\Rightarrow\ 2 \times \frac{n\,!}{2!\,(n-2)!} = \frac{n\,!}{(n-1)!\,1!} + \frac{n\,!}{(n-3)!\,3!}$

$\Rightarrow\ \frac{2 \times n\,(n-1)}{2} = n + \frac{n\,(n-1)\,(n-2)}{3 \times 2}$

$\Rightarrow\ 6\,(n^2 - n) = 6n + n\,(n^2 - 3n + 2)$

$\Rightarrow\ 6n^2 - 6n = 6n + n^3 - 3n^2 + 2n \ \Rightarrow\ n^3 - 9n^2 + 14n = 0$

$\Rightarrow\ n\,(n^2 - 9n + 14) = 0 \ \Rightarrow\ n\,(n-2)\,(n-7) = 0$

$\Rightarrow\ n = 2$ or **7.**

Rejecting $n = 2$ as there are only three terms in the expansion of $(1 + x)^2$, we have $n = 7$.

15. $(1 + x)^n = {}^nC_0 + {}^nC_1\,x + {}^nC_2\,x^2 + ... + {}^nC_n\,x^n$

$= C_0 + C_1x + C_2\,x^2 + ... + C_n\,x^n$

$\therefore\ C_1 + 2.C_2 + 3.C_3 + ... + n.C_n$

$= n + 2 \times \frac{n\,(n-1)}{2!} + 3 \times \frac{n(n-1)\,(n-2)}{3!} + ... + n$

$= n\left[1 + (n-1) + \frac{(n-1)\,(n-2)}{2!} + ... + 1\right]$

$= n\,[{}^{n-1}C_0 + {}^{n-1}C_1 + {}^{n-1}C_2 + ... + {}^{n-1}C_{n-1}] = \mathbf{n\,.\,2^{n-1}}$.

16. $C_0 + 3C_1 + 5C_2 + ...\,(2n + 1)\,C_n$

$= (C_0 + C_1 + C_2 + ... + C_n) + (2C_1 + 4C_2 + ... + 2n\,C_n)$

$= 2^n + 2\,(C_1 + 2C_2 + 3.C_3 + ... + n\,C_n)$

$= 2^n + 2 \times n\,.\,2^{n-1}$ (Proved in Q.15)

$= 2^n + n\,.\,2^n = \mathbf{2^n\,(n + 1)}$.

17. $\left(1 + \frac{C_1}{C_0}\right)\left(1 + \frac{C_2}{C_1}\right)\left(1 + \frac{C_3}{C_2}\right)....\left(1 + \frac{C_n}{C_{n-1}}\right)$

$= \left(1 + \frac{n}{1}\right)\left(1 + \frac{n\,(n-1)}{2n}\right)\left(1 + \frac{2n\,(n-1)\,(n-2)}{6n\,(n-1)}\right)....\left(1 + \frac{1}{n}\right)$

$= \left(\frac{n+1}{1}\right)\left(\frac{n+1}{2}\right)\left(\frac{n+1}{3}\right)....\left(\frac{n+1}{n}\right) = \mathbf{\frac{(n+1)^n}{n!}}$.

18. Given, $(1 + x)^n = C_0 + C_1x + C_2\,x^2 + ... + C_n\,x^n$ ...(*i*)

Also, $(x + 1)^n = C_0\,x^n + C_1x^{n-1} + C_2\,x^{n-2} + ... + C_{n-1}\,x + C_n$ ...(*ii*)

Multiplying (*i*) by (*ii*) we get

$(1 + x)^{2n} = (C_0 + C_1x + C_2\,x^2 + ... + C_{n-1}\,x^{n-1} + C_n\,x^n)$
$\times\,(C_0\,x^n + C_1x^{n-1} + C_2\,x^{n-2} + ... + C_{n-1}\,x + C_n)$

Now equating the coefficient of x^{n-1} on both the sides, we get

$C_0C_1 + C_1C_2 + ... + C_{n-1}C_n = {}^{2n}C_{n-1} = \mathbf{\frac{(2n)!}{(n-1)!\,(n+1)!}}$.

SELF ASSESSMENT SHEET

1. If the rth term in the expansion of $\left(\frac{x}{3} - \frac{2}{x^2}\right)^{10}$ contains x^4, then r is equal to

(*a*) 3 (*b*) 0 (*c*) – 3 (*d*) 5

2. If the coefficient of rth and $(r + 1)$th terms in the expansion of $(3 + 7x)^{29}$ are equal, then r equals

(*a*) 15 (*b*) 21 (*c*) 14 (*d*) None of these

3. The sum of coefficients of the expansion $\left(\frac{1}{x}+2x\right)^n$ is 6561.

The coefficient of term independent of x is

(*a*) $16.\,{}^8C_4$ (*b*) ${}^8C_4$ (*c*) ${}^8C_5$ (*d*) None of these

(*BCECE 2009*)

4. The value of $\left(\frac{{}^{50}C_0}{1}+\frac{{}^{50}C_2}{3}+\frac{{}^{50}C_4}{5}+....+\frac{{}^{50}C_{50}}{51}\right)$ is

(*a*) $\frac{2^{50}}{51}$ (*b*) $\frac{2^{50}-1}{51}$ (*c*) $\frac{2^{51}-1}{50}$ (*d*) $\frac{2^{50}-1}{50}$

(*Kerala PET 2007*)

5. ${}^{47}C_4+\sum_{r=1}^{5}{}^{52-r}C_3$ is equal to

(*a*) ${}^{47}C_6$ (*b*) ${}^{52}C_5$ (*c*) ${}^{52}C_4$ (*d*) None of these

6. The coefficent of x^5 in $(1+2x+3x^2+....)^{3/2}$ is

(*a*) 19 (*b*) 20 (*c*) 21 (*d*) 22

7. If $|x|<|$, then the coefficient of x^n in the expansion of $(1+x+x^2+x^3+.....)^2$ is

(*a*) $n-1$ (*b*) n (*c*) $n+1$ (*d*) $n+2$

8. What is the coefficient of x^5 in the expansion of $(1-2x+3x^2-4x^3+.....\infty)^{-5}$.

(*a*) $\frac{10!}{(5!)^2}$ (*b*) 5^{-5} (*c*) 5^5 (*d*) $\frac{10!}{6!\,4!}$

ANSWERS

1. (*a*) **2.** (*b*) **3.** (*a*) **4.** (*a*) **5.** (*c*) **6.** (*c*) **7.** (*c*) **8.** (*a*)

HINTS AND SOLUTIONS

1. General Term $= T_{r+1} = {}^nC_r\,(x)^{n-r}\,(a)^r$ in the expansion of $(x+a)^n$.

$\therefore$ T_r in the expansion of $\left(\frac{x}{3}-\frac{2}{x^2}\right)^{10}$

$= {}^{10}C_{r-1}\left(\frac{x}{3}\right)^{10-(r-1)}\left(-\frac{2}{x^2}\right)^{r-1}$

$= {}^{10}C_{r-1}\,(x)^{13-3r}\,(3)^{-11+r}\,(-1)^r\,.\,(2)^{r-1}$

For x^4, $13-3r=4 \Rightarrow 3r=9 \Rightarrow r=$ **3.**

2. In the expansion of $(3+7x)^{29}$

$T_{r+1} = {}^{29}C_r(3)^{29-r}(7x)^r = {}^{29}C_r\,3^{29-r}\,.\,7^r\,.\,x^r$

$\therefore$ Coefficient of $(r+1)$th term $= {}^{29}C_r\,3^{29-r}\,7^r$

$\therefore$ Coefficient of rth term $= {}^{29}C_{r-1}\,3^{29-(r-1)}\,7^{r-1}$

$= {}^{29}C_{r-1}\,3^{30-r}\,7^{r-1}$

Given, ${}^{29}C_r\times 3^{29-r}\times 7^r = {}^{29}C_{r-1}\times 3^{30-r}\times 7^{r-1}$

$\Rightarrow \frac{{}^{29}C_r}{{}^{29}C_{r-1}}=\frac{3}{7} \Rightarrow \frac{30-r}{r}=\frac{3}{7}$

$\Rightarrow 210-7r=3r \Rightarrow 210=10r \Rightarrow r=$ **21.**

3. Sum of the coefficients of the expansion $\left(\frac{1}{x}+2x\right)^n = 6561$

Putting $x=1$,

$(1+2)^n = 6561 \Rightarrow 3^n = 3^8 \Rightarrow n=8$

$\therefore$ T_{r+1} in the expansion of $\left(\frac{1}{x}+2x\right)^8$

$= {}^8C_r\left(\frac{1}{x}\right)^{8-r}(2x)^r = {}^8C_r\,2^r\,x^{2r-8}$

Since this term is independent of x, $2r-8=0 \Rightarrow r=4$.

$\therefore$ Reqd. term $= {}^8C_4\,.\,2^4 =$ **16 . ${}^8C_4$.**

4. $\frac{{}^{50}C_0}{1}+\frac{{}^{50}C_2}{3}+\frac{{}^{50}C_4}{5}+......+\frac{{}^{50}C_{50}}{51}$

$=\frac{1}{1}+\frac{50\times 49}{3\times 2!}+\frac{50\times 49\times 48\times 47}{5\times 4!}+.....+\frac{1}{51}\Big)$

$=\frac{1}{51}\left[51+\frac{51\times 50\times 49}{3!}+\frac{51\times 50\times 49\times 48\times 47}{5!}+.....\right.$

$=\frac{1}{51}\left[{}^{51}C_1+{}^{51}C_3+{}^{51}C_5+.....\right]=\frac{1}{51}\times 2^{51-1}=\frac{2^{50}}{51}$

($\because$ Sum of odd coefficient $= 2^{n-1}$)

5. ${}^{47}C_4+\sum_{r=1}^{5}{}^{52-r}C_3$

$= {}^{51}C_3+{}^{50}C_3+{}^{49}C_3+{}^{48}C_3+{}^{47}C_3+{}^{47}C_4$

$= {}^{51}C_3+{}^{50}C_3+{}^{49}C_3+{}^{48}C_3+{}^{48}C_4$

($\because {}^nC_r+{}^nC_{r+1}={}^{n+1}C_{r+1}$)

$= {}^{51}C_3+{}^{50}C_3+{}^{49}C_3+{}^{49}C_4 = {}^{51}C_3+{}^{50}C_3+{}^{50}C_4$

$= {}^{51}C_3+{}^{51}C_4 =$ **${}^{52}C_4$.**

6. $(1+2x+3x^2+.....)^{3/2} = ((1-x)^{-2})^{3/2} = (1-x)^{-3}$

$= 1+3x+\frac{3.4}{2!}x^2+\frac{3.4.5}{3!}x^3+\frac{3.4.5.6}{4!}x^4+\frac{3.4.5.6.7}{5!}x^5+....$

$\therefore$ Coefficient of $x^5 = \frac{3.4.5.6.7}{5!} =$ **21.**

7. $(1+x+x^2+x^3+.....)^2 = \{(1-x)^{-1}\}^2 = (1-x)^{-2}$

$= 1+2x+3x^2+4x^3+.....+(n+1)x^n+.....$

$\therefore$ Coefficient of x^n in this expansion = **$(n+1)$.**

8. $(1-2x+3x^2-4x^3+.....\infty)^{-5} = \{(1+x)^{-2}\}^{-5} = (1+x)^{10}$

$\therefore$ Coefficient of $x^5 = {}^{10}C_5 = \frac{10!}{5!\,5!} = \mathbf{\frac{10!}{(5!)^2}}$.

($\because (1+x)^n = 1+{}^nC_1\,x+{}^nC_2\,x^2+{}^nC_3\,x^2+....$)