CHAPTER 3

Trigonometric Ratio and Identities

RELATION BETWEEN SYSTEM OF MEASUREMENT OF ANGLES

$$\frac{D}{90}=\frac{G}{100}=\frac{2C}{\pi}$$

$$1 \text{ Radian} = \frac{180}{\pi} \text{ degree} \approx 57^\circ\ 17'\ 15'' \text{ (approximately)}$$

$$1 \text{ degree} = \frac{\pi}{180} \text{ radian} \approx 0.0175 \text{ radian}$$

BASIC TRIGONOMETRIC IDENTITIES

(*a*) $\sin^2\theta + \cos^2\theta = 1$ or $\sin^2\theta = 1 - \cos^2\theta$ or $\cos^2\theta = 1 - \sin^2\theta$

(*b*) $\sec^2\theta - \tan^2\theta = 1$ or $\sec^2\theta = 1 + \tan^2\theta$ or $\tan^2\theta = \sec^2\theta - 1$

(*c*) If $\sec\theta + \tan\theta = \text{k} \Rightarrow \sec\theta - \tan\theta = \frac{1}{k} \Rightarrow 2\sec\theta = \text{k} + \frac{1}{k}$

(*d*) $\text{cosec}^2\theta - \cot^2\theta = 1$ or $\text{cosec}^2\theta = 1 + \cot^2\theta$ or $\cot^2\theta = \text{cosec}^2\theta - 1$

(*e*) If $\text{cosec}\,\theta + \cot\theta = k \Rightarrow \text{cosec}\,\theta - \cot\theta = \frac{1}{k} \Rightarrow 2\,\text{cosec}\,\theta = k + \frac{1}{k}$

TRIGONOMETRIC FUNCTIONS OF ALLIED ANGLES

(*a*) $\sin(2n\pi + \theta) = \sin\theta$, $\cos(2n\pi + \theta) = \cos\theta$, where $n \in I$.

(*b*) $\sin(-\theta) = -\sin\theta$ $\qquad$ $\cos(-\theta) = \cos\theta$

$\sin(90^\circ - \theta) = \cos\theta$ $\qquad$ $\cos(90^\circ - \theta) = \sin\theta$

$\sin(90° + \theta) = \cos\theta$	$\cos(90° + \theta) = -\sin\theta$
$\sin(180° - \theta) = \sin\theta$	$\cos(180° - \theta) = -\cos\theta$
$\sin(180° + \theta) = -\sin\theta$	$\cos(180° + \theta) = -\cos\theta$
$\sin(270° - \theta) = -\cos\theta$	$\cos(270° - \theta) = -\sin\theta$
$\sin(270° + \theta) = -\cos\theta$	$\cos(270° + \theta) = \sin\theta$

Note:

(i) $\sin n\pi = 0$; $\cos n\pi = (-1)^n$; $\tan n\pi = 0$ where $n \in I$.

(ii) $\sin(2n + 1)\dfrac{\pi}{2} = (-1)^n$; $\cos(2n + 1)\dfrac{\pi}{2} = 0$ where $n \in I$.

IMPORTANT TRIGONOMETRIC FORMULA

(i) $\sin(A + B) = \sin A \cos B + \cos A \sin B$.

(ii) $\sin(A - B) = \sin A \cos B - \cos A \sin B$.

(iii) $\cos(A + B) = \cos A \cos B - \sin A \sin B$.

(iv) $\cos(A - B) = \cos A \cos B + \sin A \sin B$.

(v) $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

(vi) $\tan(A + B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

(vii) $\cot(A + B) = \dfrac{\cot B \cot A - 1}{\cot B + \cot A}$

(viii) $\cot(A - B) = \dfrac{\cot B \cot A + 1}{\cot B - \cot A}$

(ix) $2 \sin A \cos B = \sin(A + B) + \sin(A - B)$.

(x) $2 \cos A \sin B = \sin(A + B) - \sin(A - B)$.

(xi) $2 \cos A \cos B = \cos(A + B) + \cos(A - B)$.

(xii) $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$.

(xiii) $\sin C + \sin D = 2 \sin\left(\dfrac{C + D}{2}\right) \cos\left(\dfrac{C - D}{2}\right)$

(*xiv*) $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

(*xv*) $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(*xvi*) $\cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$

(*xvii*) $\sin 2\theta = 2\sin\theta\cos\theta = \frac{2\tan\theta}{1+\tan^2\theta}$

(*xviii*) $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$

(*xix*) $1 + \cos 2\theta = 2\cos^2\theta$ or $\cos\theta = \pm\sqrt{\frac{1+\cos 2\theta}{2}}$

(*xx*) $1 - \cos 2\theta = 2\sin^2\theta$ or $\sin\theta = \pm\sqrt{\frac{1-\cos 2\theta}{2}}$

(*xxi*) $\tan\theta = \frac{1-\cos 2\theta}{\sin 2\theta} = \frac{\sin 2\theta}{1+\cos 2\theta} = \pm\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}$

(*xxii*) $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

(*xxiii*) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

(*xxiv*) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

(*xxv*) $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$

(*xxvi*) $\sin^2 A - \sin^2 B = \sin(A+B)\cdot\sin(A-B) = \cos^2 B - \cos^2 A$.

(*xxvii*) $\cos^2 A - \sin^2 B = \cos(A+B)\cdot\cos(A-B)$.

(*xxviii*) $\sin(A+B+C)$

$= \sin A\cos B\cos C + \sin B\cos A\cos C + \sin C\cos A\cos B$

$\quad - \sin A \sin \mathrm{B} \sin \mathrm{C}$

$= \Sigma \sin A\cos B\cos C - \Pi \sin A$

$= \cos A\cos B\cos C\,[\tan A + \tan B + \tan C - \tan A\tan B\tan C]$

(*xxix*) $\cos(A + B + C)$

$$= \cos A \cos B \cos C - \sin A \sin B \cos C - \sin A \cos B \sin C - \cos A \sin B \sin C$$

$$= \Pi \cos A - \Sigma \sin A \sin B \cos C$$

$$= \cos A \cos B \cos C\,[1 - \tan A \tan B - \tan B \tan C - \tan C \tan A]$$

(*xxx*) $\tan(A + B + C)$

$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A} = \frac{S_1 - S_3}{1 - S_2}$$

(*xxxi*) $\sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \ldots \sin(\alpha + \overline{n-1}\beta)$

$$= \frac{\sin\left\{\alpha + \left(\frac{n-1}{2}\right)\beta\right\}\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

(*xxxii*) $\cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \ldots + \cos(\alpha + \overline{n-1}\beta)$

$$= \frac{\cos\left\{\alpha + \left(\frac{n-1}{2}\right)\beta\right\}\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

VALUES OF SOME T-RATIOS FOR ANGLES 18°, 36°, 15°, 22.5°, 67.5° ETC.

(*a*) $\sin 18° = \frac{\sqrt{5} - 1}{4} = \cos 72° = \sin\frac{\pi}{10}$

(*b*) $\cos 36° = \frac{\sqrt{5} + 1}{4} = \sin 54° = \cos\frac{\pi}{5}$

(*c*) $\sin 15° = \frac{\sqrt{3} - 1}{2\sqrt{2}} = \cos 75° = \sin\frac{\pi}{12}$

(*d*) $\cos 15° = \frac{\sqrt{3} + 1}{2\sqrt{2}} = \sin 75° = \cos\frac{\pi}{12}$

(*e*) $\tan\frac{\pi}{12} = 2 - \sqrt{3} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = \cot\frac{5\pi}{12}$

(*f*) $\tan\dfrac{5\pi}{12} = 2+\sqrt{3} = \dfrac{\sqrt{3}+1}{\sqrt{3}-1} = \cot\dfrac{\pi}{12}$

(*g*) $\tan(22.5°) = \sqrt{2}-1 = \cot(67.5°) = \cot\dfrac{3\pi}{8} = \tan\dfrac{\pi}{8}$

(*h*) $\tan(67.5°) = \sqrt{2}+1 = \cot(22.5°)$

MAXIMUM & MINIMUM VALUES OF TRIGONOMETRIC EXPRESSIONS

(*a*) $a\cos\theta + b\sin\theta$ will always lie in the interval $\left[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}\right]$

i.e. the maximum and minimum values are $\sqrt{a^2+b^2}, -\sqrt{a^2+b^2}$ respectively.

(*b*) Minimum value of $a^2\tan^2\theta + b^2\cot^2\theta = 2ab$, where $a, b > 0$.

(*c*) $-\sqrt{a^2+b^2+2ab\cos(\alpha-\beta)} \le a\cos(\alpha+\theta) + b\cos(\beta+\theta)$

$\le \sqrt{a^2+b^2+2ab\cos(\alpha-\beta)}$ where α and β are known angles.

(*d*) Minimum value of $a^2\cos^2\theta + b^2\sec^2\theta$ is either $2ab$ or a^2+b^2, if for some real θ equation $a\cos\theta = b\sec\theta$ is true or not true $\{a, b > 0\}$.

(*e*) Minimum value of $a^2\sin^2\theta + b^2\operatorname{cosec}^2\theta$ is either $2ab$ or a^2+b^2, if for some real θ equation $a\sin\theta = b\operatorname{cosec}\theta$ is true or not true $\{a, b > 0\}$.

IMPORTANT RESULTS

(*a*) $\sin\theta\sin(60°-\theta)\sin(60°+\theta) = \dfrac{1}{4}\sin 3\theta$

(*b*) $\cos\theta\cos(60°-\theta)\cos(60°+\theta) = \dfrac{1}{4}\cos 3\theta$

(*c*) $\tan\theta\tan(60°-\theta)\tan(60°+\theta) = \tan 3\theta$

(*d*) $\cot\theta\cot(60°-\theta)\cot(60°+\theta) = \cot 3\theta$

(*e*) (*i*) $\sin^2\theta + \sin^2(60°+\theta) + \sin^2(60°-\theta) = \dfrac{3}{2}$

(*ii*) $\cos^2\theta + \cos^2(60°+\theta) + \cos^2(60°-\theta) = \dfrac{3}{2}$

(*e*) (*i*) If $\tan A + \tan B + \tan C = \tan A\tan B\tan C$,

then $A + B + C = n\pi, n \in I$.

(*ii*) If $\tan A\tan B + \tan B\tan C + \tan C\tan A = 1$,

then $A + B + C = (2n+1)\dfrac{\pi}{2}, n \in I$.

(*g*) $\cos\theta \cos 2\theta \cos 4\theta \ldots \cos(2^{n-1}\theta) = \dfrac{\sin(2^n \theta)}{2^n \sin\theta}$

(*h*) $\cot A - \tan A = 2 \cot 2A$.

CONDITIONAL IDENTITIES

If $A + B + C = 180°$, then

(*a*) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(*b*) $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

(*c*) $\tan \dfrac{A}{2} \tan \dfrac{B}{2} + \tan \dfrac{B}{2} \tan \dfrac{C}{2} + \tan \dfrac{C}{2} \tan \dfrac{A}{2} = 1$

(*d*) $\cot \dfrac{A}{2} + \cot \dfrac{B}{2} + \cot \dfrac{C}{2} = \cot \dfrac{A}{2} \cot \dfrac{B}{2} \cot \dfrac{C}{2}$

(*e*) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

(*f*) $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$

(*g*) $\sin A + \sin B + \sin C = 4 \cos \dfrac{A}{2} \cos \dfrac{B}{2} \cos \dfrac{C}{2}$

(*h*) $\cos A + \cos B + \cos C = 1 + 4 \sin \dfrac{A}{2} \sin \dfrac{B}{2} \sin \dfrac{C}{2}$

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