Chapter 1

Relations and Functions

Exercise 1.3

Q. 1

Let f: {1, 3, 4} → {1, 2, 5} and g: {1, 2, 5} → {1, 3} be given by

f = {(1, 2), (3, 5), (4, 1)} and g = {(1, 3), (2, 3), (5, 1)}. Write down g of.

Answer:

It is given that f: {1, 3, 4} → {1, 2, 5} and g: {1, 2, 5} → {1, 3} be given by

f = {(1, 2), (3, 5), (4, 1)} and g = {(1, 3), (2, 3), (5, 1)}. Then, g of (1) = g (f (1)) = g (2) = 3 g of (3)

= g (f (3)) = g (5) = 1 gof (4) = g (f (4)) = g (1) = 3

Therefore, gof = {(1,3), (3,1), (4,3)}

Q. 2

Let f, g and h be functions from R to R. Show that

(f + g) oh = foh + goh

(f. g) oh = (foh). (goh)

Answer:

(i) (f + g) oh = foh + goh

Let us consider ((f + g) oh) (x) = (f + g) (h(x))

= f(h(x)) + g(h(x))

= (foh)(x) + (goh)(x)

= {(foh) + (goh)} (x)

Then, ((f + g) oh) (x) = {(foh) +(goh)} (x) ∀ x ϵ R

Therefore, (f + g) oh = foh + goh.

(ii) (f. g) oh = (foh). (goh)

Let us consider ((f. g) oh) (x) = (f. g) (h(x))

= f(h(x)). g(h(x))

= f (h(x)). g(h(x))

= (fog)(x). (goh)(x)

= {(fog). (goh)} (x)

Then, ((f. g) oh) (x) = {(fog). (goh)} (x) ∀ x ϵ R

Therefore, (f. g) oh = (fog). (goh)

Q. 3 A

Find gof and fog, if

(i) f (x) = | x | and g(x) = | 5x – 2 |

Answer:

(i) f(x) = |x| and g(x) = |5x -2|

Then, (gof)(x) = g (f(x)) = g(|x|) = |5|x|-2|

(fog)(x) = f(g(x)) = f (|5x -2 |) = ||5x -2 || = |5x -2 |

Q. 3 B

Find gof and fog, if

f (x) = $8x^3$ and g (x) = $x^{\frac{1}{3}}$

Answer:

(ii) f (x) = $8x^3$ and g (x) = $x^{\frac{1}{3}}$

Then, (gof) (x) = g (f(x)) = g(8x3) = $(8x^3)^{\frac{1}{3}}$ = 2

(fog)(x) = f(g(x)) = f $\left(x^{\frac{1}{3}}\right) = 8\left(x^{\frac{1}{3}}\right)^3$ = 8x

Q. 4

If f (x) = $\frac{(4x+3)}{(6x-4)}, x \neq \frac{2}{3}$, show that fof(x) = x, for all x ≠ $\frac{2}{3}$. What is the inverse of f?

Answer:

It is given that f (x) = $\frac{(4x+3)}{(6x-4)}$,x ≠ $\frac{2}{3}$,

(fof) (x) = f (f (x)) = f $\frac{(4x+3)}{(6x-4)} = \frac{4\left(\frac{4x+3}{6x-4}\right)+3}{6\left(\frac{4x+3}{6x-4}\right)-4} = \frac{16x+12+18x-12}{24x-18-24x+16} = \frac{34x}{34}$ = x

Therefore, fof(x) = x, for all x ≠ $\frac{2}{3}$

= > fof = 1

Therefore, the given function f is invertible and the inverse of f itself.

Q. 5 A

State with reason whether following functions have inverse

f: {1, 2, 3, 4} → {10} with

f = {(1, 10), (2, 10), (3, 10), (4, 10)}

Answer:

It is given that f: {1, 2, 3, 4} → {10} with

f = {(1, 10), (2, 10), (3, 10), (4, 10)}

From the given definition of f,

We can see that f is a many one function as:

F (1) = f (2) = f (3) = f (4) = 10

therefore, f is not one-one.

Therefore, function f does not have an inverse.

Q. 5 B

State with reason whether following functions have inverse

g: {5, 6, 7, 8} → {1, 2, 3, 4} with

g = {(5, 4), (6, 3), (7, 4), (8, 2)}

Answer:

It is given that: {5, 6, 7, 8} → {1, 2, 3, 4} with

g = {(5, 4), (6, 3), (7, 4), (8, 2)}

Necessary Condition for a function to have inverse: Function should be one-one and onto.

From the given definition,

We can see that f is a many one function as:

G (5) = g (7) = 4

[As at two points function have same values, the function is not one-one]

i.e. g is not one- one.

Therefore, function g does not have an inverse.

Q. 5 C

State with reason whether following functions have inverse

h: {2, 3, 4, 5} → {7, 9, 11, 13} with

h = {(2, 7), (3, 9), (4, 11), (5, 13)}

Answer:

It is given that h: {2, 3, 4, 5} → {7, 9, 11, 13} with

h = {(2, 7), (3, 9), (4, 11), (5, 13)}

We can see that all distinct elements of the set (2, 3, 4, 5) have distinct images under h.

⇒ h is one–one.

Also, h is onto since for every element of the set {7, 9, 11, 13},

there exists an element x in the set {2, 3, 4, 5} such that h(x) = y.

Therefore, h is a one–one and onto function.

Therefore, h has an inverse.

Q. 6

Show that f: [–1, 1] → R, given by f (x) = $\frac{x}{x+2}$ is one-one. Find the inverse of the function f: [–1, 1] → Range f.

(Hint: For y ∈ Range f, y = f (x) = $\frac{x}{x+2}$, for some x in [–1, 1], i.e., x = $\frac{2y}{(1-y)}$)

Answer:

It is given that f: [–1, 1] → R, given by f (x) = $\frac{x}{x+2}$

Now, Let f(x) = f(y)

⇒ $\frac{x}{x+2} = \frac{y}{y+2}$

⇒ xy + 2x = xy +2y

⇒ 2x = 2y

⇒ x = y

⇒ f is a one- one function.

Now, Let y = $\frac{x}{x+2}$, xy = x + 2y so x = $\frac{y}{1-y}$

So, for every y in the range there exists x in the domain such that f(x) = y

⇒ f is onto function.

⇒ f: [-1,1] → Range f is one-one and onto

⇒ the inverse of the function: f: [-1, 1] → Range f exists.

Let g: Range f → [-1, 1]be the inverse of range f.

Let y be an arbitrary element of range f.

Since, f: [-1, 1] → Range f is onto, we get:

y = f(x) for same x ϵ [-1, 1]

⇒ y = $\frac{x}{x+2}$

⇒ xy +2y = x

⇒ x (1 - y) = 2y

⇒ x = $\frac{2y}{1-y}$, y ≠ 1

Now, let us define g: Range f → [-1, 1]

g(y) = $\frac{2y}{1-y}$, y ≠ 1

Now, (gof)(x) = g(f(x)) = g $\left(\frac{x}{x+2}\right) = \frac{2\left(\frac{x}{x+2}\right)}{1-\frac{x}{x+2}} = \frac{2x}{x+2-x} = \frac{2x}{2}$

(fog)(y) = f(g(y)) = f $\left(\frac{2y}{1-y}\right) = \frac{\frac{2y}{1-y}}{\frac{2y}{1-y}+2} = \frac{2y}{2y+2-2y} = \frac{2y}{2}$

Thus, gof = I [-1,1] and fog = $I_{\text{Range f}}$

⇒ f-1 = g

Therefore, f-1(y) = $\frac{2y}{1-y}$, y ≠ 1

Q. 7

Consider f: R → R given by f (x) = 4x + 3. Show that f is invertible. Find the inverse of f.

Answer:

It is given that f: R → R given by f (x) = 4x + 3

Let f(x) = f(y)

⇒ 4x +3 = 4y +3

⇒ 4x = 4y

⇒ x = y

⇒ f is one- one function.

Now, for y ϵ R, Let y = 4x +3

= x = $\frac{y-3}{4}$ ∈ R

⇒ for any y ϵ R, there exists x = $\frac{Y-3}{4}$ ϵ R

such that, f(x) = f $\left(\frac{y-3}{4}\right) = 4\left(\frac{y-3}{4}\right) + 3$ = y

⇒ F is onto function.

Since, f is one –one and onto

⇒ f-1 exists.

Let us define g: R → R by g(x) = $\frac{x-3}{4}$

Now, (gof)(x) = g(f(x)) = g (4x + 3) = $\frac{(4x+3)-3}{4}$ = x

(fog)(y) = f(g(xy)) = f $\left(\frac{y-3}{4}\right) = 4\left(\frac{y-3}{4}\right) + 3$ = y – 3 + 3 = y

Therefore, gof = fog = IR

Therefore, f is invertible and the inverse of f is given by

f^{-1} (y) = g(y) = $\frac{y-3}{4}$.

Q. 8

Consider f: R+⟶ [4, ∞) given by f (x) = x2 + 4. Show that f is invertible with the inverse f^{-1} of f given by f-1 = $\sqrt{y-4}$, where R+ is the set of all non-negative real numbers.

Answer:

It is given that f: R+⟶ [4, ∞) given by f (x) = x^2 + 4.

Now, Let f(x) = f(y)

⇒ $x^2 + 4 = y^2 + 4$

⇒ $x^2 = y^2$

⇒ x = y

⇒ f is one-one function.

Now, for y ϵ [4, ∞), let y = x^2 + 4.

⇒ $x^2 = y - 4 \geq 0$

= x = $\sqrt{y-4} \geq 0$

⇒ for any y ϵ R, there exists x = $\sqrt{y-4}$ ϵ R such that

= f (x) = f $\left(\sqrt{y-4}\right) = \left(\sqrt{y-4}\right)^2$ y -4 + 4 = y.

⇒ f is onto function.

Therefore, f is one–one and onto function, so f-1 exists.

Now, let us define g: [4, ∞) ⟶ R+ by,

g(y) = $\sqrt{y-4}$

Now, gof(x) = g(f(x)) = g (x^2 + 4) = $\sqrt{(x^2+4)+4} = \sqrt{x^2}$ = x

And, fog(y) = f(g(y)) = f $\left(\sqrt{y-4}\right)$ = $\left(\sqrt{y-4}\right)^2$ + 4 = (y – 4) + 4 = y

Therefore, gof = gof = IR.

Therefore, f is invertible and the inverse of f is given by

f-1(y) = g(y) = $\sqrt{y-4}$

Q. 9

Consider f: R+ ⟶ [–5, ∞) given by f(x) = $9x^2 + 6x - 5$. Show that f is invertible with f^{-1} (y) = $\left\{\frac{(\sqrt{y+6})-1}{3}\right\}$

Answer:

It is given that f: R+ ⟶ [–5, ∞) given by f(x) = $9x^2 + 6x - 5$.

Let y be any element of [-5, ∞)

Now, let y = $9x^2 + 6x - 5$

⇒ y = $(3x+1)^2$ -1-5 = $(3x+1)^2 - 6$

⇒ 3x + 1 = $\sqrt{y+6}$

= x = $\frac{\sqrt{y+6}-1}{3}$

⇒ f is onto and its range is f = [-5, ∞)

Now, let us define g: [-5, ∞) ⟶ R+ as g(y) = $\frac{\sqrt{y+6}-1}{3}$

Now, we have:

(gof)(x) = g(f(x)) = g ($9x^2 + 6x - 5$)

= g((3x+1)2 – 6)

$= \frac{\sqrt{(3x+1)^2-6+6}-1}{3}$

$= \frac{3x+1-1}{3}$ = x

And, (fog)(y) = f(g(y)) = f $\left(\frac{\sqrt{y+6}-1}{3}\right)$

$= \left[3\left(\frac{\sqrt{y+6}-1}{3}\right)+1\right]^2 - 6$

$= \left(\sqrt{y+6}\right)^2 - 6 =$ y + 6 – 6 = y

Thus, gof = IR and fog = I (-5, ∞)

Therefore, f is invertible and the inverse of f is given by

f -1 (y) = g (y) = $\frac{\sqrt{y+6}-1}{3}$

Q. 10

Let f: X → Y be an invertible function. Show that f has unique inverse.

(Hint: suppose g_1 and g_2 are two inverses of f. Then for all y ∈ Y,

$fog_1(y)$ = 1Y(y) = $fog_2(y)$. Use one-one ness of f).

Answer:

It is given that f: X → Y be an invertible function.

Also, suppose f has two inverse

Then, for all y ϵ Y, we get:

$fog_1(y) = I_1\ (y) = fog_2(y)$

= > $f(g_1(y)) = f(g_2(y))$

= > $g_1(y) = g_2(y)$

$=> g_1 = g_2$

Therefore, f has a unique inverse.

Q. 11

Consider f: {1, 2, 3} → {a, b, c} given by f (1) = a, f (2) = b and f (3) = c. Find f^{-1} and show that $(f^{-1})^{-1} = f$.

Answer:

It is given that f: {1, 2, 3} → {a, b, c} given by

f (1) = a, f (2) = b and f (3) = c

So, if we define g:

{a, b, c} → {1, 2, 3} as

g(a) = 1, g(b) = 2, g(c) = 3, then we get:

(fog)(a) = f(g(a)) = f (1) = a

(fog)(b) = f(g(b)) = f (2) = b

(fog)(c) = f(g(c)) = f (3) = c

And

(gof) (1) = g ((1)) = g(a) = 1

(gof) (2) = g (f (2)) = g(b) = 2

(gof) (3) = g (f (3)) = g(c) = 3

Therefore, gof = IX and fog = IY, where X = {1, 2, 3} and Y = {a, b, c}

Thus, the inverse of f exists and f-1 = g.

Then, f^{-1}: {a, b, c} → {1, 2, 3} is given by

$f^{-1}(a) = 1$, $f^{-1}(b) = 2$, $f^{-1}(c) = 3$

Let us now find the inverse of f^{-1}, So, if we define h: {1, 2, 3} → {a, b, c} as

h (1) = a, h (2) = b, h (3) = c, then we get:

(goh) (1) = g (h (1)) = g(a) = 1

(goh) (2) = g (h (2)) = g(b) = 2

(goh) (3) = g (h (3)) = g(c) = 3

And,

(hog)(a) = h(g(a)) = h (1) = a

(hog)(b) = h(g(b)) = h (2) = b

(hog)(c) = h(g(c)) = h (3) = c

⇒ goh = I_X and hog = I_Y, where X = {1, 2, 3} and Y = {a, b, c}.

⇒ The inverse of g exists and $g^{-1} = h$

⇒ $(f^{-1})^{-1} = h$

⇒ h = f

⇒ $(f^{-1})^{-1} = f$

Q. 12

Let f: X → Y be an invertible function. Show that the inverse of f^{-1} is f, i.e., $(f^{-1})^{-1} = f$.

Answer:

It is given that f: X → Y be an invertible function.

Then, there exists a function g: Y → X such that gof = Ix and fog = Iy.

Then, $f^{-1} = g$.

Now, gof = Ix and fog = Iy

⇒ $f^{-1}$of = Ix and fof-1 = Iy

Thus, f^{-1}: Y→X is invertible and f is the inverse of f^{-1}.

Therefore, $(f^{-1})^{-1} = f$.

Q. 13

If f: R → R be given by f (x) = $(3 - x^3)^{\frac{1}{3}}$, then fof (x) is

A. $x^{\frac{1}{3}}$

B. x3

C. x

D. $(3 - x^3)$

Answer:

It is given that f: R → R be given by f (x) = $(3 - x^3)^{\frac{1}{3}}$,

Then, fof(x) = f(f(x)) = $f\left((3 - x^3)^{\frac{1}{3}}\right) = \left[3 - \left((3 - x^3)^{\frac{1}{3}}\right)^3\right]^{\frac{1}{3}}$

= 3 $[3 - (3 - x^3)]^{\frac{1}{3}} = (x^3)^{\frac{1}{3}}$ = x

⇒ fof(x) = x

Q. 14

Let f: R $-\left\{-\frac{4}{3}\right\}$ →R be a function defined as f (x) = $\frac{4x}{3x+4}$. The inverse of f is the map g: Range f → R $-\left\{-\frac{4}{3}\right\}$given by

A. g (y) = $\frac{3y}{3-4y}$

B. g (y) = $\frac{4y}{4-3y}$

C. g (y) = $\frac{3y}{4-3y}$

D. g (y) = $\frac{4y}{3-4y}$

Answer:

It is given that f: R $-\left\{-\frac{4}{3}\right\}$ → R be a function defined as f (x) = $\frac{4x}{3x+4}$.

Let y be any element of Range f.

Then, there exists x ϵ R -$\left\{-\frac{4}{3}\right\}$ such that y = f(x)

= y = $\frac{4x}{3x+4}$

⟹ 3xy + 4y = 4x

⟹ x (4 – 3y) = 4y

⟹ x = $\frac{4y}{4-3y}$

Let us define g: Range f ⟶ R - $\left\{-\frac{4}{3}\right\}$ as g(y) = $\frac{4y}{4-3y}$

Now, (gof)(x) = g(f(x)) = g $\left(\frac{4y}{4-3y}\right) = \frac{4\left(\frac{4x}{3x+4}\right)}{4-3\left(\frac{4y}{4-3y}\right)} = \frac{16x}{12x+16-12x} = \frac{16x}{16}$

And, (fog)(y) = f(g(y)) = f $\left(\frac{4y}{4-3y}\right) = \frac{4\left(\frac{4y}{4-3y}\right)}{3\left(\frac{4y}{4-3y}\right)+4} = \frac{16y}{12y+16-12y} = \frac{16y}{16}$ = y

Therefore, gof = ^{I} R $-\left\{-\frac{4}{3}\right\}$ and fog = $I_{\text{Range f}}$

Thus, g is the inverse of f

Therefore, the inverse of f is the map

: Range f ⟶ R - $\left\{-\frac{4}{3}\right\}$, which is given by g(y) = $\frac{4y}{4-3y}$