

Limits and Derivatives

Question1

Let for a differentiable function $f : (0, \infty) \rightarrow \mathbb{R}$, $f(x) - f(y) \geq \log_e \left(\frac{x}{y} \right) + x - y$, $\forall x, y \in (0, \infty)$. Then $\sum_{n=1}^{20} f' \left(\frac{1}{n^2} \right)$ is equal to ____

[27-Jan-2024 Shift 1]

Answer: 2890

Solution:

$$f(x) - f(y) \geq \ln x - \ln y + x - y$$

$$\frac{f(x) - f(y)}{x - y} \geq \frac{\ln x - \ln y}{x - y} + 1$$

Let $x > y$

$$\lim_{y \rightarrow x} f'(x^-) \geq \frac{1}{x} + 1 \dots (1)$$

Let $x < y$

$$\lim_{y \rightarrow x} f'(x^+) \leq \frac{1}{x} + 1 \dots (2)$$

$$f^l(x^-) = f^l(x^+)$$

$$f^l(x) = \frac{1}{x} + 1$$

$$f' \left(\frac{1}{x^2} \right) = x^2 + 1$$

$$\sum_{x=1}^{20} (x^2 + 1) = \sum_{x=1}^{20} x^2 + 20$$

$$= \frac{20 \times 21 \times 41}{6} + 20$$

$$= 2890$$

Question2

Let $f(x) = x^3 + x^2 f'(1) + x f''(2) + f'''(3)$, $x \in \mathbb{R}$. Then $f'(10)$ is equal to ____

[27-Jan-2024 Shift 1]

Answer: 202

Solution:

$$f(x) = x^3 + x^2 \cdot f'(1) + x \cdot f''(2) + f'''(3)$$

$$f'(x) = 3x^2 + 2xf'(1) + f''(2)$$

$$f''(x) = 6x + 2f'(1)$$

$$f'''(x) = 6$$

$$f'(1) = -5, f''(2) = 2, f'''(3) = 6$$

$$f(x) = x^3 + x^2 \cdot (-5) + x \cdot (2) + 6$$

$$f'(x) = 3x^2 - 10x + 2$$

$$f'(10) = 300 - 100 + 2 = 202$$

Question3

Suppose

$$f(x) = \frac{(2^x + 2^{-x}) \tan x \sqrt{\tan^{-1}(x^2 - x + 1)}}{(7x^2 + 3x + 1)^3}$$

Then the value of $f'(0)$ is equal to

[29-Jan-2024 Shift 1]

Options:

A.

π

B.

0

C.

$\sqrt{\pi}$

D.

$\pi/2$

Answer: C

Solution:

$$\begin{aligned} f'(0) &= \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(2^h + 2^{-h}) \tan h \sqrt{\tan^{-1}(h^2 - h + 1)} - 0}{(7h^2 + 3h + 1)^3 h} \\ &= \sqrt{\pi} \end{aligned}$$

Question4

Let $y = \log_e \left(\frac{1-x^2}{1+x^2} \right)$, $-1 < x < 1$. Then at $x = \frac{1}{2}$, the value of $225(y' - y'')$ is equal to

[29-Jan-2024 Shift 2]

Options:

A.

732

B.

746

C.

742

D.

736

Answer: D

Solution:

$$y = \log_e \left(\frac{1-x^2}{1+x^2} \right)$$

$$\frac{dy}{dx} = y' = \frac{-4x}{1-x^4}$$

Again,

$$\frac{d^2y}{dx^2} = y'' = \frac{-4(1+3x^4)}{(1-x^4)^2}$$

Again

$$y' - y'' = \frac{-4x}{1-x^4} + \frac{4(1+3x^4)}{(1-x^4)^2}$$

$$\text{at } x = \frac{1}{2},$$

$$y' - y'' = \frac{736}{225}$$

$$\text{Thus } 225(y' - y'') = 225 \times \frac{736}{225} = 736$$

Question5

Let $g(x)$ be a linear function and $f(x) = \begin{cases} g(x) & , x \leq 0 \\ \left(\frac{1+x}{2+x} \right)^{\frac{1}{x}} & , x > 0 \end{cases}$ is continuous at $x = 0$. If $f'(1) = f'(-1)$, then the value of $g(3)$ is

[31-Jan-2024 Shift 1]

Options:

A.

$$\frac{1}{3} \log_e \left(\frac{4}{9e^{1/3}} \right)$$

B.

$$\frac{1}{3} \log_e \left(\frac{4}{9} \right) + 1$$

C.

$$\log_e \left(\frac{4}{9} \right) - 1$$

D.

$$\log_e \left(\frac{4}{9e^{1/3}} \right)$$

Answer: D

Solution:

Let $g(x) = ax + b$

Now function $f(x)$ is continuous at $x = 0$

$$\therefore \lim_{x \rightarrow 0^-} f(x) = f(0)$$

$$\Rightarrow \lim_{x \rightarrow 0} \left(\frac{1+x}{2+x} \right)^{\frac{1}{x}} = b$$

$$\Rightarrow 0 = b$$

$$\therefore g(x) = ax$$

Now, for $x > 0$

$$\begin{aligned} f'(x) &= \frac{1}{x} \cdot \left(\frac{1+x}{2+x} \right)^{\frac{1}{x}-1} \cdot \frac{1}{(2+x)^2} \\ &+ \left(\frac{1+x}{2+x} \right)^{\frac{1}{x}} \cdot \ln \left(\frac{1+x}{2+x} \right) \cdot \left(-\frac{1}{x^2} \right) \end{aligned}$$

$$\therefore f'(1) = \frac{1}{9} - \frac{2}{3} \cdot \ln \left(\frac{2}{3} \right)$$

$$\text{And } f(-1) = g(-1) = -a$$

$$\therefore a = \frac{2}{3} \ln \left(\frac{2}{3} \right) - \frac{1}{9}$$

$$\therefore g(3) = 2 \ln \left(\frac{2}{3} \right) - \frac{1}{3}$$

$$= \ln \left(\frac{4}{9 \cdot e^{1/3}} \right)$$

Question6

If $a = \lim_{x \rightarrow 0} \frac{\sqrt{1 + \sqrt{1 + x^4}} - \sqrt{2}}{x^4}$ and $b = \lim_{x \rightarrow 0} \frac{\sin^2 x}{\sqrt{2} - \sqrt{1 + \cos x}}$, then the value of ab^3 is :

[27-Jan-2024 Shift 1]

Options:

A.

36

B.

32

C.

25

D.

30

Answer: B

Solution:

$$\begin{aligned} a &= \lim_{x \rightarrow 0} \frac{\sqrt{1 + \sqrt{1 + x^4}} - \sqrt{2}}{x^4} \\ &= \lim_{x \rightarrow 0} \frac{\sqrt{1 + x^4} - 1}{x^4 (\sqrt{1 + \sqrt{1 + x^4}} + \sqrt{2})} \\ &= \lim_{x \rightarrow 0} \frac{x^4}{x^4 (\sqrt{1 + \sqrt{1 + x^4}} + \sqrt{2}) (\sqrt{1 + x^4} + 1)} \end{aligned}$$

$$\text{Applying limit } a = \frac{1}{4\sqrt{2}}$$

$$\begin{aligned} b &= \lim_{x \rightarrow 0} \frac{\sin^2 x}{\sqrt{2} - \sqrt{1 + \cos x}} \\ &= \lim_{x \rightarrow 0} \frac{(1 - \cos^2 x)(\sqrt{2} + \sqrt{1 + \cos x})}{2 - (1 + \cos x)} \\ b &= \lim_{x \rightarrow 0} (1 + \cos x)(\sqrt{2} + \sqrt{1 + \cos x}) \end{aligned}$$

$$\text{Applying limits } b = 2(\sqrt{2} + \sqrt{2}) = 4\sqrt{2}$$

$$\text{Now, } ab^3 = \frac{1}{4\sqrt{2}} \times (4\sqrt{2})^3 = 32$$

Question7

If $\lim_{x \rightarrow 0} \frac{3 + a \sin x + \beta \cos x + \log_e(1 - x)}{3 \tan^2 x} = \frac{1}{3}$, then $2\alpha - \beta$ is equal to :

[27-Jan-2024 Shift 2]

Options:

A.

2

B.

7

C.

5

D.

1

Answer: C

Solution:

$$\lim_{x \rightarrow 0} \frac{3 + \alpha \sin x + \beta \cos x + \log_e(1-x)}{3 \tan^2 x} = \frac{1}{3}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{3 + \alpha \left[x - \frac{x^3}{3!} + \dots \right] + \beta \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots \right] + \left(-x - \frac{x^2}{2} - \frac{x^3}{3} \dots \right)}{3 \tan^2 x} = \frac{1}{3}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{(3 + \beta) + (\alpha - 1)x + \left(-\frac{1}{2} - \frac{\beta}{2} \right) x^2 + \dots}{3x^2} \times \frac{x^2}{\tan^2 x} = \frac{1}{3}$$

$$\Rightarrow \beta + 3 = 0, \alpha - 1 = 0 \text{ and } \frac{-\frac{1}{2} - \frac{\beta}{2}}{3} = \frac{1}{3}$$

$$\Rightarrow \beta = -3, \alpha = 1$$

$$\Rightarrow 2\alpha - \beta = 2 + 3 = 5$$

Question8

$$\lim_{x \rightarrow \frac{\pi}{2}} \left(\frac{1}{\left(x - \frac{\pi}{2} \right)^2} \int_{x^3}^{\left(\frac{\pi}{2} \right)^3} \cos \left(\frac{1}{t^3} \right) dt \right) \text{ is equal to}$$

[29-Jan-2024 Shift 1]

Options:

A.

$3\pi/8$

B.

$3\pi^2/4$

C.

$$3\pi^2/8$$

D.

$$3\pi/4$$

Answer: C

Solution:

Using L'hospital rule

$$\begin{aligned} &= \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{0 - \cos x \times 3x^2}{2\left(x - \frac{\pi}{2}\right)} \\ &= \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{\sin\left(x - \frac{\pi}{2}\right)}{2\left(x - \frac{\pi}{2}\right)} \times \frac{3\pi^2}{4} \\ &= \frac{3\pi^2}{8} \end{aligned}$$

Question9

Let the slope of the line $45x + 5y + 3 = 0$ be $27r_1 + \frac{9r_2}{2}$ for some $r_1, r_2 \in \mathbb{R}$.

Then $\lim_{x \rightarrow 3} \left(\int_3^x \frac{8t^2}{\frac{3r_2x}{2} - r_2x^2 - r_1x^3 - 3x} dt \right)$ is equal

[29-Jan-2024 Shift 2]

Answer: 12

Solution:

According to the question,

$$27r_1 + \frac{9r_2}{2} = -9$$

$$\lim_{x \rightarrow 3} \frac{\int_3^x 8t^2 dt}{\frac{3r_2x}{2} - r_2x^2 - r_1x^3 - 3x}$$

$$= \lim_{x \rightarrow 3} \frac{8x^2}{\frac{3r_2}{2} - 2r_2x - 3r_1x^2 - 3} \quad (\text{using LH' Rule})$$

$$= \frac{72}{\frac{3r_2}{2} - 6r_2 - 27r_1 - 3}$$

$$= \frac{72}{-\frac{9r_2}{2} - 27r_1 - 3}$$

$$= \frac{72}{9 - 3} = 12$$

Question10

Let $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow \mathbf{R}$ be a differentiable function such that $f(0) = 1/2$, If the $\lim_{x \rightarrow 0} \frac{x \int_0^x f(t) dt}{e^{x^2} - 1} = \alpha$, then $8\alpha^2$ is equal to :

[30-Jan-2024 Shift 1]

Options:

A.

16

B.

2

C.

1

D.

4

Answer: B

Solution:

$$\lim_{x \rightarrow 0} \frac{x \int_0^x f(t) dt}{\left(\frac{e^{x^2} - 1}{x^2} \right) \times x^2}$$

$$\lim_{x \rightarrow 0} \frac{\int_0^x f(t) dt}{x} \left(\lim_{x \rightarrow 0} \frac{e^{x^2} - 1}{x^2} = 1 \right)$$

$$= \lim_{x \rightarrow 0} \frac{f(x)}{1} \text{ (using L Hospital)}$$

$$f(0) = \frac{1}{2}$$

$$\alpha = \frac{1}{2}$$

$$8\alpha^2 = 2$$

Question11

Let a be the sum of all coefficients in the expansion of

$(1 - 2x + 2x^2)^{2023}(3 - 4x^2 + 2x^3)^{2024}$ and $b = \lim_{x \rightarrow 0} \left(\frac{\int_0^x \frac{\log(1+t)}{1+t^{2024}} dt}{x^2} \right)$. If the equations $cx^2 + dx + e = 0$ and $2bx^2 + ax + 4 = 0$ have a common root, where $c, d, e \in \mathbb{R}$, then $d:c : e$ equals

[31-Jan-2024 Shift 1]

Options:

A.

2:1:4

B.

4:1:4

C.

1:2:4

D.

1:1:4

Answer: D

Solution:

Put $x = 1$

$\therefore a = 1$

$$b = \lim_{x \rightarrow 0} \frac{\int_0^x \frac{\ln(1+t)}{1+t^{2024}} dt}{x^2}$$

Using L' HOPITAL Rule

$$b = \lim_{x \rightarrow 0} \frac{\ln(1+x)}{(1+x^{2024})} \times \frac{1}{2x} = \frac{1}{2}$$

Now, $cx^2 + dx + e = 0$, $x^2 + x + 4 = 0$

(D < 0)

$$\therefore \frac{c}{1} = \frac{d}{1} = \frac{e}{4}$$

Question12

$$\lim_{x \rightarrow 0} \frac{e^{2|\sin x|} - 2|\sin x| - 1}{x^2}$$

[31-Jan-2024 Shift 1]

Options:

A.

is equal to -1

B.

does not exist

C.

is equal to 1

D.

is equal to 2

Answer: D

Solution:

$$\lim_{x \rightarrow 0} \frac{e^{2|\sin x|} - 2|\sin x| - 1}{x^2}$$

$$\lim_{x \rightarrow 0} \frac{e^{2|\sin x|} - 2|\sin x| - 1}{|\sin x|^2} \times \frac{\sin^2 x}{x^2}$$

Let $|\sin x| = t$

$$\lim_{t \rightarrow 0} \frac{e^{2t} - 2t - 1}{t^2} \times \lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2}$$

$$= \lim_{t \rightarrow 0} \frac{2e^{2t} - 2}{2t} \times 1 = 2 \times 1 = 2$$

Question13

Let $f : \mathbb{R} \rightarrow (0, \infty)$ be strictly increasing function such that $\lim_{x \rightarrow \infty} \frac{f(7x)}{f(x)} = 1$. Then, the value of $\lim_{x \rightarrow \infty} \left[\frac{f(5x)}{f(x)} - 1 \right]$ is equal to

[31-Jan-2024 Shift 2]

Options:

A.

4

B.

0

C.

7/5

D.

1

Answer: B

Solution:

$$f : \mathbb{R} \rightarrow (0, \infty)$$

$$\lim_{x \rightarrow \infty} \frac{f(7x)}{f(x)} = 1$$

$\because f$ is increasing

$$\therefore f(x) < f(5x) < f(7x)$$

$$\therefore \frac{f(x)}{f(x)} < \frac{f(5x)}{f(x)} < \frac{f(7x)}{f(x)}$$

$$1 < \lim_{x \rightarrow \infty} \frac{f(5x)}{f(x)} < 1$$

$$\therefore \left[\frac{f(5x)}{f(x)} - 1 \right]$$

$$\Rightarrow 1 - 1 = 0$$

Question14

If $\lim_{x \rightarrow 0} \frac{ax^2 e^x - b \log_e(1+x) + cxe^{-x}}{x^2 \sin x} = 1$, then $16(a^2 + b^2 + c^2)$ is equal to _____

[31-Jan-2024 Shift 2]

Answer: 81

Solution:

$$\lim_{x \rightarrow 0} \frac{ax^2 \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right) - b \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \right) + cx \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots \right)}{x^3 \cdot \frac{\sin x}{x}}$$

$$= \lim_{x \rightarrow \infty} \frac{(c-b)x + \left(\frac{b}{2} - c + a \right)x^2 + \left(a - \frac{b}{3} + \frac{c}{2} \right)x^3 + \dots}{x^3} = 1$$

$$c - b = 0, \quad \frac{b}{2} - c + a = 0$$

$$a - \frac{b}{3} + \frac{c}{2} = 1 \quad a = \frac{3}{4} \quad b = c = \frac{3}{2}$$

$$a^2 + b^2 + c^2 = \frac{9}{16} + \frac{9}{4} + \frac{9}{4}$$

$$16(a^2 + b^2 + c^2) = 81$$

Question15

Let $\{x\}$ denote the fractional part of x and $f(x) = \frac{\cos^{-1}(1 - \{x\}^2) \sin^{-1}(1 - \{x\})}{\{x\} - \{x\}^3}$, $x \neq 0$. If L and R respectively denotes the left hand limit and the right hand limit of $f(x)$ at $x = 0$, then $\frac{32}{\pi^2}(L^2 + R^2)$ is equal to ____

[1-Feb-2024 Shift 1]

Answer: 18

Solution:

Finding right hand limit

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} f(0 + h)$$

$$= \lim_{h \rightarrow 0} f(h)$$

$$= \lim_{h \rightarrow 0} \frac{\cos^{-1}(1-h^2) \sin^{-1}(1-h)}{h(1-h^2)}$$

$$= \lim_{h \rightarrow 0} \frac{\cos^{-1}(1-h^2)}{h} \left(\frac{\sin^{-1} 1}{1} \right)$$

$$\text{Let } \cos^{-1}(1-h^2) = \theta \Rightarrow \cos \theta = 1-h^2$$

$$= \frac{\pi}{2} \lim_{\theta \rightarrow 0} \frac{\theta}{\sqrt{1-\cos \theta}}$$

$$= \frac{\pi}{2} \lim_{\theta \rightarrow 0} \frac{1}{\sqrt{\frac{1-\cos \theta}{\theta^2}}}$$

$$= \frac{\pi}{2} \frac{1}{\sqrt{1/2}}$$

$$R = \frac{\pi}{\sqrt{2}}$$

Now finding left hand limit

$$L = \lim_{x \rightarrow 0^-} f(x)$$

$$= \lim_{h \rightarrow 0} f(-h)$$

$$= \lim_{h \rightarrow 0} \frac{\cos^{-1}(1-\{-h\}^2) \sin^{-1}(1-\{-h\})}{\{-h\} - \{-h\}^3}$$

$$= \lim_{h \rightarrow 0} \frac{\cos^{-1}(1-(-h+1)^2) \sin^{-1}(1-(-h+1))}{(-h+1) - (-h+1)^3}$$

$$= \lim_{h \rightarrow 0} \frac{\cos^{-1}(-h^2+2h) \sin^{-1} h}{(1-h)(1-(1-h)^2)}$$

$$= \lim_{h \rightarrow 0} \left(\frac{\pi}{2} \right) \frac{\sin^{-1} h}{(1-(1-h)^2)}$$

$$= \frac{\pi}{2} \lim_{h \rightarrow 0} \left(\frac{\sin^{-1} h}{-h^2+2h} \right)$$

$$= \frac{\pi}{2} \lim_{h \rightarrow 0} \left(\frac{\sin^{-1} h}{h} \right) \left(\frac{1}{-h+2} \right)$$

$$L = \frac{\pi}{4}$$

$$\frac{32}{\pi^2} (L^2 + R^2) = \frac{32}{\pi^2} \left(\frac{\pi^2}{2} + \frac{\pi^2}{16} \right)$$

$$= 16 + 2$$

$$= 18$$

Question 16

Let $f(x) = \begin{cases} x-1 & x \text{ is even,} \\ 2x & x \text{ is odd,} \end{cases}$. $a \in N, f(f(f(a))) = 21$, then $\lim_{x \rightarrow a^-} \left\{ \frac{|x|^3}{a} - \left[\frac{x}{a} \right] \right\}$, where $[t]$ denotes the greatest integer less than or equal to t , is equal to :

[1-Feb-2024 Shift 2]

Options:

A.

121

B.

144

C.

169

D.

225

Answer: B

Solution:

$$f(x) = \begin{cases} x-1; & x = \text{even} \\ 2x; & x = \text{odd} \end{cases}$$

$$f(f(f(a))) = 21$$

C - 1 : If $a = \text{even}$

$$f(a) = a - 1 = \text{odd}$$

$$f(f(a)) = 2(a - 1) = \text{even}$$

$$f(f(f(a))) = 2a - 3 = 21 \Rightarrow a = 12$$

C - 2 : If $a = \text{odd}$

$$f(a) = 2a = \text{even}$$

$$f(f(a)) = 2a - 1 = \text{odd}$$

$$f(f(f(a))) = 4a - 2 = 21 \text{ (Not possible)}$$

Hence $a = 12$

Now

$$\lim_{x \rightarrow 12^-} \left(\frac{|x|^3}{2} - \left[\frac{x}{12} \right] \right)$$

$$= \lim_{x \rightarrow 12^-} \frac{|x|^3}{12} - \lim_{x \rightarrow 12^-} \left[\frac{x}{12} \right]$$

$$= 144 - 0 = 144.$$

Question17

If $y = \frac{(\sqrt{x}+1)(x^2-\sqrt{x})}{x\sqrt{x}+x+\sqrt{x}} + \frac{1}{15}(3\cos^2 x - 5)\cos^3 x$ then $96y' \left(\frac{\pi}{6} \right)$ is equal to :

[1-Feb-2024 Shift 2]

Answer: 105

Solution:

$$y = \frac{(\sqrt{x}+1)(x^2-\sqrt{x})}{x\sqrt{x}+x+\sqrt{x}} + \frac{1}{15}(3\cos^2 x - 5)\cos^3 x$$

$$y = \frac{(\sqrt{x}+1)(\sqrt{x})((\sqrt{x})^3-1)}{(\sqrt{x})((\sqrt{x})^2+(\sqrt{x})+1)} + \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x$$

$$y = (\sqrt{x}+1)(\sqrt{x}-1) + \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x$$

$$y' = 1 - \cos^4 x \cdot (\sin x) + \cos^2 x (\sin x)$$

$$y' \left(\frac{\pi}{6} \right) = 1 - \frac{9}{16} \times \frac{1}{2} + \frac{3}{4} \times \frac{1}{2}$$

$$= \frac{32-9+12}{32} = \frac{35}{32}$$

$$= 96y' \left(\frac{\pi}{6} \right) = 105$$

Question18

$\lim_{t \rightarrow 0} \left(1^{\frac{1}{\sin^2 t}} + 2^{\frac{1}{\sin^2 t}} + \dots + n^{\frac{1}{\sin^2 t}} \right)^{\sin^2 t}$ is equal to

[24-Jan-2023 Shift 1]

Options:

A. $n^2 + n$

B. n

C. $\frac{n(n+1)}{2}$

D. n^2

Answer: B

Solution:

Solution:

$$\lim_{t \rightarrow 0} \left(1^{\operatorname{cosec}^2 t} + 2^{\operatorname{cosec}^2 t} + \dots + n^{\operatorname{cosec}^2 t} \right)^{\sin^2 t}$$

$$= \lim_{t \rightarrow 0} n \left(\left(\frac{1}{n} \right)^{\operatorname{cosec}^2 t} + \left(\frac{2}{n} \right)^{\operatorname{cosec}^2 t} + \dots + 1 \right)^{\sin^2 t}$$

$$= n$$

Question19

The set of all values of a for which $\lim_{x \rightarrow a} ([x - 5] - [2x + 2]) = 0$, where $[\alpha]$ denotes the greater integer less than or equal to α is equal to [24-Jan-2023 Shift 2]

Options:

- A. $(-7.5, -6.5)$
- B. $(-7.5, -6.5]$
- C. $[-7.5, -6.5]$
- D. $[-7.5, -6.5)$

Answer: A

Solution:

Solution:
 $\lim_{x \rightarrow a} ([x - 5] - [2x + 2]) = 0$
 $\lim_{x \rightarrow a} ([x] - 5 - [2x] - 2) = 0$
 $\lim_{x \rightarrow a} ([x] - [2x]) = 7$
 $[a] - [2a] = 7$
 $a \in I, a = -7$
 $a \notin I, a = I + f$
 Now, $[a] - [2a] = 7$
 $-I - [2f] = 7$
 Case-I: $f \in \left(0, \frac{1}{2} \right)$
 $2f \in (0, 1)$
 $-I = 7$
 $I = -7 \Rightarrow a \in (-7, -6.5)$
 Case-II: $f \in \left(\frac{1}{2}, 1 \right)$
 $2f \in (1, 2)$
 $-I - 1 = 7$
 $I = -8 \Rightarrow a \in (-7.5, -7)$
 Hence, $a \in (-7.5, -6.5)$

Question20

Let $x = 2$ be a root of the equation $x^2 + px + q = 0$ and

$f(x) = \begin{cases} \frac{1 - \cos(x^2 - 4pp + q^2 + 8q + 16)}{(x - 2p)^4} & x \neq 2p \\ 0 & x = 2p \end{cases}$. Then $\lim_{x \rightarrow 22^+} [f(x)]$ where $[.]$ denotes greatest integer function, is

[29-Jan-2023 Shift 1]

Options:

A. 2

B. 1

C. 0

D. -1

Answer: C

Solution:

Solution:

$$\lim_{x \rightarrow 2p^+} \left(\frac{1 - \cos(x^2 - 4px + q^2 + 8q + 16)}{(x^2 - 4px + q^2 + 8q + 16)^2} \right) \left(\frac{(x^2 - 4px + q^2 + 8q + 16)^2}{(x - 2p)^2} \right)$$

$$\lim_{h \rightarrow 0} \frac{1}{2} \left(\frac{(2p + h)^2 - 4p(2p + h) + q^2 + 8q + 16}{h^2} \right)^2 = \frac{1}{2}$$

Using L'Hospital's

$$\lim_{x \rightarrow 2p^+} [f(x)] = 0$$

Question21

Let f , g and h be the real valued functions defined on $\mathbb{R}$ as

$$f(x) = \begin{cases} \frac{x}{|x|} & x \neq 0 \\ 1 & x = 0 \end{cases}, \quad g(x) = \begin{cases} \frac{\sin(x+1)}{(x+1)} & x \neq -1 \\ 1 & x = -1 \end{cases} \text{ and } h(x) = 2[x] - f(x),$$

where $[x]$ is the greatest integer $\leq x$. Then the value of $\lim_{x \rightarrow 1} g(h(x-1))$ is

[30-Jan-2023 Shift 2]

Options:

A. 1

B. $\sin(1)$

C. -1

D. 0

Answer: A

Solution:

Solution:

$$\text{LHL} = \lim_{k \rightarrow 0} g(h(-k)), \quad k > 0$$

$$= \lim_{k \rightarrow 0} g(-2 + 1) \because f(x) = -1 \quad \forall x < 0$$

$$= g(-1) = 1$$

$$\text{RHL} = \lim_{k \rightarrow 0} g(h(k)), \quad k > 0$$

$$= \lim_{k \rightarrow 0} g(-1), \because f(x) = 1, \forall x > 0$$

$$= 1$$

Question22

$$\lim_{x \rightarrow \infty} \frac{(\sqrt{3x+1} + \sqrt{3x-1})^6 + (\sqrt{3x+1} - \sqrt{3x-1})^6}{(x + \sqrt{x^2-1})^6 + (x - \sqrt{x^2-1})^6} \times 3$$

[31-Jan-2023 Shift 2]

Options:

- A. is equal to 9
- B. is equal to 27
- C. does not exist
- D. is equal to $\frac{27}{2}$

Answer: B

Solution:

Solution:

$$\lim_{x \rightarrow \infty} \frac{(\sqrt{3x+1} + \sqrt{3x-1})^6 + (\sqrt{3x+1} - \sqrt{3x-1})^6}{(x + \sqrt{x^2-1})^6 + (x - \sqrt{x^2-1})^6} \times 3$$

$$\lim_{x \rightarrow \infty} x^3 \times \left\{ \frac{x^3 \left\{ \left(\sqrt{3 + \frac{1}{x}} + \sqrt{3 - \frac{1}{x}} \right)^6 + \left(\sqrt{3 + \frac{1}{x}} - \sqrt{3 - \frac{1}{x}} \right)^6 \right\}}{x^6 \left\{ \left(1 + \sqrt{1 - \frac{1}{x^2}} \right)^6 + \left(1 - \sqrt{1 - \frac{1}{x^2}} \right)^6 \right\}} \right\}$$

$$= \frac{(2\sqrt{3})^6 + 0}{2^6 + 0} = 3^3 = (27)$$

Question23

If $f(x) = x^3 - x^2 f'(1) + x f''(2) - f'''(3)$, $x \in R$, then
[24-Jan-2023 Shift 2]

Options:

- A. $3f(1) + f(2) = f(3)$
- B. $f(3) - f(2) = f(1)$
- C. $2f(0) - f(1) + f(3) = f(2)$
- D. $f(1) + f(2) + f(3) = f(0)$

Answer: C

Solution:

Solution:

$$f(x) = x^3 - x^2 f'(1) + x f''(2) - f'''(3), x \in \mathbb{R}$$

$$\text{Let } f'(1) = a, f''(2) = b, f'''(3) = c$$

$$f(x) = x^3 - x^2 + bx - c$$

$$f'(x) = 3x^2 - 2ax + b$$

$$f''(x) = 6x - 2a$$

$$f'''(x) = 6$$

$$c = 6, a = 3, b = 6$$

$$f(x) = x^3 - 3x^2 + 6x - 6$$

$$f(1) = -2, f(2) = 2, f(3) = 12, f(0) = -6$$

$$2f(0) - f(1) + f(3) = 2 = f(2)$$

Question24

Let

$$y(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16}).$$

Then $y' - y''$ at $x = -1$ is equal to

[25-Jan-2023 Shift 1]

Options:

A. 976

B. 464

C. 496

D. 944

Answer: C

Solution:

Solution:

$$y = \frac{1 - x^{32}}{1 - x} \Rightarrow y - xy = 1 - x^{32}$$

$$y' - xy' - y = -32x^{31}$$

$$y'' - xy'' - y' - y' = -(32)(31)x^{30}$$

$$\text{at } x = -1 \Rightarrow y - y'' = 496$$

Question25

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function that satisfies the relation $f(x+y) = f(x) + f(y) - 1, \forall x, y \in \mathbb{R}$. If $f'(0) = 2$, then $|f(-2)|$ is equal to

[29-Jan-2023 Shift 1]

Answer: 3

Solution:

Solution:

$$f(x+y) = f(x) + f(y) - 1$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = f'(0) = 2$$

$$f'(x) = 2 \Rightarrow dy = 2 dx$$

$$y = 2x + C$$

$$x = 0, y = 1, c = 1$$

$$y = 2x + 1$$

$$|f(-2)| = |-4 + 1| = |-3| = 3$$

Question 26

Let f and g be twice differentiable functions on $\mathbb{R}$ such that

$$f''(x) = g''(x) + 6x$$

$$f'(1) = 4g'(1) - 3 = 9$$

$$f(2) = 3g(2) = 12$$

Then which of the following is NOT true?

[29-Jan-2023 Shift 2]

Options:

A. $g(-2) - f(-2) = 20$

B. If $-1 < x < 2$, then $|f(x) - g(x)| < 8$

C. $|f'(x) - g'(x)| < 6 \Rightarrow -1 < x < 1$

D. There exists $x_0 \in \left(1, \frac{3}{2}\right)$ such that $f(x_0) = g(x_0)$

Answer: B

Solution:

Solution:

$$f''(x) = g''(x) + 6x \dots (1)$$

$$f'(1) = 4g'(1) - 3 = 9 \dots (2)$$

$$f(2) = 3g(2) = 12 \dots (3)$$

By integrating (1)

$$f'(x) = g'(x) + 6 \frac{x^2}{2} + C$$

$$\text{At } x = 1$$

$$f'(1) = g'(1) + 3 + C$$

$$\Rightarrow 9 = 4 + 3 + C \Rightarrow C = 3$$

$$\therefore f'(x) = g'(x) + 3x^2 + 3$$

Again by integrating,

$$f(x) = g(x) + \frac{3x^3}{3} + 3x + D$$

$$\text{At } x = 2$$

$$f(2) = g(2) + 8 + 3(2) + D$$

$$\Rightarrow 12 = 4 + 8 + 6 + D \Rightarrow D = -6$$

$$\text{So, } f(x) = g(x) + x^3 + 3x - 6$$

$$\Rightarrow f(x) - g(x) = x^3 + 3x - 6$$

$$\text{At } x = -2$$

$$\Rightarrow g(-2) - f(-2) = 20 \quad (\text{Option (1) is true})$$

Now, for $-1 < x < 2$

$$h(x) = f(x) - g(x) = x^3 + 3x - 6$$

$\Rightarrow h'(x) = 3x^2 + 3$
 $\Rightarrow h(x) \uparrow$
 So, $h(-1) < h(x) < h(2)$
 $\Rightarrow -10 < h(x) < 8$
 $\Rightarrow |h(x)| < 10$ (option (2) is NOT true)
 Now, $h'(x) = f'(x) - g'(x) = 3x^2 + 3$
 If $|h'(x)| < 6 \Rightarrow |3x^2 + 3| < 6$
 $\Rightarrow 3x^2 + 3 < 6$
 $\Rightarrow x^2 < 1$
 $\Rightarrow -1 < x < 1$ (option (3) is True)
 If $x \in (-1, 1) | f'(x) - g'(x) | < 6$
 option (3) is true and now to solve
 $f(x) - g(x) = 0$
 $\Rightarrow x^3 + 3x - 6 = 0$
 $h(x) = x^3 + 3x - 6$
 here, $h(1) = -ve$ and $h\left(\frac{3}{2}\right) = +ve$
 So there exists $x_0 \in \left(1, \frac{3}{2}\right)$ such that $f(x_0) = g(x_0)$
 (option (4) is true)

Question27

Let

$$y = f(x) = \sin^3 \left(\frac{\pi}{3} \left(\cos \left(\frac{\pi}{3\sqrt{2}} (-4x^3 + 5x^2 + 1) \right)^{\frac{3}{2}} \right) \right)$$

Then, at $x = 1$,
[31-Jan-2023 Shift 1]

Options:

A. $2y' + \sqrt{3}\pi^2 y = 0$

B. $2y' + 3\pi^2 y = 0$

C. $\sqrt{2}y' - 3\pi^2 y = 0$

D. $y' + 3\pi^2 y = 0$

Answer: B

Solution:

$$y = \sin^3(\pi / 3 \cos g(x))$$

$$g(x) = \frac{\pi}{3\sqrt{2}}(-4x^3 + 5x^2 + 1)^{3/2}$$

$$g(1) = 2\pi / 3$$

$$y' = 3\sin^2 \left(\frac{\pi}{3} \cos g(x) \right) \times \cos \left(\frac{\pi}{3} \cos g(x) \right)$$

$$y'(1) = 3\sin^2 \left(-\frac{\pi}{6} \right) \cdot \cos \left(\frac{\pi}{6} \right) \cdot \frac{\pi}{3} \left(-\sin \frac{2\pi}{3} \right) g'(1)$$

$$g'(x) = \frac{\pi}{3\sqrt{2}}(-4x^3 + 5x^2 + 1)^{1/2}(-12x^2 + 10x)$$

$$g'(1) = \frac{\pi}{2\sqrt{2}}(\sqrt{2})(-2) = -\pi$$

$$y'(1) = \frac{\text{not } 6}{4} \cdot \frac{\sqrt{3}}{2} \cdot \frac{\pi}{\text{not } 2} \left(\frac{-\sqrt{3}}{2} \right) (-\pi) = \frac{3\pi^2}{16}$$

$$y'(1) = \sin^3(\pi / 3 \cos 2\pi / 3) = -\frac{1}{8}$$

$$2y'(1) + 3\pi^2 y(1) = 0$$

Question28

If $\int_0^1 (x^{21} + x^{14} + x^7)(2x^{14} + 3x^7 + 6)^{1/7} dx = \frac{1}{l}(11)^{m/n}$

where $1, m, n \in \mathbb{N}$, m and n are coprime then $l + m + n$ is equal to _____.

[1-Feb-2023 Shift 1]

Answer: 63

Solution:

$$\int (x^{20} + x^{13} + x^6)(2x^{21} + 3x^{14} + 6x^7)^{1/7} dx$$

$$2x^{21} + 3x^{14} + 6x^7 = t$$

$$42(x^{20} + x^{13} + x^6) dx = dt$$

$$\frac{1}{42} \int_0^{11} t^{\frac{1}{7}} dt = \left(\frac{t^{\frac{8}{7}}}{\frac{8}{7}} \times \frac{1}{42} \right)_0^{11}$$

$$= \frac{1}{48} \left(t^{\frac{8}{7}} \right)_0^{11} = \frac{1}{48} (11)^{8/7}$$

$$l = 48, m = 8, n = 7$$

$$l + m + n = 63$$

Question29

$\lim_{n \rightarrow \infty} \left\{ \left(2^{\frac{1}{2}} - 2^{\frac{1}{3}} \right) \left(2^{\frac{1}{2}} - 2^{\frac{1}{5}} \right) \dots \left(2^{\frac{1}{2}} - 2^{\frac{1}{2n+1}} \right) \right\}$ is equal to

[6-Apr-2023 shift 2]

Options:

A. $\frac{1}{\sqrt{2}}$

B. $\sqrt{2}$

C. 1

D. 0

Answer: D

Solution:

$$P = \lim_{n \rightarrow \infty} \left(2^{\frac{1}{2}} - 2^{\frac{1}{3}} \right) \left(2^{\frac{1}{2}} - 2^{\frac{1}{5}} \right) \dots \left(2^{\frac{1}{2}} - 2^{\frac{1}{2n+1}} \right)$$

Let

$$2^{\frac{1}{2}} - 2^{\frac{1}{3}} \rightarrow \text{Smallest}$$

$$2^{\frac{1}{2}} - 2^{\frac{1}{2n+1}} \rightarrow \text{Largest}$$

Sandwich th.

$$\left(2^{\frac{1}{2}} - 2^{\frac{1}{3}} \right)^n \leq P \leq \left(2^{\frac{1}{2}} - 2^{\frac{1}{2n+1}} \right)^n$$

$$\left(\begin{matrix} \text{lieb / w} \\ 0 \text{ and } 1 \end{matrix} \right)^n$$

$$\lim_{n \rightarrow \infty} \left(2^{\frac{1}{2}} - 2^{\frac{1}{3}} \right)^n = 0$$

$$\lim_{n \rightarrow \infty} \left(2^{\frac{1}{2}} - 2^{\frac{1}{2n+1}} \right)^n = 0$$

$$\therefore P = 0$$

Question30

$$\lim_{x \rightarrow 0} \left(\left(\frac{1 - \cos^2(3x)}{\cos^3(4x)} \right) \left(\frac{\sin^3(4x)}{(\log_e(2x + 1))^5} \right) \right) \text{ is equal to } \underline{\hspace{2cm}}$$

[8-Apr-2023 shift 1]

Options:

A. 24

B. 9

C. 18

D. 15

Answer: C

Solution:

Solution:

$$\lim_{x \rightarrow 0} \left[\frac{1 - \cos^2 3x}{9x^2} \right] \frac{9x^2}{\cos^3 4x} \cdot \frac{\left(\frac{\sin 4x}{4x} \right)^3 \times 64x^3}{\left[\frac{\ln(1 + 2x)}{2x} \right]^5 \times 32x^5}$$

$$\lim_{x \rightarrow 0} 2 \left(\frac{1}{2} \times \frac{9}{1} \times \frac{1 \times 64}{1 \times 32} \right) = 18$$

Question31

If $\alpha > \beta > 0$ are the roots of the equation $ax^2 + bx + 1 = 0$, and

$$\lim_{x \rightarrow \frac{1}{\alpha}} \left(\frac{1 - \cos(x^2 + bx + a)}{2(1 - ax)^2} \right)^{\frac{1}{2}} = \frac{1}{k} \left(\frac{1}{\beta} - \frac{1}{\alpha} \right), \text{ then k is equal to}$$

[8-Apr-2023 shift 2]

Options:

- A. β
- B. 2α
- C. 2β
- D. α

Answer: B

Solution:

Solution:

$$\therefore ax^2 + bx + 1 = a(x - \alpha)(x - \beta) \therefore \alpha\beta = \frac{1}{a}$$

$$\therefore x^2 + bx + a = a(1 - \alpha x)(1 - \beta x)$$

$$\therefore \lim_{x \rightarrow \frac{1}{\alpha}} \left\{ \frac{1 - \cos(x^2 + bx + a)}{2(1 - \alpha x)^2} \right\}^{\frac{1}{2}} = \lim_{x \rightarrow \frac{1}{2}} \left\{ \frac{1 - \cos a(1 - \alpha x)(1 - \beta x)}{2\{a(1 - \alpha x)(1 - \beta x)\}^2} \cdot a^2(1 - \beta x)^2 \right\}^{\frac{1}{2}}$$

$$= \left[\frac{1}{2} \cdot \frac{1}{2} a^2 \left(1 - \frac{\beta}{\alpha} \right)^2 \right]^{\frac{1}{2}}$$

$$= \frac{1}{2} \frac{1}{\alpha\beta} \left(1 - \frac{\beta}{\alpha} \right) = \frac{1}{2} \left(\frac{1}{\alpha\beta} - \frac{1}{\alpha^2} \right)$$

$$= \frac{1}{2\alpha} \left(\frac{1}{\beta} - \frac{1}{\alpha} \right) = \frac{1}{k} \left(\frac{1}{\beta} - \frac{1}{\alpha} \right)$$

$$\therefore k = 2\alpha \text{ Ans.}$$

Question32

Among

(S1): $\lim_{n \rightarrow \infty} \frac{1}{n^2} (2 + 4 + 6 + \dots + 2n) = 1$

(S2): $\lim_{n \rightarrow \infty} \frac{1}{n^{16}} (1^{15} + 2^{15} + 3^{15} + \dots + n^{15}) = \frac{1}{16}$

[13-Apr-2023 shift 1]

Options:

- A. Only (S1) is true
- B. Both (S1) and (S2) are true
- C. Both (S1) and (S2) are false
- D. Only (S2) is true

Answer: B

Solution:

Solution:

$$S_1 : \lim_{n \rightarrow \infty} \frac{n(n+1)}{n^2} = 1 \Rightarrow \text{True}$$

$$S_2 : \lim_{n \rightarrow \infty} \frac{1}{n^{16}} (\sum r^{15}) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum \left(\frac{r}{n} \right)^{15}$$

$$= \int_0^1 x^{15} dx = \frac{1}{16} \Rightarrow \text{True}$$

Question33

If $\lim_{x \rightarrow 0} \frac{e^{ax} - \cos(bx) - \frac{cxe^{-cx}}{2}}{1 - \cos(2x)} = 17$, then $5a^2 + b^2$ is equal to
[13-Apr-2023 shift 2]

Options:

A. 76

B. 72

C. 64

D. 68

Answer: D

Solution:

Solution:

$$\lim_{x \rightarrow 0} \frac{e^{ax} - \cos bx - \frac{cxe^{-cx}}{2}}{1 - \cos 2x} = 17$$

On expansion

$$\lim_{x \rightarrow 0} \frac{\left(1 + ax + \frac{(ax)^2}{2!} + \dots\right) - \left(1 - \frac{(bx)^2}{2!} + \dots\right) - \frac{cx}{2} \left(1 - cx + \frac{(cx)^2}{2!}\right)}{\left(\frac{1 - \cos 2x}{(2x)^2}\right) \times (2x)^2} = 17$$

$$\lim_{x \rightarrow 0} \frac{x \left(a - \frac{c}{2}\right) + x^2 \left(\frac{a^2}{2} + \frac{b^2}{2} + \frac{c^2}{2}\right)}{\frac{1}{2}(4x^2)} = 17$$

For limit to be exist

$$a - \frac{c}{2} = 0 \Rightarrow c = 2a$$

$$\Rightarrow \frac{\frac{a^2}{2} + \frac{b^2}{2} + \frac{c^2}{2}}{2} = 17$$

$$\Rightarrow \frac{a^2}{2} + \frac{b^2}{2} + \frac{4a^2}{2} = 34$$

$$\Rightarrow 5a^2 + b^2 = 68$$

Question34

If $2x^y + 3y^x = 20$, then $\frac{dy}{dx}$ at (2, 2) is equal to:
[6-Apr-2023 shift 1]

Options:

A. $-\left(\frac{3 + \log_e 8}{2 + \log_e 4}\right)$

B. $-\left(\frac{2 + \log_e 8}{3 + \log_e 4}\right)$

C. $-\left(\frac{3 + \log_e 4}{2 + \log_e 8}\right)$

D. $-\left(\frac{3 + \log_e 16}{4 + \log_e 8}\right)$

Answer: B

Solution:

Solution:

$$2x^y + 3y^x = 20$$

$$v_1^{v_2} \left(v_2 \frac{1}{v_1} + \ln v_1 \cdot v_2^1 \right)$$

$$2x^y \left(y \cdot \frac{1}{x} + \ln x \cdot \frac{dy}{dx} \right) + 3y^x \left(x \cdot \frac{1}{y} \cdot \frac{dy}{dx} + \ln y \cdot 1 \right) = 0$$

Put (2, 2)

$$2 \cdot 2 \left(1 + \ln 2 \cdot \frac{dy}{dx} \right) + 3 \cdot 2 \left(1 \cdot \frac{dy}{dx} + \ln 2 \right) = 0$$

$$\frac{dy}{dx} [8 \ln 2 + 12] + 8 + 12 \ln 2 = 0$$

$$\frac{dy}{dx} = - \left[\frac{2 + 3 \ln 2}{3 + 2 \ln 2} \right] = - \left[\frac{2 + \ln 8}{3 + \ln 4} \right]$$

Question 35

Let $f(x) = \frac{\sin x + \cos x - \sqrt{2}}{\sin x - \cos x}$, $x \in [0, \pi] - \left\{ \frac{\pi}{4} \right\}$. Then $f\left(\frac{7\pi}{12}\right) f''\left(\frac{7\pi}{12}\right)$ is equal to
[8-Apr-2023 shift 1]

Options:

A. $\frac{-2}{3}$

B. $\frac{2}{9}$

C. $\frac{-1}{3\sqrt{3}}$

D. $\frac{2}{3\sqrt{3}}$

Answer: B

Solution:

Solution:

$$f(x) = -\tan\left(\frac{x}{2} - \frac{\pi}{8}\right)$$

$$f'(x) = -\frac{1}{2} \sec^2\left(\frac{x}{2} - \frac{\pi}{8}\right)$$

$$f''(x) = -\sec^2\left(\frac{x}{2} - \frac{\pi}{8}\right) \cdot \tan\left(\frac{x}{2} - \frac{\pi}{8}\right) \cdot \frac{1}{2}$$

$$f\left(\frac{7\pi}{12}\right) = -\tan\left(\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}}$$

$$f''\left(\frac{7\pi}{12}\right) = -\frac{1}{2} \sec^2\frac{\pi}{6} \cdot \tan\frac{\pi}{6} = \frac{-1}{2} \cdot \frac{4}{3} \times \frac{1}{\sqrt{3}} = \frac{-2}{3\sqrt{3}}$$

$$f\left(\frac{7\pi}{12}\right) \cdot f''\left(\frac{7\pi}{12}\right) = \frac{2}{9}$$

Question36

Let k and m be positive real numbers such that the function

$$f(x) = \begin{cases} 3x^2 + k\sqrt{x+1} & 0 < x < 1 \\ mx^2 + k^2 & x \geq 1 \end{cases} \text{ is differentiable for all } x > 0. \text{ Then } \frac{8f'(8)}{f'\left(\frac{1}{8}\right)}$$

is equal to _____.

[8-Apr-2023 shift 2]

Answer: 309

Solution:

Solution:

function is differentiable $\forall x < 0$

so $f(1^-) = f(1) \dots (1)$

$$3 + \sqrt{2}k = m + k^2$$

$$\text{and } f_+^{-1}(1^-) = f_-^{-1}(1^+)$$

$$2m = 6 + \frac{k}{2\sqrt{2}}$$

$$m = 3 + \frac{k}{4\sqrt{2}} \dots (2)$$

$$k^2 + 3 + \frac{k}{4\sqrt{2}} = 3 + \sqrt{2}k$$

$$k = \frac{7}{4\sqrt{2}}, 0$$

$$m = 3 + \frac{7}{32}$$

$$m = \frac{103}{32}$$

$$= \frac{8 \times 2 \times 8 \times \frac{103}{32}}{\frac{16}{12}}$$

$$= 103 \times 3 = 309$$

Question37

$\lim_{x \rightarrow \frac{\pi}{2}} \left(\tan^2 x \left((2\sin^2 x + 3\sin x + 4)^{\frac{1}{2}} - (\sin^2 x + 6\sin x + 2)^{\frac{1}{2}} \right) \right)$ is equal to

[25-Jun-2022-Shift-2]

Options:

A. $\frac{1}{12}$

B. $-\frac{1}{18}$

C. $-\frac{1}{12}$

D. $\frac{1}{6}$

Answer: A

Solution:

Solution:

$$\begin{aligned} & \lim_{x \rightarrow \frac{\pi}{2}} \tan^2 x \{ \sqrt{2\sin^2 x + 3\sin x + 4} - \sqrt{\sin^2 x + 6\sin x + 2} \} \\ &= \lim_{x \rightarrow \frac{\pi}{2}} \frac{\tan^2 x (\sin^2 x - 3\sin x + 2)}{\sqrt{2\sin^2 x + 3\sin x + 4} + \sqrt{\sin^2 x + 6\sin x + 2}} \\ &= \frac{1}{6} \lim_{x \rightarrow \frac{\pi}{2}} \frac{(1 - \sin x)(2 - \sin x)}{\cos^2 x} \cdot \sin^2 x \\ &= \frac{1}{6} \lim_{x \rightarrow \frac{\pi}{2}} \frac{(2 - \sin x)\sin^2 x}{1 + \sin x} \\ &= \frac{1}{12} \end{aligned}$$

Question38

$\lim_{x \rightarrow \frac{1}{\sqrt{2}}} \frac{\sin(\cos^{-1} x) - x}{1 - \tan(\cos^{-1} x)}$ is equal to :

[26-Jun-2022-Shift-1]

Options:

A. $\sqrt{2}$

B. $-\sqrt{2}$

C. $\frac{1}{\sqrt{2}}$

D. $-\frac{1}{\sqrt{2}}$

Answer: D

Solution:

$$\lim_{x \rightarrow \frac{1}{\sqrt{2}}} \frac{\sin(\cos^{-1}x) - x}{1 - \tan(\cos^{-1}x)}$$

$$\text{Let } \cos^{-1}x = t$$

$$\Rightarrow x = \cos t$$

$$\text{When } x \rightarrow \frac{1}{\sqrt{2}}, \text{ then } t \rightarrow \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) \rightarrow \frac{\pi}{4}$$

$$\therefore \lim_{t \rightarrow \frac{\pi}{4}} \frac{\sin t - \cos t}{1 - \tan(t)}$$

$$= \lim_{t \rightarrow \frac{\pi}{4}} \frac{\sin t - \cos t}{1 - \frac{\sin t}{\cos t}}$$

$$= \lim_{t \rightarrow \frac{\pi}{4}} \frac{(\sin t - \cos t)(\cos t)}{(\cos t - \sin t)}$$

$$= \lim_{t \rightarrow \frac{\pi}{4}} -\cos t$$

$$= -\lim_{t \rightarrow \frac{\pi}{4}} \cos t$$

$$= -\frac{1}{\sqrt{2}}$$

Question39

$\lim_{x \rightarrow 0} \frac{\cos(\sin x) - \cos x}{x^4}$ is equal to :

[26-Jun-2022-Shift-2]

Options:

A. $\frac{1}{3}$

B. $\frac{1}{4}$

C. $\frac{1}{6}$

D. $\frac{1}{12}$

Answer: C

Solution:

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{\cos(\sin x) - \cos x}{x^4} &= \lim_{x \rightarrow 0} \frac{2 \sin(x + \sin x) \cdot \sin\left(\frac{x - \sin x}{2}\right)}{x^4} \\
&= \lim_{x \rightarrow 0} 2 \cdot \left(\frac{\left(\frac{x + \sin x}{2}\right) \left(\frac{x - \sin x}{2}\right)}{x^4} \right) \\
&= \lim_{x \rightarrow 0} \frac{1}{2} \cdot \left(\frac{\left(x + x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots\right) \left(x - x + \frac{x^3}{3!} \dots\right)}{x^4} \right) \\
&= \lim_{x \rightarrow 0} \frac{1}{2} \cdot \left(2 - \frac{x^2}{3!} + \frac{x^4}{5!} \dots \right) \left(\frac{1}{3!} - \frac{x^2}{5!} - 1 \right) \\
&= \frac{1}{6}
\end{aligned}$$

Question40

Let a be an integer such that $\lim_{x \rightarrow 7} \frac{18 - [1 - x]}{[x - 3a]}$ exists, where [t] is greatest integer $\leq t$. Then a is equal to :
[27-Jun-2022-Shift-1]

Options:

- A. -6
- B. -2
- C. 2
- D. 6

Answer: A

Solution:

$$\lim_{x \rightarrow 7} \frac{18 - [1 - x]}{[x - 3a]} \text{ exist } \&a \in I.$$

$$= \lim_{x \rightarrow 7} \frac{17 - [-x]}{[x] - 3a} \text{ exist}$$

$$\text{RH L} = \lim_{x \rightarrow 7^+} \frac{17 - [-x]}{[x] - 3a} = \frac{25}{7 - 3a} \left[a \neq \frac{7}{3} \right]$$

$$\text{LH L} = \lim_{x \rightarrow 7^-} \frac{17 - [-x]}{[x] - 3a} = \frac{24}{6 - 3a} [a \neq 2]$$

For limit to exist

$$\text{LH L} = \text{RH L}$$

$$\frac{25}{7 - 3a} = \frac{24}{6 - 3a}$$

$$\Rightarrow \frac{25}{7 - 3a} = \frac{8}{2 - a}$$

$$\therefore a = -6$$

Question41

Let $[t]$ denote the greatest integer $\leq t$ and $\{t\}$ denote the fractional part of t . The integral value of α for which the left hand limit of the function

$f(x) = [1 + x] + \frac{\alpha^{2[x] + \{x\}} + [x] - 1}{2[x] + \{x\}}$ at $x = 0$ is equal to $\alpha - \frac{4}{3}$, is ____

[27-Jun-2022-Shift-2]

Answer: 3

Solution:

Solution:

$$f(x) = [1 + x] + \frac{\alpha^{2[1] + \{x\}} + [x] - 1}{2[x] + \{x\}}$$

$$\lim_{x \rightarrow 0^-} f(x) = \alpha - \frac{4}{3}$$

$$\Rightarrow \lim_{x \rightarrow 0^-} 1 + [x] + \frac{\alpha^{x + [x]} + [x] - 1}{x + [x]} = \alpha - \frac{4}{3}$$

$$\Rightarrow \lim_{h \rightarrow 0^-} 1 - 1 + \frac{\alpha^{-h - 1} - 1 - 1}{-h - 1} = \alpha - \frac{4}{3}$$

$$\therefore \frac{\alpha^{-1} - 2}{-1} = \alpha - \frac{4}{3}$$

$$\Rightarrow 3\alpha^2 - 10\alpha + 3 = 0$$

$$\therefore \alpha = 3 \text{ or } \frac{1}{3}$$

$$\therefore \alpha \text{ in integer, hence } \alpha = 3$$

Question42

The value of

$$\lim_{n \rightarrow \infty} 6 \tan \left\{ \sum_{r=1}^n \tan^{-1} \left(\frac{1}{r^2 + 3r + 3} \right) \right\} \text{ is equal to :}$$

[28-Jun-2022-Shift-2]

Options:

- A. 1
- B. 2
- C. 3
- D. 6

Answer: C

Solution:

Solution:

$$\begin{aligned} & \lim_{n \rightarrow \infty} 6 \tan \left\{ \sum_{r=1}^{-1} \tan \left(\frac{1}{r^2 + 3r + 3} \right) \right\} \\ &= \lim_{n \rightarrow \infty} 6 \tan \left\{ \sum_{r=1}^{-1} \tan \left(\frac{(r+2) - (r+1)}{1 + (r+2)(r+1)} \right) \right\} \\ &= \lim_{n \rightarrow \infty} 6 \tan \left\{ \sum_{r=1}^{-1} \tan^{-1}(r+2) - \tan^{-1}(r+1) \right\} \\ &= \lim_{n \rightarrow \infty} 6 \tan \{ \tan^{-1}(n+2) - \tan^{-1}2 \} \\ &= 6 \tan \left\{ \frac{\pi}{2} - \cot^{-1} \left(\frac{1}{2} \right) \right\} \\ &= 6 \tan \left(\tan^{-1} \left(\frac{1}{2} \right) \right) \\ &= 3 \end{aligned}$$

Question43

If $\lim_{x \rightarrow 1} \frac{\sin(3x^2 - 4x + 1) - x^2 + 1}{2x^3 - 7x^2 + ax + b} = -2$, then the value of (a – b) is equal to_____
[28-Jun-2022-Shift-2]

Answer: 11

Solution:

$$\begin{aligned} & \lim_{x \rightarrow 1} \frac{\left(\frac{\sin(3x^2 - 4x + 1)}{3x^2 - 4x + 1} \right) (3x^2 - 4x + 1) - x^2 + 1}{2x^3 - 7x^2 + ax + b} = -2 \\ \Rightarrow & \lim_{x \rightarrow 1} \frac{3x^2 - 4x + 1 - x^2 + 1}{2x^3 - 7x^2 + ax + b} = -2 \\ \Rightarrow & \lim_{x \rightarrow 1} \frac{2(x-1)^2}{2x^3 - 7x^2 + ax + b} = -2 \\ \text{So } & f(x) = 2x^3 - 7x^2 + ax + b = 0 \text{ has } x = 1 \text{ as repeated root, therefore } f(1) = 0 \text{ and } f'(1) = 0 \text{ gives} \\ & a + b + 5 \text{ and } a = 8 \\ \text{So, } & a - b = 11 \end{aligned}$$

Question44

The value of $\lim_{x \rightarrow 1} \frac{(x^2 - 1)\sin^2(\pi x)}{x^4 - 2x^3 + 2x - 1}$ is equal to:
[29-Jun-2022-Shift-2]

Options:

A. $\frac{\pi^2}{6}$

B. $\frac{\pi^2}{3}$

C. $\frac{\pi^2}{2}$

D. π^2

Answer: D

Solution:

Solution:

$$S = 1 + \frac{5}{6} + \frac{12}{6^2} + \frac{22}{6^3} + \dots \quad (1)$$

$$\frac{1}{6}S = \frac{1}{6} + \frac{5}{6^2} + \frac{12}{6^3} + \dots \quad (2)$$

$$S - \frac{1}{6}S = 1 + \frac{4}{6} + \frac{7}{6^2} + \frac{10}{6^3} + \dots$$

$$\Rightarrow \frac{5S}{6} = 1 + \frac{4}{6} + \frac{7}{6^2} + \frac{10}{6^3} + \dots \quad (3)$$

Now, multiplying both sides by $\frac{1}{6}$, we get

$$\Rightarrow \frac{5S}{36} = \frac{1}{6} + \frac{4}{6^2} + \frac{7}{6^3} + \frac{10}{6^4} + \dots$$

Subtract equation (4) from equation (3), we get

$$\frac{25}{36}S = 1 + \frac{3}{6} + \frac{3}{6^2} + \frac{3}{6^3} + \dots$$

$$\Rightarrow \frac{25S}{36} = 1 + \frac{\frac{3}{5}}{1 - \frac{1}{6}}$$

$$= 1 + \frac{3}{6} \times \frac{6}{5}$$

$$= 1 + \frac{3}{5} = \frac{8}{5}$$

$$\Rightarrow S = \frac{8}{5} \times \frac{36}{25} = \frac{288}{125}$$

Question45

If $y = \tan^{-1}(\sec x^3 - \tan x^3)$, $\frac{\pi}{2} < x^3 < \frac{3\pi}{2}$, then
[24-Jun-2022-Shift-2]

Options:

A. $xy'' + 2y' = 0$

B. $x^2 y'' - 6y + \frac{3\pi}{2} = 0$

C. $x^2 y'' - 6y + 3\pi = 0$

D. $xy'' - 4y' = 0$

Answer: B

Solution:

Solution:

$$\text{Let } x^3 = \theta \Rightarrow \frac{\theta}{2} \in \left(\frac{\pi}{4}, \frac{3\pi}{4} \right)$$

$$\therefore y = \tan^{-1}(\sec \theta - \tan \theta)$$

$$= \tan^{-1} \left(\frac{1 - \sin \theta}{\cos \theta} \right)$$

$$\therefore y = \frac{\pi}{4} - \frac{\theta}{2}$$

$$y = \frac{\pi}{4} - \frac{x^3}{2}$$

$$\therefore y' = \frac{-3x^2}{2}$$

$$y'' = -3x$$

$$\therefore x^2 y'' - 6y + \frac{3\pi}{2} = 0$$

Question46

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f(x) = x^3 + x - 5$. If $g(x)$ is a function such that $f(g(x)) = x$, $\forall x \in \mathbb{R}$, then $g'(63)$ is equal to
[25-Jun-2022-Shift-1]

Options:

A. $\frac{1}{49}$

B. $\frac{3}{49}$

C. $\frac{43}{49}$

D. $\frac{91}{49}$

Answer: A

Solution:

Solution:

$$f(x) = 3x^2 + 1$$

$f'(x)$ is bijective function

and $f(g(x)) = x \Rightarrow g(x)$ is inverse of $f(x)$ $g(f(x)) = x$

$$g'(f(x)) \cdot f'(x) = 1$$

$$g'(f(x)) = \frac{1}{3x^2 + 1}$$

Put $x = 4$ we get

$$g'(63) = \frac{1}{49}$$

Question47

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfy $f(x + y) = 2^x f(y) + 4^y f(x)$, $\forall x, y \in \mathbb{R}$. If $f(2) = 3$, then

14. $\frac{f'(4)}{f'(2)}$ is equal to ____

[26-Jun-2022-Shift-2]

Answer: 248

Solution:

$$\therefore f(x + y) = 2^x f(y) + 4^y f(x) \dots \dots (1)$$

$$\text{Now, } f(y + x) = 2^y f(x) + 4^x f(y) \dots \dots (2)$$

$$\therefore 2^x f(y) + 4^y f(x) = 2^y f(x) + 4^x f(y)$$

$$(4^y - 2^y) f(x) = (4^x - 2^x) f(y)$$

$$\frac{f(x)}{4^x - 2^x} = \frac{f(y)}{4^y - 2^y} = k$$

$$\therefore f(x) = k(4^x - 2^x)$$

$$\because f(2) = 3 \text{ then } k = \frac{1}{4}$$

$$\therefore f(x) = \frac{4^x - 2^x}{4}$$

$$\therefore f'(x) = \frac{4^x \ln 4 - 2^x \ln 2}{4}$$

$$f'(x) = \frac{(2 \cdot 4^x - 2^x) \ln 2}{4}$$

$$\therefore \frac{f'(4)}{f'(2)} = \frac{2 \cdot 256 - 16}{2 \cdot 16 - 4}$$

$$\therefore 14 \frac{f'(4)}{f'(2)} = 248$$

Question48

If $\cos^{-1}\left(\frac{y}{2}\right) = \log_e\left(\frac{x}{5}\right)^5$, $|y| < 2$, then :
[27-Jun-2022-Shift-1]

Options:

A. $x^2y'' + xy' - 25y = 0$

B. $x^2y'' - xy' - 25y = 0$

C. $x^2y'' - xy' + 25y = 0$

D. $x^2y'' + xy' + 25y = 0$

Answer: D

Solution:

Solution:

$$\cos^{-1}\left(\frac{y}{2}\right) = \log_e\left(\frac{x}{5}\right)^5 \quad |y| < 2$$

Differentiating on both side

$$-\frac{1}{\sqrt{1 - \left(\frac{y}{2}\right)^2}} \times \frac{y'}{2} = \frac{5}{\frac{x}{5}} \times \frac{1}{5}$$

$$\frac{-xy'}{2} = 5 \sqrt{1 - \left(\frac{y}{2}\right)^2}$$

Square on both side

$$\frac{x^2y'^2}{4} = 25 \left(\frac{4 - y^2}{4} \right)$$

Diff on both side

$$2xy'^2 + 2y'y''x^2 = -25 \times 2yy'$$

$$xy' + y''x^2 + 25y = 0$$

Question49

Let $f(x) = \begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{vmatrix}$, $a \in \mathbb{R}$. Then the sum of the squares of all the values of a , for which $2f'(10) - f'(5) + 100 = 0$, is
[27-Jun-2022-Shift-2]

Options:

A. 117

B. 106

C. 125

D. 136

Answer: C

Solution:

Solution:

$$f(x) = \begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{vmatrix}, a \in \mathbb{R}$$

$$f(x) = a(a^2 + ax) + 1(a^2x + ax^2) \\ = a(x + a)^2$$

$$f'(x) = 2a(x + a)$$

$$\text{Now, } 2f'(10) - f'(5) + 100 = 0$$

$$\Rightarrow 2 \cdot 2a(10 + a) - 2a(5 + a) + 100 = 0$$

$$\Rightarrow 2a(a + 15) + 100 = 0$$

$$\Rightarrow a^2 + 15a + 50 = 0$$

$$\Rightarrow a = -10, -5$$

$$\therefore \text{Sum of squares of values of } a = 125$$

Question 50

If $y(x) = (x^x)^x$, $x > 0$, then $\frac{d^2x}{dy^2} + 20$ at $x = 1$ is equal to ____
[27-Jun-2022-Shift-2]

Answer: 16

Solution:

Solution:

$$\because y(x) = (x^x)^x$$

$$\therefore y = x^{x^2}$$

$$\therefore \frac{dy}{dx} = x^2 \cdot x^{x^2-1} + x^{x^2} \ln x \cdot 2x$$

$$\therefore \frac{dx}{dy} = \frac{1}{x^{x^2+1}(1 + 2 \ln x)} \dots \text{ (i)}$$

$$\text{Now, } \frac{d^2x}{dy^2} = \frac{dx}{dx} \left((x^{x^2+1}(1 + 2 \ln x))^{-1} \right) \cdot \frac{dx}{dy}$$

$$= \frac{-x(x^{x^2+1}(1 + 2 \ln x))^{-2} \cdot x^{x^2}(1 + 2 \ln x)(x^2 + 2x^2 \ln x + 3)}{x^{x^2}(1 + 2 \ln x)}$$

$$= \frac{-x^{x^2}(1 + 2 \ln x)(x^3 + 3 + 2x^2 \ln x)}{(x^{x^2}(1 + 2 \ln x))^3}$$

$$\frac{d^2x}{dy^2} \text{ at } x = 1 = -4$$

$$\therefore \frac{d^2x}{dy^2(atx=1)} + 20 = 16$$

Question51

If $\lim_{n \rightarrow \infty} (\sqrt{n^2 - n - 1} + n\alpha + \beta) = 0$, then $8(\alpha + \beta)$ is equal to:

[25-Jul-2022-Shift-1]

Options:

- A. 4
- B. -8
- C. -4
- D. 8

Answer: C

Solution:

Solution:

$$\lim_{n \rightarrow \infty} (\sqrt{n^2 - n - 1} + n\alpha + \beta) = 0$$

[This limit will be zero when $\alpha < 0$ as when $\alpha > 0$ then overall limit will be ∞ .]

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{(\sqrt{n^2 - n - 1} + n\alpha + \beta)(\sqrt{n^2 - n - 1} - (n\alpha + \beta))}{\sqrt{n^2 - n - 1} - (n\alpha + \beta)} = 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{(n^2 - n - 1) - (n\alpha + \beta)^2}{\sqrt{n^2 - n - 1} - (n\alpha + \beta)} = 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{n^2 - n - 1 - n^2\alpha^2 - 2n\alpha\beta - \beta^2}{\sqrt{n^2 - n - 1} - (n\alpha + \beta)} = 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{n^2(1 - \alpha^2) - n(1 + 2\alpha\beta) - (1 + \beta^2)}{\sqrt{n^2 - n - 1} - (n\alpha + \beta)}$$

Here power of " n " in the numerator is 2 and power of " n " in the denominator is 1.

To get the value of limit equal to zero power of " n " should be equal in both numerator and denominator, otherwise value of limit will be infinite (∞).

$\therefore$ Coefficient of n^2 should be 0 in this case.

$$\therefore 1 - \alpha^2 = 0$$

$$\Rightarrow \alpha = \pm 1$$

But α should be < 0

$$\therefore \alpha = +1 \text{ not possible}$$

$$\therefore \alpha = -1$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{0 - n(1 + 2\alpha\beta) - (1 + \beta^2)}{n \left[\sqrt{1 - \frac{1}{n} - \frac{1}{n^2}} - \alpha - \frac{\beta}{n} \right]} = 0$$

Divide numerator and denominator by n then we get,

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{-(1 + 2\alpha\beta) - \frac{(1 + \beta^2)}{n}}{\sqrt{1 - \frac{1}{n} - \frac{1}{n^2}} - \alpha - \frac{\beta}{n}} = 0$$

$$\Rightarrow \frac{-(1 + 2\alpha\beta) - 0}{\sqrt{1 - 0 - 0} - \alpha - 0} = 0$$

$$\Rightarrow \frac{-(1 + 2\alpha\beta)}{1 - \alpha} = 0$$

$$\Rightarrow -(1 + 2\alpha\beta) = 0$$

$$\Rightarrow 1 + 2\alpha\beta = 0$$

$$\Rightarrow 2\alpha\beta = -1$$

$$\Rightarrow \beta = -\frac{1}{2\alpha} = -\frac{1}{2(-1)} = \frac{1}{2}$$

$$\therefore 8(\alpha + \beta)$$

$$= 8\left(-1 + \frac{1}{2}\right)$$

$$= 8 \times -\frac{1}{2}$$

$$= -4$$

Question 52

If $\lim_{n \rightarrow \infty} (\sqrt{n^2 - n - 1} + n\alpha + \beta) = 0$, then $8(\alpha + \beta)$ is equal to:
[25-Jul-2022-Shift-1]

Options:

- A. 4
- B. -8
- C. -4
- D. 8

Answer: C

Solution:

Solution:

$$\lim_{n \rightarrow \infty} (\sqrt{n^2 - n - 1} + n\alpha + \beta) = 0$$

[This limit will be zero when $\alpha < 0$ as when $\alpha > 0$ then overall limit will be ∞ .]

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{(\sqrt{n^2 - n - 1} + n\alpha + \beta)(\sqrt{n^2 - n - 1} - (n\alpha + \beta))}{\sqrt{n^2 - n - 1} - (n\alpha + \beta)} = 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{(n^2 - n - 1) - (n\alpha + \beta)^2}{\sqrt{n^2 - n - 1} - (n\alpha + \beta)} = 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{n^2 - n - 1 - n^2\alpha^2 - 2n\alpha\beta - \beta^2}{\sqrt{n^2 - n - 1} - (n\alpha + \beta)} = 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{n^2(1 - \alpha^2) - n(1 + 2\alpha\beta) - (1 + \beta^2)}{\sqrt{n^2 - n - 1} - (n\alpha + \beta)}$$

Here power of "n" in the numerator is 2 and power of "n" in the denominator is 1.

To get the value of limit equal to zero power of "n" should be equal in both numerator and denominator, otherwise value of limit will be infinite (∞).

$\therefore$ Coefficient of n^2 should be 0 in this case.

$$\therefore 1 - \alpha^2 = 0$$

$$\Rightarrow \alpha = \pm 1$$

But α should be < 0

$$\therefore \alpha = +1 \text{ not possible}$$

$$\therefore \alpha = -1$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{0 - n(1 + 2\alpha\beta) - (1 + \beta)}{n \left[\sqrt{1 - \frac{1}{n} - \frac{1}{n^2}} - \alpha - \frac{\beta}{n} \right]} = 0$$

Divide numerator and denominator by n then we get,

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{-(1 + 2\alpha\beta) - \frac{(1 + \beta)}{n}}{\sqrt{1 - \frac{1}{n} - \frac{1}{n^2}} - \alpha - \frac{\beta}{n}} = 0$$

$$\Rightarrow \frac{-(1 + 2\alpha\beta) - 0}{\sqrt{1 - 0 - 0} - \alpha - 0} = 0$$

$$\Rightarrow \frac{-(1+2\alpha\beta)}{1-\alpha} = 0$$

$$\Rightarrow -(1+2\alpha\beta) = 0$$

$$\Rightarrow 1+2\alpha\beta = 0$$

$$\Rightarrow 2\alpha\beta = -1$$

$$\Rightarrow \beta = -\frac{1}{2\alpha} = -\frac{1}{2(-1)} = \frac{1}{2}$$

$$\therefore 8(\alpha + \beta)$$

$$= 8\left(-1 + \frac{1}{2}\right)$$

$$= 8 \times -\frac{1}{2}$$

$$= -4$$

Question53

If $\lim_{n \rightarrow \infty} \frac{(n+1)^{k-1}}{n^{k+1}} [(nk+1) + (nk+2) + \dots + (nk+n)]$, then the integral value of k is $= 33 \cdot \lim_{n \rightarrow \infty} \frac{1}{n^{k+1}} \cdot [1^k + 2^k + 3^k + \dots + n^k]$ equal to
[25-Jul-2022-Shift-1]

Answer: 5

Solution:

Solution:

$$\lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^{k-1} \frac{1}{n} \sum_{r=1}^n \left(k + \frac{r}{n} \right) = 33 \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \left(\frac{r}{n} \right)^k$$

$$\Rightarrow \int_0^1 (k+x) dx = 33 \int_0^1 x^k dx$$

$$\Rightarrow \frac{2k+1}{2} = \frac{33}{k+1}$$

$$\Rightarrow k = 5$$

Question54

$\lim_{x \rightarrow \frac{\pi}{4}} \frac{8\sqrt{2} - (\cos x + \sin x)^7}{\sqrt{2} - \sqrt{2} \sin 2x}$ is equal to
[25-Jul-2022-Shift-2]

Options:

A. 14

B. 7

C. $14\sqrt{2}$

D. $7\sqrt{2}$

Answer: A

Solution:

Solution:

$$\begin{aligned} \lim_{x \rightarrow \frac{\pi}{4}} \frac{8\sqrt{2} - (\cos x + \sin x)^7}{\sqrt{2} - \sqrt{2} \sin 2x} & \left(\frac{0}{0} \text{ form} \right) \\ &= \lim_{x \rightarrow \frac{\pi}{4}} \frac{-7(\cos x + \sin x)^6(-\sin x + \cos x)}{-2\sqrt{2} \cos 2x} \text{ using L - H Rule} \\ &= \lim_{x \rightarrow \frac{\pi}{4}} \frac{56(\cos x - \sin x)}{2\sqrt{2} \cos 2x} \left(\frac{0}{0} \right) \\ &= \lim_{x \rightarrow \frac{\pi}{4}} \frac{-56(\sin x + \cos x)}{-4\sqrt{2} \sin 2x} \text{ using L - H Rule} \\ &= 7\sqrt{2} \cdot \sqrt{2} = 14 \end{aligned}$$

Question55

If the function

$$f(x) = \begin{cases} \frac{\log_e(1-x+x^2) + \log_e(1+x+x^2)}{\sec x - \cos x}, & x \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right) - \{0\} \\ k, & x = 0. \end{cases}$$

is continuous at $x = 0$, then k is equal to:
[26-Jul-2022-Shift-1]

Options:

- A. 1
- B. -1
- C. e
- D. 0

Answer: A

Solution:

$$f(x) = \begin{cases} \frac{\log_e(1-x+x^2) + \log_e(1+x+x^2)}{\sec x - \cos x} & x \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right) - \{0\} \\ k & x = 0. \end{cases}$$

for continuity at $x = 0$

$$\lim_{x \rightarrow 0} f(x) = k$$

$$\therefore k = \lim_{x \rightarrow 0} \frac{\log_e(x^4 + x^2 + 1)}{\sec x - \cos x} \left(\frac{0}{0} \text{ form} \right)$$

$$= \lim_{x \rightarrow 0} \frac{\cos x \log_e(x^4 + x^2 + 1)}{\sin^2 x}$$

$$= \lim_{x \rightarrow 0} \frac{\log_e(x^4 + x^2 + 1)}{x^2}$$

$$= \lim_{x \rightarrow 0} \frac{\ln(1 + x^2 + x^4)}{x^2 + x^4} \cdot \frac{x^2 + x^4}{x^2}$$

$$= 1$$

Question56

Let $\beta = \lim_{x \rightarrow 0} \frac{\alpha x - (e^{3x} - 1)}{\alpha x(e^{3x} - 1)}$ for some $\alpha \in \mathbb{R}$. Then the value of $\alpha + \beta$ is :

[26-Jul-2022-Shift-2]

Options:

A. $\frac{14}{5}$

B. $\frac{3}{2}$

C. $\frac{5}{2}$

D. $\frac{7}{2}$

Answer: C

Solution:

Solution:

$$\beta = \lim_{x \rightarrow 0} \frac{\alpha x - (e^{3x} - 1)}{\alpha x (e^{3x} - 1)}, \alpha \in \mathbb{R}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{\alpha}{3} - \left(\frac{e^{3x} - 1}{3x} \right)}{\alpha x \left(\frac{e^{3x} - 1}{3x} \right)}$$

So, $\alpha = 3$ (to make independent form)

$$\begin{aligned} \beta &= \lim_{x \rightarrow 0} \frac{1 - \left(\frac{e^{3x} - 1}{3x} \right)}{3x} = \frac{1 - \left(3x + \frac{9x^2}{2} + \dots \right)}{3x} \\ &= \frac{-\left(\frac{9}{2}x^2 + \frac{(3x)^3}{31} + \dots \right)}{9x^2} = \frac{-1}{2} \end{aligned}$$

$$\therefore \alpha + \beta = 3 - \frac{1}{2} = \frac{5}{2}$$

Question57

$$\lim_{x \rightarrow 0} \left(\frac{(x + 2 \cos x)^3 + 2(x + 2 \cos x)^2 + 3 \sin(x + 2 \cos x)}{(x + 2)^3 + 2(x + 2)^2 + 3 \sin(x + 2)} \right)^{\frac{100}{x}} \text{ is equal to } \underline{\hspace{2cm}}.$$

[28-Jul-2022-Shift-1]

Answer: 1

Solution:

Let $x + 2 \cos x = a$

$x + 2 = b$

as $x \rightarrow 0$, $a \rightarrow 2$ and $b \rightarrow 2$

$$\lim_{x \rightarrow 0} \left(\frac{a^3 + 2a^2 + 3 \sin a}{b^3 + 2b^2 + 3 \sin b} \right)^{\frac{100}{x}}$$

$$= e^{\lim_{x \rightarrow 0} \frac{100}{x} \cdot \frac{(a^3 - b^3) + 2(a^2 - b^2) + 3(\sin a - \sin b)}{b^3 + 2b^2 + 3 \sin b}}$$

$$\begin{aligned} \therefore \lim_{x \rightarrow 0} \frac{a - b}{x} &= \lim_{x \rightarrow 0} \frac{2(\cos x - 1)}{x} = 0 \\ &= e^0 \\ &= 1 \end{aligned}$$

Question58

If $\lim_{x \rightarrow 0} \frac{e^x + \beta e^{-x} + \gamma \sin x}{x \sin^2 x} = \frac{2}{3}$, where $\alpha, \beta, \gamma \in \mathbb{R}$, then which of the following is

NOT correct?

[29-Jul-2022-Shift-1]

Options:

A. $\alpha^2 + \beta^2 + \gamma^2 = 6$

B. $\alpha\beta + \beta\gamma + \gamma\alpha + 1 = 0$

C. $\alpha\beta^2 + \beta\gamma^2 + \gamma\alpha^2 + 3 = 0$

D. $\alpha^2 - \beta^2 + \gamma^2 = 4$

Answer: C

Solution:

Solution:

$$\lim_{x \rightarrow 0} \frac{\alpha e^x + \beta e^{-x} + \gamma \sin x}{x \sin^2 x} = \frac{2}{3}$$

$\Rightarrow \alpha + \beta = 0$ (to make indeterminate form) (i)

Now,

$$\lim_{x \rightarrow 0} \frac{\alpha e^x - \beta e^{-x} + \gamma \cos x}{3x^2} = \frac{2}{3} \text{ (Using L-H Rule)}$$

$\Rightarrow \alpha - \beta + \gamma = 0$ (to make indeterminate form) (ii)

Now,

$$\lim_{x \rightarrow 0} \frac{\alpha e^x + \beta e^{-x} - \gamma \sin x}{6x} = \frac{2}{3} \text{ (Using L-H Rule)}$$

$$\Rightarrow \frac{\alpha - \beta + \gamma}{6} = \frac{2}{3}$$

$\Rightarrow \alpha - \beta + \gamma = 4$ (iii)

$\Rightarrow \gamma = -2$

and (i) + (ii)

$$2\alpha = -\gamma$$

$\Rightarrow \alpha = 1$ and $\beta = -1$

$$\text{and } \alpha\beta^2 + \beta\gamma^2 + \gamma\alpha^2 + 3 = 1 - 4 - 2 + 3 = -2$$

Question59

The value of $\log_e 2 \frac{d}{dx}(\log_{\cos x} \operatorname{cosec} x)$ at $x = \frac{\pi}{4}$ is
[26-Jul-2022-Shift-2]

Options:

A. $-2\sqrt{2}$

B. $2\sqrt{2}$

C. -4

D. 4

Answer: D

Solution:

Solution:

Let $f(x) = \log_{\cos x} \operatorname{cosec} x$

$$= \frac{\log \operatorname{cosec} x}{\log \cos x}$$

$$\Rightarrow f'(x) = \frac{\log \cos x \cdot \sin x \cdot \left(-\operatorname{cosec} x \cot x - \log \operatorname{cosec} x \cdot \frac{1}{\cos x} \cdot -\sin x \right)}{(\log \cos x)^2}$$

$$\text{at } x = \frac{\pi}{4}$$

$$f'\left(\frac{\pi}{4}\right) = \frac{-\log\left(\frac{1}{\sqrt{2}}\right) + \log \sqrt{2}}{\left(\log \frac{1}{\sqrt{2}}\right)^2} = \frac{2}{\log \sqrt{2}}$$

$$\therefore \log_e 2 f'(x) \text{ at } x = \frac{\pi}{4} = 4$$

Question60

Let $x(t) = 2\sqrt{2} \cos t \sqrt{\sin 2t}$ and

$y(t) = 2\sqrt{2} \sin t \sqrt{\sin 2t}$, $t \in \left(0, \frac{\pi}{2}\right)$

Then $\frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}$ at $t = \frac{\pi}{4}$ is equal to:

[28-Jul-2022-Shift-2]

Options:

A. $\frac{-2\sqrt{2}}{3}$

B. $\frac{2}{3}$

C. $\frac{1}{3}$

D. $\frac{-2}{3}$

Answer: D

Solution:

Solution:

$$x = 2\sqrt{2} \cos t \sqrt{\sin 2t}, y = 2\sqrt{2} \sin t \sqrt{\sin 2t}$$

$$\therefore \frac{dx}{dt} = \frac{2\sqrt{2} \cos 3t}{\sqrt{\sin 2t}}, \frac{dy}{dt} = \frac{2\sqrt{2} \sin 3t}{\sqrt{\sin 2t}}$$

$$\therefore \frac{dy}{dx} = \tan 3t, \left(\text{at } t = \frac{\pi}{4}, \frac{dy}{dx} = -1 \right)$$

$$\text{and } \frac{d^2y}{dx^2} = 3 \sec^2 3t \cdot \frac{dt}{dx} = \frac{3 \sec^2 3t \cdot \sqrt{\sin 2t}}{2\sqrt{2} \cos 3t}$$

$$\left(\therefore \text{At } t = \frac{\pi}{4}, \frac{d^2y}{dx^2} = -3 \right)$$

$$\therefore \frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} = \frac{2}{-3} = \frac{-2}{3}$$

Question61

For the curve C : $(x^2 + y^2 - 3) + (x^2 - y^2 - 1)^5 = 0$, the value of $3y' - y^3y''$, at the point (α, α) , $\alpha > 0$, on C, is equal to _____.
[27-Jul-2022-Shift-2]

Answer: 16

Solution:

Solution:

$\because C : (x^2 + y^2 - 3) + (x^2 - y^2 - 1)^5 = 0$ for point (α, α)

$$\alpha^2 + \alpha^2 - 3 + (\alpha^2 - \alpha^2 - 1)^5 = 0$$

$$\therefore \alpha = \sqrt{2}$$

On differentiating $(x^2 + y^2 - 3) + (x^2 - y^2 - 1)^5 = 0$ we get

$$x + yy' + 5(x^2 - y^2 - 1)^4(x - yy') = 0$$

$$\text{When } x = y = \sqrt{2} \text{ then } y' = \frac{3}{2}$$

Again on differentiating eq. (i) we get :

$$1 + (y')^2 + yy'' + 20(x^2 - y^2 - 1)(2x - 2yy')(x - yy') + 5(x^2 - y^2 - 1)^4(1 - y'^2 - yy'') = 0$$

$$\text{For } x = y = \sqrt{2} \text{ and } y' = \frac{3}{2} \text{ we get } y'' = -\frac{23}{4\sqrt{2}}$$

$$\therefore 3y' - y^3y'' = 3 \cdot \frac{3}{2} - (\sqrt{2})^3 \cdot \left(-\frac{23}{4\sqrt{2}}\right) = 16$$

Question62

If $\lim_{x \rightarrow 0} \frac{ax - (e^{4x} - 1)}{ax(e^{4x} - 1)}$ exists and is equal to b, then the value of $a - 2b$ is
[2021, 25 Feb. Shift-11]

Answer: 5

Solution:

$$\lim_{x \rightarrow 0} \frac{ax - (e^{4x} - 1)}{ax(e^{4x} - 1)} = L \text{ (say) } \left[\frac{0}{0} \text{ form} \right]$$

Apply L - Hospital rule,

$$L = \lim_{x \rightarrow 0} \frac{a - 4e^{4x}}{a(e^{4x} - 1) + ax(4e^{4x})}$$

[Limit exist everywhere except $a = 4$]

Again, apply L-Hospital rule,

$$L = \lim_{x \rightarrow 0} \frac{-16e^{4x}}{a(4e^{4x}) + a(4e^{4x}) + ax(16e^{4x})}$$

$$= \frac{-16}{4a + 4a} = \frac{-2}{a}$$

$$= \frac{-2}{4} = \frac{-1}{2} \quad (\text{use } a = 4)$$

Given, $L = b$

$$\Rightarrow \frac{-2}{a} = \frac{-1}{2} = b$$

Then, $a - 2b = 4 - 2\left(\frac{-1}{2}\right) = 4 + 1 = 5$

Question63

$\lim_{n \rightarrow \infty} 1 + \left(\frac{1 + \frac{1}{2} + \dots + \frac{1}{n}}{n^2} \right)^n$ is equal to

[2021, 25 Feb. Shift-1]

Options:

A. $\frac{1}{2}$

B. 0

C. $\frac{1}{e}$

D. 1

Answer: D

Solution:

Solution:

Given, limit form is 1^∞ .

$$L = e^{\lim_{n \rightarrow \infty} \left(\frac{1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}}{n} \right)}$$

$$S = 1 + \left(\frac{1}{2} + \frac{1}{3} \right) + \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} \right) + \dots$$

Clearly,

$$S < 1 + \left(\frac{1}{2} + \frac{1}{2} \right) + \left(\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} \right) + \dots +$$

$$\left(\left(\frac{1}{2^n} + \dots + \frac{1}{2^n} \right)_{2^n \text{ times}} \right)$$

$$S < 1 + 1 + 1 + 1 + \dots + 1$$

$$S < n + 1$$

$$\therefore L = e^{\lim_{n \rightarrow \infty} \left(\frac{n+1}{2^{n+1}-1} \right)} \Rightarrow L = e^0$$

$$\therefore L = 1$$

Question64

$\lim_{x \rightarrow 0} \frac{\int_0^x (\sin \sqrt{t}) dt}{x^3}$ is equal to

[2021, 24 Feb. Shift-1]

Options:

A. $\frac{2}{3}$

B. $\frac{3}{2}$

C. $\frac{1}{15}$

D. 0

Answer: A

Solution:

Solution:

Given, $\lim_{x \rightarrow 0} \frac{\int_0^{x^2} (\sin \sqrt{t}) dt}{x^3}$

$\therefore$ It is of the form $\frac{0}{0}$.

By differentiating numerator and denominator,

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin \sqrt{x^2} \cdot 2x}{3x^2} &= \lim_{x \rightarrow 0} \frac{\sin x \cdot 2x}{3x^2} \\ &= \frac{2}{3} \lim_{x \rightarrow 0} \frac{\sin x}{x} = \frac{2}{3}(1) = \frac{2}{3} \end{aligned}$$

Question65

$\lim_{n \rightarrow \infty} \tan \left\{ \sum_{r=1}^n \tan^{-1} \left(\frac{1}{1+r+r^2} \right) \right\}$ is equal to

[2021, 24 Feb. Shift-I]

Answer: 1

Solution:

Solution:

Given, $\lim_{n \rightarrow \infty} \tan \left\{ \sum_{r=1}^n \tan^{-1} \left(\frac{1}{1+r+r^2} \right) \right\}$

$$= \tan \left(\lim_{n \rightarrow \infty} \sum_{r=1}^n [\tan^{-1}(r+1) - \tan^{-1}r] \right)$$

$$= \tan \left(\lim_{n \rightarrow \infty} \left(\tan^{-1}(n+1) - \frac{\pi}{4} \right) \right)$$

$$= \tan \left(\frac{\pi}{2} - \frac{\pi}{4} \right) = \tan \frac{\pi}{4} = 1$$

Hence, the required value is 1.

Question66

The value of $\lim_{h \rightarrow 0} 2 \left\{ \frac{\sqrt{3} \sin\left(\frac{\pi}{6} + h\right) - \cos\left(\frac{\pi}{6} + h\right)}{\sqrt{3}h(\sqrt{3} \cos h - \sin h)} \right\}$

[2021, 26 Feb. Shift-1]

Options:

- A. $\frac{4}{3}$
- B. $\frac{2}{\sqrt{3}}$
- C. $\frac{3}{4}$
- D. $\frac{2}{3}$

Answer: A

Solution:

Solution:

$$\begin{aligned}
 & \lim_{h \rightarrow 0} 2 \left\{ \frac{\sqrt{3} \sin\left(\frac{\pi}{6} + h\right) - \cos\left(\frac{\pi}{6} + h\right)}{\sqrt{3}h(\sqrt{3} \cos h - \sin h)} \right\} \\
 &= \lim_{h \rightarrow 0} 2 \left\{ \frac{2 \left(\frac{\sqrt{3}}{2} \sin\left(\frac{\pi}{6} + h\right) - \frac{1}{2} \cos\left(\frac{\pi}{6} + h\right) \right)}{2 \times \sqrt{3}h \left(\frac{\sqrt{3}}{2} \cos h - \frac{1}{2} \sin h \right)} \right\} \\
 &= \lim_{h \rightarrow 0} 2 \left\{ \frac{\cos\left(\frac{\pi}{6}\right) \sin\left(\frac{\pi}{6} + h\right)}{\sqrt{3}h \left(\cos \frac{\pi}{6} \cos h - \sin \frac{\pi}{6} \sin h \right)} \right\} \\
 &= \lim_{h \rightarrow 0} 2 \left\{ \frac{\cos\left(\frac{\pi}{6} + h\right)}{\frac{\cos\left(\frac{\pi}{6} + h\right)}{6}} \right\} \\
 &= \lim_{h \rightarrow 0} 2 \left\{ \frac{\sin\left(\frac{\pi}{6} + h - \frac{\pi}{6}\right)}{\sqrt{3}h \cos\left(h + \frac{\pi}{6}\right)} \right\} \\
 &= \lim_{h \rightarrow 0} \frac{2}{\sqrt{3}} \left\{ \frac{\sinh}{h \cos(h + \pi/6)} \right\} \\
 &= \frac{2}{\sqrt{3}} \cdot \lim_{h \rightarrow 0} \frac{\sinh}{h} \cdot \lim_{h \rightarrow 0} \frac{1}{\cos(h + \pi/6)} \\
 &= \frac{2}{\sqrt{3}} \cdot (1) \cdot \frac{1}{\cos(\pi/6)} \\
 &= \frac{2}{\sqrt{3}} \cdot 1 \cdot \frac{2}{\sqrt{3}} = \frac{4}{3}
 \end{aligned}$$

Question67

The value of $\lim_{n \rightarrow \infty} \frac{[r] + [2r] + \dots + [nr]}{n^2}$, where r is non-zero real number and [r] denotes the greatest integer less than or equal to r, is equal to
[2021, 17 March Shift-11]

Options:

A. $\frac{r}{2}$

B. r

C. $2r$

D. 0

Answer: A

Solution:

Solution:

As, we know that,

$$r \leq [r] < r + 1$$

$$2r \leq [2r] < 2r + 1$$

$$3r \leq [3r] < 3r + 1$$

$$\vdots \quad \vdots \quad \vdots$$

$$nr \leq [nr] < nr + 1$$

$$\text{Adding } (r + 2r + 3r + 4r + \dots + nr) \leq [r] + [2r] + [3r]$$

$$+ [4r] + \dots + [nr] < (r + 1) + (2r + 1) + (3r + 1)$$

$$+ (4r + 1) + \dots + (nr + 1)$$

$$\Rightarrow r(1 + 2 + 3 + 4 + \dots + n) \leq [r] + [2r]$$

$$+ [3r] + \dots + [nr]$$

$$< (r + 2r + 3r + \dots + nr) + (1 + 1 + 1 + \dots + 1)_{n- \text{ times}}$$

$$\Rightarrow r \cdot \frac{n(n+1)}{2} \leq [r] + [2r] + [3r] + \dots + [nr]$$

$$< \frac{r \cdot (n(n+1))}{2} + n$$

$$\Rightarrow \frac{r \cdot \left(\frac{n(n+1)}{2} \right)}{n^2}$$

$$\leq \frac{[r] + [2r] + [3r] + \dots + [nr]}{n^2}$$

$$\text{Now, } \lim_{n \rightarrow \infty} \frac{n(n+1) \cdot r}{2 \cdot n^2} = \lim_{n \rightarrow \infty} \frac{n \cdot n \left(1 + \frac{1}{n} \right) \cdot r}{2n^2}$$

$$\text{and } \lim_{n \rightarrow \infty} \frac{\frac{(1+0) \cdot r}{2}}{2} = \frac{r}{2}$$

$$= \lim_{n \rightarrow \infty} \frac{n \left(n + \frac{n(n+1)}{2} \right) + n}{n^2}$$

$$= \lim_{n \rightarrow \infty} \frac{n^2}{2n^2 \left\{ \left(1 + \frac{1}{n} \right) \cdot r + 2n + \frac{2}{n} \right\}} \dots \left(i \right)$$

$$= \frac{(1+0) \cdot r + 0}{2} = \frac{r}{2} \dots (ii)$$

$$\text{From Eqs. (i) and (ii), by Sandwich theorem, we conclude that, } \lim_{n \rightarrow \infty} \frac{[r] + [2r] + [3r] + \dots + [nr]}{n^2} = \frac{r}{2}$$

Sandwich Theorem

$$\Rightarrow \text{Let } g(x) \leq f(x) \leq h(x)$$

$$\text{and } \lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} h(x) = 1$$

$$\therefore \lim_{x \rightarrow a} f(x) = 1$$

Question68

If $\lim_{x \rightarrow 0} \frac{ae^x - b \cos x + ce^{-x}}{x \sin x} = 2$, then $a + b + c$ is equal to

[2021, 16 March Shift-1]

Answer: 4

Solution:

Solution:

We have, $\lim_{x \rightarrow 0} \frac{ae^x - b \cos x + ce^{-x}}{x \sin x} = 2$

$$\Rightarrow a \left(1 + x + \frac{x^2}{2!} \dots \right) - b \left(1 - \frac{x^2}{2} + \frac{x^4}{4!} \dots \right)$$

$$+ c \left(1 - x + \frac{x^2}{2!} \dots \right)$$

$$\lim_{x \rightarrow 0} \frac{\quad}{x \left(x - \frac{x^3}{3!} + \dots \right)}$$

$$(a - b + c) + (a - c)x$$

$$+ \left(\frac{a}{2} + \frac{b}{2} + \frac{c}{2} \right) x^2 + \dots$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\quad}{x^2 - \frac{x^4}{6} + \dots}$$

Here, in numerator, all the coefficients of x^k , where $k < 2$ has to be zero, then only limit will exist.

$$a - b + c = 0$$

$$a - c = 0$$

$$\Rightarrow a = c$$

$$\Rightarrow b = 2a$$

After solving limit,

$$\frac{a + b + c}{2} = 2$$

$$\text{So, } a + 2a + a = 4$$

$$\Rightarrow a = 1$$

$$\therefore a = 1, b = 2 \text{ and } c = 1$$

$$a + b + c = 1 + 2 + 1 = 4$$

Question69

If $\lim_{x \rightarrow 0} \frac{\sin^{-1}x - \tan^{-1}x}{3x^3}$ is equal to L, then the value of $(6L + 1)$ is

[2021, 18 March Shift-1]

Options:

A. $\frac{1}{6}$

B. $\frac{1}{2}$

C. 6

D. 2

Answer: D

Solution:

Solution:

$$\text{Given, } L = \lim_{x \rightarrow 0} \frac{\sin^{-1}x - \tan^{-1}x}{3x^3}$$

$$\begin{aligned}
&\Rightarrow L \\
&\left(x + \frac{1^2 \cdot x^3}{3!} + \frac{1^2 \cdot 3^2 \cdot x^5}{5!} + \dots \right) \\
&= \lim_{x \rightarrow 0} \frac{- \left(x - \frac{x^3}{3} + \frac{x^5}{5} \dots \right)}{3x^3} \\
&\Rightarrow (\text{ using expansion of } \sin^{-1}x \text{ and } \tan^{-1}x) \\
&= \lim_{x \rightarrow 0} \frac{\left(\frac{x^3}{3!} + \frac{9x^5}{5!} + \dots \right) - \left(-\frac{x^3}{3} + \frac{x^5}{5} \dots \right)}{3x^3} \\
&\Rightarrow L \\
&= \lim_{x \rightarrow 0} \frac{x^3 \left[\left(\frac{1}{6} + \frac{9x^2}{120} + \dots \right) + \left(\frac{1}{3} - \frac{x^2}{5} + \dots \right) \right]}{3x^3} \\
&\Rightarrow L = \frac{\frac{1}{6} + \frac{1}{3}}{3} = \frac{\frac{1+2}{6}}{3} = \frac{1}{6} \\
&\therefore 6L + 1 = 6 \times \frac{1}{6} + 1 = 2
\end{aligned}$$

Question70

The value of $\lim_{x \rightarrow 0^+} \frac{\cos^{-1}(x - [x]^2) \cdot \sin^{-1}(x - [x]^2)}{x - x^3}$ where $[x]$ denotes the greatest integer $\leq x$ is
[2021, 17 March Shift-1]

Options:

A. π

B. 0

C. $\frac{\pi}{4}$

D. $\frac{\pi}{2}$

Answer: D

Solution:

Solution:

$$\begin{aligned}
&\lim_{x \rightarrow 0^+} \frac{\cos^{-1}(x - [x]^2) \cdot \sin^{-1}(x - [x]^2)}{x - x^3} \\
&x \rightarrow 0 + h \\
&= \lim_{h \rightarrow 0} \frac{\cos^{-1}(h - 0) \cdot \sin^{-1}(h - 0)}{h - h^3} \\
&= \lim_{h \rightarrow 0} \frac{\cos^{-1}h \cdot \sin^{-1}h}{h(1 - h)(1 + h)} \\
&= \lim_{h \rightarrow 0} \left(\frac{\sin^{-1}h}{h} \right) \left[\frac{\cos^{-1}h}{(1 - h)(1 + h)} \right] = 1 \cdot \frac{\pi}{2} \\
&\text{RHL} = \frac{\pi}{2}
\end{aligned}$$

Question71

The value of the limit $\lim_{\theta \rightarrow 0} \frac{\tan(\pi \cos^2 \theta)}{\sin(2\pi \sin^2 \theta)}$ is equal to
[2021, 17 March Shift-II]

Options:

A. $-\frac{1}{2}$

B. $-\frac{1}{4}$

C. 0

D. $\frac{1}{4}$

Answer: A

Solution:

Solution:

Method (I)

$$\begin{aligned} \text{Let } L &= \lim_{\theta \rightarrow 0} \left(\frac{\tan(\pi \cos^2 \theta)}{\sin(2\pi \sin^2 \theta)} \right) \\ &= \lim_{\theta \rightarrow 0} \left(\frac{\tan[\pi(1 - \sin^2 \theta)]}{\sin(2\pi \sin^2 \theta)} \right) \\ &= \lim_{\theta \rightarrow 0} \left(\frac{\tan(\pi - \pi \sin^2 \theta)}{\sin(2\pi \sin^2 \theta)} \right) \\ &= \lim_{\theta \rightarrow 0} \left(\frac{-\tan(\pi \sin^2 \theta)}{\sin(2\pi \sin^2 \theta)} \right) \\ &= \lim_{\theta \rightarrow 0} \left[\frac{\frac{-\tan(\pi \sin^2 \theta)}{(\pi \sin^2 \theta)} \times (\pi \sin^2 \theta)}{\frac{\sin(2\pi \sin^2 \theta)}{(2\pi \sin^2 \theta)} \times (2\pi \sin^2 \theta)} \right] = \frac{-1}{2} \end{aligned}$$

Method (II)

$$\begin{aligned} \text{Let } L &= \lim_{\theta \rightarrow 0} \left[\frac{\tan(\pi \cos^2 \theta)}{\sin(2\pi \sin^2 \theta)} \right] \left(\text{Form } \frac{0}{0} \right) \\ [\text{Using L-Hospital Rule}] \\ L &= \lim_{\theta \rightarrow 0} \frac{\sec^2(\pi \cos^2 \theta)(-2\pi \cos \theta \cdot \sin \theta)}{\cos(2\pi \sin^2 \theta) \cdot (4\pi \sin \theta \cdot \cos \theta)} \\ &= \frac{-1}{2} \times \frac{(-1)^2}{1} = \frac{-1}{2} \end{aligned}$$

Question 72

If $f(x) = \sin \left(\cos^{-1} \left(\frac{1 - 2^{2x}}{1 + 2^{2x}} \right) \right)$ and its first derivative with respect to x is $-\frac{b}{a} \log_e 2$ when $x = 1$, where a and b are integers, then the minimum value of $|a^2 - b^2|$ is

[2021, 17 March Shift-1]

Answer: 481

Solution:

Solution:

$$f(x) = \sin \left(\cos^{-1} \left(\frac{1 - 2^{2x}}{1 + 2^{2x}} \right) \right)$$

Let 2^{2x} be $\tan^2 \theta$.

$$\therefore f(x) = \sin \left[\cos^{-1} \left(\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \right) \right]$$

$$= \sin[\cos^{-1}(\cos 2\theta)] = \sin 2\theta$$

$$= \frac{2 \tan \theta}{1 + \tan^2 \theta} = \frac{2 \cdot 2^x}{1 + 2^{2x}}$$

$$f(x) = 2 \cdot \left(\frac{2^x}{1 + 2^{2x}} \right)$$

$$f'(x) = 2 \cdot \left[\frac{(1 + 2^{2x})(2^x \log 2) - 2^x(2^{2x} \log 2 \cdot 2)}{(1 + 2^{2x})^2} \right]$$

$$f'(1) = 2 \left(\frac{5 \cdot 2 \log 2 - 2 \cdot 8 \log 2}{5^2} \right)$$

$$= \left(-\frac{12}{25} \right) \log 2$$

$$= \frac{-b}{a} \log e 2$$

$$\Rightarrow a = 25 \text{ and } b = 12$$

$$\therefore |a^2 - b^2|_{\min} = |25^2 - 12^2| = 481$$

Question 73

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function such that $f(2) = 4$ and $f'(2) = 1$. Then, the value of $\lim_{x \rightarrow 2} \frac{x^2 f(2) - 4f(x)}{x - 2}$ is equal to

[2021, 27 July Shift-1]

Options:

A. 4

B. 8

C. 16

D. 12

Answer: D

Solution:

Solution:

$$f(2) = 4, f'(2) = 1$$

$$\text{Now, } \lim_{x \rightarrow 2} \frac{x^2 f(2) - 4f(x)}{x - 2}$$

Applying L-Hospital Rule as $\frac{0}{0}$ form on putting $x = 2$

$$\text{So, } \lim_{x \rightarrow 2} \frac{2xf(2) - 4f'(x)}{1}$$

$$= 2 \cdot 2 \cdot f(2) - 4f'(2)$$

$$= 4 \cdot 4 - 4 \cdot 1 = 12$$

Question74

If $\lim_{x \rightarrow 0} \frac{\alpha x e^x - \beta \log_e(1+x) + \gamma x^2 e^{-x}}{x \sin^2 x} = 10$, $\alpha, \beta, \gamma \in \mathbb{R}$, then the value of $\alpha + \beta + \gamma$ is
[2021, 20 July Shift-2]

Answer: 3

Solution:

Solution:

$$\lim_{x \rightarrow 0} \frac{\alpha x e^x - \beta \log_e(1+x) + \gamma x^2 e^{-x}}{x \sin^2 x} = 10$$

$$\begin{aligned} \text{Now, } \lim_{x \rightarrow 0} \frac{\alpha x e^x - \beta \log_e(1+x) + \gamma x^2 e^{-x}}{x \sin^2 x} \\ \alpha x(1+x+x^2/2+\dots) - \beta \\ = \lim_{x \rightarrow 0} \frac{\left(x - \frac{x^2}{2} + \frac{x^3}{3} \dots\right) - \gamma x^2(1-x+x^2/2\dots)}{x \sin^2 x} \end{aligned}$$

$$\begin{aligned} x(\alpha + \beta) + x^2(\alpha + \beta/2 + \gamma) + x^3 \\ = \lim_{x \rightarrow 0} \frac{\left(\frac{\alpha}{2} - \frac{\beta}{3} - \gamma\right)x \dots}{x \sin^2 x} \end{aligned}$$

For limit to exist, the numerator must have degree greater than or equal to denominator.

Degree of denominator = 3

$\therefore$ For limit to exist, $\alpha - \beta = 0 \dots$ (i)

and $\alpha + \frac{\beta}{2} + \gamma = 0 \dots$ (ii)

Also, for terms greater than degree '3', gives 0 as $x \rightarrow 0$

$\therefore \frac{\alpha}{2} - \frac{\beta}{3} - \gamma = 10 \dots$ (iii)

From Eq. (i), $\beta = \alpha$

From Eq. (ii),

$$\gamma = \left(\frac{\alpha}{2} + \alpha\right) = -\frac{3\alpha}{2}$$

Putting these in Eq. (iii),

$$\frac{\alpha}{2} - \frac{\alpha}{3} + \frac{3\alpha}{2} = 10 \Rightarrow \frac{3\alpha - 2\alpha + 9\alpha}{6} = 10$$

$$\Rightarrow \frac{10\alpha}{6} = 10 \Rightarrow \alpha = 6$$

Now, $\alpha = \beta \Rightarrow \beta = 6$

$$\text{Again, } \gamma = \frac{-3\alpha}{2} \Rightarrow \gamma = -9$$

$$\therefore \alpha + \beta + \gamma = 6 + 6 - 9 = 12 - 9 = 3$$

Question75

The value of $\lim_{x \rightarrow 0} \left(\frac{x}{\sqrt[8]{1-\sin x} - \sqrt[8]{1+\sin x}} \right)$ is equal to
[2021, 27 July Shift-II]

Options:

- A. 0
- B. 4
- C. -4
- D. -1

Answer: C

Solution:

Solution:

$$\lim_{x \rightarrow 0} \left(\frac{x}{\sqrt[8]{1 - \sin x} - \sqrt[8]{1 + \sin x}} \right)$$

Rationalise denominator three times, $\lim_{x \rightarrow 0} \left(\frac{x}{\sqrt[8]{1 - \sin x} - \sqrt[8]{1 + \sin x}} \right)$

$$\left(\frac{\sqrt[8]{1 - \sin x} + \sqrt[8]{1 + \sin x}}{\sqrt[8]{1 - \sin x} + \sqrt[8]{1 + \sin x}} \right)$$

$$\left(\frac{\sqrt[4]{1 - \sin x} + \sqrt[4]{1 + \sin x}}{\sqrt[4]{1 - \sin x} + \sqrt[4]{1 + \sin x}} \right)$$

$$\left(\frac{\sqrt{1 - \sin x} + \sqrt{1 + \sin x}}{\sqrt{1 - \sin x} + \sqrt{1 + \sin x}} \right)$$

$$= \lim_{x \rightarrow 0} \left[\frac{x}{(1 - \sin x) - (1 + \sin x)} \right]$$

$$\left(\frac{\sqrt[8]{1 - \sin x} + \sqrt[8]{1 + \sin x}}{\sqrt[4]{1 - \sin x} + \sqrt[4]{1 + \sin x}} \right) (\sqrt{1 - \sin x} + \sqrt{1 + \sin x})$$

$$= \lim_{x \rightarrow 0} \left[\frac{x}{-2 \sin x} \right] (\sqrt[8]{1 - \sin x} + \sqrt[8]{1 + \sin x})$$

$$\left(\sqrt[4]{1 - \sin x} \sqrt[4]{1 + \sin x} \right) (\sqrt{1 - \sin x} + \sqrt{1 + \sin x})$$

$$= \left(-\frac{1}{2} \right) (2)(2)(2) \left[\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right]$$

$$= -4$$

Question 76

If the value of $\lim_{x \rightarrow 0} (2 - \cos x \sqrt{\cos 2x}) \left(\frac{x+2}{x^2} \right)$ is equal to e^a , then a is equal to

[2021, 20 July Shift-1]

Answer: 3

Solution:

Solution:

$$\lim_{x \rightarrow 0} (2 - \cos x \sqrt{\cos 2x}) \frac{x+2}{x^2} = 1^\infty$$

$$\Rightarrow \lim_{x \rightarrow 0} (1 + 1 - \cos x \sqrt{\cos 2x}) \frac{x+2}{x^2}$$

$$\Rightarrow e^{x \rightarrow 0} (1 - \cos x \sqrt{\cos 2x}) \left(\frac{x+2}{x^2} \right)$$

$$\Rightarrow \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} -$$

$$\Rightarrow \cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} -$$

$$= \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots \right) \left(1 - 2x^2 + \frac{2}{3}x^4 \dots \right) \frac{1}{2}$$

We have to extract till the coefficient of x^2 as denominator is x^2 .

$$\text{So, } \left(1 - \frac{x^2}{2} \right) (1 - 2x^2) \frac{1}{2} = \left(1 - \frac{x^2}{2} \right) (1 - x^2)$$

$$= \left(1 - \frac{x^2}{2} - x^2 + \frac{x^4}{2} \right) = \left(1 - \frac{3}{2}x^2 \right)$$

$$\text{So, } e^{x \rightarrow 0} \lim_{(1 - \cos x \sqrt{\cos 2x})} \left(\frac{x+2}{x^2} \right)$$

$$= e^{x \rightarrow 0} \left[1 - \left(\frac{3x^2}{2} \right) \right] \left(\frac{x+2}{x^2} \right)$$

$$= e^{x \rightarrow 0} \left(\frac{3x^2}{2} \right) \left(\frac{x+2}{x^2} \right) = e^3$$

$$\therefore e^a = e^3 \Rightarrow a = 3$$

Question77

If α_1, β are the distinct roots of $x^2 + bx + c = 0$, then $\lim_{x \rightarrow \beta} \frac{e^{2(x^2 + bx + c)} - 1 - 2(x^2 + bx + c)}{(x - \beta)^2}$ is equal to
[2021, 27 Aug. Shift-1]

Options:

A. $b^2 + 4c$

B. $2(b^2 + 4c)$

C. $2(b^2 - 4c)$

D. $b^2 - 4c$

Answer: C

Solution:

Solution:

$$\because \alpha, \beta \text{ are distinct roots of } x^2 + bx + c = 0$$

$$\Rightarrow x^2 + bx + c = (x - \alpha)(x - \beta) = 0$$

$$\text{Now, } \lim_{x \rightarrow \beta} \frac{e^{2(x^2 + bx + c)} - 1 - 2(x^2 + bx + c)}{(x - \beta)^2}$$

$$= \lim_{x \rightarrow \beta} \frac{e^{2(x - \alpha)(x - \beta)} - 1 - 2(x - \alpha)(x - \beta)}{(x - \beta)^2}$$

$$= \lim_{h \rightarrow 0} \frac{e^{2(\beta - \alpha + h)h} - 1 - 2(\beta - \alpha + h)h}{h^2}$$

$$1 + 2(\beta - \alpha + h)h + \frac{[2(\beta - \alpha + h)h]^2}{2!}$$

$$= \lim_{h \rightarrow 0} \frac{+ \dots - 1 - 2h(\beta - \alpha + h)}{h^2}$$

$$= \lim_{h \rightarrow 0} \frac{2(\beta - \alpha + h)^2 h^2 + \dots}{h^2}$$

$$= 2(\beta - \alpha)^2 = 2(b^2 - 4c)$$

Question78

If $\lim_{x \rightarrow \infty} (\sqrt{x^2 - x + 1} - ax) = b$, then the ordered pair (a, b) is
[2021, 27 Aug. Shift-2]

Options:

- A. $\left(1, \frac{1}{2}\right)$
- B. $\left(1, -\frac{1}{2}\right)$
- C. $\left(-1, \frac{1}{2}\right)$
- D. $\left(-1, -\frac{1}{2}\right)$

Answer: B

Solution:

Solution:

$$\begin{aligned} \text{Given, } \lim_{x \rightarrow \infty} (\sqrt{x^2 - x + 1} - ax) &= b \\ \Rightarrow \lim_{x \rightarrow \infty} (\sqrt{x^2 - x + 1} - ax) &= b \\ \frac{(\sqrt{x^2 - x + 1}) + ax}{(\sqrt{x^2 - x + 1} + ax)} &= b \\ \Rightarrow \lim_{x \rightarrow \infty} \frac{x^2 - x + 1 - a^2x^2}{\sqrt{x^2 - x + 1} + ax} &= b \\ \text{Limit exists only if } a^2 &= 1 \\ \therefore \lim_{x \rightarrow \infty} \frac{-x + 1}{\sqrt{x^2 - x + 1} + ax} &= \pm \\ \Rightarrow \lim_{x \rightarrow \infty} \frac{-1 + \frac{1}{x}}{\sqrt{1 - \frac{1}{x} + \frac{1}{x^2}} + a} &= b \Rightarrow \frac{-1}{1 + a} = b \\ \text{But } a &\neq -1 \\ a &= 1 \\ b &= -\frac{1}{2} \quad (a, b) = \left(1, -\frac{1}{2}\right) \end{aligned}$$

Question79

$\lim_{x \rightarrow 2} \left(\sum_{n=1}^9 \frac{x}{n(n+1)x^2 + 2(2n+1)x + 4} \right)$ is equal to
[2021, 26 Aug. Shift-II]

Options:

A. $\frac{9}{44}$

B. $\frac{5}{24}$

C. $\frac{1}{5}$

D. $\frac{7}{36}$

Answer: A

Solution:

Solution:

We have,

$$\begin{aligned} S &= \lim_{x \rightarrow 2} \sum_{n=1}^9 \frac{x}{n(n+1)x^2 + 2(2n+1)x + 4} \\ &= \sum_{n=1}^9 \frac{2}{4(n^2 + 3n + 2)} \\ &= \frac{1}{2} \sum_{n=1}^9 \frac{(n+2) - (n+1)}{(n+1)(n+2)} \\ &= \frac{1}{2} \sum_{n=1}^9 \left[\frac{1}{n+1} - \frac{1}{n+2} \right] \\ &= \frac{1}{2} \left[\left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \dots + \left(\frac{1}{10} - \frac{1}{11} \right) \right] \\ &= \frac{1}{2} \left(\frac{1}{2} - \frac{1}{11} \right) = \frac{1}{2} \times \left(\frac{11-2}{2 \times 11} \right) = \frac{9}{44} \end{aligned}$$

Question80

$\lim_{x \rightarrow 0} \frac{\sin^2(\pi \cos^4 x)}{x^4}$ is equal to
[2021, 31 Aug. Shift-1]

Options:

A. π^2

B. $2\pi^2$

C. $4\pi^2$

D. 4π

Answer: C

Solution:

Solution:

$$\begin{aligned} &= \lim_{x \rightarrow 0} \frac{\sin^2(\pi \cos^4 x)}{x^4} \\ &= \lim_{x \rightarrow 0} \frac{\sin^2[\pi(1 - \cos^4 x)]}{[\pi(1 - \cos^4 x)]^2} \cdot \frac{\pi^2(1 - \cos^4 x)^2}{x^4} \\ &= \lim_{x \rightarrow 0} \pi^2 \frac{\sin^4 x(1 + \cos^2 x)^2}{x^4} \end{aligned}$$

$$= \lim_{x \rightarrow 0} \pi^2 (1 + \cos^2 x)^2 = 4\pi^2$$

Question81

If $\alpha = \lim_{x \rightarrow \frac{\pi}{4}} \frac{\tan^3 x - \tan x}{\cos\left(x + \frac{\pi}{4}\right)}$ and $\beta = \lim_{x \rightarrow 0} (\cos x)^{\cot x}$ are the roots of the quation,

$ax^2 + bx - 4 = 0$, then the ordered pair (a, b) is
[2021, 31 Aug. Shift-II]

Options:

- A. (1, -3)
- B. (-1, 3)
- C. (-1, -3)
- D. (1, 3)

Answer: D

Solution:

Solution:

$$\begin{aligned} \alpha &= \lim_{x \rightarrow \frac{\pi}{4}} \frac{\tan^3 x - \tan x}{\cos\left(x + \frac{\pi}{4}\right)} \\ &= \lim_{x \rightarrow \frac{\pi}{4}} \frac{\tan x(\tan x + 1)(\tan x - 1)}{\cos\left(x + \frac{\pi}{4}\right)} \\ &= \lim_{x \rightarrow \frac{\pi}{4}} \frac{\frac{\sin x}{\cos x} \cdot \left(\frac{\sin x - \cos x}{\cos x}\right) \left(\frac{\sin x + \cos x}{\cos x}\right)}{\frac{1}{\sqrt{2}}(\cos x - \sin x)} \\ &= \lim_{x \rightarrow \frac{\pi}{4}} \frac{-\sqrt{2} \sin x(\sin x + \cos x)}{\cos 3x} \\ &= \frac{-\sqrt{2} \times \frac{1}{\sqrt{2}} \times \sqrt{2}}{\frac{1}{2\sqrt{2}}} = -4 \end{aligned}$$

$$\begin{aligned} \text{and } \beta &= \lim_{x \rightarrow 0} (\cos x)^{\cot x} \\ &= e^{x \rightarrow 0} \frac{\cos x - 1}{\tan x} \\ &= e^{x \rightarrow 0} - \frac{\sin x}{\sec x} = e^0 = 1 \end{aligned}$$

Equation whose roots are x and β , is

$$\begin{aligned} x^2 + 3x - 4 &= 0 \\ \therefore 1 \cdot b &= 3 \end{aligned}$$

Question82

If $y^{1/4} + y^{-1/4} = 2x$, and $(x^2 - 1) \frac{d^2 y}{d x^2} + \alpha x \frac{d y}{d x} + \beta y = 0$ then $|\alpha - \beta|$ is equal to

[2021, 27 Aug. Shift-1]

Answer: 17

Solution:

Solution:

$$\begin{aligned}\text{Given, } y^{\frac{1}{4}} + y^{-\frac{1}{4}} &= 2x \\ \Rightarrow (y^{\frac{1}{4}} + y^{-\frac{1}{4}})^2 &= (2x)^2 \\ \Rightarrow (y^{\frac{1}{4}} + y^{-\frac{1}{4}})^2 &= 4x^2\end{aligned}$$

Differentiating w.r.t. x , we get

$$\begin{aligned}\frac{1}{4y} \left(y^{\frac{1}{4}} - y^{-\frac{1}{4}} \right) \frac{dy}{dx} &= 2 \\ \Rightarrow \left(y^{\frac{1}{4}} - y^{-\frac{1}{4}} \right) \frac{dy}{dx} &= 8y \dots (i)\end{aligned}$$

$$\begin{aligned}\text{Now, } y^{\frac{1}{4}} - y^{-\frac{1}{4}} &= \sqrt{\left(y^{\frac{1}{4}} + y^{-\frac{1}{4}} \right)^2 - 4} \\ \Rightarrow y^{\frac{1}{4}} - y^{-\frac{1}{4}} &= 2\sqrt{x^2 - 1} \dots (ii) \\ \Rightarrow \left(\sqrt{x^2 - 1} \right) \frac{dy}{dx} &= 4y\end{aligned}$$

[using Eqs. (i) and (ii)] Squaring on both sides,

$$(x^2 - 1) \left(\frac{dy}{dx} \right)^2 = 16y^2$$

Again, differentiating w.r.t. x

$$(x^2 - 1) \cdot 2 \frac{dy}{dx} \cdot \frac{d^2y}{dx^2} + 2x \left(\frac{dy}{dx} \right)^2 = 32y \frac{dy}{dx}$$

On dividing by $\frac{2dy}{dx}$, we get

$$(x^2 - 1) \frac{d^2y}{dx^2} + x \frac{dy}{dx} = 16y$$

$$\text{or } (x^2 - 1) \frac{d^2y}{dx^2} + x \frac{dy}{dx} - 16y = 0$$

Comparing with

$$(x^2 - 1) \frac{d^2y}{dx^2} + \alpha x \frac{dy}{dx} + \beta y = 0$$

$$\therefore \alpha = 1, \beta = -16$$

$$\therefore |\alpha - \beta| = |1 + 16| = 17$$

Question83

. If $y = y(x)$ is an implicit function of x such that $\log_e(x + y) = 4xy$, then

$\frac{d^2y}{dx^2}$ at $x = 0$ is equal to

[2021, 26 Aug. Shift-1]

Answer: 40

Solution:

Solution:

We have, $\ln(x + y) = 4xy$

$$\Rightarrow x + y = e^{4xy}$$

$$\Rightarrow 1 + \frac{dy}{dx} = \left(4x \frac{dy}{dx} + 4y \right) e^{4xy}$$

If $x = 0$, then $y = 1$

$$\text{At } (0, 1), \quad \frac{dy}{dx} = 3$$

$$\frac{d^2y}{dx^2} = e^{4xy} \left(4x \frac{dy}{dx} + 4y \right)^2$$

$$+ e^{4xy} \left(4x \frac{d^2y}{dx^2} + \frac{4dy}{dx} + 4 \frac{dy}{dx} \right)$$

At $x = 0$,

$$\frac{d^2y}{dx^2} = 16 + 24 = 40$$

Question 84

Let $f(x) = \cos \left(2 \tan^{-1} \sin \left(\cot^{-1} \sqrt{\frac{1-x}{x}} \right) \right)$ $0 < x < 1$. Then,
[2021, 26 Aug. Shift-1]

Options:

A. $(1-x)^2 f'(x) - 2(f(x))^2 = 0$

B. $(1+x)^2 f'(x) + 2(f(x))^2 = 0$

C. $(1-x)^2 f'(x) + 2(f(x))^2 = 0$

D. $(1+x)^2 f'(x) - 2f(x)^2 = 0$

Answer: C

Solution:

Solution:

$$f(x) = \cos \left(2 \tan^{-1} \sin \left(\cot^{-1} \sqrt{\frac{1-x}{x}} \right) \right)$$

$$\cot^{-1} \sqrt{\frac{1-x}{x}} = \sin^{-1} \sqrt{x}$$

$$\therefore f(x) = \cos(2 \tan^{-1} \sin \sin^{-1} \sqrt{x})$$

$$\text{or } f(x) = \cos(2 \tan^{-1} \sqrt{x})$$

$$= \cos \tan^{-1} \left(\frac{2\sqrt{x}}{1-x} \right)$$

$$f(x) = \cos \cos^{-1} \left(\frac{1-x}{1+x} \right)$$

$$f(x) = \frac{1-x}{1+x}$$

$$f'(x) = \frac{-(1+x) - (1-x)}{(1+x)^2} = \frac{-2}{(1+x)^2}$$

$$f'(x)(1-x)^2 = -2 \left(\frac{1-x}{1+x} \right)^2$$

$$(1-x)^2 f'(x) + 2[f(x)]^2 = -2 \left(\frac{1-x}{1+x} \right)^2$$

$$+2\left(\frac{1-x}{1+x}\right)^2 = 0$$

Question85

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Then, $\lim_{x \rightarrow \frac{\pi}{4}} \frac{\frac{\pi}{4} \int^{\sec^2 x} f(x) dx}{x^2 - \frac{\pi^2}{16}}$ is equal to

[2021, 01 Sep. Shift-II]

Options:

- A. $f(2)$
- B. $2f(2)$
- C. $2f(\sqrt{2})$
- D. $4f(2)$

Answer: B

Solution:

Solution:

Using L-Hopital's rule

$$\lim_{x \rightarrow \frac{\pi}{4}} \frac{\frac{\pi}{4} \cdot 2 \sec x \cdot \sec x \cdot \tan x \cdot f(\sec^2 x) - 0}{2x}$$

[using Leibnitz theorem]

$$= \frac{\frac{\pi}{4} \cdot 2(\sqrt{2})^2 \cdot (1)f(2)}{2 \cdot \frac{\pi}{4}} = 2f(2)$$

Question86

$\lim_{x \rightarrow 2} \frac{3^x + 3^{3-x} - 12}{3^{\frac{x}{2}} - 3^{1-x}}$ is equal to _____.

[NA Jan. 7, 2020 (I)]

Answer: 36

Solution:

Solution:

Let $3^x = t^2$

$$\lim_{t \rightarrow 3} \frac{t^2 + \frac{27}{t^2} - 12}{\frac{1}{t} - \frac{3}{t^2}} = \lim_{t \rightarrow 3} \frac{t^4 - 12t^2 + 27}{t - 3}$$

$$= \lim_{t \rightarrow 3} \frac{(t^2 - 3)(t + 3)(t - 3)}{t - 3}$$

$$= (3^2 - 3)(3 + 3) = 36$$

Question87

$\lim_{x \rightarrow 0} \left(\frac{3x^2 + 2}{7x^2 + 2} \right)^{1/x^2}$ is equal to:

[Jan. 8, 2020 (I)]

Options:

- A. $\frac{1}{e}$
- B. $\frac{1}{e^2}$
- C. e^2
- D. e

Answer: B

Solution:

Solution:

$$\text{Let } R = \lim_{x \rightarrow 0} \left(\frac{3x^2 + 2}{7x^2 + 2} \right)^{\frac{1}{x^2}} = e^{\lim_{x \rightarrow 0} \left\{ \frac{3x^2 + 2}{7x^2 + 2} - 1 \right\}}$$

$$= e^{\lim_{x \rightarrow 0} \frac{1}{x^2} \left\{ \frac{-4x^2}{7x^2 + 2} \right\}} = e^{\frac{-4}{2}} = e^{-2} = \frac{1}{e^2}$$

Question88

$\lim_{x \rightarrow 0} \int_0^x \frac{t \sin(10t) dt}{x}$ is equal to:

[Jan. 8, 2020 (II)]

Options:

- A. 0
- B. $\frac{1}{10}$
- C. $-\frac{1}{5}$
- D. $-\frac{1}{10}$

Answer: A

Solution:

Solution:

Using L' Hospital rule,

$$\lim_{x \rightarrow 0} \frac{x \sin(10x)}{1} = 0$$

Question89

If $\lim_{x \rightarrow 1} \frac{x + x^2 + x^3 + \dots + x^n - n}{x - 1} = 820$, ($n \in \mathbb{N}$) then the value of n is equal to

[NA Sep. 02, 2020 (I)]

Answer: 40

Solution:

Solution:

$$\lim_{x \rightarrow 1} \frac{x + x^2 + x^3 + \dots + x^n - n}{x - 1} = 820 \left(\frac{0}{0} \text{ case} \right)$$

$$\lim_{x \rightarrow 1} \frac{1 + 2x + 3x^2 + \dots + nx^{n-1}}{1} = 820 \text{ (Using L' Hospital rule)}$$

$$\Rightarrow 1 + 2 + 3 + \dots + n = 820$$

$$\Rightarrow \frac{n(n+1)}{2} = 820 \Rightarrow n^2 + n - 1640 = 0$$

$$\Rightarrow n = 40, n \in \mathbb{N}$$

Question90

$\lim_{x \rightarrow 0} \left(\tan \left(\frac{\pi}{4} + x \right) \right)^{1/x}$ is equal to :

[Sep. 02, 2020 (II)]

Options:

A. e

B. 2

C. 1

D. e^2

Answer: D

Solution:

Solution:

$$\begin{aligned} \lim_{x \rightarrow 0} \left(\frac{1 + \tan x}{1 - \tan x} \right)^{1/x} \\ \Rightarrow \lim_{e^x \rightarrow 0} \frac{1}{x} \left[\tan \left(\frac{\pi}{4} + x \right) - 1 \right] &= \lim_{e^x \rightarrow 0} \frac{1}{x} \left(\frac{1 + \tan x}{1 - \tan x} - 1 \right) \\ \Rightarrow \lim_{e^x \rightarrow 0} \left(\frac{2 \tan x}{1 - \tan x} \right) \frac{1}{x} &= \lim_{e^x \rightarrow 0} \left(\frac{\tan x}{x} \right) \left(\frac{2}{1 - \tan x} \right) = e^2 \end{aligned}$$

Question91

If

$$\lim_{x \rightarrow 0} \left\{ \frac{1}{x^8} \left(1 - \cos \frac{x^2}{2} - \cos \frac{x^2}{4} + \cos \frac{x^2}{2} \cos \frac{x^2}{4} \right) \right\} = 2^{-k},$$

then the value of k is _____.

[NA Sep. 03, 2020 (I)]

Answer: 8

Solution:

Solution:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\left(1 - \cos \frac{x^2}{2} \right) \left(1 - \cos \frac{x^2}{4} \right)}{x^4} &= 2^{-k} \\ \Rightarrow \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{x^2}{4}}{\frac{x^4}{16} \times 16} \times \frac{2 \sin^2 \frac{x^2}{8}}{\frac{x^4}{64} \times 64} &= 2^{-k} \\ \Rightarrow \frac{4}{16 \times 64} = 2^{-8} = 2^{-k} \left[\because \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \right] \\ \therefore k &= 8 \end{aligned}$$

Question92

Let [t] denote the greatest integer $\leq t$. If for some

$\lambda \in \mathbb{R} - \{0, 1\}$, $\lim_{x \rightarrow 0} \left| \frac{1 - x + |x|}{\lambda - x + [x]} \right| = L$, then L is equal to:

[Sep. 03, 2020 (I)]

Options:

A. 1

B. 2

C. $\frac{1}{2}$

D. 0

Answer: B

Solution:

Solution:

$$\text{Given } \lim_{x \rightarrow 0} \left| \frac{1-x+|x|}{\lambda-x+|x|} \right| = L$$

$$\text{Here, L.H.L.} = \lim_{h \rightarrow 0} \left| \frac{1+h+h}{\lambda+h-1} \right| = \left| \frac{1}{\lambda-1} \right|$$

$$\text{R.H.L.} = \lim_{h \rightarrow 0} \left| \frac{1-h+h}{\lambda+h+0} \right| = \left| \frac{1}{\lambda} \right|$$

Given that limit exists. Hence L.H.L. = R.H.L.

$$\Rightarrow |\lambda - 1| = |\lambda|$$

$$\Rightarrow \lambda = \frac{1}{2} \text{ and } L = \left| \frac{1}{\lambda} \right| = 2$$

Question93

$$\lim_{x \rightarrow a} \frac{(a+2x)^{\frac{1}{3}} - (3x)^{\frac{1}{3}}}{(3a+x)^{\frac{1}{3}} - (4x)^{\frac{1}{3}}} \text{ (a} \neq 0 \text{) is equal to}$$

[Sep. 03, 2020 (II)]

Options:

A. $\left(\frac{2}{3}\right)^{\frac{4}{3}}$

B. $\left(\frac{2}{3}\right)\left(\frac{2}{9}\right)^{\frac{1}{3}}$

C. $\left(\frac{2}{9}\right)^{\frac{4}{3}}$

D. $\left(\frac{2}{9}\right)\left(\frac{2}{3}\right)^{\frac{1}{3}}$

Answer: B

Solution:

Solution:

$$\lim_{x \rightarrow a} \frac{(a+2x)^{\frac{1}{3}} - (3x)^{\frac{1}{3}}}{(3a+x)^{\frac{1}{3}} - (4x)^{\frac{1}{3}}} \quad \left[\frac{0}{0} \text{ case} \right]$$

Apply L'Hospital rule

$$\lim_{x \rightarrow a} \frac{\frac{1}{3}(a+2x)^{-2/3} \cdot 2 - \frac{1}{3} \cdot (3x)^{-2/3} \cdot 3}{\frac{1}{3}(3a+x)^{-2/3} \cdot 1 - \frac{1}{3}(4x)^{-2/3} \cdot 4}$$

$$= \frac{\frac{1}{3}(3a)^{-2/3} \cdot (2-3)}{\frac{1}{3}(4a)^{-2/3} \cdot (1-4)} = \frac{3^{-2/3} \cdot \frac{1}{3}}{4^{-2/3} \cdot \frac{1}{3}} = \frac{2^{4/3}}{9^{1/3}} \cdot \frac{1}{3} = \frac{2}{3} \cdot \left(\frac{2}{9}\right)^{1/3}$$

Question94

$\lim_{n \rightarrow \infty} \tan \left\{ \sum_{r=1}^n \tan^{-1} \left(\frac{1}{1+r+r^2} \right) \right\}$ is equal to__

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Answer: 1

Solution:

Solution:

$$\begin{aligned} & \lim_{n \rightarrow \infty} \tan \left(\sum_{r=1}^n \tan^{-1} \left(\frac{1}{1+r(r+1)} \right) \right) \\ &= \lim_{n \rightarrow \infty} \tan \left(\sum_{r=1}^n \tan^{-1} \left(\frac{r+1-r}{1+r(r+1)} \right) \right) \\ &= \tan \left(\lim_{n \rightarrow \infty} \sum_{r=1}^n [\tan^{-1}(r+1) - \tan^{-1}(r)] \right) \\ &= \tan \left(\lim_{n \rightarrow \infty} \left(\tan^{-1}(n+1) - \frac{\pi}{4} \right) \right) \\ &= \tan \left(\frac{\pi}{4} \right) = 1 \end{aligned}$$

Question95

Substitution & Rationalisation If α is the positive root of the equation, $p(x) = x^2 - x - 2 = 0$, then $\lim_{x \rightarrow \alpha^+} \frac{\sqrt{1 - \cos(p(x))}}{x + \alpha - 4}$ is equal to:

[Sep. 05, 2020 (I)]

Options:

- A. $\frac{3}{2}$
- B. $\frac{3}{\sqrt{2}}$
- C. $\frac{1}{\sqrt{2}}$
- D. $\frac{1}{2}$

Answer: B

Solution:

Solution:

$$\begin{aligned} x^2 - x - 2 &= 0 \Rightarrow (x-2)(x+1) = 0 \\ \Rightarrow x &= 2, -1 \Rightarrow \alpha = 2 \end{aligned}$$

$$\begin{aligned}
& \therefore \lim_{x \rightarrow 2^+} \frac{\sqrt{1 - \cos(x^2 - x - 2)}}{x - 2} \\
&= \lim_{x \rightarrow 2^+} \frac{\sqrt{2} \left| \sin \left(\frac{x^2 - x - 2}{2} \right) \right|}{x - 2} \\
&= \lim_{x \rightarrow 2^+} \frac{\sqrt{2} \sin(x^2 - x - 2) \cdot 2}{\left(\frac{x^2 - x - 2}{2} \right)} \times \frac{(x^2 - x - 2)}{2(x - 2)} \\
&= \frac{1}{\sqrt{2}} \lim_{x \rightarrow 2^+} \left(\frac{\sin \left(\frac{x^2 - x - 2}{2} \right)}{\frac{x^2 - x - 2}{2}} \right) \times \lim_{x \rightarrow 2^+} \frac{(x - 2)(x + 1)}{(x - 2)} \\
&= \frac{1}{\sqrt{2}} \times 1 \times 3 = \frac{3}{\sqrt{2}}
\end{aligned}$$

Question96

$$\lim_{x \rightarrow 0} \frac{x \left(e^{(\sqrt{1+x^2+x^4}-1)/x} - 1 \right)}{\sqrt{1+x^2+x^4}-1}$$

[Sep. 05, 2020 (II)]

Options:

A. is equal to $\sqrt{e}$

B. is equal to

C. is equal to 0

D. does not exist

Answer: B

Solution:

Solution:

$$\begin{aligned}
\text{Let } L &= \lim_{x \rightarrow 0} \frac{x \left(e^{\frac{\sqrt{1+x^2+x^4}-1}{x}} - 1 \right)}{\sqrt{1+x^2+x^4}-1} \\
&= \lim_{x \rightarrow 0} \frac{e^{\frac{\sqrt{1+x^2+x^4}-1}{x}} - 1}{\frac{\sqrt{1+x^2+x^4}-1}{x}} \\
\text{Put } \frac{\sqrt{1+x^2+x^4}-1}{x} &= t \text{ when } x \rightarrow 0 \Rightarrow t \rightarrow 0 \\
\therefore L &= \lim_{t \rightarrow 0} \frac{e^t - 1}{t} = 1
\end{aligned}$$

Question97

$$\lim_{x \rightarrow \frac{\pi}{4}} \frac{\cot^3 x - \tan x}{\cos \left(x + \frac{\pi}{4} \right)} \text{ is:}$$

[Jan. 12, 2019 (I)]

Options:

A. 4

B. $4\sqrt{2}$

C. $8\sqrt{2}$

D. 8

Answer: D

Solution:

Solution:

$$\begin{aligned}\lim_{x \rightarrow \frac{\pi}{4}} \frac{\cot^3 x - \tan x}{\cos\left(x + \frac{\pi}{4}\right)} &= \lim_{x \rightarrow \frac{\pi}{4}} \frac{\cot^3 x \left(1 - \frac{\tan x}{\cos^3 x}\right)}{\cos(x + \pi/4)} \\&= \lim_{x \rightarrow \frac{\pi}{4}} \frac{(1 - \tan^4 x)}{\tan^3 x \cos(x + \pi/4)} \\&= \lim_{x \rightarrow \frac{\pi}{4}} \frac{(1 + \tan^2 x)(1 - \tan x)(1 + \tan x)}{\tan^3 x \left(\frac{\cos x - \sin x}{\sqrt{2}}\right)} \\&= \lim_{x \rightarrow \frac{\pi}{4}} \frac{(1 + \tan^2 x)(1 + \tan x)(\cos x - \sin x)}{\frac{\sin^3 x}{\cos^2 x} \left(\frac{\cos x - \sin x}{\sqrt{2}}\right)} \\&= \frac{(2)(2)}{\frac{1}{(\sqrt{2})(\sqrt{2})}} = 8\end{aligned}$$

Question98

$\lim_{x \rightarrow 1^-} \frac{\sqrt{\pi} - \sqrt{2\sin^{-1}x}}{\sqrt{1-x}}$ is equal to:

[Jan. 12, 2019 (II)]

Options:

A. $\frac{1}{\sqrt{2\pi}}$

B. $\sqrt{\frac{2}{\pi}}$

C. $\sqrt{\frac{\pi}{2}}$

D. $\sqrt{\pi}$

Answer: B

Solution:

$$\begin{aligned}
\lim_{x \rightarrow 1^-} \frac{\sqrt{\pi} - \sqrt{2\sin^{-1}x}}{\sqrt{1-x}} &= \lim_{h \rightarrow 0} f(1-h) \\
&= \lim_{h \rightarrow 0} \frac{\sqrt{\pi} - \sqrt{2\sin^{-1}(1-h)}}{\sqrt{1-(1-h)}} \\
&= \lim_{h \rightarrow 0} \frac{\sqrt{\pi} - \sqrt{2\sin^{-1}(1-h)}}{\sqrt{h}} \\
&= \lim_{h \rightarrow 0} \frac{-\frac{1}{2\sqrt{2\sin^{-1}(1-h)}} \times 2 \times \frac{1}{\sqrt{1-(1-h)^2}}(-1)}{\frac{1}{2\sqrt{h}}} \\
&= \lim_{h \rightarrow 0} \frac{\frac{1}{\sqrt{2\sin^{-1}(1-h)}} \frac{1}{\sqrt{h(2-h)}}}{\frac{1}{2\sqrt{h}}} \\
&= 2 \times \frac{1}{\sqrt{\pi}} \times \frac{1}{\sqrt{2}} = \sqrt{\frac{2}{\pi}}
\end{aligned}$$

Question99

Let $[x]$ denote the greatest integer less than or equal to x . Then :

$$\lim_{x \rightarrow 0} \frac{\tan(\pi \sin^2 x) + (|x| - \sin(x[x]))^2}{x^2} :$$

[Jan. 11, 2019 (I)]

Options:

- A. does not exist
- B. equals π
- C. equals $\pi + 1$
- D. equals 0

Answer: A

Solution:

Solution:

$$\text{RHL is, } \lim_{x \rightarrow 0^+} \frac{\tan(\pi \sin^2 x) + (x - 0)^2}{x^2}$$

$$= \lim_{x \rightarrow 0^+} \left(\frac{\tan(\pi \sin^2 x)}{x^2} + 1 \right) = 1 + \pi$$

$$\text{And LHL is, } \lim_{x \rightarrow 0^-} \frac{\tan(\pi \sin^2 x) + (-x + \sin x)^2}{x^2}$$

$$= \lim_{x \rightarrow 0^-} \frac{\tan(\pi \sin^2 x) + x^2 + \sin^2 x - 2x \sin x}{x^2}$$

$$= \pi + 1 + 1 - 2 = \pi$$

Since, $LH L \neq RH L$

Hence, limit does not exist.

Question100

$$\lim_{x \rightarrow 0} \frac{x \cot(4x)}{\sin^2 x \cot^2(2x)} \text{ is equal to}$$

[Jan. 11, 2019 (II)]

Options:

- A. 0
- B. 2
- C. 4
- D. 1

Answer: D

Solution:

Solution:

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{x \cot 4x}{\sin^2 x \cdot \cot^2 2x} &= \lim_{x \rightarrow 0} \frac{x \cdot \tan^2 2x}{\sin^2 x \cdot \tan 4x} \\ &= \lim_{x \rightarrow 0} \left(\frac{x}{\sin x} \right)^2 \cdot \left(\frac{\tan 2x}{2x} \right)^2 \cdot \left(\frac{4x}{\tan 4x} \right) \cdot \frac{4}{2^2} = 1\end{aligned}$$

Question101

For each $t \in \mathbb{R}$, let $[t]$ be the greatest integer less than or equal to t . Then,

$$\lim_{x \rightarrow 1^+} \frac{(1 - |x| + \sin |1 - x|) \sin \left(\frac{\pi}{2} [1 - x] \right)}{|1 - x| [1 - x]}$$

[Jan. 10, 2019 (I)]

Options:

- A. equals 1
- B. equals 0
- C. equals - 1
- D. does not exist

Answer: B

Solution:

Solution:

$$\begin{aligned}&\lim_{x \rightarrow 1^+} \frac{(1 - |x| + \sin(|1 - x|)) \sin \left(\frac{\pi}{2} [1 - x] \right)}{|1 - x| [1 - x]} \\ &= \lim_{h \rightarrow 0} \frac{(1 - |1 + h| + \sin(|1 - 1 - h|)) \sin \left(\frac{\pi}{2} [1 - 1 - h] \right)}{|1 - 1 - h| [1 - 1 - h]} \\ &= \lim_{h \rightarrow 0} \frac{(1 - 1 - h + \sinh) \sin \left(\frac{\pi}{2} (-1) \right)}{h([0 - h])}\end{aligned}$$

$$= \lim_{h \rightarrow 0} \frac{(-h + \sin h) \sin\left(-\frac{\pi}{2}\right)}{h(-1)} = 0$$

Question102

$$\lim_{y \rightarrow 0} \frac{\sqrt[4]{1 + \sqrt{1 + y^4}} - \sqrt{2}}{y^4}$$

[Jan. 9, 2019 (I)]

Options:

- A. exists and equals $\frac{1}{4\sqrt{2}}$
- B. exists and equals $\frac{1}{2\sqrt{2}(\sqrt{2} + 1)}$
- C. exists and equals $\frac{1}{2\sqrt{2}}$
- D. does not exist

Answer: A

Solution:

Solution:

$$\begin{aligned} L &= \lim_{y \rightarrow 0} \frac{\sqrt[4]{1 + \sqrt{1 + y^4}} - \sqrt{2}}{y^4} \\ &= \lim_{y \rightarrow 0} \frac{(\sqrt[4]{1 + \sqrt{1 + y^4}} - \sqrt{2})(\sqrt[4]{1 + \sqrt{1 + y^4}} + \sqrt{2})}{y^4(\sqrt[4]{1 + \sqrt{1 + y^4}} + \sqrt{2})} \\ &= \lim_{y \rightarrow 0} \frac{1 + \sqrt{1 + y^4} - 2}{y^4(\sqrt[4]{1 + \sqrt{1 + y^4}} + \sqrt{2})} \\ &= \lim_{y \rightarrow 0} \frac{(\sqrt{1 + y^4} - 1)(\sqrt{1 + y^4} + 1)}{y^4(\sqrt[4]{1 + \sqrt{1 + y^4}} + \sqrt{2})(\sqrt{1 + y^4} + 1)} \\ &= \lim_{y \rightarrow 0} \frac{1 + y^4 - 1}{y^4(\sqrt[4]{1 + \sqrt{1 + y^4}} + \sqrt{2})(\sqrt{1 + y^4} + 1)} \\ &= \frac{1}{2\sqrt{2} \times 2} = \frac{1}{4\sqrt{2}} \end{aligned}$$

Question103

For each $x \in \mathbb{R}$, let $[x]$ be greatest integer less than or equal to x . Then

$\lim_{x \rightarrow 0} \frac{x([x] + |x|) \sin[x]}{x}$ is equal to:

[Jan. 09, 2019 (II)]

Options:

- A. $-\sin 1$
- B. 1

C. $\sin 1$

D. 0

Answer: A

Solution:

Solution:

$$\begin{aligned} & \lim_{x \rightarrow 0^-} \frac{x([x] + |x|) \cdot \sin[x]}{|x|} \\ &= \lim_{x \rightarrow 0} \frac{(0-h)([0-h] + |0-h|) \cdot \sin[0-h]}{|0-h|} \\ &= \lim_{h \rightarrow 0} \frac{(-h)(-1+h) \sin(-1)}{h} \\ &= \lim_{h \rightarrow 0} (1-h) \sin(-1) = -\sin 1 \end{aligned}$$

Question104

Let $f(x) = 5 - |x - 2|$ and $g(x) = |x + 1|$, $x \in \mathbb{R}$. If $f(x)$ attains maximum value at α and $g(x)$ attains minimum value at β then $\lim_{x \rightarrow -\alpha\beta} \frac{(x-1)(x^2-5x+6)}{x^2-6x+8}$ is equal to :
[April 12, 2019(II)]

Options:

A. $1/2$

B. $-3/2$

C. $-1/2$

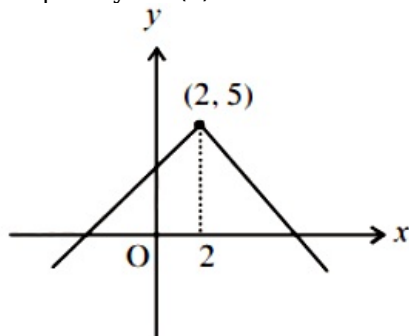
D. $3/2$

Answer: A

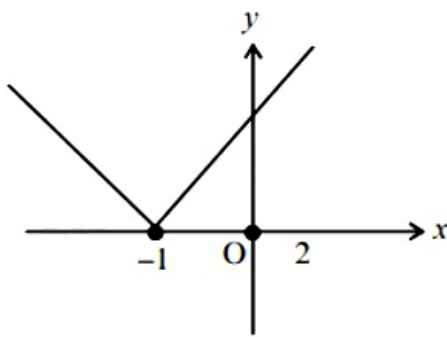
Solution:

Solution:

$f(x) = 5 - |x - 2|$
Graph of $y = f(x)$



By the graph $f(x)$ is maximum at $x = 2$
 $\therefore \alpha = 2$
 $g(x) = |x + 1|$
Graph of $y = g(x)$



By the graph $g(x)$ is minimum at $x = -1$
 $\therefore \beta = -1$

$$\begin{aligned}\text{Now, } \lim_{x \rightarrow 2} \frac{(x-1)(x-2)(x-3)}{(x-2)(x-4)} \\ = \lim_{x \rightarrow 2} \frac{(x-1)(x-3)}{x-4} = \frac{1}{2}\end{aligned}$$

Question 105

$\lim_{x \rightarrow 0} \frac{x + 2 \sin x}{\sqrt{x^2 + 2 \sin x + 1} - \sqrt{\sin^2 x - x + 1}}$ is

[April 12, 2019(II)]

Options:

- A. 6
- B. 2
- C. 3
- D. 1

Answer: B

Solution:

Solution:

Given limit is,

$$\lim_{x \rightarrow 0} \frac{x + 2 \sin x}{\sqrt{x^2 + 2 \sin x + 1} - \sqrt{\sin^2 x - x + 1}}$$

On rationalising,

$$\begin{aligned}&= \lim_{x \rightarrow 0} \frac{(x + 2 \sin x) [\sqrt{x^2 + 2 \sin x + 1} + \sqrt{\sin^2 x - x + 1}]}{(x^2 - \sin^2 x) + (x + 2 \sin x)} \\&= \lim_{x \rightarrow 0} \frac{\left[1 + 2 \left(\frac{\sin x}{x} \right) \right] [\sqrt{x^2 + 2 \sin x + 1} + \sqrt{\sin^2 x - x + 1}]}{\left(x - \frac{\sin^2 x}{x} \right) + \left(1 + 2 \left(\frac{\sin x}{x} \right) \right)} \\&= \frac{3 \times 2}{3} = 2 \quad \left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right]\end{aligned}$$

Question 106

If $\lim_{x \rightarrow 1} \frac{x^4 - 1}{x - 1} = \lim_{x \rightarrow k} \frac{x^3 - k^3}{x^2 - k^2}$, then k is:

[April 10, 2019 (I)]

Options:

A. $\frac{8}{3}$

B. $\frac{3}{8}$

C. $\frac{3}{2}$

D. $\frac{4}{3}$

Answer: A

Solution:

Solution:

Given, $\lim_{x \rightarrow 1} x^4 - 1x - 1 = \lim_{x \rightarrow K} \left(\frac{x^3 - k^3}{x^2 - k^2} \right)$

Taking L.H.S. $\lim_{x \rightarrow 1} \frac{x^4 - 1}{x - 1} \left(\frac{0}{0} \text{ form} \right)$

$\lim_{x \rightarrow 1} \frac{4x^3}{1} = 4$ [Using L Hospital's Rule]

$\therefore \lim_{x \rightarrow K} \frac{x^3 - k^3}{x^2 - k^2} = 4$

$\Rightarrow \lim_{x \rightarrow K} \frac{3x^2}{2x} = 4$ [Using L Hospital's Rule]

$\Rightarrow \frac{3}{2}k = 4 \Rightarrow k = \frac{8}{3}$

Question107

If $\lim_{x \rightarrow 1} \frac{x^2 - ax + b}{x - 1} = 5$, then a + b is equal to :

[April 10, 2019 (II)]

Options:

A. -4

B. 5

C. -7

D. 1

Answer: C

Solution:

Solution:

$\lim_{x \rightarrow 1} \frac{x^2 - ax + b}{x - 1} = 5$

$\therefore$ limit is finite $\therefore 1 - a + b = 0$

$\Rightarrow \lim_{x \rightarrow 1} \frac{2x - a}{1} = 5 \left(\frac{0}{0} \text{ form} \right)$ (By L Hospital's rule)

$\Rightarrow 2 - a = 5 \Rightarrow a = -3$ and $b = -4$

Then $a + b = -3 - 4 = -7$

Question108

$\lim_{x \rightarrow 0} \frac{\sin^2 x}{\sqrt{2} - \sqrt{1 + \cos x}}$ equals:

[April 8, 2019 (I)]

Options:

A. $4\sqrt{2}$

B. $\sqrt{2}$

C. $2\sqrt{2}$

D. 4

Answer: A

Solution:

Solution:

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{\sin^2 x}{\sqrt{2} - \sqrt{1 + \cos x}} \\ &= \lim_{x \rightarrow 0} \frac{\sin^2 x}{\sqrt{2} - \sqrt{2 \cos^2 \frac{x}{2}}} \left[\frac{0}{0} \right] \\ &= \lim_{x \rightarrow 0} \frac{\sin^2 x}{\sqrt{2} \left[1 - \cos \frac{x}{2} \right]} = \lim_{x \rightarrow 0} \frac{\sin^2 x}{2\sqrt{2} \sin^2 \frac{x}{4}} \\ &= \lim_{x \rightarrow 0} \frac{\left(\frac{\sin x}{x} \right)^2 \cdot 16}{2\sqrt{2} \left(\frac{\sin \frac{x}{4}}{\frac{x}{4}} \right)^2} = \frac{16}{2\sqrt{2}} = 4\sqrt{2} \end{aligned}$$

Question109

If α and β are the roots of the equation $375x^2 - 25x - 2 = 0$, then

$\lim_{n \rightarrow \infty} \sum_{r=1}^n \alpha^r + \lim_{n \rightarrow \infty} \sum_{r=1}^n \beta^r$ is equal to :

[April 12, 2019 (I)]

Options:

A. $\frac{21}{346}$

B. $\frac{29}{358}$

C. $\frac{1}{12}$

D. $\frac{7}{116}$

Answer: C

Solution:

Solution:

Given equation is, $375x^2 - 25x - 2 = 0$

Sum and product of the roots are,

$$\alpha + \beta = \frac{25}{375} \text{ and } \alpha\beta = \frac{-2}{375}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{r=1}^n (\alpha^r + \beta^r) \\ &= (\alpha + \alpha^2 + \alpha^3 + \dots \dots \dots \infty (\beta + \beta^2 + \beta^3 \dots \dots \dots + \infty) \\ &= \alpha 1 - \alpha + \beta 1 - \beta = \frac{\alpha + \beta - 2\alpha\beta}{1 - (\alpha + \beta) + \alpha\beta} \\ &= \frac{\frac{25}{375} + \frac{4}{375}}{1 - \frac{25}{375} - \frac{2}{375}} = \frac{29}{375 - 25 - 2} = \frac{29}{348} = \frac{1}{12} \end{aligned}$$

Question 110

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function satisfying $f'(3) + f'(2) = 0$. Then

$\lim_{x \rightarrow 0} \left(\frac{1 + f(3+x) - f(3)}{1 + f(2-x) - f(2)} \right)^{\frac{1}{x}}$ is equal to :

[April 08, 2019 (II)]

Options:

A. 1

B. e^{-1}

C. e

D. e^2

Answer: A

Solution:

Solution:

$$I = \lim_{x \rightarrow 0} \left(\frac{1 + f(3+x) - f(3)}{1 + f(2-x) - f(2)} \right)^{\frac{1}{x}} \text{ [} 1^\infty \text{ form]}$$

$\Rightarrow I = e^{I_1}$, where

$$\begin{aligned} I_1 &= \lim_{x \rightarrow 0} \left(\left(\frac{1 + f(3+x) - f(3)}{1 + f(2-x) - f(2)} - 1 \right) \right) \left(\frac{1}{x} \right) \\ &= \lim_{x \rightarrow 0} \left(\frac{1}{x} \right) \left(\frac{f(3+x) - f(3) - f(2-x) + f(2)}{1 + f(2-x) - f(2)} \right) \text{ (} \frac{0}{0} \text{ form)} \end{aligned}$$

By L. Hospital Rule,

$$\begin{aligned} I_1 &= \lim_{x \rightarrow 0} \left(\frac{f'(3+x) + f'(2-x)}{1} \right) \lim_{x \rightarrow 0} \left(\frac{1}{1 + f(2-x) - f(2)} \right) \\ &= f'(3) + f'(2) = 0 \\ \Rightarrow I &= e^{I_1} = e^0 = 1 \end{aligned}$$

Question111

$\lim_{x \rightarrow 0} \frac{x \tan 2x - 2x \tan x}{(1 - \cos 2x)^2}$ equals.

[Online April 15, 2018]

Options:

A. 1

B. $-\frac{1}{2}$

C. $\frac{1}{4}$

D. $\frac{1}{2}$

Answer: D

Solution:

Solution:

$$\begin{aligned}\text{Let, } L &= \lim_{x \rightarrow 0} \frac{(x \tan 2x - 2x \tan x)}{(1 - \cos 2x)^2} = \lim_{x \rightarrow 0} K \quad (\text{say}) \\ \Rightarrow K &= \frac{x \left[\frac{2 \tan x}{1 - (\tan x)^2} \right] - 2x \tan x}{(1 - (1 - 2\sin^2 x))^2} \\ &= \frac{2x \tan x - [2x \tan x - 2x \tan^3 x]}{4\sin^4 x \times (1 - \tan^2 x)} \\ &= \frac{2x \tan^3 x}{4\sin^4 x \times (1 - \tan^2 x)} = \frac{2x \tan^3 x}{4\sin^4 x \times \left(\frac{\cos^2 x - \sin^2 x}{\cos^2 x} \right)} \\ &= \frac{2x \frac{\sin^3 x}{\cos^3 x}}{4\sin^4 x \times \left(\frac{\cos^2 x - \sin^2 x}{\cos^2 x} \right)} \\ \Rightarrow K &= \frac{x}{2 \sin x \times (\cos^2 x - \sin^2 x) \cos x} \\ \therefore L &= \lim_{x \rightarrow 0} \frac{x}{2 \sin x} \times \lim_{x \rightarrow 0} \frac{1}{\cos x (\cos^2 x - \sin^2 x)} \\ &= \lim_{x \rightarrow 0} \frac{x}{2 \sin x} \times \lim_{x \rightarrow 0} \frac{1}{\cos 0 (\cos^2 0 - \sin^2 0)} = \frac{1}{2}\end{aligned}$$

Question112

For each $t \in \mathbb{R}$, let $[t]$ be the greatest integer less than or equal to t . Then

$$\lim_{x \rightarrow 0^+} x \left(\left[\frac{1}{x} \right] + \left[\frac{2}{x} \right] + \dots + \left[\frac{15}{x} \right] \right)$$

[2018]

Options:

A. is equal to 15.

- B. is equal to 120.
- C. does not exist (in R).
- D. is equal to 0.

Answer: B

Solution:

Solution:

$$\begin{aligned} \text{Since, } \lim_{x \rightarrow 0^+} x \left(\left\lfloor \frac{1}{x} \right\rfloor + \left\lfloor \frac{2}{x} \right\rfloor + \dots + \left\lfloor \frac{15}{x} \right\rfloor \right) \\ = \lim_{x \rightarrow 0^+} x \left(\frac{1+2+3+\dots+15}{x} \right) - \left(\left\{ \frac{1}{x} \right\} + \left\{ \frac{2}{x} \right\} + \dots + \left\{ \frac{15}{x} \right\} \right) \\ \because 0 \leq \left\{ \frac{r}{x} \right\} < 1 \Rightarrow 0 \leq x \left\{ \frac{r}{x} \right\} < x \\ \therefore \lim_{x \rightarrow 0^+} x \left(\frac{1+2+3+\dots+15}{x} \right) = \frac{15 \times 16}{2} = 120 \end{aligned}$$

Question 113

$$\lim_{x \rightarrow 0} \frac{(27+x)^{\frac{1}{3}} - 3}{9 - (27+x)^{\frac{2}{3}}} \text{ equals.}$$

[Online April 16, 2018]

Options:

- A. $-\frac{1}{3}$
- B. $\frac{1}{6}$
- C. $-\frac{1}{6}$
- D. $\frac{1}{3}$

Answer: C

Solution:

Solution:

$$\text{Let } L = \lim_{x \rightarrow 0} \frac{(27+x)^{\frac{1}{3}} - 3}{9 - (27+x)^{\frac{2}{3}}}$$

Here 'L' is in the indeterminate form i.e., $\frac{0}{0}$

$\therefore$ using the L'Hospital rule we get:

$$L = \lim_{x \rightarrow 0} \frac{\frac{1}{3}(27+x)^{\frac{-2}{3}}}{-\frac{2}{3}(27+x)^{\frac{-1}{3}}} = \frac{\frac{1}{3} \times (27)^{\frac{-2}{3}}}{-\frac{2}{3} \times 27^{\frac{-1}{3}}} = -\frac{1}{6}$$

Question 114

Let $f(x)$ be a polynomial of degree 4 having extreme values at $x = 1$ and $x = 2$. If $\lim_{x \rightarrow 0} \left(\frac{f(x)}{x^2} + 1 \right) = 3$ then $f(-1)$ is equal to
[Online April 15, 2018]

Options:

A. $\frac{1}{2}$

B. $\frac{3}{2}$

C. $\frac{5}{2}$

D. $\frac{9}{2}$

Answer: D

Solution:

Solution:

$\because f(x)$ has extremum values at $x = 1$ and $x = 2$

$\therefore f'(1) = 0$ and $f'(2) = 0$

As, $f(x)$ is a polynomial of degree 4.

Suppose $f(x) = Ax^4 + Bx^3 + Cx^2 + Dx + E$

$$\because \lim_{x \rightarrow 0} \left(\frac{f(x)}{x^2} + 1 \right) = 3$$

$$\Rightarrow \lim_{x \rightarrow 0} \left(\frac{Ax^4 + Bx^3 + Cx^2 + Dx + E}{x^2} + 1 \right) = 3$$

$$\Rightarrow \lim_{x \rightarrow 0} \left(Ax^2 + Bx + C + \frac{D}{x} + \frac{E}{x^2} + 1 \right) = 3$$

As limit has finite value, so $D = 0$ and $E = 0$

$$\text{Now } A(0)^2 + B(0) + C + 0 + 0 + 1 = 3$$

$$\Rightarrow C + 1 = 3 \Rightarrow C = 2$$

$$f'(x) = 4Ax^3 + 3Bx^2 + 2Cx + D$$

$$f'(1) = 0 \Rightarrow 4A(1) + 3B(1) + 2C(1) + D = 0$$

$$\Rightarrow 4A + 3B = -4 \dots\dots(i)$$

$$f'(2) = 0 \Rightarrow 4A(8) + 3B(4) + 2C(2) + D = 0$$

$$\Rightarrow 8A + 3B = -2 \dots\dots\dots(ii)$$

From equations (i) and (ii), we get

$$A = \frac{1}{2} \text{ and } B = -2$$

$$\text{So, } f(x) = \frac{x^4}{2} - 2x^3 + 2x^2$$

$$\text{Therefore, } f(-1) = \frac{(-1)^4}{2} - 2(-1)^3 + 2(-1)^2$$

$$= \frac{1}{2} + 2 + 2 = \frac{9}{2}. \text{ Hence } f(-1) = \frac{9}{2}$$

Question 115

$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\cot x - \cos x}{(\pi - 2x)^3}$ equals:

[2017]

Options:

A. $\frac{1}{4}$

B. $\frac{1}{24}$

C. $\frac{1}{16}$

D. $\frac{1}{8}$

Answer: C

Solution:

Solution:

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\cot x(1 - \sin x)}{-8 \left(x - \frac{\pi}{2} \right)^3} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cot x(1 - \sin x)}{8 \left(\frac{\pi}{2} - x \right)^3}$$

$$\text{Put } \frac{\pi}{2} - x = t \Rightarrow \text{as } x \rightarrow \frac{\pi}{2} \Rightarrow t \rightarrow 0$$

$$= \lim_{t \rightarrow 0} \frac{\cot \left(\frac{\pi}{2} - t \right) \left(1 - \sin \left(\frac{\pi}{2} - t \right) \right)}{8t^3}$$

$$= \lim_{t \rightarrow 0} \frac{\tan t(1 - \cos t)}{8t^3} = \lim_{t \rightarrow 0} \frac{\tan t}{8t} \cdot \frac{1 - \cos t}{t^2}$$

$$= \frac{1}{8} \cdot 1 \cdot \frac{1}{2} = \frac{1}{16}$$

Question116

$\lim_{x \rightarrow 3} \frac{\sqrt{3x} - 3}{\sqrt{2x - 4} - \sqrt{2}}$ is equal to:

[Online April 8, 2017]

Options:

A. $\sqrt{3}$

B. $\frac{1}{\sqrt{2}}$

C. $\frac{\sqrt{3}}{2}$

D. $\frac{1}{2\sqrt{2}}$

Answer: B

Solution:

Solution:

$$\text{Let } A = \lim_{x \rightarrow 3} \frac{\sqrt{3x} - 3}{\sqrt{2x - 4} - \sqrt{2}}$$

Rationalise

$$\Rightarrow A = \lim_{x \rightarrow 3} \frac{(3x-9) \times (2x-4+\sqrt{2})}{\{(2x-4)-2\} \times (\sqrt{3x}+3)}$$
$$= \lim_{x \rightarrow 3} \frac{3(x-3)}{2(x-3)} \times \frac{\sqrt{2x-4}+\sqrt{2}}{(\sqrt{3x}+3)} = \frac{3}{2} \times \frac{2\sqrt{2}}{6} = \frac{1}{\sqrt{2}}$$

Question117

Let $p = \lim_{x \rightarrow 0^+} (1 + \tan^2 \sqrt{x})^{\frac{1}{2x}}$ then log p is equal to
[2016]

Options:

A. $\frac{1}{2}$

B. $\frac{1}{4}$

C. 2

D. 1

Answer: A

Solution:

Solution:

$$\ln p = \lim_{x \rightarrow 0^+} \frac{1}{2x} \ln(1 + \tan^2 \sqrt{x})$$

$$\lim_{x \rightarrow 0^+} \frac{1}{x} \ln(\sec \sqrt{x})$$

Applying L hospital's rule :

$$= \lim_{x \rightarrow 0^+} \frac{\sec \sqrt{x} \tan \sqrt{x}}{\sec \sqrt{x} \cdot 2\sqrt{x}} = \lim_{x \rightarrow 0^+} \frac{\tan \sqrt{x}}{2\sqrt{x}} = \frac{1}{2}$$

Question118

$$\lim_{x \rightarrow 0} \frac{(1 - \cos 2x)^2}{2x \tan x - x \tan 2x} \text{ is}$$

[Online April 10, 2016]

Options:

A. 2

B. $-\frac{1}{2}$

C. -2

D. $\frac{1}{2}$

Answer: C

Solution:

Solution:

$$\lim_{x \rightarrow 0} \frac{(1 - \cos 2x)^2}{2x \tan x - x \tan 2x}$$

$$\begin{aligned} &= \lim_{x \rightarrow 0} \frac{(2\sin^2 x)^2}{2x \left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots \right) - x \left(2x + \frac{2^3 x^3}{3} + 2 \frac{2^5 x^5}{15} + \dots \right)} \\ &= \lim_{x \rightarrow 0} \frac{4 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right)^4}{x^4 \left(\frac{2}{3} - \frac{8}{5} \right) + x^6 \left(\frac{4}{15} - \frac{64}{15} \right)} \\ &= \lim_{x \rightarrow 0} \frac{4 \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots \right)^4}{-2 + x^2 \left(-\frac{60}{15} \right) + \dots} \end{aligned}$$

(dividing numerator & denominator by x^4)
= -2

Question119

If $\lim_{x \rightarrow \infty} \left(1 + \frac{a}{x} - \frac{4}{x^2} \right)^{2x} = e^3$, then 'a' is equal to:

[Online April 9, 2016]

Options:

A. 2

B. $\frac{3}{2}$

C. $\frac{1}{2}$

D. $\frac{2}{3}$

Answer: B

Solution:

Solution:

$$\begin{aligned} &\lim_{x \rightarrow \infty} \left(1 + \frac{a}{x} - \frac{4}{x^2} \right)^{2x} \text{ (} 1^\infty \text{ form)} \\ &= e \left[\lim_{x \rightarrow \infty} \left(1 + \frac{a}{x} - \frac{4}{x^2} - 1 \right) 2x \right] \\ &= e \lim_{x \rightarrow \infty} \left(2a - \frac{8}{x} \right) = e^{2a} \\ \therefore 2a = 3 \Rightarrow a &= \frac{3}{2} \end{aligned}$$

Question120

$\lim_{x \rightarrow 0} \frac{(1 - \cos 2x)(3 + \cos x)}{x \tan 4x}$ is equal to

[2015]

Options:

A. 2

B. $\frac{1}{2}$

C. 4

D. 3

Answer: A

Solution:

Solution:

Multiply and divide by x in the given expression, we get

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{(1 - \cos 2x)(3 + \cos x)}{x^2} \cdot \frac{x}{\tan 4x} \\ &= \lim_{x \rightarrow 0} \frac{2\sin^2 x}{x^2} \cdot \frac{3 + \cos x}{1} \cdot \frac{x}{\tan 4x} \\ &= 2 \lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2} \cdot \lim_{x \rightarrow 0} 3 + \cos x \cdot \lim_{x \rightarrow 0} \frac{x}{\tan 4x} \\ &= 2.4 \cdot \frac{1}{4} \lim_{x \rightarrow 0} \frac{4x}{\tan 4x} = 2.4 \cdot \frac{1}{4} = 2 \end{aligned}$$

Question121

$\lim_{x \rightarrow 0} \frac{e^{x^2} - \cos x}{\sin^2 x}$ is equal to:

[Online April 10, 2015]

Options:

A. 2

B. 3

C. $\frac{3}{2}$

D. $\frac{5}{4}$

Answer: C

Solution:

Solution:

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{2xe^{x^2} + \sin x}{2 \sin x \cos x} \\ & \lim_{x \rightarrow 0} \left(\frac{x}{\sin x} e^{x^2} + \frac{1}{2} \right) \frac{1}{\cos x} = 1 + \frac{1}{2} = \frac{3}{2} \end{aligned}$$

Question122

Let $f(x)$ be a polynomial of degree four having extreme values at $x = 1$ and $x = 2$. If $\lim_{x \rightarrow 0} \left[1 + \frac{f(x)}{x^2} \right] = 3$, then $f(2)$ is equal to :
[2015]

Options:

- A. 0
- B. 4
- C. - 8
- D. - 4

Answer: A

Solution:

Solution:

$$\lim_{x \rightarrow 0} \left[1 + \frac{f(x)}{x^2} \right] = 3$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{f(x)}{x^2} = 2$$

So, $f(x)$ contain terms in x^2 , x^3 and x^4 .

$$\text{Let } f(x) = a_1x^2 + a_2x^3 + a_3x^4$$

$$\text{Since } \lim_{x \rightarrow 0} \frac{f(x)}{x^2} = 2 \Rightarrow a_1 = 2$$

$$\text{Hence, } f(x) = 2x^2 + a_2x^3 + a_3x^4$$

$$f'(x) = 4x + 3a_2x^2 + 4a_3x^3$$

$$\text{As given : } f'(1) = 0 \text{ and } f'(2) = 0$$

$$\text{Hence, } 4 + 3a_2 + 4a_3 = 0 \dots (i)$$

$$\text{and } 8 + 12a_2 + 32a_3 = 0 \dots (ii)$$

By $4 \times (i) - (ii)$, we get

$$16 + 12a_2 + 16a_3 - (8 + 12a_2 + 32a_3) = 0$$

$$\Rightarrow 8 - 16a_3 = 0 \Rightarrow a_3 = 1/2$$

$$\text{and by eqn. (i), } 4 + 3a_2 + 4/2 = 0 \Rightarrow a_2 = -2$$

$$\Rightarrow f(x) = 2x^2 - 2x^3 + \frac{1}{2}x^4$$

$$f(2) = 2 \times 4 - 2 \times 8 + \frac{1}{2} \times 16 = 0$$

Question123

$\lim_{x \rightarrow 0} \frac{\sin(\pi \cos^2 x)}{x^2}$ is equal to:

[2014]

Options:

- A. $-\pi$
- B. π
- C. $\frac{\pi}{2}$

D. 1

Answer: B

Solution:

Solution:

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin(\pi \cos^2 x)}{x^2} \\&= \lim_{x \rightarrow 0} \frac{\sin[\pi(1 - \sin^2 x)]}{x^2} \\&= \lim_{x \rightarrow 0} \sin \frac{(\pi - \pi \sin^2 x)}{x^2} [\because \sin(\pi - \theta) = \sin \theta] \\&= \lim_{x \rightarrow 0} \sin \frac{(\pi \sin^2 x)}{\pi \sin^2 x} \times \frac{\pi \sin^2 x}{x^2} \\&= \lim_{x \rightarrow 0} 1 \times \pi \left(\frac{\sin x}{x} \right)^2 = \pi\end{aligned}$$

Question124

If

$$\lim_{x \rightarrow 2} \frac{\tan(x-2)\{x^2 + (k-2)x - 2k\}}{x^2 - 4x + 4} = 5,$$

then k is equal to:

[Online April 11, 2014]

Options:

A. 0

B. 1

C. 2

D. 3

Answer: D

Solution:

Solution:

$$\begin{aligned}\lim_{x \rightarrow 2} \frac{\tan(x-2)\{x^2 + (k-2)x - 2k\}}{x^2 - 4x + 4} &= 5 \\ \Rightarrow \lim_{x \rightarrow 2} \frac{\tan(x-2)\{x^2 + kx - 2x - 2k\}}{(x-2)^2} &= 5 \\ \Rightarrow \lim_{x \rightarrow 2} \frac{\tan(x-2)\{x(x-2) + k(x-2)\}}{(x-2) \times (x-2)} &= 5 \\ \Rightarrow \lim_{x \rightarrow 2} \left(\frac{\tan(x-2)}{(x-2)} \right) \times \lim_{x \rightarrow 2} \left(\frac{(k+x)(x-2)}{(x-2)} \right) &= 5 \\ \Rightarrow 1 \times \lim_{x \rightarrow 2} (k+x) = 5 \left\{ \because \lim_{h \rightarrow 0} \frac{\tan h}{h} = 1 \right\} \\ \text{or } k+2 &= 5 \\ \Rightarrow k &= 3\end{aligned}$$

Question125

$\lim_{x \rightarrow 0} \frac{(1 - \cos 2x)(3 + \cos x)}{x \tan 4x}$ is equal to
[2013]

Options:

A. $-\frac{1}{4}$

B. $\frac{1}{2}$

C. 1

D. 2

Answer: D

Solution:

Solution:

Multiply and divide by x in the given expression, we get

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{(1 - \cos 2x)(3 + \cos x)}{x^2} \cdot \frac{x}{\tan 4x} & \left[\because 1 - \cos 2x = 2\sin^2 \frac{x}{2} \right] \\ &= \lim_{x \rightarrow 0} \frac{2\sin^2 x}{x^2} \cdot \frac{3 + \cos x}{1} \cdot \frac{x}{\tan 4x} \\ &= 2 \lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2} \cdot \lim_{x \rightarrow 0} (3 + \cos x) \cdot \lim_{x \rightarrow 0} \frac{4x}{\tan 4x} \times \frac{1}{4} \\ &= 2 \cdot 4 \cdot \frac{1}{4} = 2 \end{aligned}$$

Question126

Let $f(1) = -2$ and $f'(x) \geq 4.2$ for $1 \leq x \leq 6$. The possible value of $f(6)$ lies in the interval :
[April 25, 2013]

Options:

A. $[15, 19)$

B. $(-\infty, 12)$

C. $[12, 15)$

D. $[19, \infty)$

Answer: D

Solution:

Solution:

Given $f(1) = -2$ and $f'(x) \geq 4.2$ for $1 \leq x \leq 6$

$$\text{Consider } f'(x) = \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow f(x+h) - f(x) = f'(x) \cdot h \geq (4.2)h$$

So, $f(x+h) \geq f(x) + (4.2)h$
 put $x = 1$ and $h = 5$, we get
 $f(6) \geq f(1) + 5(4.2) \Rightarrow f(6) \geq 19$
 Hence $f(6)$ lies in $[19, \infty)$

Question 127

$\lim_{x \rightarrow 0} \frac{\sin(\pi \cos^2 x)}{x^2}$ equals

[Online May 26, 2012]

Options:

A. $-\pi$

B. 1

C. -1

D. π

Answer: D

Solution:

Solution:

$$\begin{aligned} \text{Consider, } \lim_{x \rightarrow 0} \frac{\sin(\pi \cos^2 x)}{x^2} \\ &= \lim_{x \rightarrow 0} \frac{\sin(\pi - \pi \sin^2 x)}{x^2} [\because \sin(\pi - \theta) = \sin \theta] \\ &= \lim_{x \rightarrow 0} \frac{\sin(\pi \sin^2 x)}{\pi \sin^2 x} \times \frac{(\pi \sin^2 x)}{x^2} = \pi \end{aligned}$$

Question 128

$\lim_{x \rightarrow 0} \left(\frac{x - \sin x}{x} \right) \sin \left(\frac{1}{x} \right)$

[Online May 7, 2012]

Options:

A. equals 1

B. equals 0

C. does not exist

D. equals - 1

Answer: B

Solution:

Solution:

$$\begin{aligned}
 &\text{Consider } \lim_{x \rightarrow 0} \left(\frac{x - \sin x}{x} \right) \sin \left(\frac{1}{x} \right) \\
 &= \lim_{x \rightarrow 0} \left[\frac{x \left(1 - \frac{\sin x}{x} \right)}{x} \right] \times \lim_{x \rightarrow 0} \sin \left(\frac{1}{x} \right) \\
 &= \lim_{x \rightarrow 0} \left[1 - \frac{\sin x}{x} \right] \times \lim_{x \rightarrow 0} \sin \left(\frac{1}{x} \right) \\
 &= \left[1 - \lim_{x \rightarrow 0} \frac{\sin x}{x} \right] \times \lim_{x \rightarrow 0} \sin \left(\frac{1}{x} \right) \\
 &= 0 \times \lim_{x \rightarrow 0} \sin \left(\frac{1}{x} \right) = 0
 \end{aligned}$$

Question 129

If $f(x) = 3x^{10} - 7x^8 + 5x^6 - 21x^3 + 3x^2 - 7$, then $\lim_{\alpha \rightarrow 0} \frac{f(1-\alpha) - f(1)}{\alpha^3 + 3\alpha}$ is
[Online May 19, 2012]

Options:

A. $-\frac{53}{3}$

B. $\frac{53}{3}$

C. $-\frac{55}{3}$

D. $\frac{55}{3}$

Answer: B

Solution:

Solution:

$$\text{Let } f(x) = 3x^{10} - 7x^8 + 5x^6 - 21x^3 + 3x^2 - 7$$

$$f'(x) = 30x^9 - 56x^7 + 30x^5 - 63x^2 + 6x$$

$$f'(1) = 30 - 56 + 30 - 63 + 6$$

$$= 66 - 63 - 56 = -53$$

$$\text{Consider } \lim_{\alpha \rightarrow 0} \frac{f(1-\alpha) - f(1)}{\alpha^3 + 3\alpha}$$

$$= \lim_{\alpha \rightarrow 0} \frac{f'(1-\alpha)(-1) - 0}{3\alpha^2 + 3} \quad (\text{By using L'hospital rule})$$

$$= \frac{f'(1-0)(-1)}{3(0)^2 + 3} = \frac{-f'(1)}{3} = \frac{53}{3}$$

Question 130

Let $f : \mathbb{R} \rightarrow [0, \infty)$ be such that $\lim_{x \rightarrow 5} f(x)$ exists and $\lim_{x \rightarrow 5} \frac{(f(x))^2 - 9}{\sqrt{|x-5|}} = 0$

Then $\lim_{x \rightarrow 5} f(x)$ equals :

[2011 RS]

Options:

A. 0

B. 1

C. 2

D. 3

Answer: D

Solution:

Solution:

$$\text{Given that } \lim_{x \rightarrow 5} \frac{(f(x))^2 - 9}{\sqrt{|x - 5|}} = 0$$

$$\Rightarrow \lim_{x \rightarrow 5} [(f(x))^2 - 9] = 0$$

$$\Rightarrow \left[\lim_{x \rightarrow 5} f(x) \right]^2 = 9 \Rightarrow \lim_{x \rightarrow 5} f(x) = 3$$

Question 131

$$\lim_{x \rightarrow 2} \left(\frac{\sqrt{1 - \cos\{2(x - 2)\}}}{x - 2} \right)$$

[2011]

Options:

A. equals $\sqrt{2}$

B. equals $-\sqrt{2}$

C. equals $\frac{1}{\sqrt{2}}$

D. does not exist

Answer: D

Solution:

Solution:

$$\lim_{x \rightarrow 2} \frac{\sqrt{1 - \cos\{2(x - 2)\}}}{x - 2} \left[\because 1 - \cos \theta = 2 \sin^2 \frac{\theta}{2} \right]$$

$$= \lim_{x \rightarrow 2} \frac{\sqrt{2} |\sin(x - 2)|}{x - 2}$$

$$\text{L.H.L.} = - \lim_{x \rightarrow 2} \frac{\sqrt{2} \sin(x - 2)}{(x - 2)} = -\sqrt{2}$$

$$\text{R.H.L.} = \lim_{x \rightarrow 2} \frac{\sqrt{2} \sin(x - 2)}{(x - 2)} = \sqrt{2}$$

Thus $\text{L.H.L.} \neq \text{R.H.L.}$

Hence, $\lim_{x \rightarrow 2} \frac{\sqrt{1 - \cos\{2(x - 2)\}}}{x - 2}$ does not exist.

Question132

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a positive increasing function with $\lim_{x \rightarrow \infty} \frac{f(3x)}{f(x)} = 1$ then

$$\lim_{x \rightarrow \infty} \frac{f(2x)}{f(x)} =$$

[2010]

Options:

A. $\frac{2}{3}$

B. $\frac{3}{2}$

C. 3

D. 1

Answer: D

Solution:

Solution:

Given that $f(x)$ is a positive increasing function.

$$\therefore 0 < f(x) < f(2x) < f(3x)$$

Divided by $f(x)$

$$\Rightarrow 0 < 1 < \frac{f(2x)}{f(x)} < \frac{f(3x)}{f(x)}$$

$$\Rightarrow \lim_{x \rightarrow \infty} 1 \leq \lim_{x \rightarrow \infty} \frac{f(2x)}{f(x)} \leq \lim_{x \rightarrow \infty} \frac{f(3x)}{f(x)}$$

By Sandwich Theorem.

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{f(2x)}{f(x)} = 1$$

Question133

Let α and β be the distinct roots of $ax^2 + bx + c = 0$, then $\lim_{x \rightarrow \alpha} \frac{1 - \cos(ax^2 + bx + c)}{(x - \alpha)^2}$

is equal to

[2005]

Options:

A. $\frac{a^2}{2}(\alpha - \beta)^2$

B. 0

C. $\frac{-a^2}{2}(\alpha - \beta)^2$

D. $\frac{1}{2}(\alpha - \beta)^2$

Answer: A

Solution:

Solution:

Given that

$$ax^2 + bx + c = a(x - \alpha)(x - \beta)$$

$$\lim_{x \rightarrow \alpha} \frac{1 - \cos a(x - \alpha)(x - \beta)}{(x - \alpha)^2}$$

$$= \lim_{x \rightarrow \alpha} \frac{2\sin^2 \left(a \frac{(x - \alpha)(x - \beta)}{2} \right)}{(x - \alpha)^2}$$

$$= \lim_{x \rightarrow \alpha} \frac{2}{(x - \alpha)^2} \times \frac{\sin^2 \left(a \frac{(x - \alpha)(x - \beta)}{2} \right)}{\frac{a^2(x - \alpha)^2(x - \beta)^2}{4}} \times \frac{a^2(x - \alpha)^2(x - \beta)^2}{4}$$

$$= \frac{a^2}{2}(\alpha - \beta)^2.$$

Question134

If $\lim_{x \rightarrow \infty} \left(1 + \frac{a}{x} + \frac{b}{x^2} \right)^{2x} = e^2$, then the values of a and b, are
[2004]

Options:

A. a = 1 and b = 2

B. a = 1, b ∈ R

C. a ∈ R, b = 2

D. a ∈ R, b ∈ R

Answer: B

Solution:**Solution:**

We know that $\lim_{x \rightarrow \infty} (1 + \frac{1}{x})^x = e$

Given that $\lim_{x \rightarrow \infty} \left(1 + \frac{a}{x} + \frac{b}{x^2} \right)^{2x} = e^2$

$$\Rightarrow \lim_{x \rightarrow \infty} \left[\left(1 + \frac{a}{x} + \frac{b}{x^2} \right) \left(\frac{1}{\frac{a}{x} + \frac{b}{x^2}} \right) \right]^{2x \left(\frac{a}{x} + \frac{b}{x^2} \right)} = e^2$$

$$\Rightarrow e^{\lim_{x \rightarrow \infty} \left[a + \frac{b}{x} \right]} = e^2 \Rightarrow e^{2a} = e^2$$

$$\Rightarrow a = 1 \text{ and } b \in \mathbb{R}$$

Question135

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\left[1 - \tan \left(\frac{x}{2} \right) \right] [1 - \sin x]}{\left[1 + \tan \left(\frac{x}{2} \right) \right] [\pi - 2x]^3} \text{ is}$$

[2003]

Options:

A. ∞

B. $\frac{1}{8}$

C. 0

D. $\frac{1}{32}$

Answer: D

Solution:

Solution:

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\tan\left(\frac{\pi}{4} - \frac{x}{2}\right) \cdot (1 - \sin x)}{(\pi - 2x)^3}$$

$$\text{Let } x = \frac{\pi}{2} + y; y \rightarrow 0$$

$$= \lim_{y \rightarrow 0} \frac{\tan\left(-\frac{y}{2}\right) \cdot (1 - \cos y)}{(-2y)^3}$$

$$= \lim_{y \rightarrow 0} \frac{-\tan \frac{y}{2} \cdot 2 \sin^2 \frac{y}{2}}{(-8) \cdot \frac{y^3}{8} \cdot 8} \left[\because 1 - \cos \theta = 2 \sin^2 \frac{\theta}{2} \right]$$

$$= \lim_{y \rightarrow 0} \frac{1}{32} \frac{\tan \frac{y}{2}}{\left(\frac{y}{2}\right)} \cdot \left[\frac{\sin y / 2}{y / 2}\right]^2 = \frac{1}{32} \left[\because \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = \lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta} = 1 \right]$$

Question 136

$\lim_{x \rightarrow 0} \frac{\log x^n - [x]}{[x]}$, $n \in \mathbb{N}$, ($[x]$ denotes greatest integer less than or equal to x)
[2002]

Options:

A. has value -1

B. has value 0

C. has value 1

D. does not exist

Answer: D

Solution:

Solution:

Since, $\lim_{x \rightarrow 0^-} [x] = -1 \neq \lim_{x \rightarrow 0^+} [x] = 0$. So $\lim_{x \rightarrow 0} [x]$ does not exist, hence the required limit does not exist.

Question137

$$\lim_{x \rightarrow \infty} \left(\frac{x^2 + 5x + 3}{x^2 + x + 2} \right)^x$$

[2002]

Options:

A. e^4

B. e^2

C. e^3

D. 1

Answer: A

Solution:

Solution:

$$\begin{aligned} \lim_{x \rightarrow \infty} \left(\frac{x^2 + 5x + 3}{x^2 + x + 2} \right)^x &= \lim_{x \rightarrow \infty} \left(1 + \frac{4x + 1}{x^2 + x + 2} \right)^x \\ &= \lim_{x \rightarrow \infty} \left[\left(1 + \frac{4x + 1}{x^2 + x + 2} \right)^{\frac{x^2 + x + 2}{4x + 1}} \right]^{\frac{(4x + 1)x}{x^2 + x + 2}} \\ &= e^{\lim_{x \rightarrow \infty} \frac{4x^2 + x}{x^2 + x + 2}} \left[\because \lim_{x \rightarrow \infty} \left(1 + \frac{1}{\lambda x} \right)^{\lambda x} = e^\lambda \right] \\ &= e^{\lim_{x \rightarrow \infty} \frac{4 + \frac{1}{x}}{1 + \frac{1}{x} + \frac{2}{x^2}}} = e^4 \left[\because \frac{1}{\infty} = 0 \right] \end{aligned}$$

Question138

$$\lim_{x \rightarrow 0} \frac{\sqrt{1 - \cos 2x}}{\sqrt{2x}} \text{ is}$$

[2002]

Options:

A. 1

B. -1

C. zero

D. does not exist

Answer: D

Solution:

Solution:

$$\lim_{x \rightarrow 0} \frac{\sqrt{1 - \cos 2x}}{\sqrt{2}x};$$

$$\lim_{x \rightarrow 0} \frac{\sqrt{2\sin^2 x}}{\sqrt{2}x} \Rightarrow \lim_{x \rightarrow 0} \frac{|\sin x|}{x} \left[\because \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \right]$$

The limit of above does not exist as

$$\text{LH S} = -1 \neq \text{RH L} = 1$$

Question 139

Let $f(x) = 4$ and $f'(x) = 4$. Then $\lim_{x \rightarrow 2} \frac{xf(2) - 2f(x)}{x - 2}$ is given by [2002]

Options:

- A. 2
- B. -2
- C. -4
- D. 3

Answer: C

Solution:

Given that $f(2) = 4$ and $f'(2) = 4$

We have, $\lim_{x \rightarrow 2} \frac{xf(2) - 2f(x)}{x - 2}, \left(\frac{0}{0} \right)$

Applying L-Hospital's rule, we get

$$\begin{aligned} &= \lim_{x \rightarrow 2} f(2) - 2f'(x) = f(2) - 2f'(2) \\ &= 4 - 2 \times 4 = -4 \end{aligned}$$
