MATHEMATICS

ELLIPSE

INTRODUCTION TO ELLIPSE

NOTES

What you already know

- Basic understanding of geometrical figures
- Basic algebraic formulas
- Basics about conics
- Conditions for different conics
- Analytic interpretation of different conics

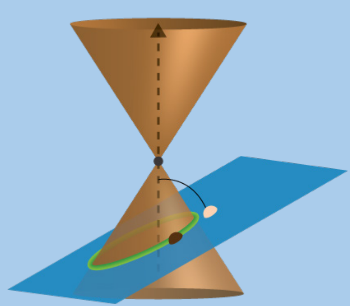
What you will learn

- To find the different parameters associated with an ellipse

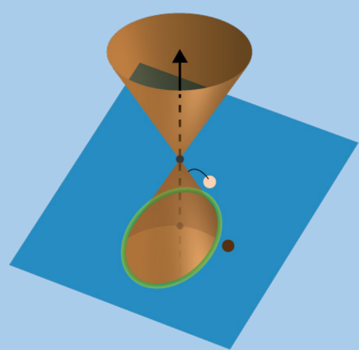
Visualising Ellipse

As we know that when the cutting plane is slightly tilted with respect to the base of the double sided right circular cone, we get the conic ellipse.

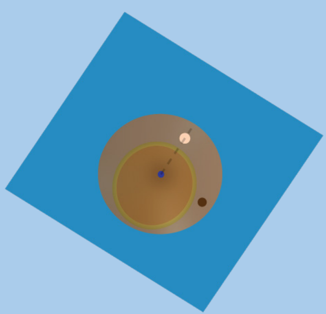
Plane and double cone arrangement



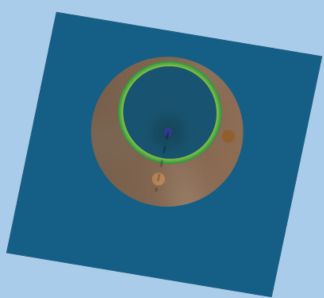
Isometric view



Top view



Bottom view



Analytic interpretation

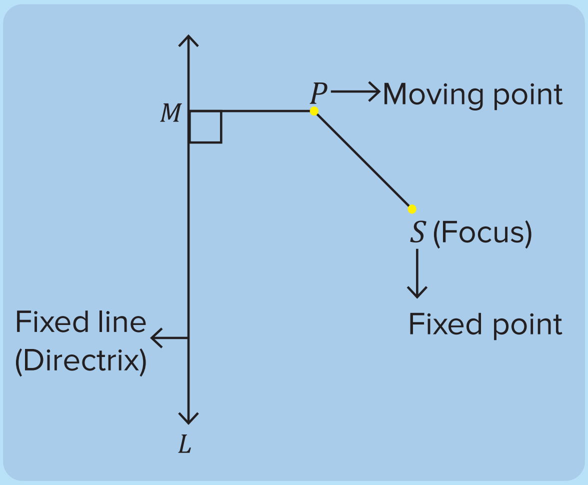
A conic is the locus of a moving point such that the ratio of its distance from a fixed point and a fixed line is always constant. I.e.,

$$\frac{\text{Distance from a fixed point (Focus)}}{\text{Distance from a fixed line (Directrix)}} = \text{Constant}$$

Or $\frac{PS}{PM}$ = Constant = Eccentricity (e)

For parabola, e = 1

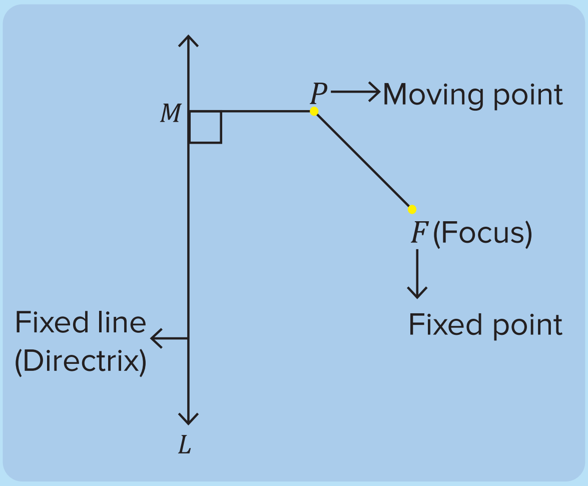
For ellipse, 0 < e < 1



Ellipse

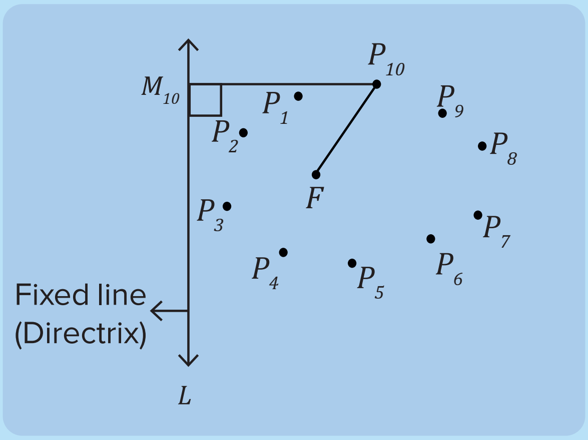
An ellipse is the locus of a moving point such that the ratio of its distance from a fixed point (focus) and a fixed line (directrix) is a positive constant which is always less than 1 (0 < e < 1).

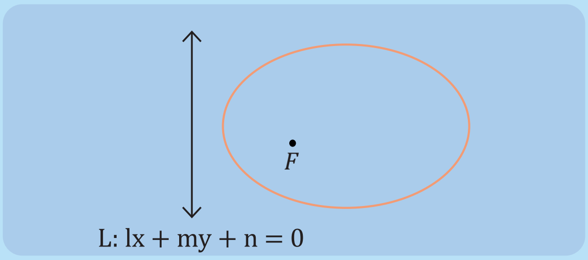
Let the fixed point be F (focus) and the fixed line (directrix) be L: $lx + my + n = 0$



Let us consider, e = 0.5 Let M_i be the respective distances from the points for i = 1 to 10

Or $\frac{P_1F}{P_1M_1} = \frac{P_2F}{P_2M_2} = ... = 0.5$

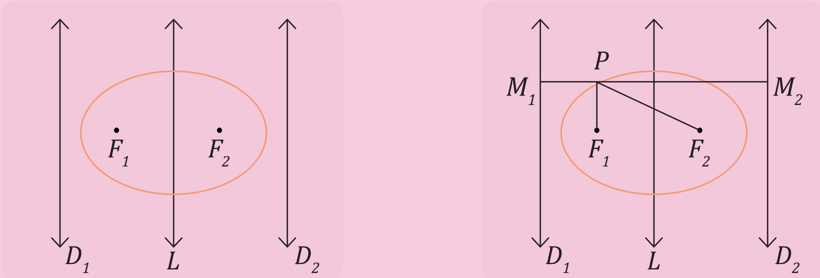




Note

Here, the first directrix and first focus are given. We will take a parallel line L passing through the ellipse such that it divides the ellipse into two congruent halves (line L acts as a plane mirror). After taking the images with respect to this line, we get the second directrix and second focus.

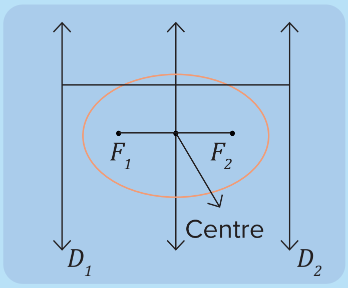
Also, if we take a point P(h, k), then $\frac{PF_1}{PM_1} = \frac{PF_2}{PM_2} = e$



Hence, for an ellipse, there are two foci (F_1 and F_2) and two directrices.

Basic terminologies related to an ellipse

1. Centre of ellipse: It is the midpoint of the line joining its two foci.
2. Both directrices D_1 and D_2 are parallel to each other.
   (D_1: $lx + my + n = 0$ and D_2: $l\,x + my + n' = 0$)



Find the equation of the ellipse whose focus is (-1, 1), directrix is x - y + 3 = 0, and eccentricity is $\frac{1}{2}$.

Solution

Step 1:

Given, Focus (F) ≡ (-1, 1)

Directrix: $l \equiv x - y + 3 = 0$ and $e = \frac{1}{2}$

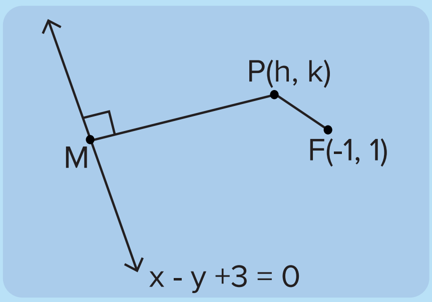
Let us consider P(h, k) be a moving point.

Now, by definition, $\frac{PF}{PM} = e \Rightarrow \frac{PF}{PM} = \frac{1}{2}$

$\Rightarrow PF = \frac{1}{2}PM$

Here, after using suitable distance formulas, we get,

$\Rightarrow \sqrt{(h + 1)^2 + (k - 1)^2} = \frac{1}{2}\left(\frac{|h - k + 3|}{\sqrt{1^2 + (-1)^2}}\right)$



Step 2:

Now, after squaring both sides, we get,

$\Rightarrow (h + 1)^2 + (k - 1)^2 = \frac{1}{8} \times (h - k + 3)^2$

$\Rightarrow 8[(h + 1)^2 + (k - 1)^2] = (h - k + 3)^2$

$\Rightarrow 8(h^2 + 2h + 1 + k^2 - 2k + 1) = (h^2 + k^2 + 9 - 2hk - 6k + 6h)$

Finally, we get,

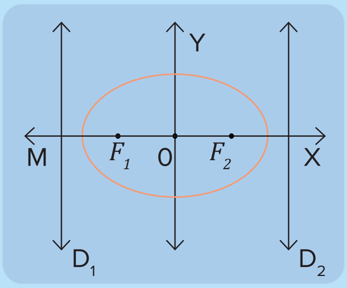
$7h^2 + 7k^2 + 2hk + 10h - 10k + 7 = 0$

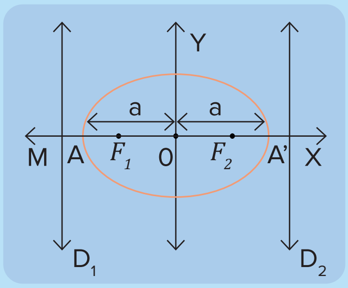
Now, to get the equation, replace $h \to x$ and $k \to y$.

$7x^2 + 2xy + 7y^2 + 10x - 10y + 7 = 0$

Standard ellipse

Let F_1 and F_2 be the foci, and D_1 and D_2 be the directrices of the ellipse.
Here,
Centre of ellipse ≡ Origin or (0, 0),
Axis of ellipse ≡ X-axis

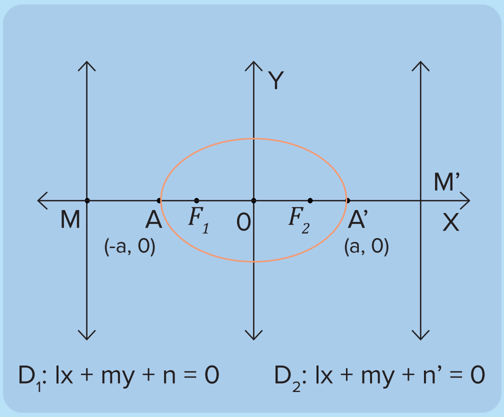




Let A ≡ (-a, 0) and A' ≡ (a, 0)
Now, by the definition of ellipse, $\frac{AF_1}{AM} = e \Rightarrow \frac{OA - OF_1}{OM - OA} = e$

As given, OA = a, therefore, $\frac{a - OF_1}{OM - a} = e$
$\Rightarrow a - OF_1 = e(OM - a)$
Or $a - OF_1 = eOM - ea$ ---------(i)
Now, for A',
$\Rightarrow \frac{A'F_1}{A'M} = e \Rightarrow \frac{OA' + OF_1}{OM + OA'} = e$
As we know that the ellipse is symmetric to Y-axis, OA'=a
$\Rightarrow \frac{a + OF_1}{OM + a} = e$

$\Rightarrow a + OF_1 = eOM + ea$ ---------(ii)
Now, after adding equation (i) and (ii), we get,
$2a = 2eOM \Rightarrow OM = \frac{a}{e}$
After subtracting equation (ii) from (i), we get,
We get, $-2OF_1 = -2ae \Rightarrow OF_1 = ae$
Therefore, we get the coordinates for different points as follows:
$M(\frac{-a}{e}, 0)$, $M'(\frac{a}{e}, 0)$, $F_1(-ae, 0)$, and $F_2(ae, 0)$
Equation of directrices
$D_1 : x = \frac{-a}{e}$ and $D_2 : x = \frac{a}{e}$



Standard equation of an ellipse

Let P ≡ (h, k) be a point on the ellipse having eccentricity e, $0 < e < 1$
We know, by definition,
$\frac{PF_1}{PM_1} = e$ and $\frac{PF_2}{PM_2} = e$ -------(i)
Now, taking the first condition, Here, PF_1 = Distance between P(h, k) and focus, $F_1(-ae, 0)$

$\Rightarrow PF_1 = \sqrt{(h+ae)^2 + k^2}$

And PM_1 = perpendicular distance from P(h, k) and directrix,

$D_1: x = \frac{-a}{e}$

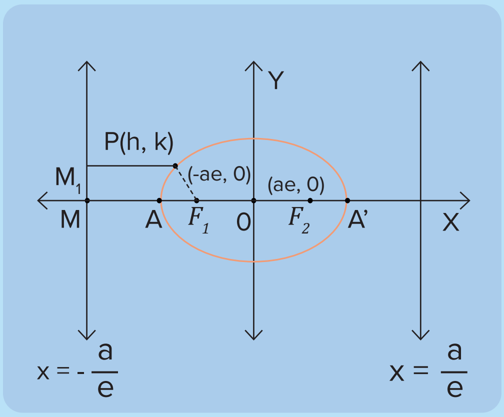
Here, $D_1: ex + 0y + a = 0$

So, $PM_1 = \left|\frac{eh + a}{\sqrt{e^2 + 0^2}}\right| \Rightarrow PM_1 = \frac{|eh + a|}{e}$

Now, using equation (i), we get,

$PF_1 = ePM_1$

$\Rightarrow \sqrt{(h+ae)^2 + k^2} = e \times \frac{|eh + a|}{e}$



Or $\sqrt{(h+ae)^2 + k^2} = |eh + a|$

Now, after squaring both sides, we get,

$(h + ae)^2 + k^2 = (eh + a)^2$

$\Rightarrow h^2 + a^2e^2 + 2aeh + k^2 = e^2h^2 + a^2 + 2aeh$

$\Rightarrow h^2(1 - e^2) + k^2 = a^2(1 - e^2)$

Now, divide by $a^2(1 - e^2)$,

$\Rightarrow \frac{h^2}{a^2} + \frac{k^2}{a^2(1 - e^2)} = 1$

Let us consider $b^2 = a^2(1 - e^2)$

$\Rightarrow \frac{h^2}{a^2} + \frac{k^2}{b^2} = 1$

Replace $h \to x$ and $k \to y$

∴ Standard equation of the ellipse is

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $b^2 = a^2(1 - e^2)$

And, as we know that $0 < e < 1$

$\Rightarrow 0 < 1 - e^2 < 1$

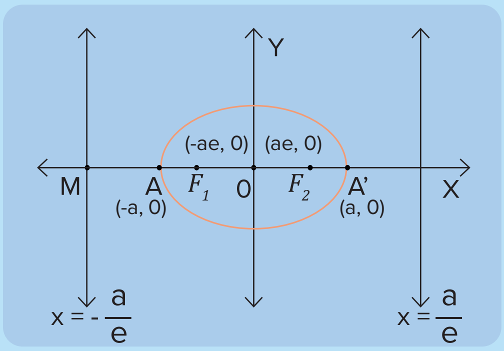
$\Rightarrow 0 < a^2(1 - e^2) < a^2$

$\Rightarrow 0 < b^2 < a^2$

Or $b < a \Rightarrow$ Horizontal ellipse

The standard equation of an ellipse is as follows:

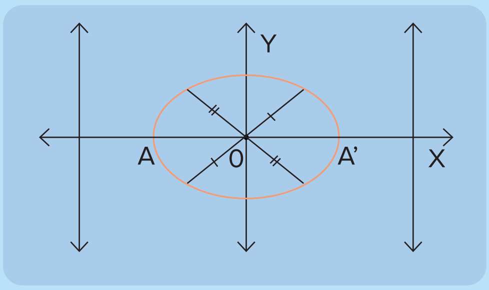
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



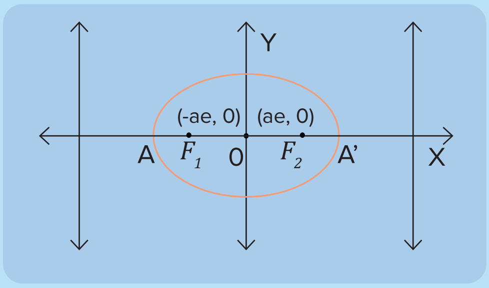
Terms associated with an ellipse

Centre: The point that bisects every chord of an ellipse drawn through it is known as the centre of the ellipse.

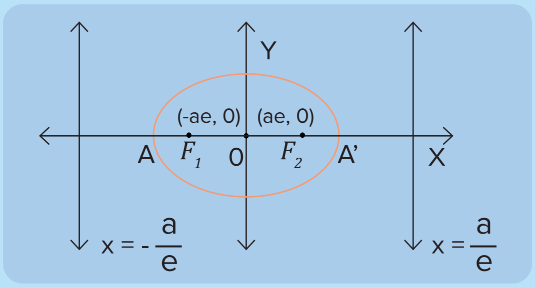
Here, centre (O) ≡ (0, 0)



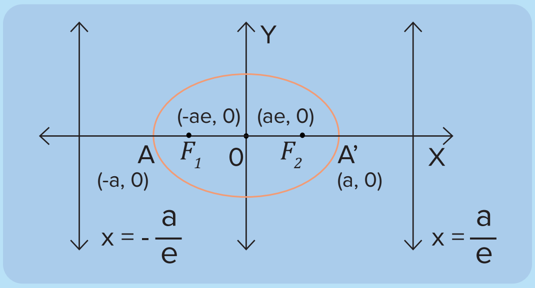
Foci: The foci are $F_1(-ae, 0)$ and $F_2(ae, 0)$.
Or combined form of foci (F) $\equiv (\pm ae, 0)$



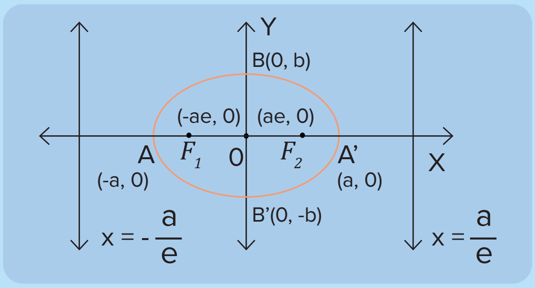
Directrices: The equation of directrices are
$x = \frac{-a}{e}$ and $x = \frac{a}{e}$
Or combined form of directrices, $x = \pm \frac{a}{e}$



Vertices: The point of intersection of an ellipse with the line passing through the foci is known as vertices.
Here, vertices are A≡ (-a, 0) and
A'≡ (a, 0).

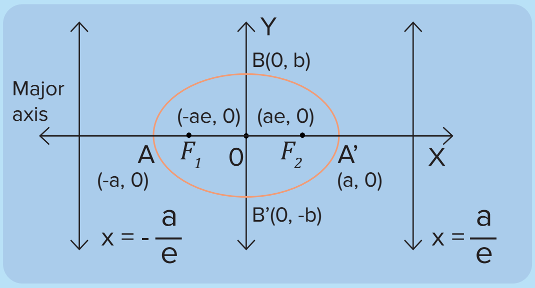


Co-vertices: The point of intersection of an ellipse with the line perpendicular to the line passing through the foci, is called co-vertices.
Here, co-vertices are B≡(0, b) and
B'≡ (0, -b).

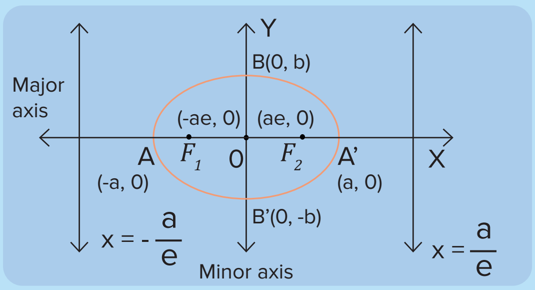


Major axis: The line that passes through the foci and is perpendicular to the directrices is known as major axis.
Here,

- The major axis intersects the ellipse at the vertices.
- The length of major axis is the distance between its vertices, i.e., AA'= 2a
- The major axis is the longest chord of an ellipse.
- Half of the major axis i.e. OA' or OA is the semi major axis of the ellipse. Its length is a.



Minor axis: The axis that is the perpendicular bisector of the major axis is known as minor axis.
Let the ellipse intersect the minor axis at B and B'.
Also, at B and B', x = 0
The standard equation will become,
$$\frac{0}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow y^2 = b^2 \text{ or } y = \pm b$$
∴ B ≡ (0, b) and B' ≡ (0, -b)



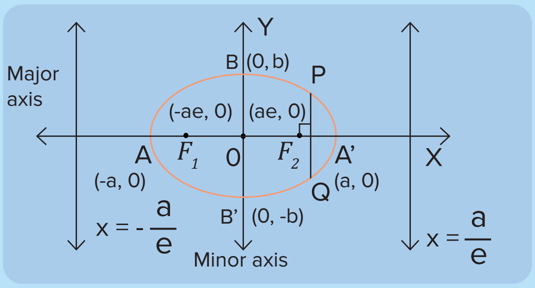
And length of minor axis is $BB' = 2b$

Where $b^2 = a^2(1 - e^2)$

Half the minor axis i.e. OB or OB' is called the semi-minor axis. Its length is b.

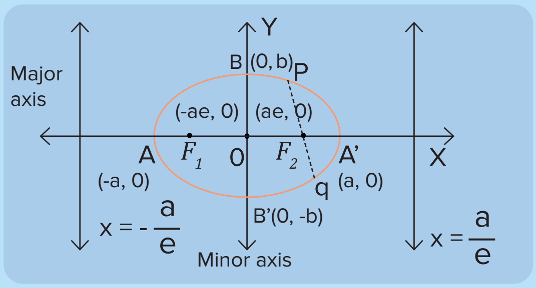
Double ordinate: A chord perpendicular to the major axis is known as double ordinate.

There can be infinite double ordinates in an ellipse.
From the figure, PQ is the double ordinate



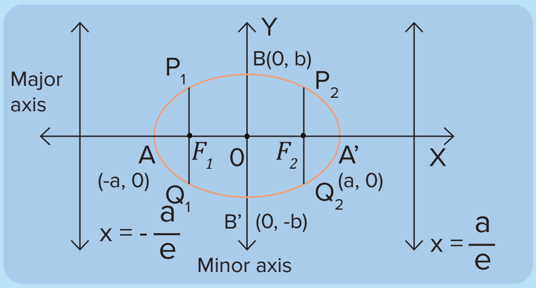
Focal chord: A chord passing through the focus is known as focal chord.

There can be infinite focal chords in an ellipse.
From the figure, PQ is the focal chord.



Latus rectum: The focal chord perpendicular to the major axis is known as latus rectum.
Here, latus rectum ➔ P_1Q_1 and P_2Q_2
And equation of $P_1Q_1 \Rightarrow x = -ae$
Equation of $P_2Q_2 \Rightarrow x = ae$

Length of Latus rectum of the ellipse
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$ is given by $\frac{2b^2}{a}$.



Proof
Equation of LR, P_2Q_2: $x = ae$
Let us consider $P_2 = (ae, y_0)$ and $Q_2 = (ae, y_1)$
Since P_2 lie on the ellipse, it will satisfy its equation :

$$\frac{(ae)^2}{a^2} + \frac{(y_0)^2}{b^2} = 1$$

$$\Rightarrow y_0^2 = b^2(1 - e^2) \text{ or } y_0^2 = b^2 \times \frac{b^2}{a^2}$$

$$\Rightarrow y_0 = \pm \frac{b^2}{a}$$

Hence, $P_2\left(ae, \frac{b^2}{a}\right)$ and $Q_2\left(ae, \frac{-b^2}{a}\right)$

Length of $|P_2Q_2| = \frac{2b^2}{a}$

Similarly, equation of LR, P_1Q_1: $x = -ae$

We get, $P_1\left(-ae, \frac{b^2}{a}\right)$ and $Q_1\left(-ae, \frac{-b^2}{a}\right)$

Therefore, length of LR, $|P_1Q_1| = \frac{2b^2}{a}$

Focal distance: The distance between the focus to any point on an ellipse is known as focal distance or focal radius.
Let P≡ (h, k) be any point on the ellipse, then the focal distance is PF_1 or PF_2.

As we know, by definition, $\frac{PF_1}{PM_1} = e$ and $\frac{PF_2}{PM_2} = e$

$\Rightarrow PF_1 = ePM_1$ and $PF_2 = ePM_2$ --(i)

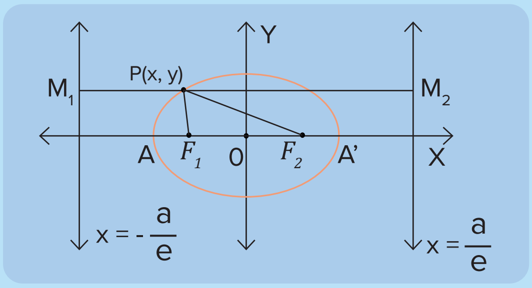
We can see that

$PF_1 = e(\frac{a}{e} - x) \Rightarrow PF_1 = a - ex$

It is the same for PF_2 .
We can see that,

$PF_2 = e(\frac{a}{e} + x) \Rightarrow PF_2 = a + ex$

Therefore, focal distance of $P(x, y) = a \pm ex$



Concept Check 1

Consider the ellipse $5x^2 + 9y^2 = 45$. Find the centre, vertices, eccentricity, foci, equation of directrices, length of major axis, equation of major axis, length of minor axis, equation of minor axis, and length of LR.

Summary Sheet

Key Takeaways

Ellipse
An ellipse is the locus of a moving point such that the ratio of its distance from a fixed point (focus) and a fixed line (directrix) is a positive constant which is always less than 1. ($0 < e < 1$)

Terms associated with an ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Centre: The point that bisects every chord of an ellipse drawn through it is known as the centre of the ellipse. Here, centre (O) ≡ (0, 0)

Foci: The foci are $F_1(-ae, 0)$ and $F_2(ae, 0)$ or combined form of foci (F) ≡ (± ae, 0)

Directrices: The equation of directrices are $x = \frac{-a}{e}$ and $x = \frac{a}{e}$ or combined form of directrices, $x = \pm \frac{a}{e}$

Vertices: The point of intersection of an ellipse with the line passing through the foci is known as vertices. Here, vertices are A≡ (-a, 0) and A'≡ (a, 0)

Co-vertices: The point of intersection of an ellipse with the line perpendicular to the line passing through the foci, is called co-vertices.
Here, co-vertices are $B \equiv (0, b)$ and $B' \equiv (0, -b)$

Major axis: The line that passes through the foci and is perpendicular to the directrices is known as major axis.
Here,
- The major axis intersects the ellipse at the vertices.
- The length of major axis is the distance between its vertices, i.e., $AA' = 2a$
- The major axis is the longest chord of an ellipse.
- Equation of major axis, $y = 0$

Minor axis: The axis that is the perpendicular bisector of the major axis is known as minor axis.
Equation of minor axis, $x = 0$

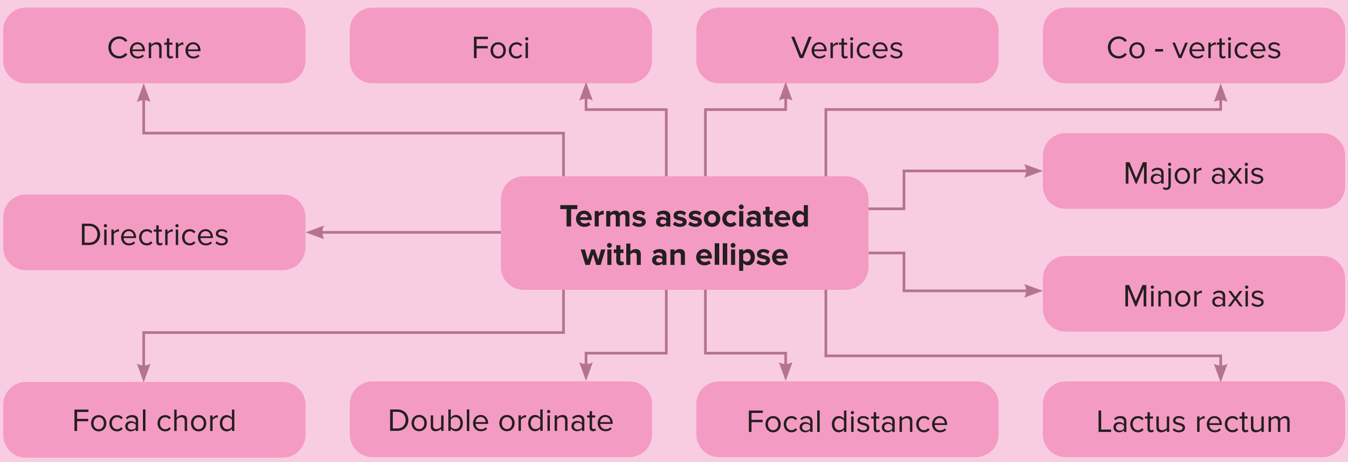
Double ordinate: A chord perpendicular to the major axis is known as double ordinate.
There can be infinite double ordinates in an ellipse.

Focal chord: A chord passing through the focus is known as focal chord. There can be infinite focal chords in an ellipse.

Latus rectum: The focal chord perpendicular to the major axis is known as latus rectum.
Length of LR = $\frac{2b^2}{a}$

Focal distance: The distance between the focus to any point on an ellipse is known as focal distance or focal radius. Focal distance of $P(x, y) = a \pm ex$

Mind Map



Self-Assessment

1. Find the eccentricity of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose LR is half of its major axis.
2. Find the equation of the ellipse whose foci are (±2, 0) and eccentricity is $\frac{1}{2}$.

Answers

Concept Check 1

Step 1:

Given, equation of an ellipse

$5x^2 + 9y^2 = 45$

Now, to convert this equation in the standard form, just divide the whole expression by 45.

Then, we get, $\frac{5x^2}{45} + \frac{9y^2}{45} = 1$

Or $\frac{x^2}{9} + \frac{y^2}{5} = 1$

After comparing the given equation with

$\frac{x^2}{9} + \frac{y^2}{5} = 1$,

We have, $a^2 = 9 \Rightarrow a = \pm 3$ and $b = \pm\sqrt{5}$

Step 2:

We know that, $b^2 = a^2(1 - e^2)$

Therefore, $5 = 9(1 - e^2)$ or $e^2 = 1 - \frac{5}{9}$

$\Rightarrow e = \frac{2}{3}$

Step 3:

Now, we have, Centre = (0, 0)

Vertices = $(\pm a, 0) = (\pm 3, 0)$ Eccentricity (e) = $\frac{2}{3}$

Foci = $(\pm ae, 0) = (\pm 2, 0)$

Equation of directrices, $x = \pm \frac{a}{e} = \pm \frac{9}{2}$

Length of major axis = 2a = 6

Equation of major axis, y = 0

Length of minor axis = $2b = 2\sqrt{5}$

Equation of minor axis, x = 0

Length of LR = $\frac{2b^2}{a} = \frac{10}{3}$

Self-Assessment 1

Step 1:

Given, LR is half of its major axis for an ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Now, let us consider a > b, then LR = $\frac{2b^2}{a}$

And the length of major axis = 2a

Given condition,

LR = Semi-major axis = $\frac{1}{2}$ major axis

$\Rightarrow \frac{2b^2}{a} = a$

$\Rightarrow b^2 = \frac{a^2}{2}$

Step 2:

We know that, $b^2 = a^2(1 - e^2)$

Now, using $b^2 = \frac{a^2}{2}$,

We get, $\frac{a^2}{2} = a^2(1 - e^2)$

$\Rightarrow e^2 = 1 - \frac{1}{2}$

or $e = \frac{1}{\sqrt{2}}$

Self-Assessment 2

Step 1:

Given, foci ≡ (± 2, 0) and eccentricity(e) = $\frac{1}{2}$

Let us consider the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

As $e = \frac{1}{2}$, foci of the ellipse are $(\pm ae, 0) = (\pm 2, 0)$

$\Rightarrow ae = 2 \Rightarrow a = 4$

We also know that, $b^2 = a^2(1 - e^2)$

$\Rightarrow b^2 = 16 \times (1 - \frac{1}{4}) = 12$

Step 2:

Now, the equation of ellipse will be,

$\frac{x^2}{16} + \frac{y^2}{12} = 1$

MATHEMATICS

ELLIPSE

EQUATION AND TERMS ASSOCIATED WITH VERTICAL ELLIPSE AND TRANSLATED ELLIPSE

NOTES

What you already know

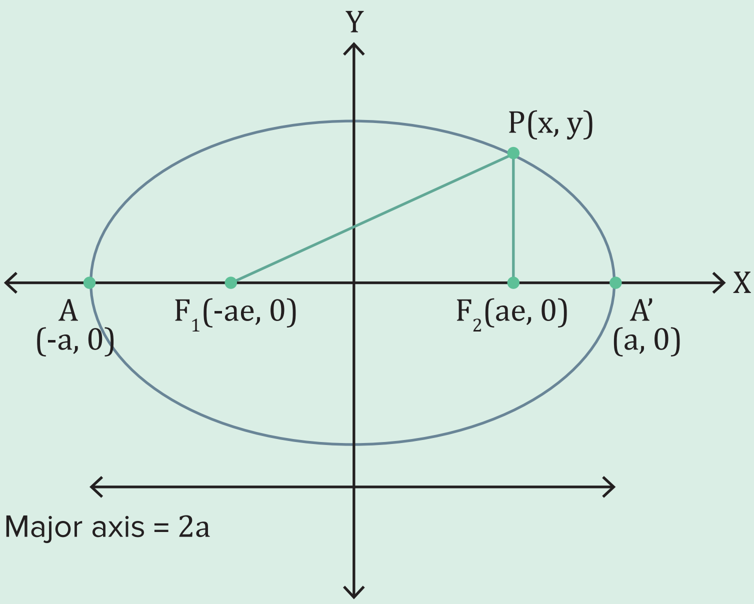
- Basic understanding of geometrical figures
- Basic algebraic formulas
- Basics about conics
- Conditions for different conics
- Analytic interpretation of different conics
- To find the different parameters associated with an ellipse

What you will learn

- Alternate definition of an ellipse
- Horizontal and vertical standard equation of an ellipse
- Translation or shifting of an ellipse

Alternate Definition of an Ellipse

An ellipse is a set of all the points in a plane; the sum of whose distances from two fixed points (foci) in a plane is a constant (length of major axis = 2a), i.e., the sum of the focal distances of any point on the ellipse is equal to the length of the major axis.



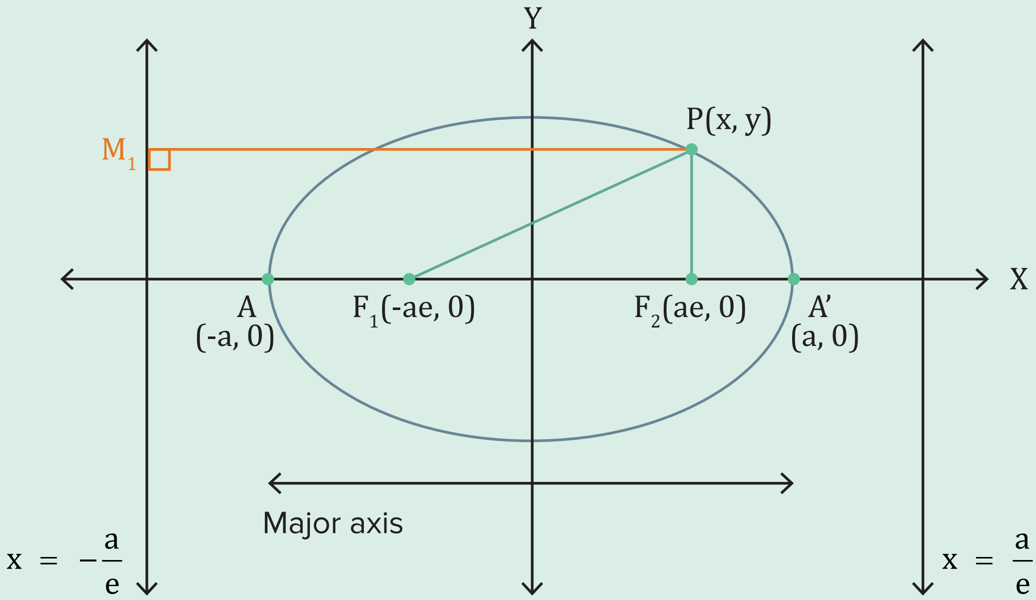
Proof

We know that,

PF_1 = Distance between point P(x, y) and foci $F_1(-ae, 0)$

$PF_1 = ePM_1 = e\left(\frac{a}{e} + x\right)$

$\Rightarrow PF_1 = a + ex$



Same as the previous case,

PF_2 = Distance between point P(x, y) and foci $F_2(ae, 0)$

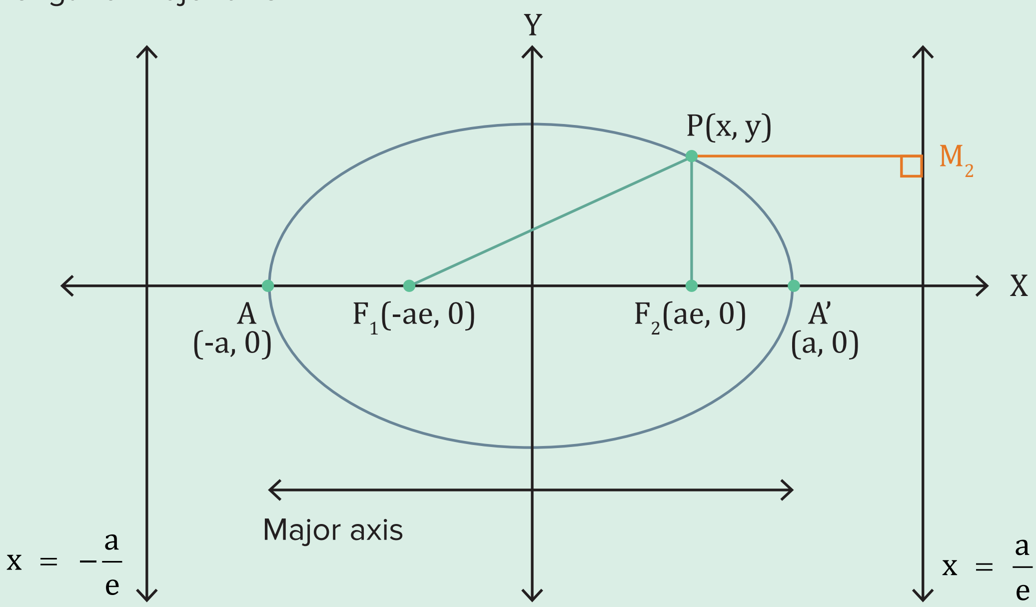
$PF_2 = e\,PM_2 = e\left(\frac{a}{e} - x\right) = a - ex$

Now, sum of both the focal distances,

$PF_1 + PF_2 = (a + ex) + (a - ex)$

$= 2a$

= Length of major axis



Note

In $\triangle PF_1F_2$, we can see that,

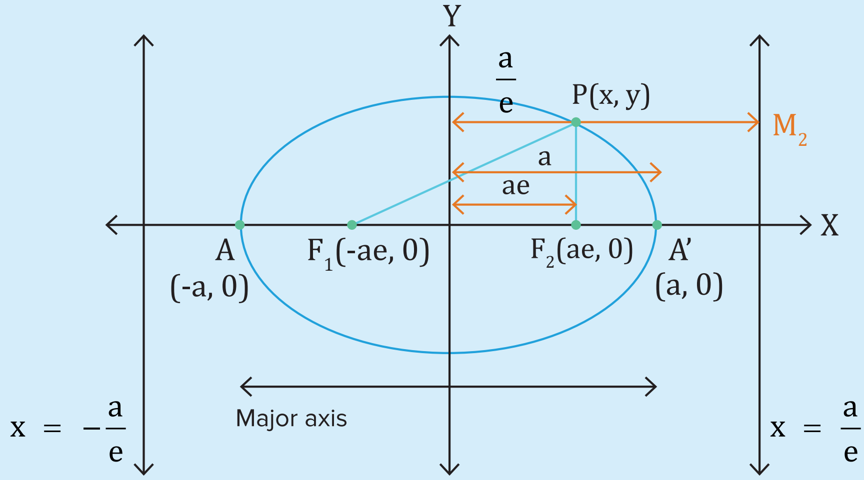
$PF_1 + PF_2 > F_1F_2$

$\Rightarrow 2a > 2ae$

$a > ae$

$\Rightarrow ae < a < \frac{a}{e}$

$(\frac{a}{e} > a$, as $0 < e < 1$, on dividing $a > ae$ by e on both sides).



Horizontal Standard Ellipse

The standard equation of horizontal ellipse is as follows:

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; $a > b$ (It is known as horizontal ellipse, as it is elongated horizontally along x-axis)

$O(0,0)$ is the centre of the ellipse.

$F_1(-ae, 0)$, $F_2(ae, 0)$ are the foci.

$x = \frac{-a}{e}$, $x = \frac{a}{e}$ are the equations of the directrices.

x- axis is the major axis and y-axis is the minor axis.

A and A' are the extremities of the major axis. Hence, 2a is the length of the major axis and a being the length of the semi-major axis.

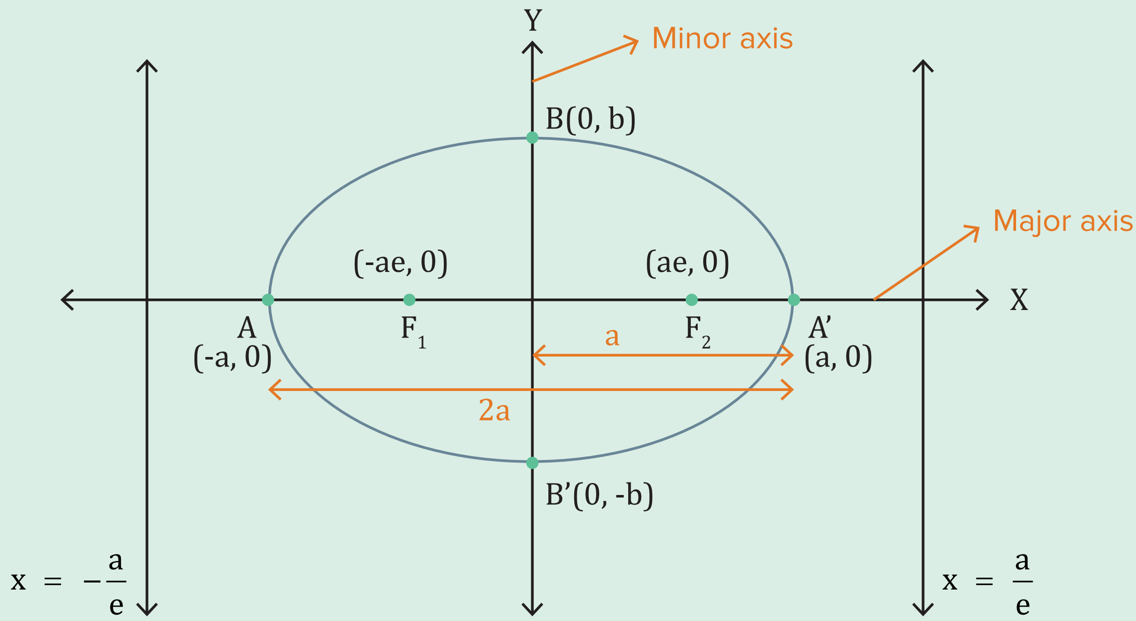
B and B' are the extremities of the minor axis. Hence, 2b is the length of the minor axis and b being the length of the semi-minor axis.

a, b, e are related as: $b^2 = a^2(1 - e^2)$

$$\Rightarrow 1 - e^2 = \frac{b^2}{a^2}$$

$$\Rightarrow e = \sqrt{1 - \frac{b^2}{a^2}}$$

$[\because e > 0$, so negative values are excluded$]$



1) In $\triangle BOF_1$, using Pythagoras theorem,

$$\Rightarrow BF_1 = \sqrt{(ae)^2 + b^2}$$

We know that, $b^2 = a^2\left(1 - e^2\right)$

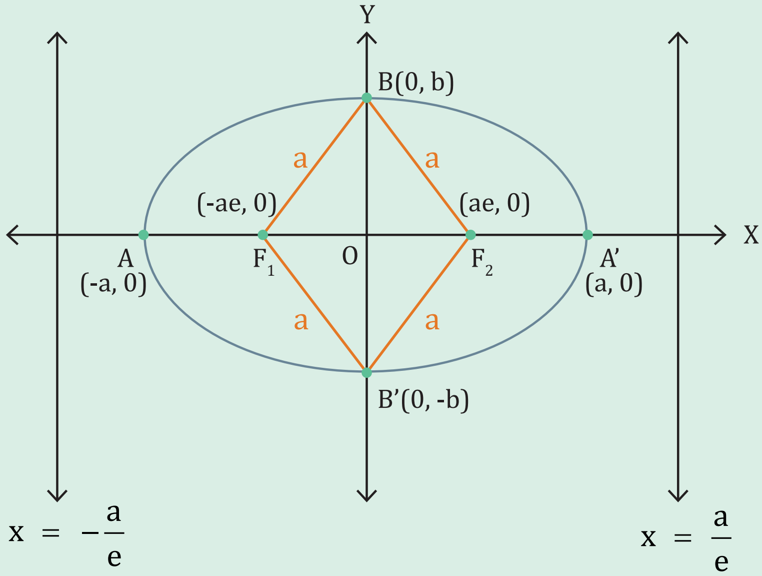
$$\Rightarrow BF_1 = \sqrt{a^2e^2 + a^2 - a^2e^2}$$

$$\Rightarrow BF_1 = |a| = a \quad [\because a > 0]$$

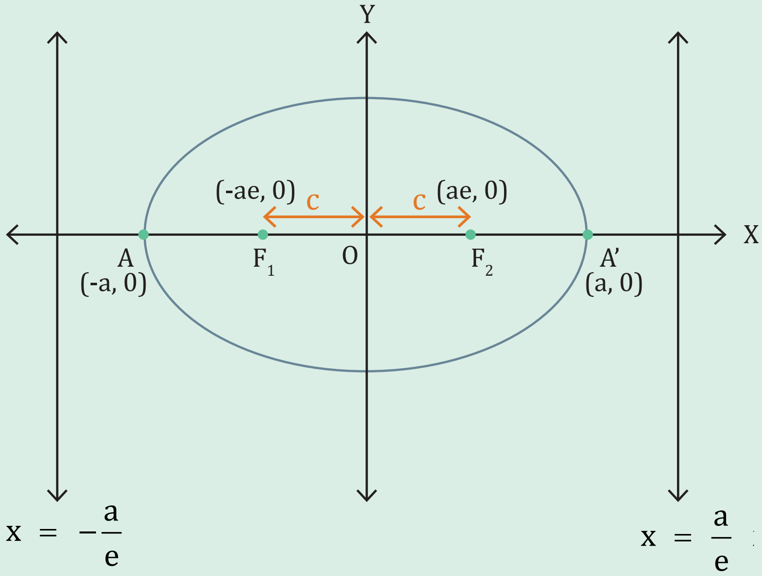
Similarly, for $\triangle BOF_2$, $\triangle B'OF_2$, and $B'OF_1$, we get,

$BF_2 = B'F_2 = B'F_1 = a$ = Length of semi-major axis

Thus, in F_1BF_2B', we have all the sides of equal length a.



2) Let us consider $OF_1 = c$ (constant)
Now, by definition,
$OF_1 = ae$
$\Rightarrow ae = c$
$\Rightarrow e = \frac{c}{a}$
Thus, we can find the eccentricity of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ by dividing the distance between the centre and one of the foci by the length of semi-major axis.

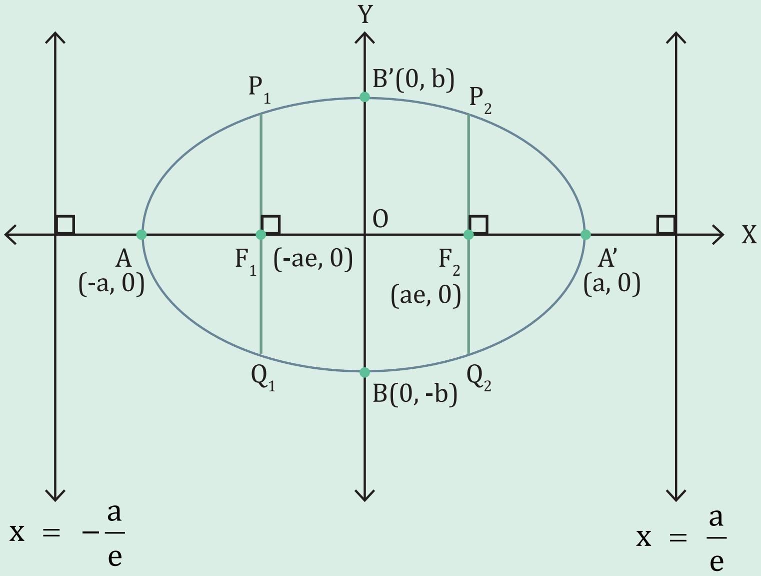


3) **Latus rectum:** The focal chord perpendicular to the major axis is known as latus rectum.

Here, latus rectum → P_1Q_1 and P_2Q_2

And equation of $P_1Q_1 \Rightarrow x = -ae$

Equation of $P_2Q_2 \Rightarrow x = ae$



Proof

Let us consider $P_2 = (ae, y_0)$

Then, the equation of ellipse will become,

$$\frac{(ae)^2}{a^2} + \frac{{y_0}^2}{b^2} = 1$$

$$\Rightarrow {y_0}^2 = b^2(1-e^2) \text{ or } {y_0}^2 = b^2 \times \frac{b^2}{a^2}$$

$$\Rightarrow y_0 = \pm\frac{b^2}{a}$$

Hence, $P_2 = \left(ae, \frac{b^2}{a}\right)$ and $Q_2\left(ae, \frac{-b^2}{a}\right)$

Length of $P_2Q_2 = \frac{2b^2}{a}$

Same for the other latus rectum,

$P_1Q_1 : x = -ae$

We get, $P_1\left(-ae, \frac{b^2}{a}\right)$ and $Q_1\left(-ae, -\frac{b^2}{a}\right)$

Therefore, the length of LR: $P_1Q_1 = \frac{2b^2}{a}$

We can also conclude that, P_1 and P_2 are mirror images about y-axis and so are Q_1 and Q_2.

4) **Focal distance:** The distance between the focus to any point on the ellipse is known as focal distance or focal radii.

Let $P \equiv (x, y)$ be any point on the ellipse.

Then, the focal distance is PF_1 or PF_2.

We know by definition that,

$\frac{PF_1}{PM_1}$ = e and $\frac{PF_2}{PM_2}$ = e

$\Rightarrow PF_1 = ePM_1$ and

$PF_2 = ePM_2$ ..... (i)

We can see that,

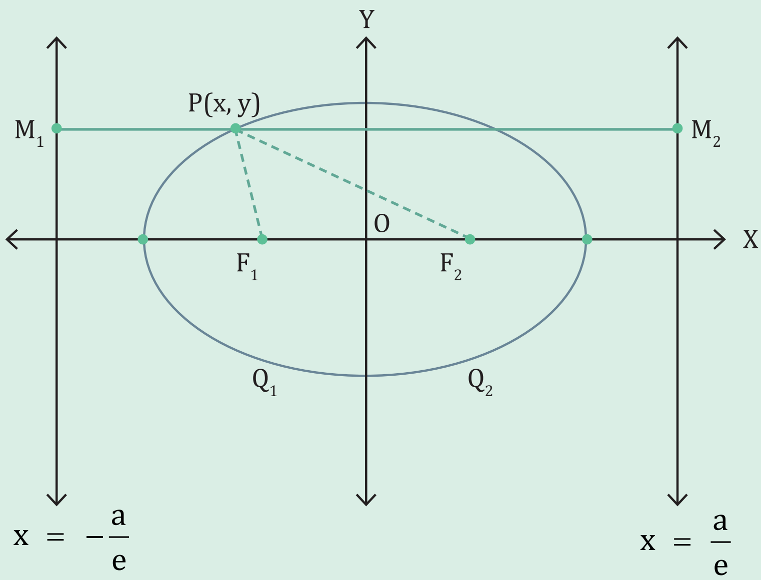
$PF_1 = e\left(\frac{a}{e} - x\right) \Rightarrow PF_1 = a - ex$

Same for PF_2 as well

We can see that,

$PF_2 = e\left(\frac{a}{e} + x\right) \Rightarrow PF_2 = a + ex$

Therefore, the focal distance of P(x, y) is $a \pm ex$.



5) We see that in ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$: a(length of semi-major axis) → x and b(length of semi-minor axis) → y

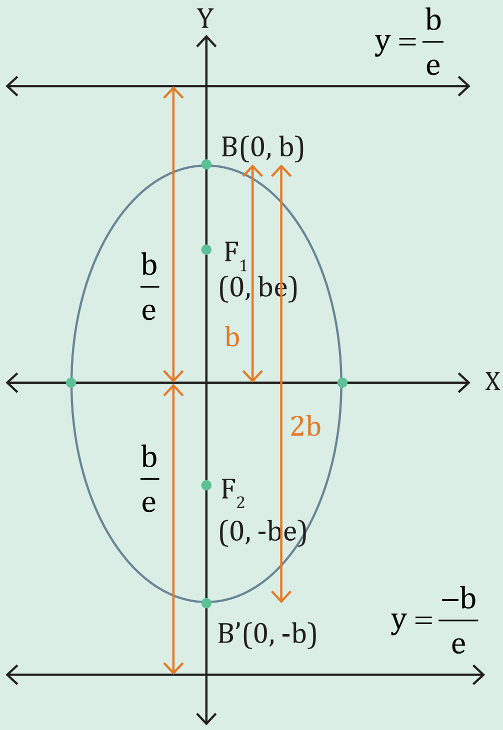
$a^2 > b^2 \Rightarrow a > b \Leftrightarrow x > y \Rightarrow$ Standard horizontal ellipse

Vertical Standard Ellipse

In the vertical ellipse, we observe that the centre is O(0, 0). It is known as vertical ellipse, as it is elongated vertically along the y-axis. The foci $F_1(0, be)$, $F_2(0, -be)$ lie on the y-axis. $y = \frac{b}{e}$, $y = \frac{-b}{e}$ are the equations of the two directrices. Major axis is along the y-axis and minor axis is along the x-axis. B(0, b), B'(0, -b) are the vertices, and 2b is the length of major axis.

Equation of a vertical ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

Where $0 < a < b$ and $a^2 = b^2(1 - e^2)$



1) In $\triangle AOF_1$, using Pythagoras theorem, we get,

$AF_1 = \sqrt{a^2 + (be)^2}$

We know that for a vertical ellipse,

$a^2 = b^2(1 - e^2)$

$\Rightarrow AF_1 = \sqrt{b^2 - b^2e^2 + b^2e^2}$

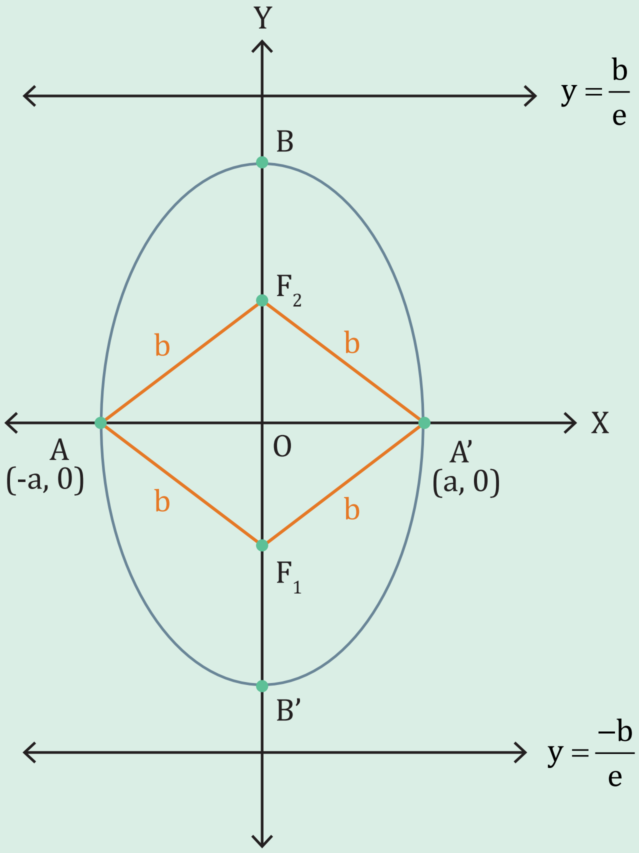
Or $AF_1 = |b| = b \qquad [\because b > 0]$

Similarly, for

$A'OF_1$, $A'OF_2$, and AOF_2, we get,

$A'F_1 = A'F_2 = AF_2 = b$ = Semi-major axis

Thus, in $F_1A'F_2A$, we have all the sides of an equal length b.



2) **Latus rectum**

Let us consider LR P_1Q_1 and coordinates for point $P_1(x_0, be)$ that will lie on the ellipse.

$$\Rightarrow \frac{x_0^2}{a^2} + \frac{b^2e^2}{b^2} = 1$$

We know that for a vertical ellipse,

$$a^2 = b^2(1 - e^2) \text{ or } (1 - e^2) = \frac{a^2}{b^2}$$

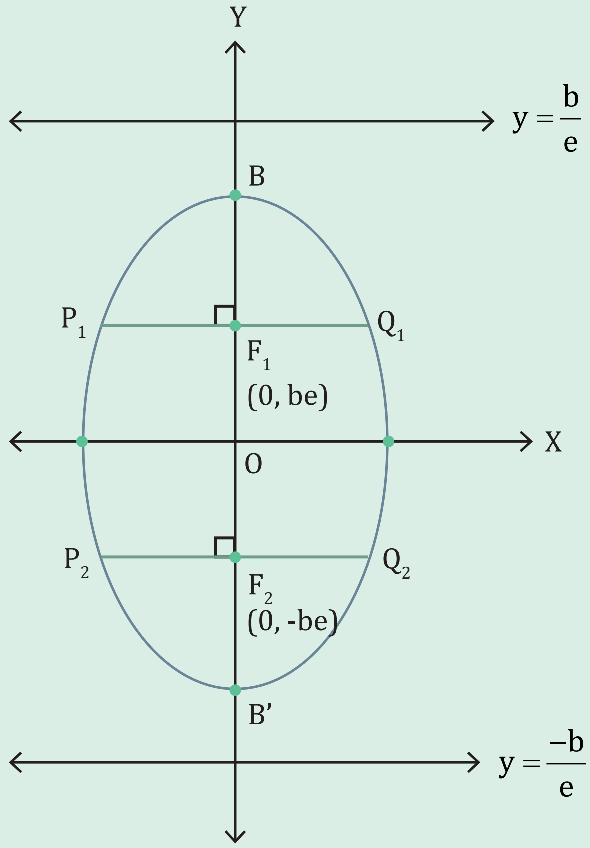
$$\Rightarrow x_0^2 = a^2 \times \frac{a^2}{b^2}$$

$$\text{Or } x_0 = \pm \frac{a^2}{b}$$

Thus, we get the coordinates of P_1 and Q_1 as follows:

$$P_1\left(-\frac{a^2}{b}, be\right) \text{ and } Q_1\left(\frac{a^2}{b}, be\right)$$

And length of LR $P_1Q_1 = \frac{2a^2}{b}$



Similarly, we get P_2 as $\left(-\frac{a^2}{b}, -be\right)$ and

Q_2 as $\left(\frac{a^2}{b}, -be\right)$.

Now, length of LR $P_2Q_2 = \frac{2a^2}{b}$

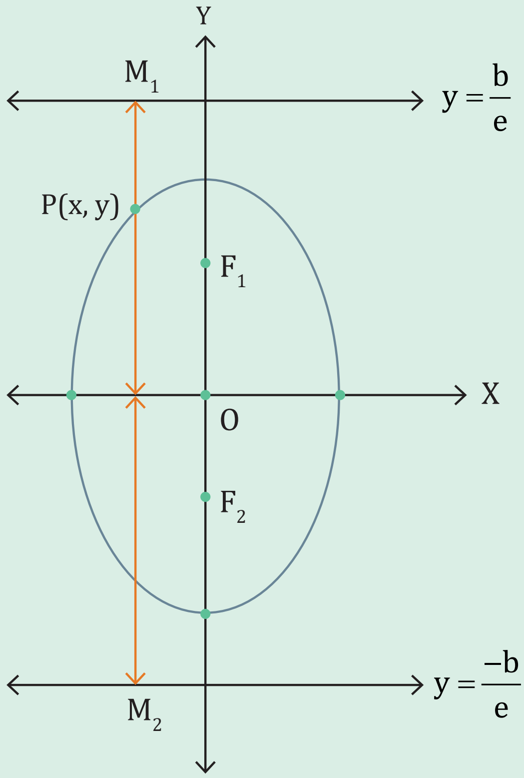
3) **Focal distance**

As we know by definition that
$PF_1 = ePM_1$ and $PF_2 = ePM_2$

$\Rightarrow PF_1 = e\left(\frac{b}{e} - y\right) = b - ey$

Similarly for PF_2

$\Rightarrow PF_2 = e\left(\frac{b}{e} + y\right) = b + ey$

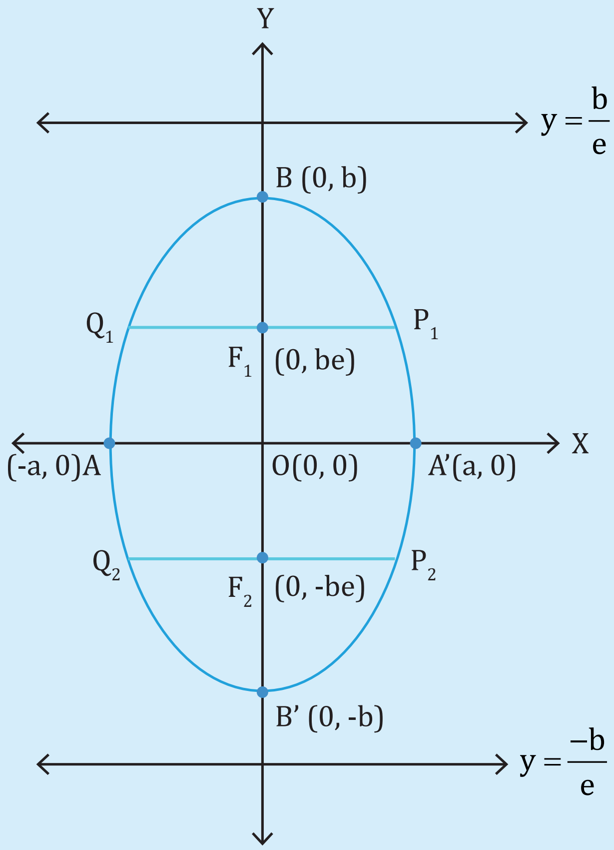


4) We see that in ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$: a (length of semi-minor axis) → x and b (length of semi-major axis) → y

$a^2 < b^2 \Rightarrow a < b \Leftrightarrow x < y \Rightarrow$ Standard vertical ellipse

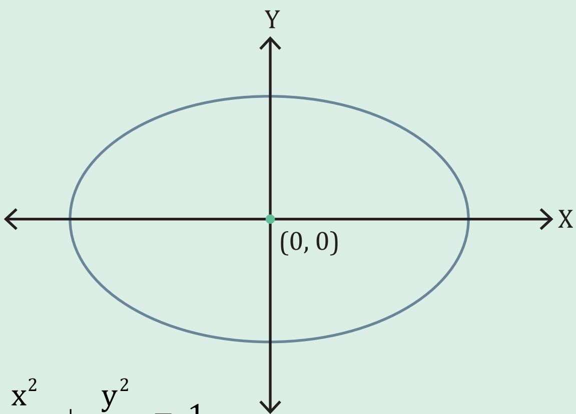
Note

Center	O(0, 0)
Foci	$F_1(0, be)$, $F_2(0, -be)$
Vertices	B(0, b), B'(0, -b)
Major axis	BB'
Length of major axis	2b
Minor axis	AA'
Length of minor axis	2a
Directrices	$y = \pm \frac{b}{e}$
Length of LR	$\frac{2a^2}{b}$
Focal distances of P(x, y)	b ± ey

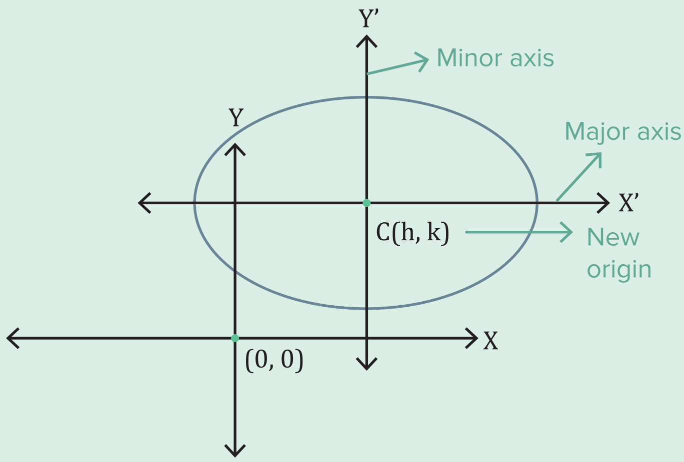


Axes of Ellipse Parallel to Coordinate Axes

Let us consider that we have a horizontal standard ellipse having the centre at the origin (0, 0). Now, we will see how different parameters will change after shifting the centre of the ellipse to a new origin (h, k).



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



Let us consider that after shifting, the equation becomes $\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1$

Now using,

Shifting of origin,

XY - plane = (0, 0) ≡ (x, y)

And X' Y'- Plane = (h, k) ≡ (x', y')

Now, relations between the two given coordinates are given by,

x' = x - h and y' = y - k

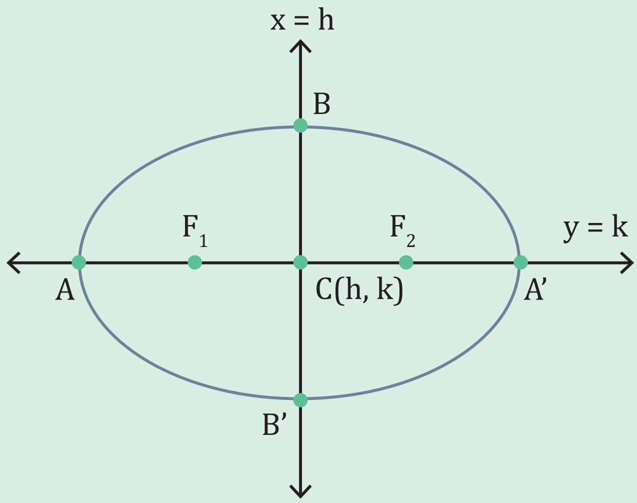
We get the equation as $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

Now, we will consider different cases and solve accordingly.

Case 1: Major axis is parallel to x-axis

The equation of ellipse becomes,

$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1$ or $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$,



Where a > b; $b^2 = a^2(1 - e^2)$

Now, centre of the new ellipse: (x', y') ≡ (0, 0)

⇒ (x - h, y - k) ≡ (0, 0) or x = h and y = k

Foci of the new ellipse: (x', y') ≡ (± ae, 0)

⇒(x - h, y - k) ≡ (± ae, 0) or

x = h + ae or h - ae and y = k

Vertices of the new ellipse: (x', y') ≡ (± a, 0)

⇒(x - h, y - k) ≡ (± a, 0) or

x = h + a or h - a and y = k

Directrices of the new ellipse: $(x') \equiv \left(\pm \frac{a}{e}\right)$

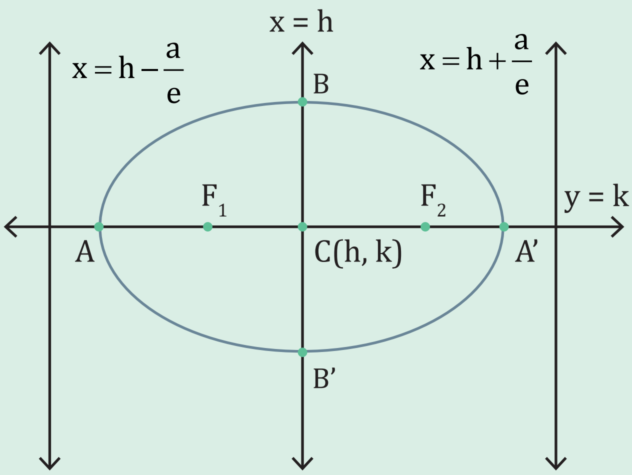
$\Rightarrow (x - h) \equiv \left(\pm \frac{a}{e}\right)$

Or $x = h + \frac{a}{e}$ or $h - \frac{a}{e}$

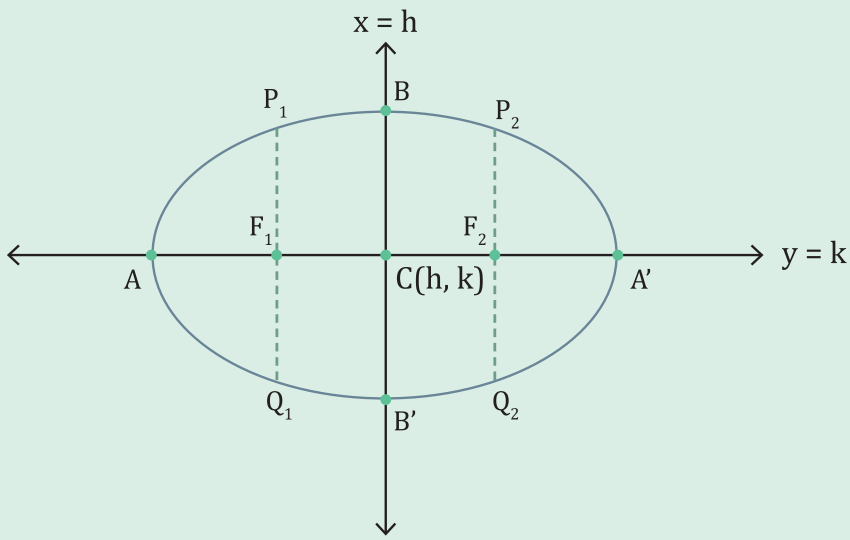
Short trick

To find the different coordinates of a translated ellipse at centre (h, k), just add h to its x-coordinate and k to its y-coordinate.

Center	$C(h, k)$
Foci	$F_1(h - ae, k), F_2(h + ae, k)$
Vertices	$A\ (h - a, k), A'\ (h + a, k)$
Length of major axis	$AA' = 2a$
Length of minor axis	$BB' = 2b$
Equation of directrix	$x = h \pm \frac{a}{e}$
Eccentricity	$e = \sqrt{1 - \frac{b^2}{a^2}}$



Ends of L.R	$P_1\left(h - ae,\ k + \frac{b^2}{a}\right),\ Q_1\left(h - ae,\ k - \frac{b^2}{a}\right)$ $P_2\left(h + ae,\ k + \frac{b^2}{a}\right),\ Q_2\left(h + ae,\ k - \frac{b^2}{a}\right)$
Length of L.R	$\frac{2b^2}{a}$



Case 2: When major axis is y-axis

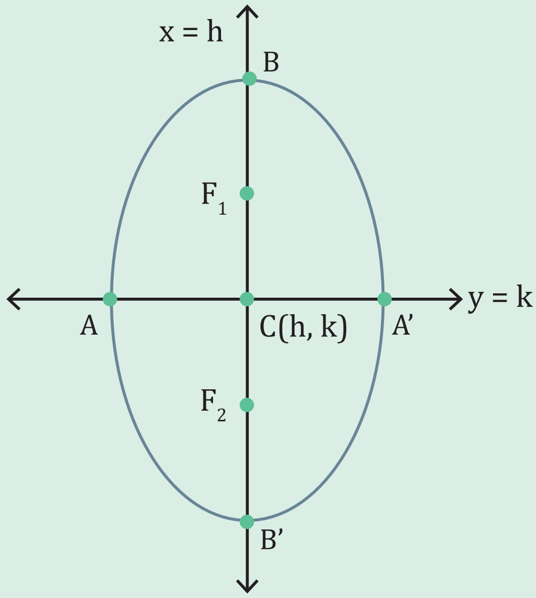
We can see that after shifting the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; a < b becomes

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1;\ a<b \text{ and}$$

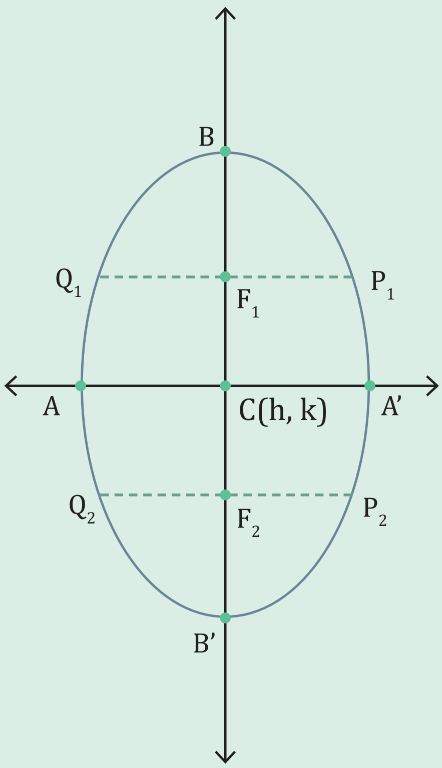
$$a^2 = b^2\left(1-e^2\right)$$

We see that after shifting the centre of the ellipse at (h, k), different parameters will get changed as follows:

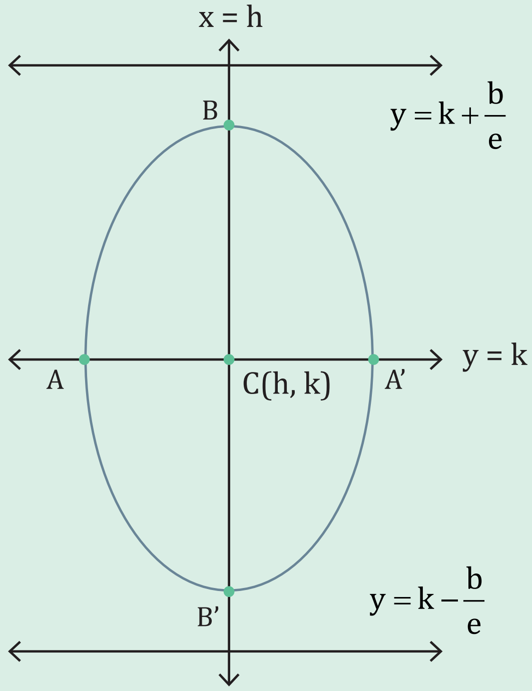
Centre	C(h, k)
Foci	$F_1(h, k + be)$, $F_2(h, k - be)$
Vertices	B(h, k + b), B'(h, k - b)



Ends of L.R	$P_1\left(h+\frac{a^2}{b}, k+be\right)$, $Q_1\left(h-\frac{a^2}{b}, k+be\right)$ $P_2\left(h+\frac{a^2}{b}, k-be\right)$, $Q_2\left(h-\frac{a^2}{b}, k-be\right)$
Length of L.R	$\frac{2a^2}{b}$



Length of major axis	$BB' = 2b$
Length of minor axis	$AA' = 2a$
Equation of directrix	$y = k \pm \frac{b}{e}$
Eccentricity	$e = \sqrt{1 - \frac{a^2}{b^2}}$



If (5, 12) and (24, 7) are the foci of an ellipse passing through the origin, then find its eccentricity.

Solution

Step 1

Given, foci $F_1 \equiv (5, 12)$ and $F_2 \equiv (24, 7)$

As the ellipse is passing through the origin $O(0, 0)$,

Using,

$2ae = F_1F_2$ and $OF_1 + OF_2 = 2a$

We get,

$$2ae = \sqrt{(24-5)^2 + (7-12)^2}$$

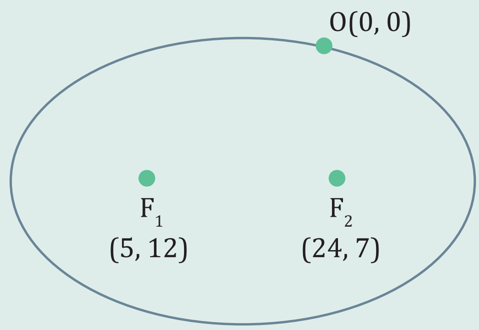
$$\Rightarrow 2ae = \sqrt{361+25} = \sqrt{386}$$

Or $2ae = \sqrt{386}$

And

$$2a = \sqrt{(5-0)^2 + (12-0)^2} + \sqrt{(24-0)^2 + (7-0)^2}$$

Or $2a = 13 + 25 \Rightarrow 2a = 38$



Step 2

Now, we have to find the eccentricity.

So, $e = \frac{2ae}{2a}$ or $e = \frac{\sqrt{386}}{38}$

Concept Check 1

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In an ellipse with the centre at the origin, if the difference of lengths of the major axis and the minor axis is 10 and one of the foci is at $(0, 5\sqrt{3})$, then find the length of its latus rectum.

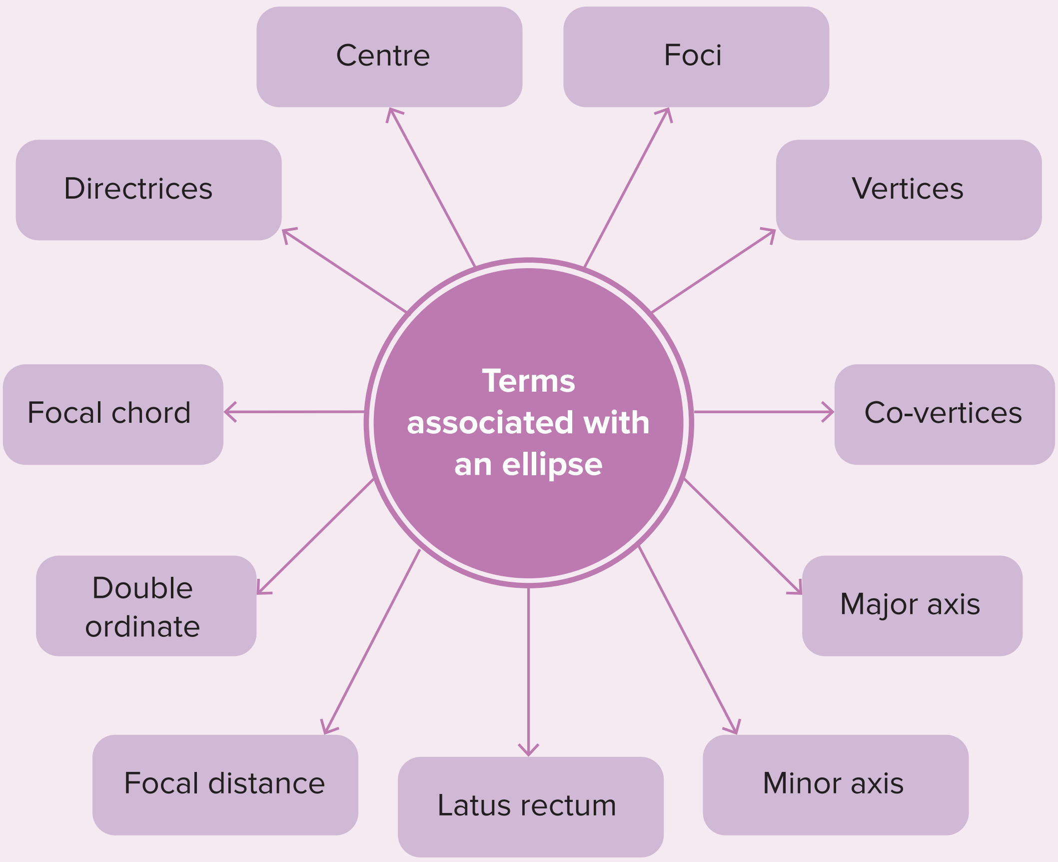
(a) 5 (b) 10 (c) 8 (d) 6

Summary Sheet

Key Results

- The sum of the focal distances of any point on an ellipse is equal to the length of the major axis.
- **Latus rectum**: The focal chord perpendicular to the major axis is known as latus rectum.
- **Focal distance**: The distance of any point on an ellipse from the focus is known as focal distance or focal radius of that point.

Mind Map



Self-Assessment

1. If the focal distance of an end of the minor axis of an ellipse (referred to its axis of x and y, respectively) is k and the distance between its foci is $2h$, then find its equation.
2. If $P(x, y)$ is any point of the ellipse $16x^2 + 25y^2 = 400$ and $F_1 \equiv (3, 0)$, $F_2 \equiv (-3, 0)$, then find the value of $PF_1 + PF_2$.

Answers

Concept Check 1

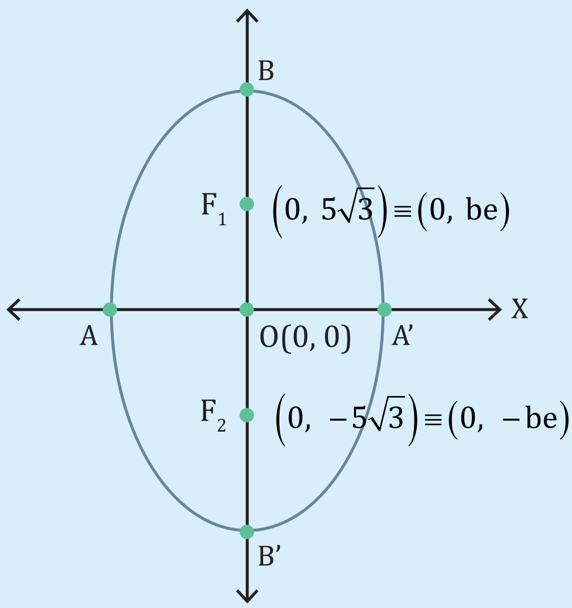
Step 1:

Given, an ellipse with the centre $\equiv (0, 0)$
Focus $\equiv (0, 5\sqrt{3})$, lies on the Y-axis
$\therefore$ Major axis is along the Y-axis.
$\therefore$ $be = 5\sqrt{3}$ ----(i)

Also, $2b - 2a = 10$
$\Rightarrow b - a = 5$ ----(ii)

Now, using $a^2 = b^2(1 - e^2)$,

Or $b^2 e^2 = b^2 - a^2 \Rightarrow (b - a)(b + a) = (be)^2$
Now, after replacing value of be and $(b - a)$,
We get, $5(a + b) = 75$ or
$a + b = 15$ --(iii)



Step 2:

Now, after adding equation (ii) and (iii),
We get, $2b = 20$ or $b = 10$
And $a = 5$

Length of LR = $\dfrac{2a^2}{b} = \dfrac{2 \times 25}{10}$

Or LR = 5

Hence, option (a) is the correct answer.

Self-Assessment 1

Step 1:

Given, distance between foci is $2h$.

Now, let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

We know that the distance between foci = $2ae$

$\Rightarrow 2ae = 2h \Rightarrow ae = h$

And focal distance of one end of minor axis is $a = k$

Step 2:

$\Rightarrow b^2 = a^2 - a^2 e^2 = k^2 - h^2$

So, the equation of ellipse is,

$$\frac{x^2}{k^2} + \frac{y^2}{(k^2 - h^2)} = 1$$

Self-Assessment 2

Step 1:

Given, we have $16x^2 + 25y^2 = 400$

$\Rightarrow \frac{x^2}{25} + \frac{y^2}{16} = 1 \Rightarrow a^2 = 25$ and $b^2 = 16$

Then, eccentricity is given by,

$e^2 = 1 - \frac{b^2}{a^2} \Rightarrow e^2 = \frac{9}{25}$ or $e = \frac{3}{5}$

Step 2:

So, the coordinates of the foci are $(\pm ae, 0)$ or $(\pm 3, 0)$.

Thus, F_1 and F_2 are the foci of the ellipse.

Since the sum of the focal distances of a point on an ellipse is equal to its major axis,

Or $PF_1 + PF_2 = 2a = 10$

$PF_1 + PF_2 = 10$

MATHEMATICS

ELLIPSE

POSITION OF A POINT WITH RESPECT TO AN ELLIPSE AND PARAMETRIC EQUATION OF ELLIPSE

NOTES

What you already know

- Introduction to ellipse
- Equation and terms associated with different ellipses

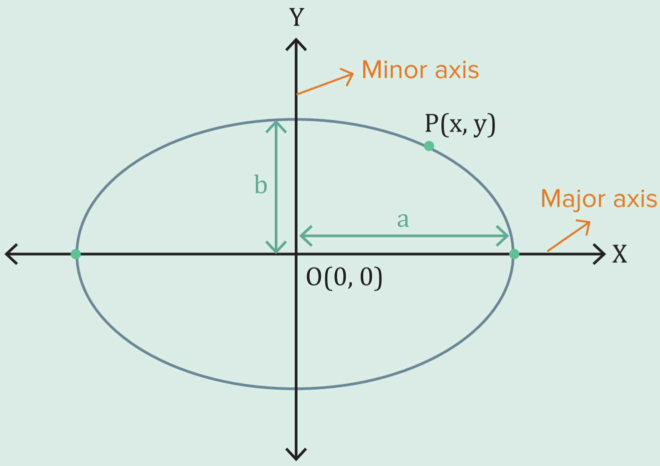
What you will learn

- Position of a point with respect to an ellipse
- Parametric equations of an ellipse
- Auxiliary circle
- Eccentric angle
- Chord joining two points

Standard horizontal ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1(a > b)$$

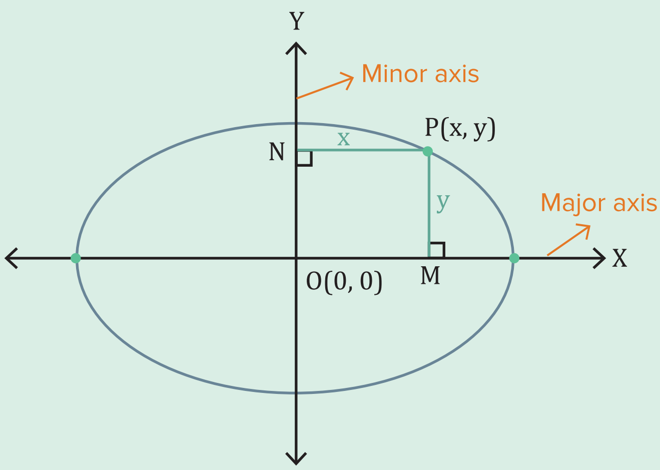
Here, a denotes the length of the semi-major axis and b denotes the length of the semi-minor axis. The major axis and minor axis are perpendicular to each other, intersecting at origin. The x and y in the equation represent any point P(x, y) on the ellipse.



Let M and N be the feet of the perpendiculars from the point P(x, y) onto the major axis and the minor axis, respectively.

We know that, PM = y and PN = x

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{PN^2}{a^2} + \frac{PM^2}{b^2} = 1$$

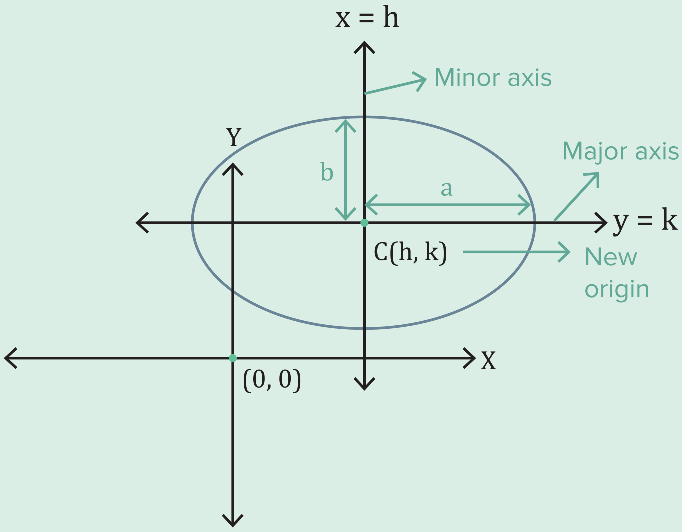


Translated horizontal ellipse

Let the centre of standard horizontal ellipse be shifted to C(h, k).

For this translated ellipse, y = k is the major axis and x = h is the minor axis.

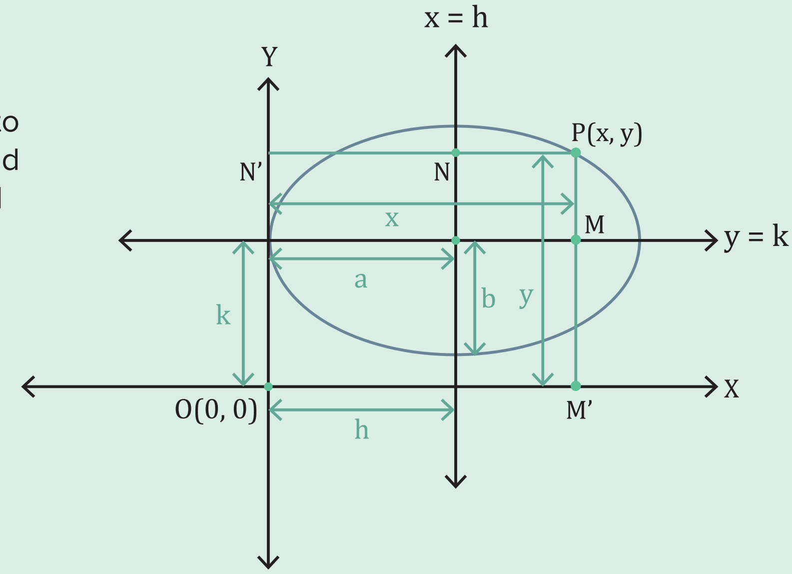
Here, a denotes the length of the semi-major axis and b denotes the length of the semi-minor axis.



Let M and N be the foot of the perpendiculars from the point P(x, y) onto the major and minor axis respectively and then extend them to meet the x-axis and y-axis respectively at M' and N'.
We know that, PM' = y, PN' = x
So, PM = y - k, PN = x - h

Therefore, the equation is as follows:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1(a>b)$$



Note

When translated, the dimensions, distances, and lengths do not change.

Equation of an ellipse referred to two perpendicular lines as axes

Let these perpendicular lines be the following:
$L_1: a_1x + b_1y + c_1 = 0$
$L_2 : b_1x - a_1y + c_2 = 0$
(Since $m_{L_1} . m_{L_2} = -1$,

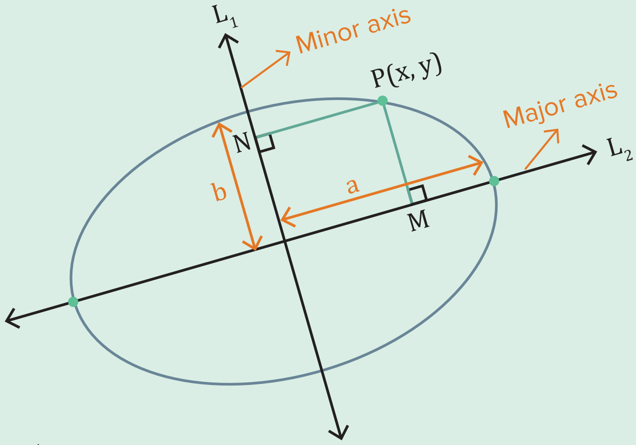
Observtion: $\frac{-a}{b} \cdot \frac{b}{a} = -1$)

$$PN = \frac{|a_1x + b_1y + c_1|}{\sqrt{a_1^2 + b_1^2}}$$

$$PM = \frac{|b_1x - a_1y + c_2|}{\sqrt{b_1^2 + a_1^2}}$$

The equation of the ellipse is as follows:

$$\frac{\left(\frac{|a_1x + b_1y + c_1|}{\sqrt{a_1^2 + b_1^2}}\right)^2}{a^2} + \frac{\left(\frac{|b_1x - a_1y + c_2|}{\sqrt{b_1^2 + a_1^2}}\right)^2}{b^2} = 1(a > b)$$



Note

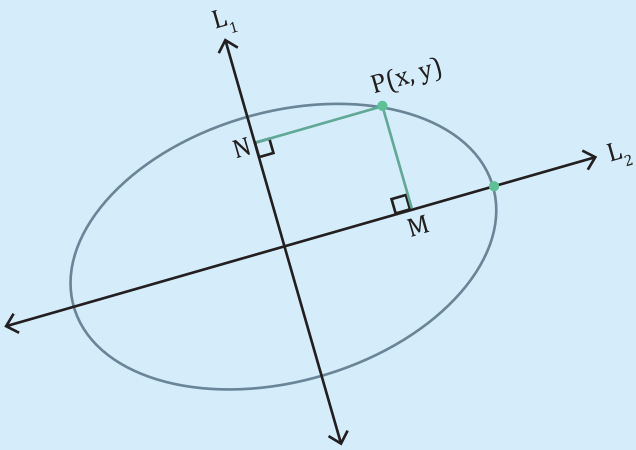
The centre is the point of intersection of L_1 and L_2.

If a > b, the major axis lies along the L_2 and the minor axis on the L_1.

If $a > b$, $b^2 = a^2(1 - e^2)$

If $a < b$, $a^2 = b^2(1 - e^2)$

The relation between a, b, and e remains the same for standard, translated, and rotated horizontal ellipse as the dimensions do not change.



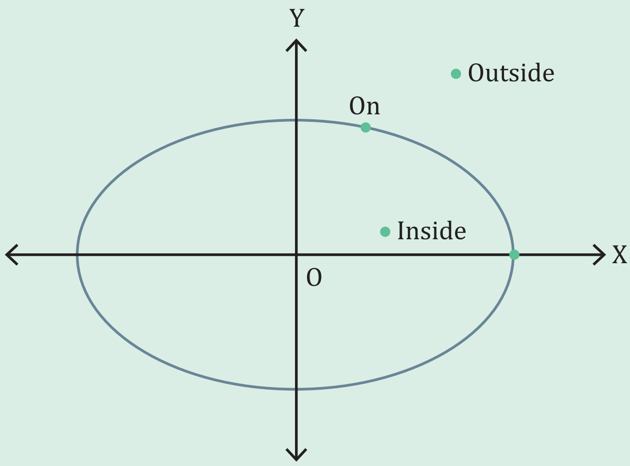
Position of a Point with respect to an Ellipse

Let $P(x_1, y_1)$ be any point in the plane of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

We have the following three cases:

1. Interior: Point lying inside the ellipse $(S_1 < 0)$
2. Point lying on the ellipse $(S_1 = 0)$
3. Exterior: Point lying outside the ellipse $(S_1 > 0)$.

Here, $S: \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$ and $S_1: \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$



Verify the position of $P(3, 2)$ with respect to $\frac{x^2}{25} + \frac{y^2}{16} = 1$.

Solution

Given,

$\frac{x^2}{25} + \frac{y^2}{16} = 1$

$\Rightarrow \frac{x^2}{5^2} + \frac{y^2}{4^2} = 1 \Rightarrow a = 5,\ b = 4$ and $5 > 4 \Rightarrow$ Horizontal ellipse

The centre of the given ellipse is $O(0,0)$.

Here, $S: \frac{x^2}{25} + \frac{y^2}{16} - 1$

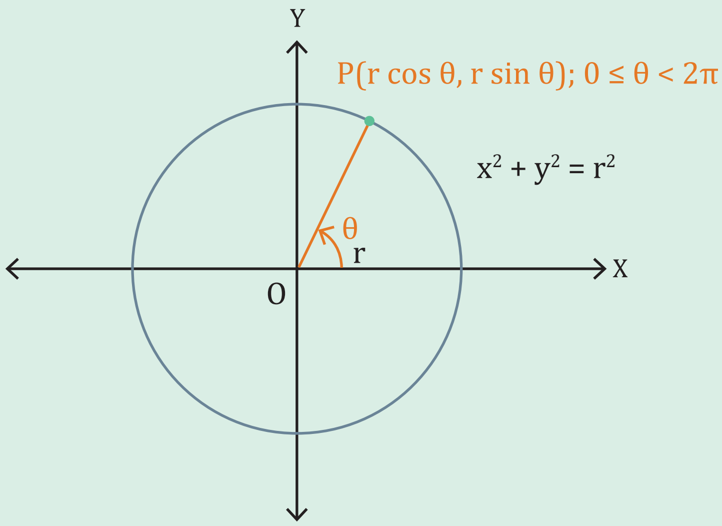
$\Rightarrow S_1: \frac{3^2}{25} + \frac{2^2}{16} - 1 = \frac{-39}{100}(<0)$

Therefore, the given point $P(3,2)$ lies inside the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$

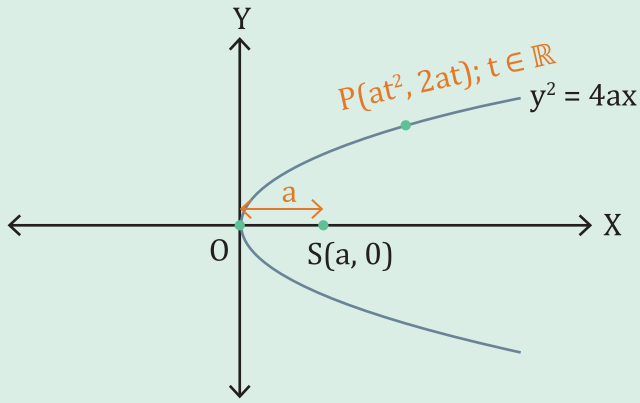
Revisiting parametric coordinates

Let us consider a circle, $x^2 + y^2 = r^2$, where r is the radius of the circle. Consider a point $P(x, y)$ on this circle such that OP makes an angle θ with respect to the positive direction of the x-axis.

The parametric coordinates of the point $P(x, y)$ will be $P(r\cos\theta, r\sin\theta)$, where θ belongs to $[0, 2\pi)$.



Let us consider the standard rightward opening parabola $y^2 = 4ax$. The parametric coordinates of any point $P(x, y)$ on this parabola will be $P(at^2, 2at)$, where $t \in \mathbb{R}$.

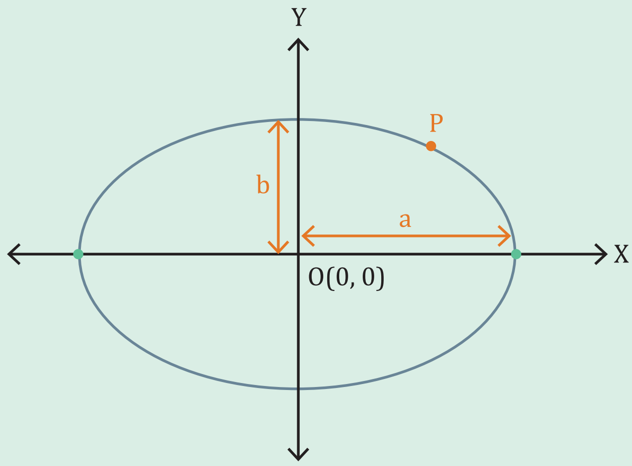


Parametric Equation of an Ellipse

Let us consider the standard horizontal ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1(a > b)$$

Let P be any point on this ellipse. Our aim is to find the parametric coordinates of this point with respect to a parameter. Therefore, to find these parametric coordinates, we are going to take additional help from the auxiliary circle of the ellipse.



Auxiliary Circle

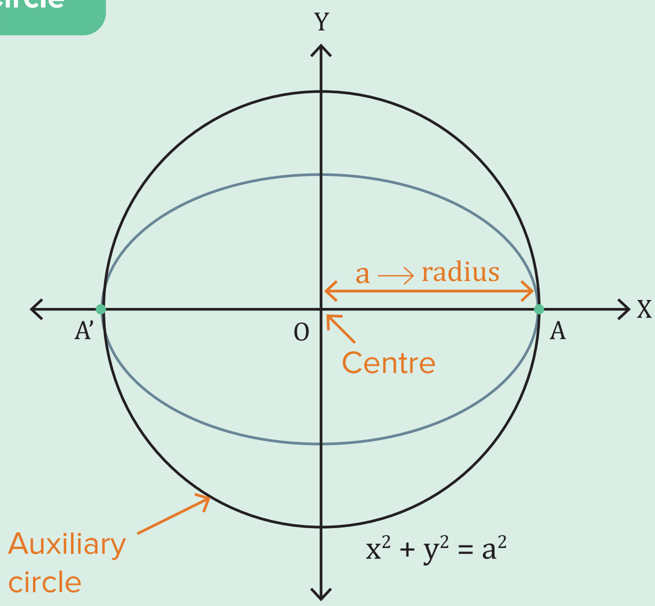
The circle described on the major axis as diameter is known as the auxiliary circle of the given ellipse.

For the standard horizontal ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1(a > b)$$

AA' is the major axis. Hence, as shown in the figure, the circle is drawn taking AA' as diameter, i.e., O as the centre with radius a gives the auxiliary circle.

The equation of the auxiliary circle for this ellipse is $x^2 + y^2 = a^2$

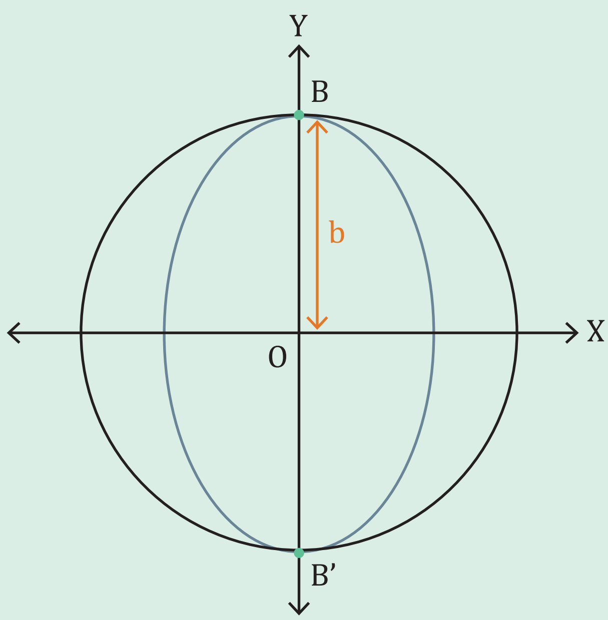


Similarly, the standard vertical ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1(a < b)$$

BB' is the major axis. Hence, as shown in the given figure, the circle is drawn taking BB' as diameter, i.e., O as the centre with radius b gives the auxiliary circle.

The equation of auxiliary circle for this ellipse is $x^2 + y^2 = b^2$



Eccentric Angle and Parametric Equation

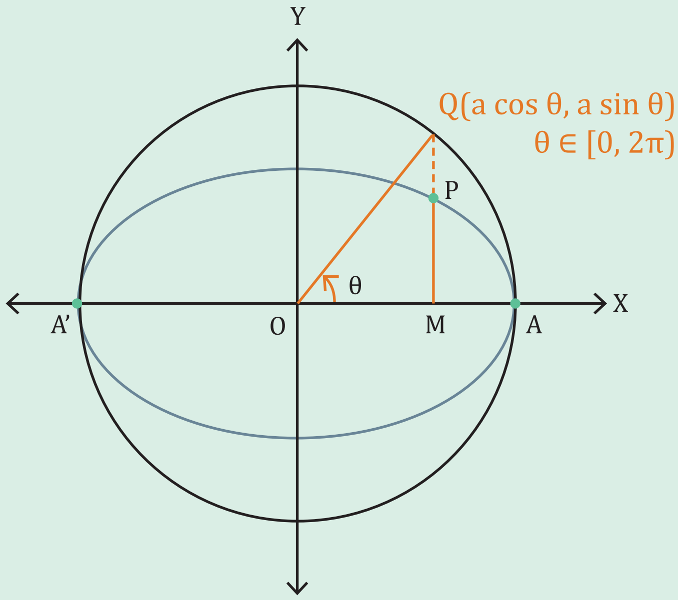
Consider the standard horizontal ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1(a > b)$$

Now, we know that the auxiliary circle equation for this ellipse is

$x^2 + y^2 = a^2$

Let us go back to find the parametric coordinates of a point P on the ellipse. Let M be the foot of the perpendicular drawn from point P onto the x-axis and Q be the point on the auxiliary circle obtained by extending this perpendicular PM in the opposite direction. Let OQ make an angle θ in anticlockwise sense with respect to the positive direction of the x-axis. The parametric coordinates of Q will be (a cos θ, a sin θ), where $\theta \in [0, 2\pi)$ P and Q are on the same line, i.e., perpendicular to the x-axis. Hence, P and Q have the same x-coordinate. Let P be (a cos θ, y). As P is a point on the ellipse, it has to satisfy its equation.



i.e., $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\Rightarrow \frac{(a\cos\theta)^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow y^2 = b^2 \sin^2\theta$$

$$\Rightarrow y = \pm b \sin\theta$$

$$\Rightarrow y = b \sin\theta \qquad (\theta \in [0, 2\pi))$$

Hence, $P(\theta) \equiv (a\cos\theta, b\sin\theta)$ is the parametric form of any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1(a > b)$, where $\theta \in [0, 2\pi)$ is known as the eccentric angle of point P.

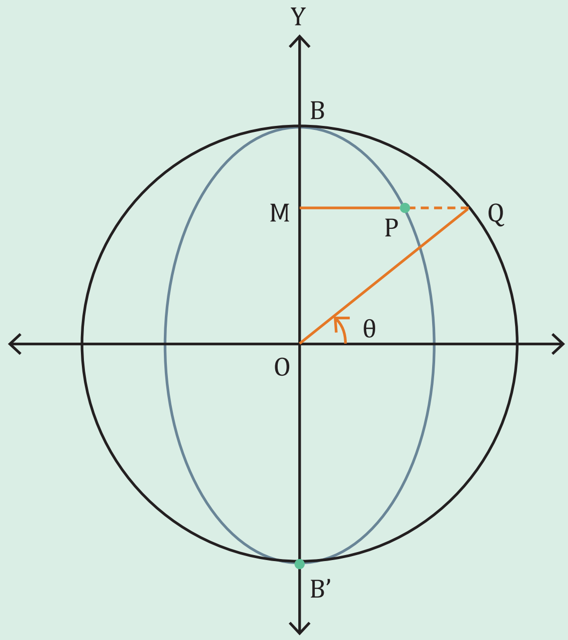
Also, $x = a\cos\theta$ and $y = b\sin\theta$ represent the parametric equations for this ellipse.

Consider the standard vertical ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1(a < b)$$

Now, we know that the auxiliary circle equation for this ellipse is $x^2 + y^2 = b^2$

Let us find the parametric coordinates of a point P on this ellipse. Let M be the foot of the perpendicular drawn from point P onto the y-axis and Q be the point on the auxiliary circle obtained by extending this perpendicular PM in the opposite direction. Let OQ make an angle θ in the anticlockwise sense with respect to the positive direction of the x-axis. The parametric coordinates of Q will be $(b\cos\theta, b\sin\theta)$, where $\theta \in [0, 2\pi)$



P and Q are on the same line, i.e., perpendicular to the y-axis. Hence, P and Q have the same y-coordinate. Let P be $(x, b\sin\theta)$. As P is a point on the ellipse, it has to satisfy its equation.

i.e., $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\Rightarrow \frac{x^2}{a^2} + \frac{(b\sin\theta)^2}{b^2} = 1$$

$\Rightarrow x = \pm a\cos\theta$

$\Rightarrow x = a\cos\theta \ \ (\theta \in [0, 2\pi))$

Hence, $P(\theta) \equiv (a\cos\theta, b\sin\theta)$ is the parametric form of any point on the ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 (a < b)$ where $\theta \in [0, 2\pi)$ is known as the eccentric angle of point P.

Also, $x = a\cos\theta$ and $y = b\sin\theta$ represent the parametric equations for this ellipse.

Consider a standard horizontal ellipse whose centre is translated/shifted to (h, k).

The equation of this ellipse is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 (a > b)$

Let $x - h = x'$, $y - k = y'$

$\Rightarrow \frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1$, for this the parametric form of any point on it will be

$P(x', y') = (a\cos\theta, b\sin\theta)$

$\Rightarrow x' = x - h = a\cos\theta, y' = y - k = b\sin\theta$

$\Rightarrow x = h + a\cos\theta, y = k + b\sin\theta$

Hence, $P(\theta) \equiv (h + a\cos\theta, k + b\sin\theta)$ is the parametric form of any point on the translated horizontal ellipse, where $\theta \in [0, 2\pi)$ is known as the eccentric angle of point P.

Also, $x = h + a\cos\theta$ and $y = k + b\sin\theta$ represent the parametric equations for this translated horizontal ellipse.

Similarly, $x = h + a\cos\theta, y = k + b\sin\theta$ also represent the parametric equations for the translated vertical ellipse obtained by shifting the centre of standard vertical ellipse to (h, k), where $\theta \in [0, 2\pi)$.

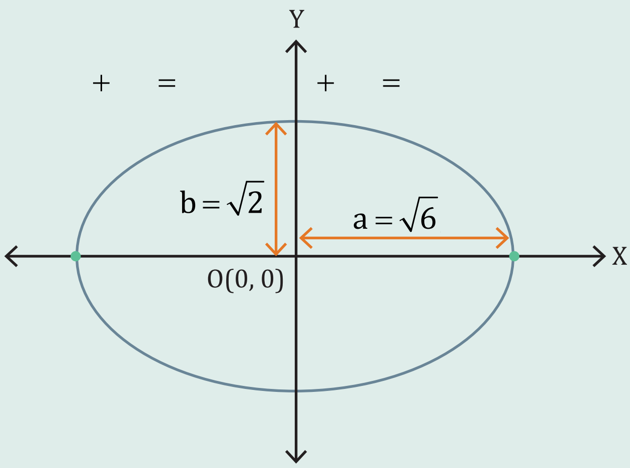
Note

We observe that the parametric form of any point is the same for both the standard horizontal and vertical ellipses. However, the significant point is that for $P(\theta) \equiv (a\cos\theta, b\sin\theta)$, a and b denote the length of the semi-major and semi-minor axes, respectively, for a standard horizontal ellipse, whereas a and b denote the length of semi-minor and semi-major axes, respectively, for a standard vertical ellipse.

Find the eccentric angle of a point on the ellipse $\frac{x^2}{6} + \frac{y^2}{2} = 1$ whose distance from the centre is 2.

Solution

Step 1



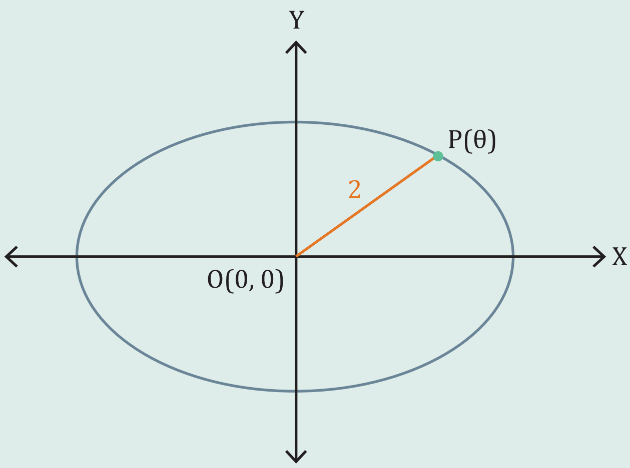
On comparing $\frac{x^2}{6} + \frac{y^2}{2} = 1$ with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

we get the following:

$a^2 = 6,\ b^2 = 2 \Rightarrow a = \sqrt{6},\ b = \sqrt{2} \Rightarrow a > b$

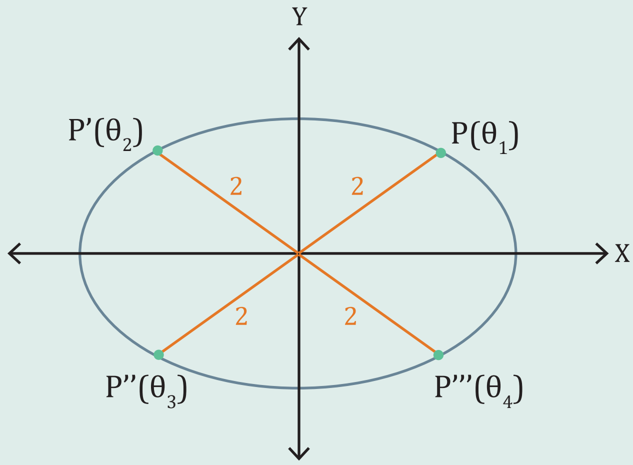
Hence, the given equation represents a standard horizontal ellipse whose centre is at $O(0, 0)$.

Step 2



We have to find θ of a point $P(\theta)$, i.e., 2 units from the origin.

As the ellipse is symmetric about both the axes, we get, P', P'', and P''' symmetrically 2 units from the origin in all the four quadrants. P, P', P'', and P'''are pairwise mirror images of each other in this cyclic order, i.e., P and P' are mirror images of each other with respect to the y-axis, P' and P'' with respect to the x-axis, P'' and P''' with respect to the y-axis, and P''' and P with respect to the x-axis. Hence, we can get four angles θ_1, θ_2, θ_3, and θ_4 that can satisfy the given condition.



Step 3

We know that the parametric form of any point on a standard horizontal ellipse is as follows:
$P(\theta) \equiv (a\cos\theta, b\sin\theta)$

$$\Rightarrow P(\theta) = \left(\sqrt{6}\ \cos\theta,\ \sqrt{2}\ \sin\theta\right)$$

Given, $OP = 2$

$$\sqrt{\left(\sqrt{6}\ \cos\theta - 0\right)^2 + \left(\sqrt{2}\ \sin\theta - 0\right)^2} = 2$$

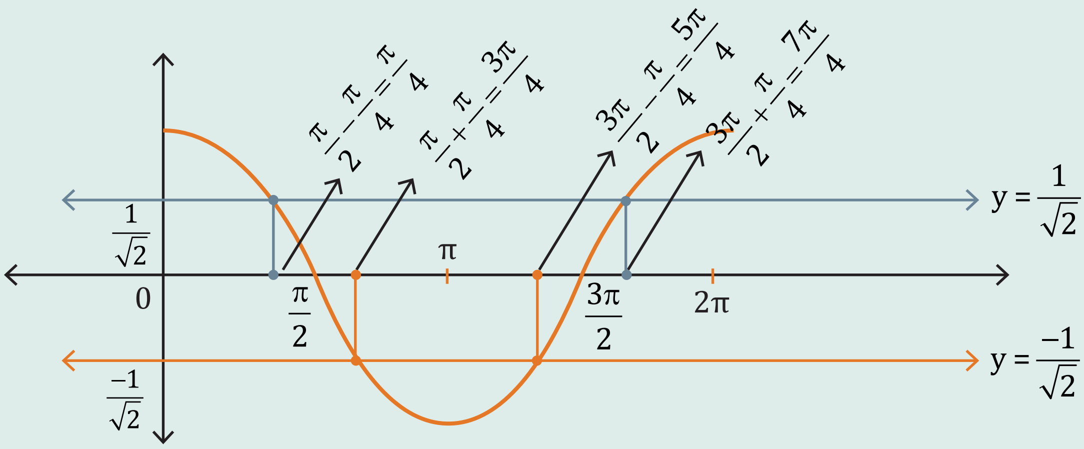
$$\Rightarrow \sqrt{6\cos^2\theta + 2\sin^2\theta} = 2$$

$$\Rightarrow \sqrt{6\cos^2\theta + 2\left(1 - \cos^2\theta\right)} = 2$$

On squaring and solving for θ, we get the following:

$$\cos\theta = \pm\frac{1}{\sqrt{2}}$$

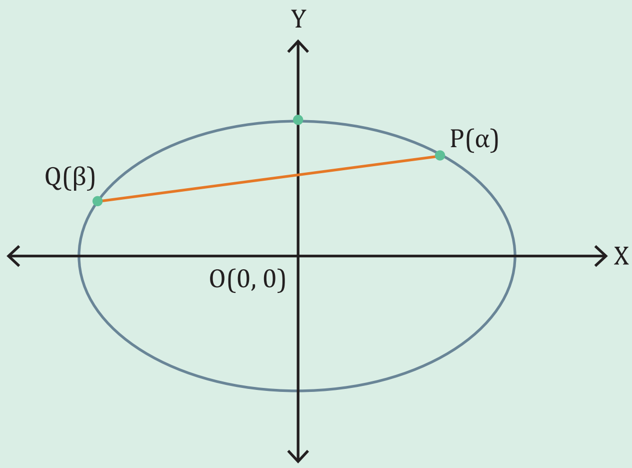
$$\Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \qquad \left(\text{As } \theta \in [0, 2\pi)\right)$$



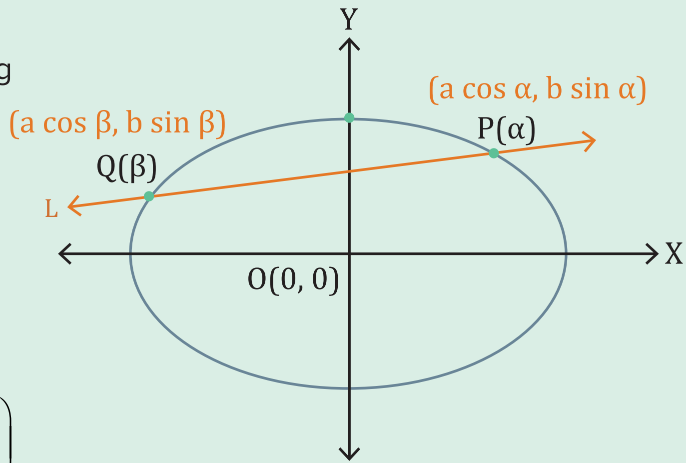
$\theta_1, \theta_2, \theta_3, \theta_4$ are $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$, respectively, that are the maximum possible eccentric angles of a point on the given ellipse satisfying the given conditions.

Chord joining two points

Consider two points P(α) and Q(β) on the ellipse $S = 0$, where S is $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \left(a > b\right)$ for a standard horizontal ellipse.



Parametric coordinates of these points are as follows:
$P(\alpha) \equiv (a\cos\alpha, b\sin\alpha)$, $Q(\beta) \equiv (a\cos\beta, b\sin\beta)$
Let us extend this chord on both sides and say, L.
Now,
Equation of L = Equation of the chord PQ
We know two points on this line, hence, by using the two-point form of a line, let us find the equation of L.
Let $P \equiv (x_1, y_1)$ and $Q \equiv (x_2, y_2)$.
By using two-point form of a line, we get the following:



$$\Rightarrow y - b\sin\alpha = \frac{b\sin\beta - b\sin\alpha}{a\cos\beta - a\cos\alpha}\left(x - a\cos\alpha\right)$$

$$\left(\text{Two-point form of a line: } y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}\left(x - x_1\right)\right)$$

$$\Rightarrow \frac{x}{a}\left(\sin\alpha - \sin\beta\right) + \frac{y}{b}\left(-\cos\alpha + \cos\beta\right)$$
$$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

On applying sum to product transformations, we get:

$$\Rightarrow \frac{x}{a}\left(2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}\right) + \frac{y}{b}\left(-2\sin\frac{\beta-\alpha}{2}\sin\frac{\alpha+\beta}{2}\right)$$
$$= 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha-\beta}{2}$$

On simplification, we get the equation of the chord joining P(α) and Q(β) as follows:

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$$

Find the angle subtended by the chord joining $P(\alpha)$ and $Q(\beta)$ on the ellipse $S=0$ at its centre, if $\tan\alpha \cdot \tan\beta = -\frac{a^2}{b^2}$.

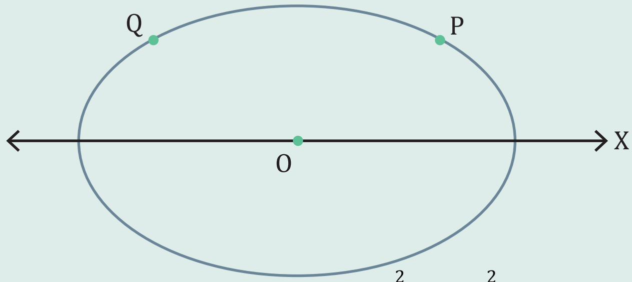
Solution

Step 1

Let S: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1(a>b)$, and P and Q are the two points on this ellipse.

With respect to the parametric form, we get the following:

$P(\alpha) \equiv (a \cos \alpha, b \sin \alpha)$
$Q(\beta) \equiv (a \cos \beta, b \sin \beta)$



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Step 2

Let θ be the angle between OP and OQ whose slopes are m_1 and m_2, respectively.

$$m_1 = \frac{b \sin\alpha}{a \cos\alpha} = \frac{b}{a}\tan\alpha$$

$$m_2 = \frac{b \sin\beta}{a \cos\beta} = \frac{b}{a}\tan\beta$$

$$\Rightarrow m_1 m_2 = \frac{b^2}{a^2}\tan\alpha \tan\beta$$

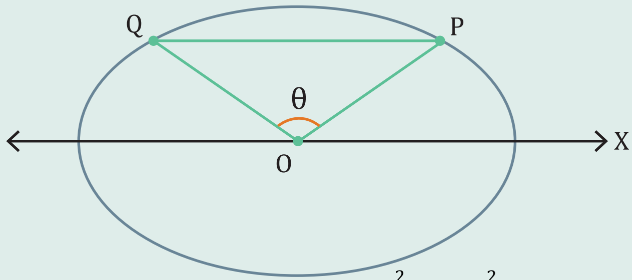
Given,

$$\tan \alpha \tan \beta = -\frac{a^2}{b^2}$$

$$\Rightarrow m_1 m_2 = -1$$

We know, if the product of the slopes of two lines is -1, then the two lines are perpendicular to each other.

$\Rightarrow \theta = 90°$



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Concept Check

Prove that any point on the ellipse with foci (-1, 0) and (7, 0), and eccentricity as $\frac{1}{2}$ is $(3 + 8\cos\theta, 4\sqrt{3}\sin\theta), \theta \in \mathbb{R}$.

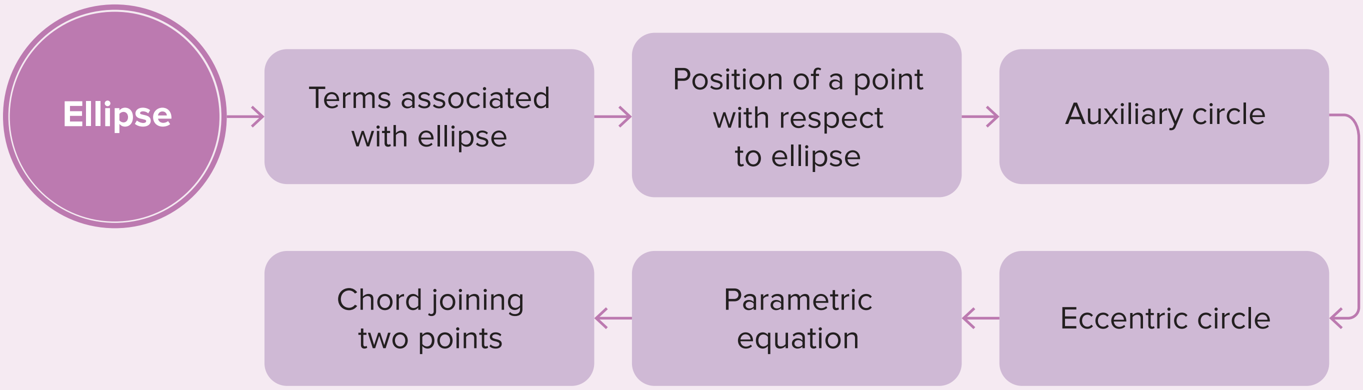
Summary Sheet

Key Takeaways

- The position of a point $P(x_1, y_1)$ in the plane of an ellipse is as follows:
  Interior: Point lying inside the ellipse $(S_1 < 0)$
  On: Point lying on the ellipse $(S_1 = 0)$
  Exterior: Point lying outside the ellipse $(S_1 > 0)$
  Here, $S: \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$ and $S_1: \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$
- For the standard horizontal ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1(a > b)$, equation of auxiliary circle is $x^2 + y^2 = a^2$.
- For the standard vertical ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1(a < b)$, equation of auxiliary circle is $x^2 + y^2 = b^2$.
- $P(\theta) \equiv (a\cos\theta, b\sin\theta)$ is the parametric form of any point on the standard horizontal and vertical ellipses where, $\theta \in [0, 2\pi)$ is known as the eccentric angle of point P.
- $P(\theta) \equiv (h + a\cos\theta, k + b\sin\theta)$ is the parametric form of any point on the translated horizontal and vertical ellipses where, $\theta \in [0, 2\pi)$ is known as the eccentric angle of point P where the equation of the translated ellipse will be $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ for both horizontal and vertical ellipses.
- The equation of the chord joining $P(\alpha)$ and $Q(\beta)$ as follows:

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$$

Mind Map



Self-Assessment

Find the equation of the ellipse whose parametric equations are $x = 1 + 4\cos\theta$ and $y = 2 + 3\sin\theta$.

Answers

Concept Check

Given, (-1, 0) and (7, 0) as foci, foci are on x-axis, hence, a > b(Horizontal ellipse)

We know that the centre of an ellipse is the mid point of the line joining two foci.

Centre $\equiv (3, 0)$

Distance between two foci = 2ae

$$\Rightarrow 2a\left(\frac{1}{2}\right) = 8 \Rightarrow a = 8$$

For an ellipse $b^2 = a^2(1 - e^2) \quad (a > b)$

$$\Rightarrow b^2 = 64\left(1 - \frac{1}{4}\right)$$

$$\Rightarrow b = 4\sqrt{3}$$

Hence the equation of the ellipse is: $\dfrac{(x-3)^2}{64} + \dfrac{(y-0)^2}{48} = 1$

The parametric form of any point on $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$ is $P(\theta) \equiv (h + a\cos\theta,\ k + b\sin\theta)$

Here, $h = 3,\ k = 0,\ a = 8,\ b = 4\sqrt{3}$

Any point on the given ellipse will be: $P(\theta) \equiv (3 + 8\cos\theta, 4\sqrt{3}\sin\theta)$

Hence proved.

Self-Assessment

Given, $x = 1 + 4\cos\theta$, $y = 2 + 3\sin\theta$

On comparing with parametric equations $x = h + a\cos\theta$ and $y = k + b\sin\theta$, we get the following:

$\Rightarrow h = 1, k = 2, a = 4, b = 3$

Here, the inferences $a > b$ and centre at (h, k) suggests that it is a translated horizontal ellipse.

Equation of the ellipse is $\frac{(x-1)^2}{4^2} + \frac{(y-2)^2}{3^2} = 1$

M A T H E M A T I C S

ELLIPSE

TANGENTS TO AN ELLIPSE AND PROBLEMS ON TANGENTS TO ELLIPSE

NOTES

What you already know

- Basic algebraic formulas
- Basics about conics
- Conditions for different conics
- Analytic interpretation of different conics
- To find the different parameters associated with an ellipse
- Horizontal and vertical standard equation of an ellipse
- Translation or shifting of an ellipse

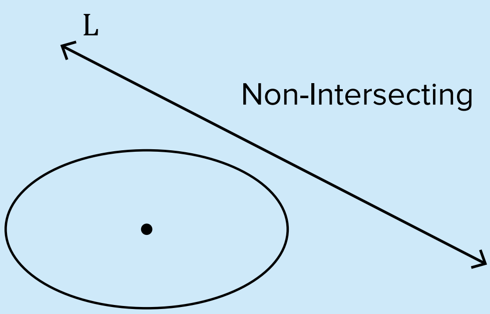
What you will learn

- Different conditions for a line to be a tangent, intersecting line, and a non-intersection of an ellipse
- Problems on tangents to an ellipse

Concept Check 1

Find the maximum length of chord of ellipse $\frac{x^2}{8}+\frac{y^2}{4}=1$ such that the eccentric angles of its extremities differ by $\frac{\pi}{2}$.

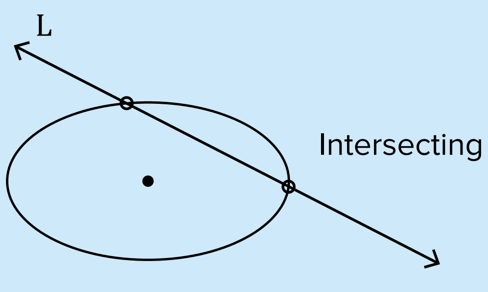
A Line and an Ellipse

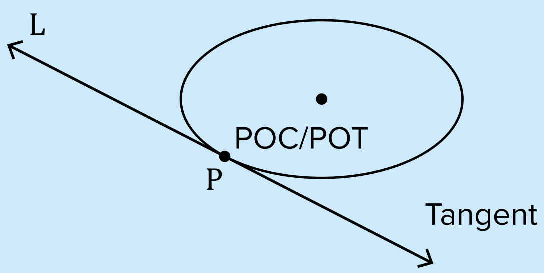


Let us consider the line, $y = mx + c \rightarrow L = 0$,

And the ellipse, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \rightarrow S = 0$,

Both the equations can be arranged as follows:





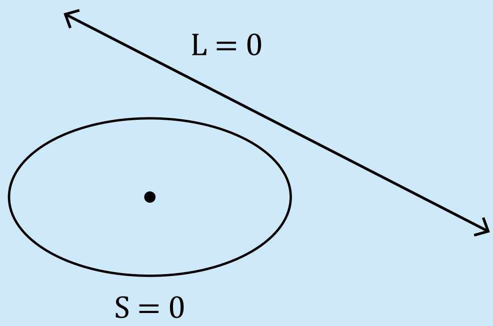
Here,
POC = Point of contact
POT = Point of tangency
To find the different conditions for different arrangements, we will solve both the equations.

By substituting the value of y from the equation of line in the equation of ellipse, we get the following:

$$\frac{x^2}{a^2} + \frac{(mx + c)^2}{b^2} = 1$$

$\Rightarrow b^2x^2 + a^2(mx + c)^2 = a^2b^2$

$\Rightarrow b^2x^2 + a^2(m^2x^2 + c^2 + 2mcx) = a^2b^2$



Or

$(b^2 + a^2m^2)x^2 + 2mca^2x + a^2c^2 - a^2b^2 = 0$ ---(i)

Equation (i) is quadratic in x.

Now, discriminant, $\Delta = B^2 - 4AC$

Or

$\Delta = 4m^2c^2a^4 - 4(b^2 + a^2m^2)(a^2c^2 - a^2b^2)$

$\Rightarrow \Delta = 4m^2c^2a^4 - 4b^2a^2c^2 + 4b^4a^2 - 4a^4m^2c^2 + 4a^4b^2m^2$

$\Rightarrow \Delta = 4b^2a^2(-c^2 + b^2 + a^2m^2)$

Or $\Delta = 4a^2b^2(a^2m^2 + b^2 - c^2)$

Now, $\Delta = 4a^2b^2(a^2m^2 + b^2 - c^2)$

Now, we will take different cases to solve for different conditions

Roots of equation (i) $\rightarrow$
- Real and distinct, if $\Delta > 0$ (Intersecting line)
- Real and equal, if $\Delta = 0$ (Tangent line)
- Imaginary, if $\Delta < 0$ (Non-intersecting line)

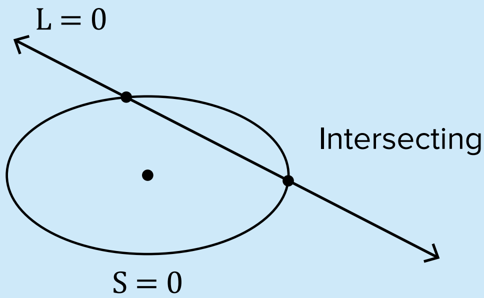
Case 1: $\Delta > 0$

$\Rightarrow 4a^2b^2(b^2 - c^2 + a^2m^2) > 0$

$\Rightarrow (b^2 - c^2 + a^2m^2) > 0$

$\Rightarrow a^2m^2 + b^2 > c^2$

The line will intersect the ellipse in two distinct points.



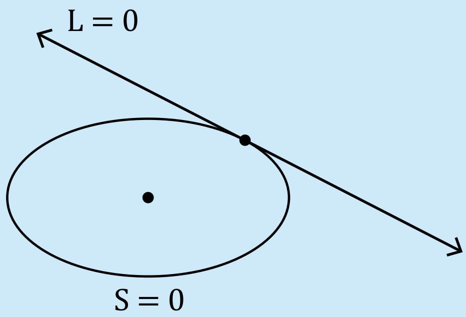
Case 2: $\Delta = 0$

$\Rightarrow 4a^2b^2(b^2 - c^2 + a^2m^2) = 0$

$\Rightarrow a^2m^2 + b^2 = c^2$

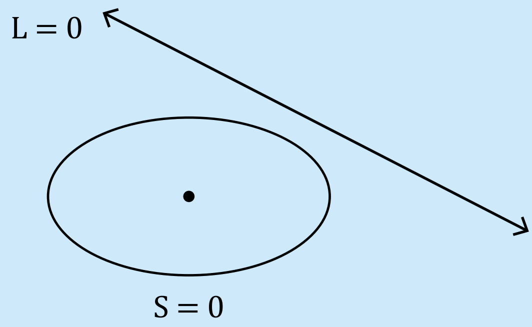
$\Rightarrow c = \pm\sqrt{a^2m^2 + b^2}$

The line touches the ellipse or is a tangent to the ellipse.



Line l: $y = mx + c$ is a tangent to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
Then,
Equation of tangent, $y = mx + \sqrt{a^2m^2 + b^2}$
And $y = mx - \sqrt{a^2m^2 + b^2}$
Also, these two equations represent parallel tangents to the ellipse.

Case 3: $\Delta < 0$
$\Rightarrow 4a^2b^2(b^2 - c^2 + a^2m^2) < 0$
$\Rightarrow a^2m^2 + b^2 < c^2$
The line does not meet the ellipse.



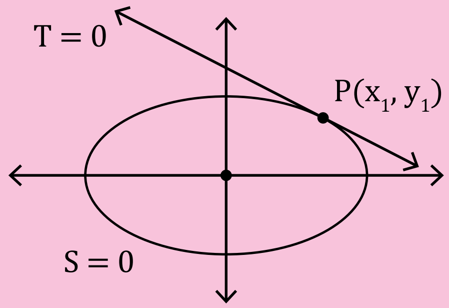
Equation of a Tangent to an Ellipse

The equation of a tangent to an ellipse can be written by the following three ways:
(a) Point form (b) Parametric form (c) Slope form

(a) Point form

The equation of tangent to an ellipse, S: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ at a point $P(x_1, y_1)$ is $T = 0$.

Representation of T.
Given,
Equation of an ellipse, S: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$
Now, as we know that to get the value of $T = 0$,
replace $x^2 \longrightarrow xx_1$, $y^2 \longrightarrow yy_1$
$\Rightarrow$ T: $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0$



Result

For a translated ellipse,
$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$
Now, to find the equation of tangent,
Let us consider $(x - h) = x'$ and
$(y - k) = y'$

$\Rightarrow \frac{(x')^2}{a^2} + \frac{(y')^2}{b^2} = 1$
Now, using the '$T = 0$' method for tangent,
$\Rightarrow \frac{x'x_1}{a^2} + \frac{y'y_1}{b^2} = 1$
Now, after replacing x' and y', we get the following:
T: $\frac{(x-h)x_1}{a^2} + \frac{(y-k)y_1}{b^2} - 1 = 0$

(b) Parametric form

The equation of tangent to an ellipse, S: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$, at a point $P(a\cos\theta, b\sin\theta)$ is as follows: T: $\left(\frac{\cos\theta}{a}\right)x + \left(\frac{\sin\theta}{b}\right)y - 1 = 0$

Proof

Let us consider point $P(\theta)$ on the ellipse,

S: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$

$\Rightarrow P(\theta) \equiv (a\cos\theta, b\sin\theta)$

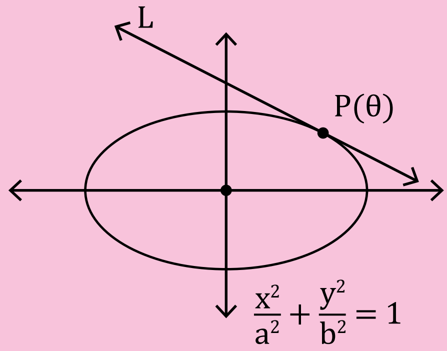
As we know that the equation of tangent to an ellipse, S: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$, is as follows:

T: $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0$

Now, by replacing x_1, y_1 by $a\cos\theta, b\sin\theta$, respectively, we get the following:

T: $\left(\frac{a\cos\theta}{a^2}\right)x + \left(\frac{b\sin\theta}{b^2}\right)y - 1 = 0$

Or $\left(\frac{\cos\theta}{a}\right)x + \left(\frac{\sin\theta}{b}\right)y - 1 = 0$



(c) Slope form

Equation of tangent to an ellipse S: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ at a point $P(-\frac{a^2m}{c}, \frac{b^2}{c})$ is as follows:

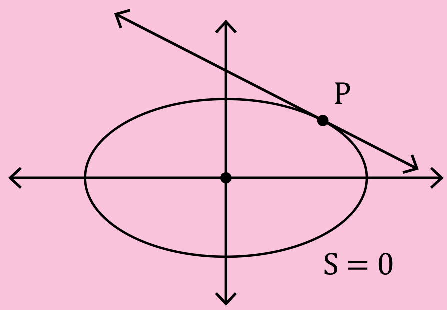
T: $y = mx \pm \sqrt{a^2m^2 + b^2}$, where $c = \pm\sqrt{a^2m^2 + b^2}$

Proof

The line, $y = mx + c$, touches the ellipse,

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $c^2 = a^2m^2 + b^2$.

So, the equation of tangent becomes the following: T: $y = mx \pm \sqrt{a^2m^2 + b^2}$

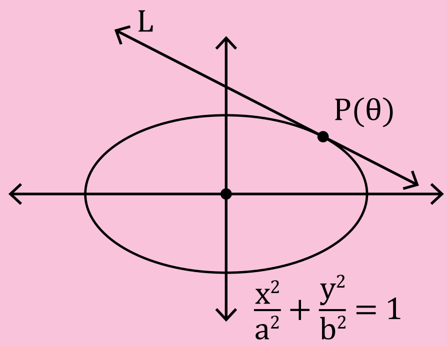


Point of contact (POC)/Point of tangency (POT)

As we know that the equations of tangent at points $P(x_1, y_1)$ and $P(a\cos\theta, b\sin\theta)$ are as follows:

L: $\left(\frac{\cos\theta}{a}\right)x + \left(\frac{\sin\theta}{b}\right)y - 1 = 0$

L: $y = mx \pm \sqrt{a^2m^2 + b^2}$



As we know that both the equations represent the same line, the ratio of their coefficients will be equal.

$$\Rightarrow \frac{\left(\frac{\cos\theta}{a}\right)}{m} = \frac{\left(\frac{\sin\theta}{b}\right)}{-1} = -\frac{1}{c}$$

Now, after solving, we get the following:

$\cos\theta = -\frac{am}{c}$ and $\sin\theta = \frac{b}{c}$

So, the point of tangency or point of contact is $P(a\cos\theta, b\sin\theta) \equiv P\left(-\frac{a^2m}{c}, \frac{b^2}{c}\right)$

Now, as we know that c has two values, we will get the two respective coordinates for P.

Solve

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If the tangents of the ellipse $4x^2 + y^2 = 8$ at the points $(1, 2)$ and (p, q) are perpendicular to each other, then what is the value of p^2?

(a) $\frac{128}{7}$ (b) $\frac{64}{7}$ (c) $\frac{4}{17}$ (d) $\frac{2}{17}$

Solution

Step 1

Given, equation of the ellipse, $4x^2 + y^2 = 8$

$\Rightarrow \frac{x^2}{2} + \frac{y^2}{8} = 1$

$\Rightarrow a < b$ (Standard vertical ellipse)

Also, T_1 and T_2 are perpendicular to each other.

Let us consider the slope of $T_1 = m_1$ and $T_2 = m_2$

$\Rightarrow m_1 \times m_2 = -1$

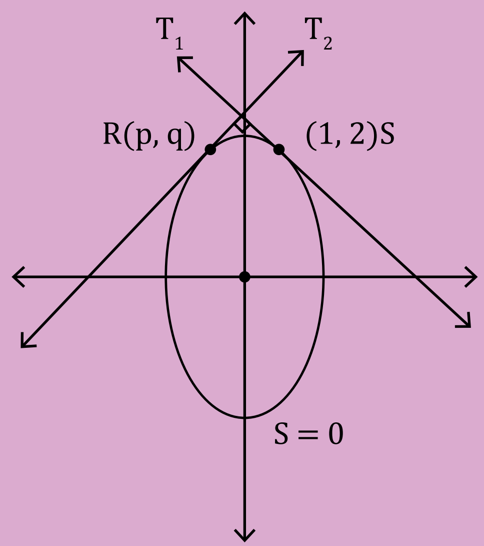
Now, by using the point form of a tangent to an ellipse, we get the following:

$T_1: 4x(1) + y(2) - 8 = 0$

$\Rightarrow 4x + 2y - 8 = 0$

$T_2: 4x(p) + y(q) - 8 = 0$

$\Rightarrow 4px + qy - 8 = 0$



Step 2

As we can see that $m_1 = -2$ and $m_2 = \frac{-4p}{q}$

$\Rightarrow (-2) \times \left(\frac{-4p}{q}\right) = -1$

$\Rightarrow q = -8p$ ---(i)

Also, point (p, q) lies on tangent T_2.

$\Rightarrow 4p^2 + q^2 - 8 = 0$ -----(ii)

Step 3

Now, by solving equations (i) and (ii), we get the following: $4p^2 + 64p^2 = 8$

$\Rightarrow 68p^2 = 8$

$\Rightarrow p^2 = \frac{8}{68} = \frac{2}{17}$

Hence, option (d) is the correct answer.

Concept Check 2

If $x - y + k = 0$ is tangent to ellipse $9x^2 + 16y^2 = 144$, then find the value of k.

Solve

Find the equation of tangent(s) to the ellipse $2x^2 + 3y^2 = 6$ whose inclination is 30°. Also, find the point of tangency.

Solution

Step 1

Given, equation of ellipse $2x^2 + 3y^2 = 6$

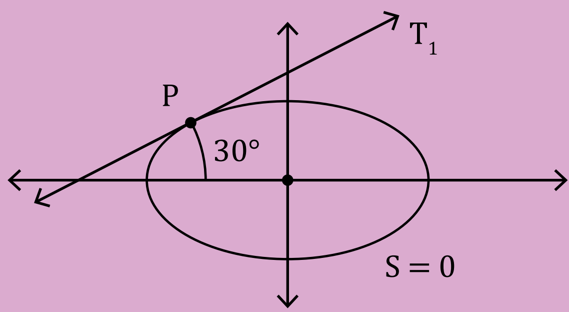
$\Rightarrow \frac{x^2}{3} + \frac{y^2}{2} = 1$

$\Rightarrow a > b$ (Standard horizontal ellipse)

Also, inclination of a tangent is 30°.

$\Rightarrow m = \tan 30° = \frac{1}{\sqrt{3}}$

Using,



Line l: $y = mx + c$ is a tangent to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then $c = \pm\sqrt{a^2m^2 + b^2}$

Step 2

We get the following:

$$c = \pm\sqrt{3 \times \left(\frac{1}{3}\right) + 2} = \pm\sqrt{3}$$

Thus, on solving the equation of tangents, we get the following:

$$y = \frac{1}{\sqrt{3}}x + \sqrt{3} \text{ and } y = \frac{1}{\sqrt{3}}x - \sqrt{3}$$

Step 3

Using,

Line l: $y = mx + c$ is a tangent to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then the point of tangency or point of contact is as follows:

$P(a \cos\theta, b \sin\theta) \equiv P\left(-\frac{a^2m}{c}, \frac{b^2}{c}\right)$

When $c = \sqrt{3}$, then $P\left(-1, \frac{2}{\sqrt{3}}\right)$

When $c = -\sqrt{3}$, then $P\left(1, -\frac{2}{\sqrt{3}}\right)$

Concept Check 3

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If tangents are drawn to ellipse $x^2 + 2y^2 = 2$ at all the points on the ellipse other than its four vertices, then the midpoints of the tangents intercepted between the coordinate axes lie on which curve?

(a) $\frac{x^2}{4} + \frac{y^2}{2} = 1$ (b) $\frac{1}{4x^2} + \frac{1}{2y^2} = 1$ (c) $\frac{x^2}{2} + \frac{y^2}{4} = 1$ (d) $\frac{1}{2x^2} + \frac{1}{4y^2} = 1$

Summary Sheet

- **A line and an ellipse**

  Let us consider a line, $y = mx + c$

  And an ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

  Case 1: If $a^2m^2 + b^2 > c^2 \Rightarrow$ The line will intersect the ellipse in two distinct points.

  Case 2: If $c = \pm\sqrt{a^2m^2 + b^2} \Rightarrow$ The line touches the ellipse or is a tangent to the ellipse.

  Case 3: If $a^2m^2 + b^2 < c^2 \Rightarrow$ The line does not meet the ellipse.

- **Equation of a tangent to an ellipse**
  - Point form

    The equation of tangent to an ellipse, $S: \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$, at a point $P(x_1, y_1)$ is as follows:

    $T: \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0$

  - Parametric form

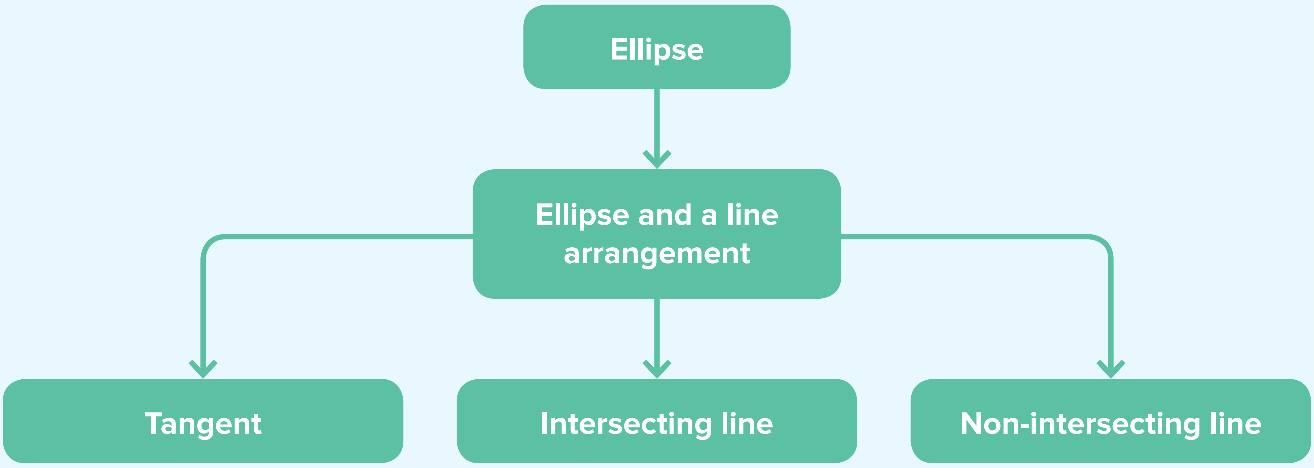
    The equation of tangent to an ellipse, $S: \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$, at a point $P\ (a\cos\theta, b\sin\theta)$ is as follows: $T: \left(\frac{\cos\theta}{a}\right)x + \left(\frac{\sin\theta}{b}\right)y - 1 = 0$

  - Slope form

    Equation of tangent to an ellipse $S: \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ at a point $P\left(-\frac{a^2m}{c}, \frac{b^2}{c}\right)$ is as follows:

    $T:\ y = mx \pm\sqrt{a^2m^2 + b^2}$, where $c = \pm\sqrt{a^2m^2 + b^2}$

Mind Map



Self-Assessment

1. Find the equations of the tangents drawn from point $(2, 3)$ to ellipse $9x^2 + 16y^2 = 144$.

2. Find the points on ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ such that the tangent at each point makes equal angles with the axes.

Answers

Concept Check 1

Step 1

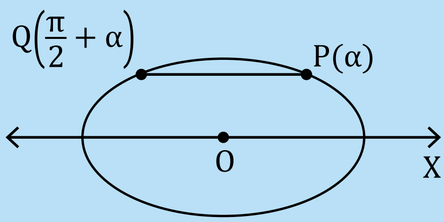
Given, equation of ellipse $\frac{x^2}{8} + \frac{y^2}{4} = 1$

$\Rightarrow a^2 = 8 \Rightarrow a = 2\sqrt{2}$

And $b^2 = 4 \Rightarrow b = 2$

$\Rightarrow$ Given, ellipse is a horizontal ellipse

Let $P(\alpha)$, $Q\left(\alpha + \frac{\pi}{2}\right)$ be the extremities of chord PQ.



Now, using parametric coordinate form for an ellipse, we get the following:

$P \equiv (2\sqrt{2} \cos \alpha, 2 \sin \alpha)$ and

$Q \equiv \left(2\sqrt{2} \cos\left(\alpha + \frac{\pi}{2}\right), 2 \sin\left(\alpha + \frac{\pi}{2}\right)\right)$

Or $Q \equiv (-2\sqrt{2} \sin \alpha, 2 \cos \alpha)$

Step 2

Now, by using the two-point distance formula, we get the following:

$PQ = \sqrt{(2\sqrt{2}\cos\alpha + 2\sqrt{2}\sin\alpha)^2 + (2\sin\alpha - 2\cos\alpha)^2}$

Now, by squaring both the sides, we get the following:

$(PQ)^2 = (2\sqrt{2}\cos\alpha + 2\sqrt{2}\sin\alpha)^2 + (2\sin\alpha - 2\cos\alpha)^2$

As we know, $(PQ)_{max} \equiv (PQ)^2_{max}$

So,

$(PQ)^2 = 8(\cos^2\alpha + \sin^2\alpha + 2\sin\alpha \times \cos\alpha) + 4(\sin^2\alpha + \cos^2\alpha - 2\sin\alpha \times \cos\alpha)$

$\Rightarrow (PQ)^2 = 8(1 + \sin 2\alpha) + 4(1 - \sin 2\alpha)$

$\Rightarrow (PQ)^2 = 8 + 8\sin 2\alpha + 4 - 4\sin 2\alpha$

Or $(PQ)^2 = 12 + 4\sin 2\alpha$

As we can see, $(PQ)^2$ will be the maximum, when $\sin 2\alpha = 1$

$\Rightarrow (PQ_{max})^2 = 16 \Rightarrow PQ_{max} = 4$

Concept Check 2

Step 1

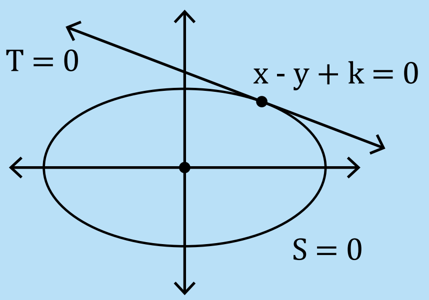
Given, equation of ellipse $9x^2 + 16y^2 = 144$

$\Rightarrow \frac{x^2}{16} + \frac{y^2}{9} = 1$

And a tangent to this ellipse is

$x - y + k = 0$ or $y = x + k$

Using,



Line l: $y = mx + c$ is a tangent to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then $c = \pm\sqrt{a^2m^2 + b^2}$

Step 2

We get $k = \pm\sqrt{16 \times 1 + 9} = \pm\sqrt{25}$

Or $k = \pm 5$

Now, the equation of tangent will be as follows:

$y = x + 5$ and $y = x - 5$

Concept Check 3

Step 1

Given, equation of ellipse $x^2 + 2y^2 = 2$

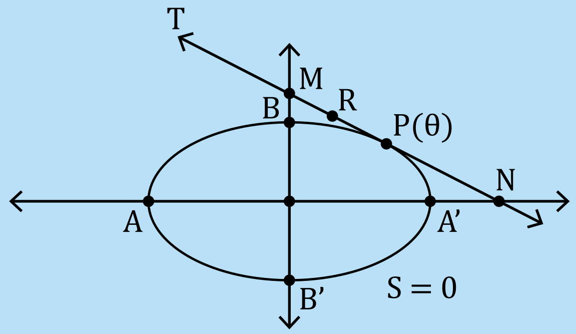
$\Rightarrow \frac{x^2}{2} + \frac{y^2}{1} = 1$

$\Rightarrow$ a > b (Standard horizontal ellipse)

Let us consider $P(\theta)$ to be any point on the ellipse (other than the vertices).

And a point $R(h, k)$, i.e., the midpoint of the segment of tangent T between the axes.

As we know, $P(\theta) \equiv (\sqrt{2}\cos\theta, \sin\theta)$



Step 2

Now,

Equation of a tangent at point P by using parametric form,

$T: \left(\frac{\cos\theta}{\sqrt{2}}\right)x + (\sin\theta)y - 1 = 0$

Now, coordinates of point $M \equiv (0, \operatorname{cosec}\theta)$ And $N \equiv (\sqrt{2}\sec\theta, 0)$

Here, by using the midpoint formula, we get the following:

$h = \frac{\sqrt{2}\sec\theta}{2}$ and $k = \frac{\operatorname{cosec}\theta}{2}$

Step 3

$\cos\theta = \frac{1}{\sqrt{2}h}$ and $\sin\theta = \frac{1}{2k}$

By adding both the terms after squaring, we get the following:

$\frac{1}{2h^2} + \frac{1}{4k^2} = 1$

Now, by replacing h by x and k by y, we get the following:

$\frac{1}{2x^2} + \frac{1}{4y^2} = 1$

Hence, option (d) is the correct answer.

Self-Assessment 1

Step 1

Given, equation of ellipse $9x^2 + 16y^2 = 144$

$\Rightarrow \frac{x^2}{16} + \frac{y^2}{9} = 1$ (Standard horizontal ellipse)

and we have to find the equations of the tangents drawn from the point $(2, 3)$ to this ellipse.

Let us consider the equation of tangent as $y = mx \pm \sqrt{16m^2 + 9}$, and that it passes through point $(2, 3)$.

$\Rightarrow 3 = 2m \pm \sqrt{16m^2 + 9}$

$\Rightarrow (3 - 2m) = \pm\sqrt{16m^2 + 9}$

Step 2

Now, after squaring both the sides, we get the following:

$\Rightarrow 9 + 4m^2 - 12m = 16m^2 + 9$

Or $12m^2 + 12m = 0$

$\Rightarrow m = 0$ or -1

Now, tangents passing through point $(2, 3)$ and having slope 0 and -1 are $y = 3$ and $y = -x + 5$

Self-Assessment 2

Step 1

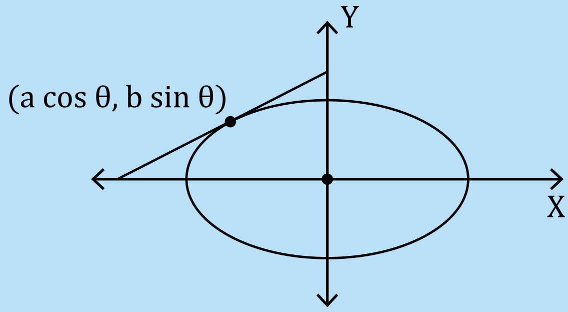
Let P be $(a \cos \theta, b \sin \theta)$, then the equation of tangent at P will be as follows:

$$\left(\frac{\cos \theta}{a}\right)x + \left(\frac{\sin \theta}{b}\right)y = 1$$

Now, the slope of tangent is as follows:

$$= -\frac{b \cos \theta}{a \sin \theta} = \pm \tan 45°$$

$$\Rightarrow \cot \theta = \pm \frac{a}{b}$$



Step 2

$$\Rightarrow \cos \theta = \pm \frac{a}{\sqrt{a^2 + b^2}}$$

Also, $\sin \theta = \pm \frac{b}{\sqrt{a^2 + b^2}}$

Therefore, the coordinates of the required points are $\left[\pm \frac{a^2}{\sqrt{a^2 + b^2}}, \pm \frac{b^2}{\sqrt{a^2 + b^2}}\right]$

MATHEMATICS

NOTES

ELLIPSE

AUXILIARY CIRCLE, DIRECTOR CIRCLE, EQUATION OF NORMAL, PAIR OF TANGENTS, CHORD OF CONTACT FOR ELLIPSE

What you already know

- Introduction to ellipse
- Standard equation of ellipse
- Parametric equation of ellipse
- Tangents to an ellipse

What you will learn

- Auxiliary circle
- Director circle
- Equation of normal
- Pair of tangents
- Chord of contact for ellipse

Quick Recap

- The equation of tangent to standard ellipse is given by $T = 0$, where $T: \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$
- **Point form**

  Equation of tangent to an ellipse, $S: \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$, at point $P(x_1, y_1)$ is given by $T: \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0$
- The condition for the line $y = mx + c$ to touch the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

  $a^2m^2 + b^2 = c^2 \Rightarrow c = \pm\sqrt{a^2m^2 + b^2}$

Solve

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If the tangents to parabola $y^2 = x$ at point (α, β), where $\beta > 0$, is also tangent to the ellipse $x^2 + 2y^2 = 1$, then what is α equal to?

(a) $\sqrt{2} + 1$ (b) $\sqrt{2} - 1$ (c) $2\sqrt{2} + 1$ (d) $2\sqrt{2} - 1$

Solution

Step 1

It is based on the concept of common tangent.

Given, tangent at $P(\alpha, \beta), (\beta > 0)$, to $S': y^2 = x$ is also the tangent to $S: x^2 + 2y^2 = 1$, i.e., $\frac{x^2}{1} + \frac{y^2}{\left(\frac{1}{2}\right)} = 1$

Comparing $y^2 = x$ with standard parabola $y^2 = 4Ax$ we will get $4A = 1 \Rightarrow A = \frac{1}{4}$

The vertex of this parabola is at the origin and focus $(A, 0) \equiv \left(\frac{1}{4}, 0\right)$.

Now, comparing $\frac{x^2}{1}+\frac{y^2}{\left(\frac{1}{2}\right)}=1$ with standard ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get,

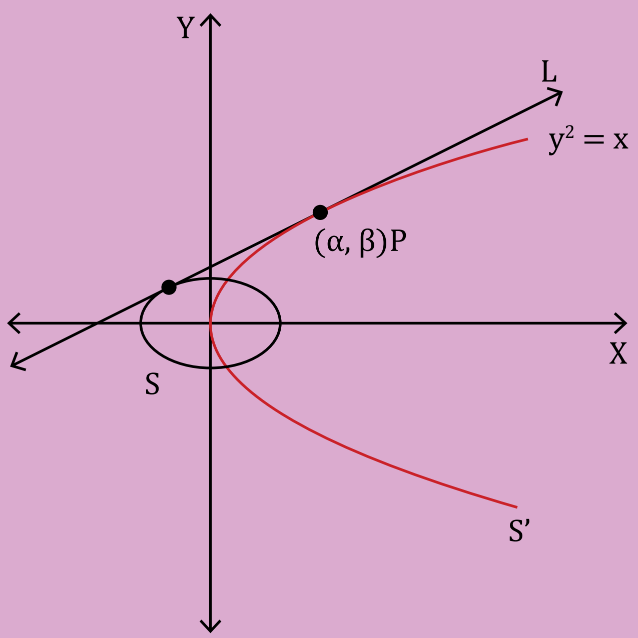
$a^2=1$ and $b^2=\frac{1}{2}$.

Step 2

As $a>b$, the ellipse is a standard horizontal ellipse having centre at the origin and major axis along the x-axis.

$P(\alpha, \beta)$ is on the parabola and the tangent to parabola will also be the tangent to ellipse.

As $P(\alpha, \beta)$ is on parabola, it will satisfy equation $y^2=x \Rightarrow \beta^2=\alpha$ ...(1)



Step 3

Equation of tangent at (α, β) to the parabola $y^2=x$ is $T=0$.

$S: y^2 - x = 0,\ T: yy_1 - \frac{x+x_1}{2}=0 \Rightarrow T: y\beta - \frac{x+\alpha}{2}=0$

Step 4

Equation of tangent, $y\beta=\left(\frac{1}{2}\right)x+\frac{\alpha}{2}$

By rewriting it in the slope form, we get the following:

$y=\left(\frac{1}{2\beta}\right)x+\frac{\alpha}{2\beta}$ ...(2)

We know that equation (2) is also a tangent to S: $\frac{x^2}{1}+\frac{y^2}{\frac{1}{2}}=1$, where slope$(m)=\frac{1}{2\beta}$

and y-intercept $(c)=\frac{\alpha}{2\beta}$

Step 5

$c^2 = a^2m^2 + b^2$

Here, $c = \frac{\alpha}{2\beta}$, $m = \frac{1}{2\beta}$, $a^2 = 1$, and $b^2 = \frac{1}{2}$.

By substituting the respective values, we get the following:

$$\left(\frac{\alpha}{2\beta}\right)^2 = 1\left(\frac{1}{2\beta}\right)^2 + \frac{1}{2}$$

$$\Rightarrow \left(\frac{\alpha^2}{4\beta^2}\right) = \left(\frac{1}{4\beta^2}\right) + \frac{1}{2} \Rightarrow \left(\frac{\alpha^2}{4\beta^2}\right) - \left(\frac{1}{4\beta^2}\right) = \frac{1}{2}$$

$$\Rightarrow \left(\frac{\alpha^2 - 1}{4\beta^2}\right) = \frac{1}{2} \Rightarrow \left(\frac{\alpha^2 - 1}{4\alpha}\right) = \frac{1}{2}$$

(From equation (1), $\beta^2 = \alpha$)

Step 6

By solving it, we get the following:

$$\alpha^2 - 2\alpha - 1 = 0 \Rightarrow \alpha = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$$

Here, $\alpha > 0$, so, $\alpha = \sqrt{2} + 1$

So, option (a) is the correct answer.

Note

Locus of foot of perpendiculars from foci upon the tangents is the **auxiliary circle**.

Proof

Step 1

For ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$

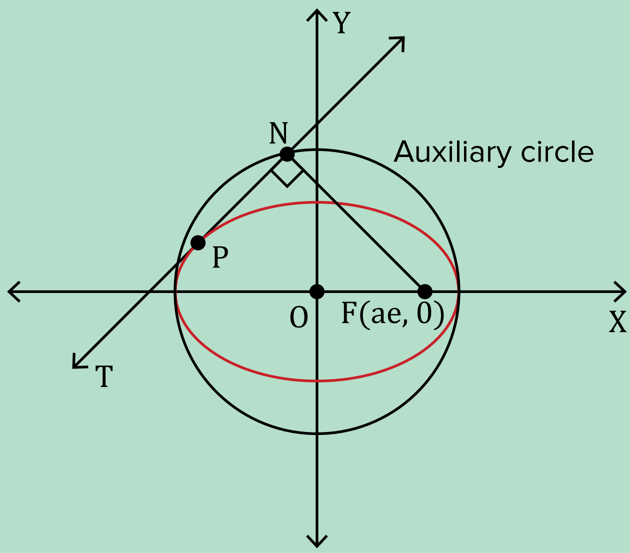
Equation of tangent at any point P is as follows:

$y = mx \pm \sqrt{a^2m^2 + b^2}$

$\Rightarrow y - mx = \pm\sqrt{a^2m^2 + b^2}$

By squaring on both the sides, we get the following:

$\Rightarrow y^2 + m^2x^2 - 2mxy = a^2m^2 + b^2$ .....(1)



Step 2

Let N be the foot of the perpendicular drawn from focus F(ae, 0)

Equation of FN is $\frac{y - 0}{x - ae} = \frac{-1}{m} \Rightarrow x + my = ae$

By squaring on both the sides, we get the following: $\Rightarrow x^2 + m^2y^2 + 2mxy = a^2e^2$ .....(2)

By adding equations (1) and (2), we get the following: $y^2(1 + m^2) + x^2(1 + m^2) = a^2m^2 + b^2 + a^2e^2$

$y^2(1 + m^2) + x^2(1 + m^2) = a^2m^2 + b^2 + a^2 - b^2$ $\quad [\because b^2 = a^2(1 - e^2)]$

$\Rightarrow (x^2 + y^2)(1 + m^2) = a^2(1 + m^2) \Rightarrow x^2 + y^2 = a^2$

This is the equation of the auxiliary circle.

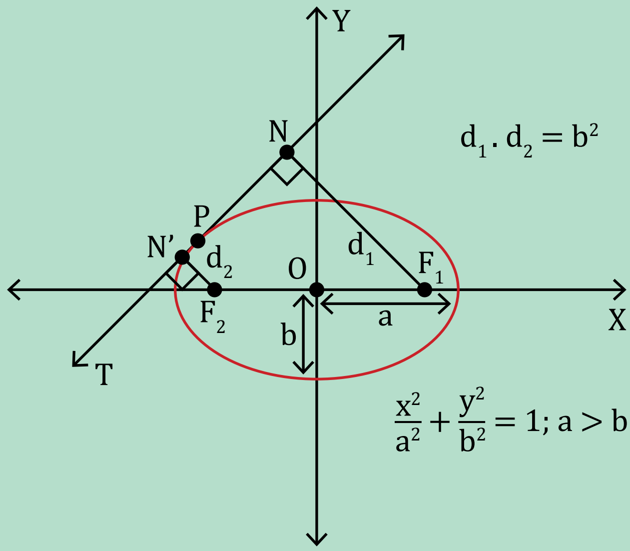
Therefore, the locus of the feet perpendiculars from foci upon any tangent is an auxiliary circle.

Note

Product of the length of the perpendicular from foci upon any tangent of an ellipse is equal to the square of the semi-minor axis, i.e., $d_1 d_2 = b^2$

Hint: $d_1 = \dfrac{mae + \sqrt{a^2m^2 + b^2}}{\sqrt{1 + m^2}}$

and $d_2 = \dfrac{-mae + \sqrt{a^2m^2 + b^2}}{\sqrt{1 + m^2}}$



Director Circle

The locus of the point of intersection of perpendicular tangents to an ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $(a > b)$ is a circle concentric to the ellipse, and it is known as director circle. It is given by the equation $x^2 + y^2 = a^2 + b^2$.

Proof

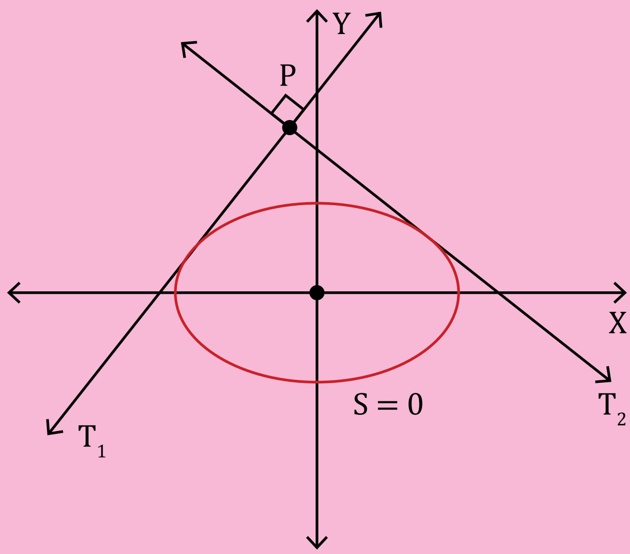
Step 1

Let $P = (h, k)$ be a point outside ellipse $S = 0$. The pair of tangents from P are mutually perpendicular. Director circle is the locus of P.

$\Rightarrow T_1(m_1), T_2(m_2)$ are distinct tangents from P,

where, m_1: Slope of T_1, m_2: Slope of T_2

$\Rightarrow$ Equation of a tangent to the ellipse of slope m: $y = mx \pm \sqrt{a^2m^2 + b^2}$



Step 2

Tangent passes through $P(h, k)$

$\Rightarrow k - mh = \pm\sqrt{a^2m^2 + b^2}$

On squaring both sides and solving, we get the following:

$k^2 + m^2h^2 - 2mhk = a^2m^2 + b^2$

$\Rightarrow m^2(h^2 - a^2) - 2mhk + (k^2 - b^2) = 0 \ldots (i)$

Quadratic equation with roots m_1 and m_2

$\Rightarrow m_1 \cdot m_2 = -1$ (Perpendicular lines)

Step 3

From (i), we get the following:

$m_1 \cdot m_2 = \frac{k^2 - b^2}{h^2 - a^2} = -1$ (Product of roots)

$h^2 + k^2 = a^2 + b^2$

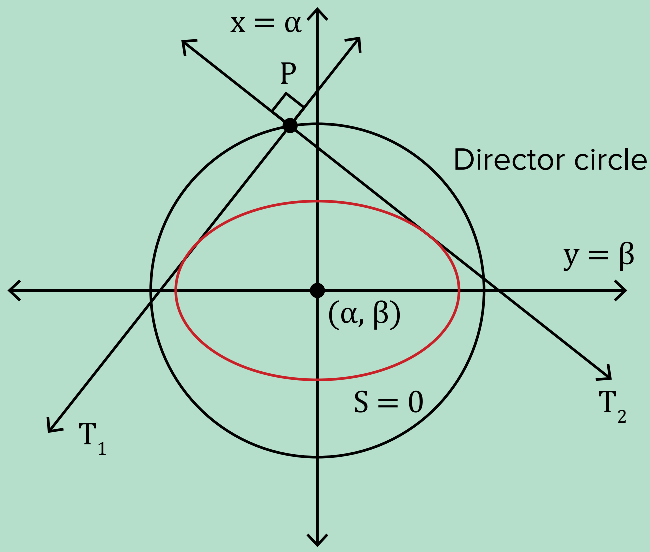
By replacing $h \to x$ and $k \to y$, we get the following:

$\Rightarrow x^2 + y^2 = a^2 + b^2$

Circle with centre at origin and radius $\sqrt{a^2 + b^2}$

Note

Director circle of $\frac{(x - \alpha)^2}{a^2} + \frac{(y - \beta)^2}{b^2} = 1$ is $(x - \alpha)^2 + (y - \beta)^2 = a^2 + b^2$.



Solve

Find the angle between the tangents drawn from any point on circle $x^2 + y^2 = 41$ to ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$.

Solution

Given, $x^2 + y^2 = 41 = 25 + 16$

So, $x^2 + y^2 = a^2 + b^2$

This is the equation of the director circle of the ellipse.

Tangents drawn from any point on the director circle of the ellipse are mutually perpendicular.

Therefore, the angle between them will be 90°.

Solve

Find the angle between the pair of tangents drawn from $(2, 3)$ to the ellipse $9x^2 + 16y^2 = 144$.

Solution

Step 1

$S: 9x^2 + 16y^2 - 144$

$S_1: 9(2)^2 + 16(3)^2 - 144 = 36 + 144 - 144 = 36 > 0$

Hence, P is the lying exterior of the ellipse.

Given, equation of ellipse $9x^2 + 16y^2 = 144$, $\Rightarrow \frac{x^2}{16} + \frac{y^2}{9} = 1$ (Where, $a^2 = 16, b^2 = 9$)

Step 2

The equation of a tangent to ellipse

$S = 0$ is $y = mx \pm \sqrt{a^2m^2 + b^2}$

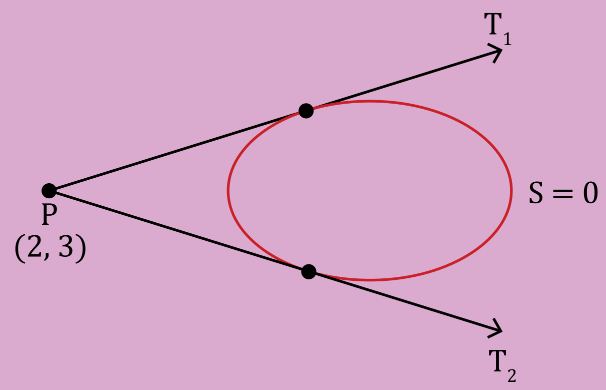
It passes through point $(2, 3)$. So, this point will satisfy the equation of tangent.

$3 = 2m \pm \sqrt{a^2m^2 + b^2}$

By putting the values of $a^2 = 16$ and $b^2 = 9$, we get the following:

$3 = 2m \pm \sqrt{16m^2 + 9}$

$\Rightarrow 3 - 2m = \pm \sqrt{16m^2 + 9}$



Step 3

On squaring both the sides and solving, we get the following:

$9 - 12m + 4m^2 = 16m^2 + 9$

$\Rightarrow 12m^2 + 12m = 0 \Rightarrow m^2 + m = 0$

$\Rightarrow m(m + 1) = 0 \Rightarrow m = 0$ or $m = -1$

Let the slope of the first tangent be $0 \Rightarrow m_1 = 0$, and the second tangent be $-1 \Rightarrow m_2 = -1$.

Step 3

If θ is the acute angle between the tangents,

then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$

$= \left|\frac{0 - (-1)}{1 + 0(-1)}\right| = 1$

$\Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

Parametric form of a tangent to an ellipse

The equation of the tangent at any point $(a\cos\theta, b\sin\theta)$ is $\left(\frac{x}{a}\right)\cos\theta + \left(\frac{y}{b}\right)\sin\theta = 1$.

Equation of Normal

Point form

The equation of normal at $P \equiv (x_1, y_1)$ on ellipse $S = 0$ is given by $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$.

Proof

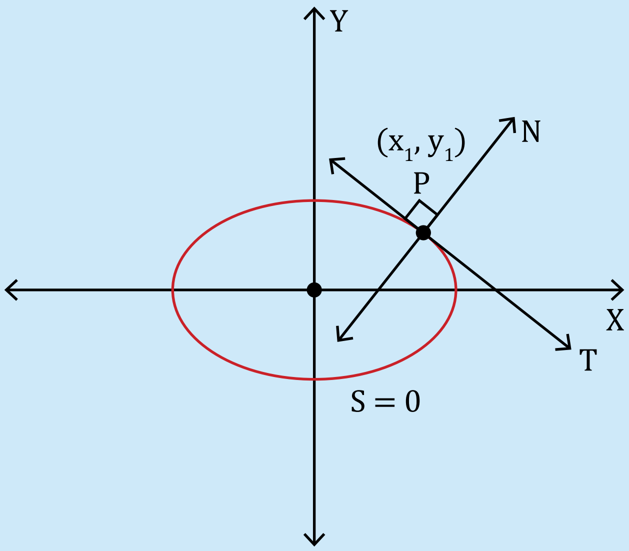
Step 1

The equation of tangent at $P(x_1, y_1)$ is $T = 0$

$\Rightarrow \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0$

So, slope of tangent $= \frac{-b^2x_1}{a^2y_1}$

$\Rightarrow$ Slope of normal $= \frac{a^2y_1}{b^2x_1}$



Step 2

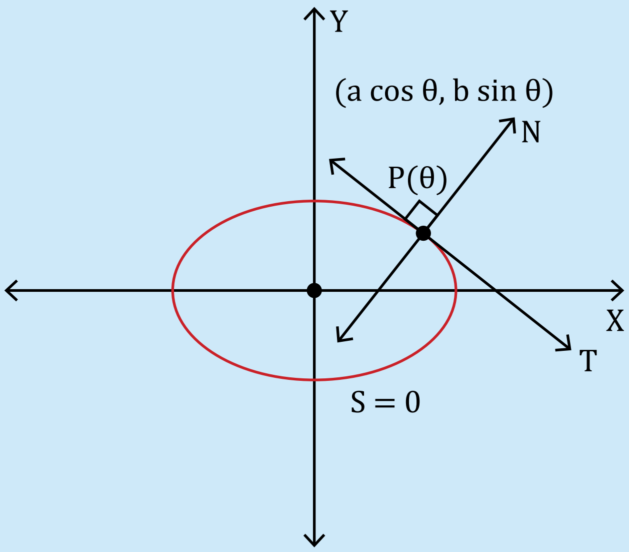
So, the equation of normal at P is given as follows:

$(y - y_1) = \frac{a^2y_1}{b^2x_1}(x - x_1)$

$\Rightarrow \frac{b^2(y - y_1)}{y_1} = \frac{a^2(x - x_1)}{x_1} \quad \Rightarrow \frac{b^2y}{y_1} - b^2 = \frac{a^2x}{x_1} - a^2 \quad \Rightarrow \frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$

Parametric form

This form is used when we know the parameter of the point upon which the normal is being drawn.
Consider the standard ellipse having θ as its eccentric angle, where N is the normal to the ellipse at point P.



Then, the equation of normal at $P(\theta)$ on ellipse $S = 0$ is given as follows:

$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$

Proof

θ is an eccentric angle of point P. The parametric coordinates are $(a \cos \theta, b \sin \theta)$, which are x_1 and y_1 of the points upon which the normal is drawn.

$x_1 = a \cos \theta, y_1 = b \sin \theta$

By writing the equation of normal in the point form, we get the following:

$$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2 = a^2e^2$$

By replacing x_1 with $a \cos \theta$ and y_1 with $b \sin \theta$, we get the following:

$$\frac{a^2x}{a \cos \theta} - \frac{b^2y}{b \sin \theta} = a^2 - b^2 = a^2e^2$$

$$\frac{ax}{\cos \theta} - \frac{by}{\sin \theta} = a^2 - b^2 = a^2e^2$$

Slope of the normal

$$m = \frac{a}{\cos \theta} \times \frac{\sin \theta}{b}$$

$$m = \frac{a \tan \theta}{b}$$

Slope form

The slope form is being used when you are aware of the slope of the normal line.

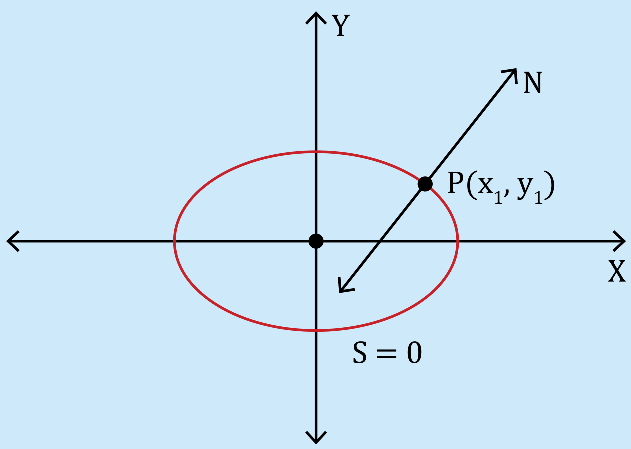
In an ellipse, there are two tangents of the same slope 'm' parallel to each other. Therefore, there will be two distinct normals of the same slope 'm' both parallel to each other.

The equation of normal to ellipse $S = 0$, whose slope 'm' is given by $y = mx \mp \frac{(a^2 - b^2)m}{\sqrt{a^2 + b^2m^2}}$. The point of contact of the normals have the following coordinates:

$$P \equiv \left(\pm \frac{a^2}{\sqrt{a^2 + b^2m^2}}, \pm \frac{mb^2}{\sqrt{a^2 + b^2m^2}}\right).$$

Proof

We have to show that the line of slope m given as, $y = mx + c$ happens to be a normal to this ellipse only when the constant c is equal to $\mp \frac{(a^2 - b^2)m}{\sqrt{a^2 + b^2m^2}}$.



Let P be the point at which the normal has been drawn and that has an eccentric angle of θ. The parametric form of the same line can be given as follows:

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$$

By writing the equation of normal in the parametric form, we get the following:

$$\Rightarrow y = \left(\frac{a\tan\theta}{b}\right)x - \frac{(a^2 - b^2)\sin\theta}{b}$$

Writing it in the slope form,

$$\Rightarrow y = m \,.\, x + c$$

Since, both the equations represent the same line $m = \frac{a\tan\theta}{b}$ and $c = \frac{-(a^2 - b^2)\sin\theta}{b}$

$$\Rightarrow \tan\theta = \frac{mb}{a}$$

By applying the trigonometric property, we get the following:

$$\Rightarrow \sin\theta = \frac{\pm bm}{\sqrt{a^2 + b^2m^2}}$$

By substituting the value of sin θ in $c = \frac{-(a^2 - b^2)\sin\theta}{b}$, we get the following:

$$\Rightarrow c = \frac{\mp(a^2 - b^2)bm}{b\sqrt{a^2 + b^2m^2}} = \frac{\mp(a^2 - b^2)m}{\sqrt{a^2 + b^2m^2}}$$

Coordinates of the Point of Contact of Both Normals

Consider N as a normal of slope m and the coordinates upon which the normal is being drawn are (x_1, y_1).

The equation of the normal in the point form is $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} - (a^2 - b^2) = 0$

Write the obtained equation of normal in the slope form.

$$y = m(x) + c \Rightarrow m(x) - y + c = 0$$

The ratio of corresponding coefficients are equal.

$$\frac{a^2}{mx_1} = \frac{b^2}{y_1} = \frac{-(a^2 - b^2)}{c} \Rightarrow \frac{a^2}{mx_1} = \frac{-(a^2 - b^2)}{c} \Rightarrow x_1 = \frac{-a^2c}{m(a^2 - b^2)} \text{.......(i)}$$

But $c = \frac{\mp(a^2 - b^2)m}{\sqrt{a^2 + b^2m^2}}$ ....(ii)

From (i) and (ii) $\Rightarrow x_1 = \pm\frac{a^2}{\sqrt{a^2 + b^2m^2}}$

The obtained coordinates are $P \equiv \left(\pm\frac{a^2}{\sqrt{a^2 + b^2m^2}}, \pm\frac{mb^2}{\sqrt{a^2 + b^2m^2}}\right)$

Solve JEE MAIN - 2019

Let $x = 4$ be a directrix to an ellipse whose centre is at origin and its eccentricity is 0.5. If $P(1, \beta)$, where $\beta > 0$ is a point on this ellipse, then what is the equation of the normal to it at P.

(a) $8x - 2y = 5$ (b) $4x - 2y = 1$ (c) $7x - 4y = 1$ (d) $4x - 3y = 2$

Solution

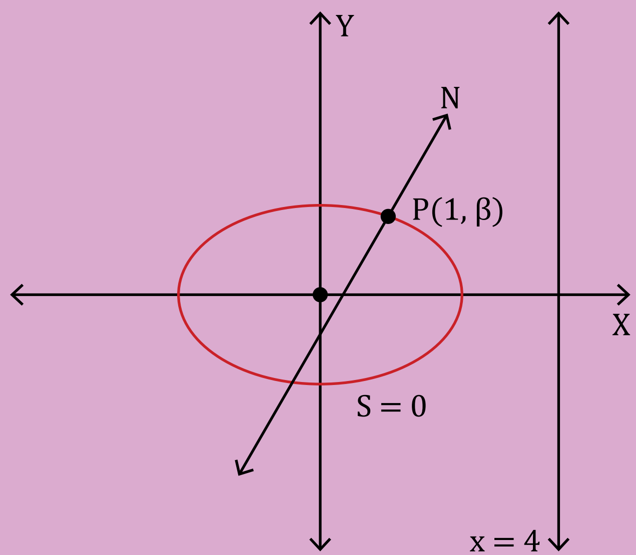
Step 1

Given, centre $\equiv (0, 0)$, directrix, $x = 4$, so the ellipse is a standard horizontal ellipse and $e = 0.5$

Equation of directrix: $x = \frac{a}{e} = 4 \Rightarrow \frac{a}{0.5} = 4 \Rightarrow a = 2$

$b^2 = a^2(1 - e^2) = 4(1 - \frac{1}{4}) = 3$

So, the equation of the ellipse is $\frac{x^2}{4} + \frac{y^2}{3} = 1$



Step 2

$P(1, \beta)$ lies on the ellipse,

$\Rightarrow \frac{1}{4} + \frac{\beta^2}{3} = 1 \Rightarrow \frac{\beta^2}{3} = 1 - \frac{1}{4} = \frac{3}{4}$

$\beta^2 = \frac{9}{4} \Rightarrow \beta = \frac{3}{2} (\beta > 0)$,

$P(1, \beta) = (1, \frac{3}{2})$;

Step 3

Equation of normal,

$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$

$\Rightarrow 4x - 2y = 1$

So, option (b) is the correct answer.

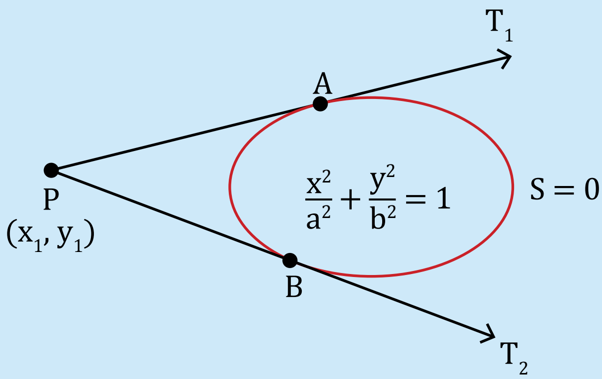
Pair of Tangents

The combined equation of the pair of tangents from an external point $P \equiv (x_1, y_1)$ to the ellipse $S = 0$ is given by $T^2 = SS_1$.

Where $S: \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$

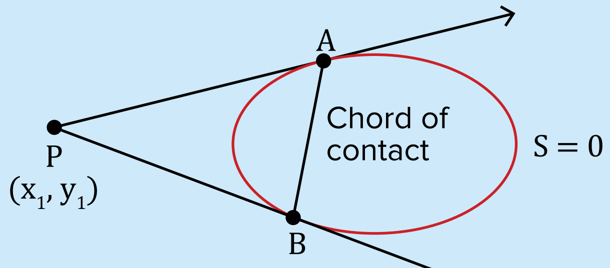
$T: \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$, when we put (x_1, y_1) in S, then

we get $S_1: \frac{(x_1)^2}{a^2} + \frac{(y_1)^2}{b^2} - 1$



Chord of contact for Ellipse

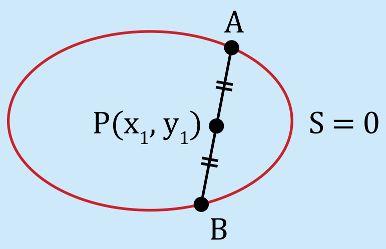
From the exterior point $P(x_1, y_1)$, we get two tangents on the ellipse and two distinct points of tangencies on the ellipse and the line joining these points of tangencies is known as the chord of contact. With respect to the point 'P' and equation of this chord AB is $T = 0$, i.e., $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0$.



If the equation of AB is $lx + my + n = 0$, then $P = \left(\frac{-a^2l}{n}, \frac{-b^2m}{n}\right)$

Chord with Given Midpoint

The equation of a chord of the ellipse $S = 0$, whose midpoint is $P(x_1, y_1)$, is $T = S_1$.



Summary Sheet

Key Takeaways

- The locus of the feet of perpendiculars from foci upon any tangent is an auxiliary circle.
- The product of the length of the perpendiculars from foci upon any tangent of an ellipse is equal to the square of the semi-minor axis, i.e., $d_1d_2 = b^2$.
- **Director circle of an ellipse**: The locus of the points of intersection of perpendicular tangents to an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $(a > b)$ is a circle concentric to the ellipse known as the director circle. It is given as follows: $x^2 + y^2 = a^2 + b^2$
- Director circle of $\frac{(x-\alpha)^2}{a^2} + \frac{(y-\beta)^2}{b^2} = 1$ is $(x-\alpha)^2 + (y-\beta)^2 = a^2 + b^2$

- **Equation of normal**

(a) **Point form:** The equation of normal at $P(x_1, y_1)$ on the ellipse $S = 0$ is given as follows:

$$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$$

(b) Equation of normal at $P(\theta)$ on the ellipse $S = 0$ is given as follows:

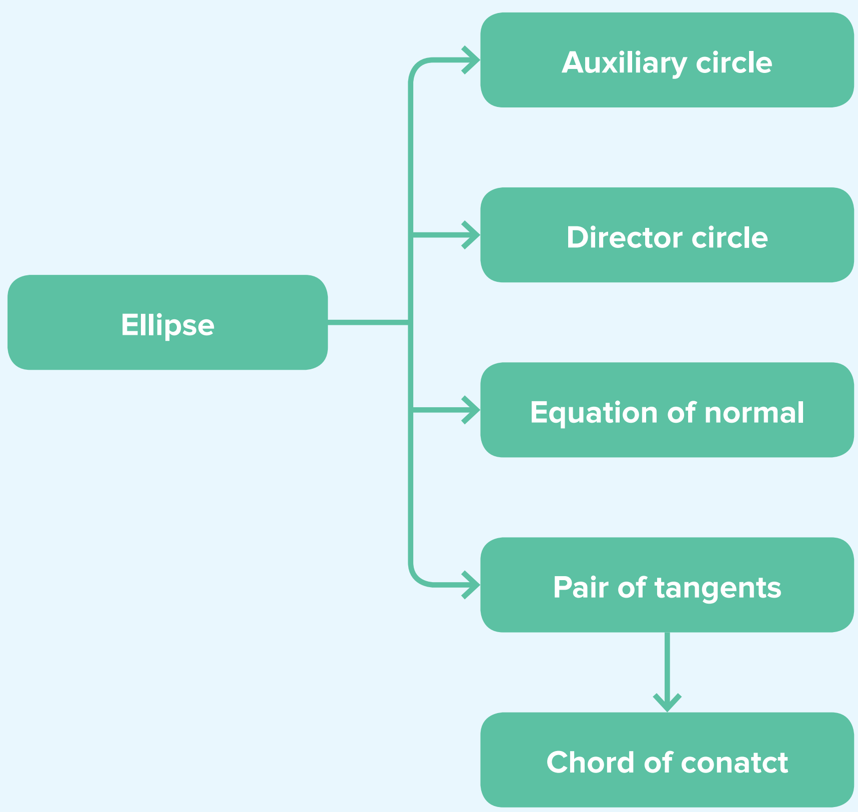
$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$$

- The coordinates of the point of contact of both the normals are as follows:

$$\left(\pm\frac{a^2}{\sqrt{a^2 + b^2m^2}}, \pm\frac{mb^2}{\sqrt{a^2 + b^2m^2}}\right).$$

- **Pair of tangents:** The combined equation of the pair of tangents from an external point $P(x_1, y_1)$ to the ellipse $S = 0$ is $T^2 = SS_1$.
- **Chord with a given midpoint:** The equation of a chord of ellipse $S = 0$, whose midpoint is $P(x_1, y_1)$, is $T = S_1$.

Mind Map



Self-Assessment

1. Find the equation of normal to the ellipse $9x^2 + 16y^2 = 144$ at the end of the latus rectum in the first quadrant.

2. What is the condition that the line, $x \cos \alpha + y \sin \alpha = p$, may be a normal to the ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$?

3. If the normal to the ellipse $3x^2 + 4y^2 = 12$ at a point P on it is parallel to the line $2x + y = 4$ and the tangent to the ellipse at P passes through $Q(4, 4)$ then PQ is equal to

(a) $\frac{5\sqrt{5}}{2}$ (b) $\frac{\sqrt{61}}{2}$ (c) $\frac{\sqrt{221}}{2}$ (d) $\frac{\sqrt{157}}{2}$

Answers

Self-Assessment 1

Step 1

Given,

Ellipse is $9x^2 + 16y^2 = 144$

I.e., $\frac{x^2}{16} + \frac{y^2}{9} - 1 = 0$ (Comparing with $S = 0, a^2 = 16, b^2 = 9$)

$$e = \sqrt{\frac{a^2 - b^2}{a^2}} = \sqrt{\frac{16 - 9}{16}} = \frac{\sqrt{7}}{4}$$

Step 2

The end of the latus rectum in the first quadrant is $P\left(ae, \frac{b^2}{a}\right) = \left(\sqrt{7}, \frac{9}{4}\right)$

Equation of normal at P is $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$

I.e., $\frac{16x}{\sqrt{7}} - \frac{9y}{\frac{9}{4}} = 16 - 9 \Rightarrow 16x - 4\sqrt{7}y = 7\sqrt{7}$

Self-Assessment 2

Step 1

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ & $P(a \cos \theta, b \sin \theta)$

Tangent at $P \Rightarrow T_p = \frac{x(a \cos \theta)}{a^2} + \frac{y(b \sin \theta)}{b^2} = 1 \Rightarrow \frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1$

$bx \cos \theta + ay \sin \theta = ab$

$\Rightarrow m_T = \frac{-b \cos \theta}{a \sin \theta} \Rightarrow m_N = \frac{a \sin \theta}{b \cos \theta}$

Step 2

Normal at P $\Rightarrow (y - b\sin\theta) = \dfrac{a\sin\theta}{b\cos\theta}(x - a\cos\theta) \Rightarrow ax\sin\theta - by\cos\theta = (a^2 - b^2)\sin\theta\cos\theta$

Compare with equation of normal $x\cos\alpha + y\sin\alpha = p$

$\Rightarrow a\sin\theta = \cos\alpha \Rightarrow \sin\theta = \dfrac{\cos\alpha}{a}$

And $-b\cos\theta = \sin\alpha \Rightarrow \cos\theta = \dfrac{-\sin\alpha}{b}$ and $p = (a^2 - b^2)\sin\theta\cos\theta$

Step 3

We know that $\sin^2\theta + \cos^2\theta = 1$

$\Rightarrow \left(\dfrac{\cos\alpha}{a}\right)^2 + \left(\dfrac{-\sin\alpha}{b}\right)^2 = 1 \Rightarrow b^2\cos^2\alpha + a^2\sin^2\alpha = a^2 b^2 \dots \text{(i)}$

$p = (a^2 - b^2)\sin\theta\cos\theta = (a^2 - b^2)\left(\dfrac{\cos\alpha}{a}\right)\left(\dfrac{-\sin\alpha}{b}\right)$

$p^2 a^2 b^2 = (a^2 - b^2)^2\cos^2\alpha\sin^2\alpha \dots \text{(ii)}$

From (i) and (ii)

$p^2(a^2\sec^2\alpha + b^2\operatorname{cosec}^2\alpha) = (a^2 - b^2)^2$

Self-Assessment 3

Equation of given ellipse is $3x^2 + 4y^2 = 12 \Rightarrow \dfrac{x^2}{4} + \dfrac{y^2}{3} = 1 \quad \text{.........(i)}$

Now, let point $P(2\cos\theta, \sqrt{3}\sin\theta)$, so equation of tangent to ellipse (i) at point P is

$\dfrac{x\cos\theta}{2} + \dfrac{y\sin\theta}{\sqrt{3}} = 1 \quad \text{.........(ii)}$

Since, tangent (ii) passes through point $Q(4, 4)$

Therefore, $2\cos\theta + \dfrac{4}{\sqrt{3}}\sin\theta = 1 \quad \text{........(iii)}$

And equation of normal to ellipse (i) at point P is

$\dfrac{4x}{2\cos\theta} - \dfrac{3y}{\sqrt{3}\sin\theta} = 4 - 3 \Rightarrow 2x\sin\theta - \sqrt{3}y\cos\theta = \sin\theta\cos\theta \quad \text{.......(iv)}$

Since, normal (iv) is parallel to line, $2x + y = 4$

Therefore slope of normal (iv) = slope of line, $2x + y = 4$

$\Rightarrow \dfrac{2}{\sqrt{3}}\tan\theta = -2 \Rightarrow \tan\theta = -\sqrt{3} \Rightarrow \theta = 120° \Rightarrow (\sin\theta, \cos\theta) = \left(\dfrac{\sqrt{3}}{2}, \dfrac{-1}{2}\right)$

Hence point $P\left(-1, \dfrac{3}{2}\right)$

Now, $PQ = \sqrt{(4+1)^2 + (4 - \frac{3}{2})^2}$ [given coordinates of $Q = (4, 4)$]

$= \sqrt{25 + \dfrac{25}{4}} = \dfrac{5\sqrt{5}}{2}$

Option (a) is the correct answer.

MATHEMATICS

ELLIPSE

SOLVING PROBLEMS OF AN ELLIPSE AND GEOMETRICAL PROPERTIES OF AN ELLIPSE

NOTES

What you already know

- Parameters associated with an ellipse
- Horizontal and vertical standard equation of an ellipse
- Translation or shifting of an ellipse
- Condition and equation of tangent
- Auxiliary and Director circle
- Equation of normal in various forms
- Equation of pair of tangents
- Chord of contact and its equation

What you will learn

- Geometrical properties of ellipse

Concept Check 1

The chord of contact of all the points on the line $x - y - 5 = 0$ with respect to the ellipse $x^2 + 4y^2 = 4$ always passes through a fixed point _____ .

The locus of the middle point of chord of the ellipse $\frac{x^2}{16}+\frac{y^2}{25}=1$ passing through Q(0, 5) is an ellipse whose centre is _____ .

Solution

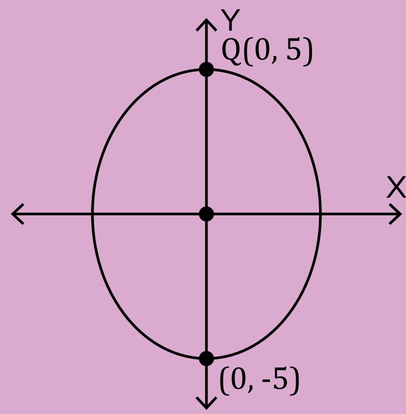
Step 1

Given, ellipse S: $\frac{x^2}{16}+\frac{y^2}{25}-1=0$

$a^2 = 16, b^2 = 25 \quad (a < b)$

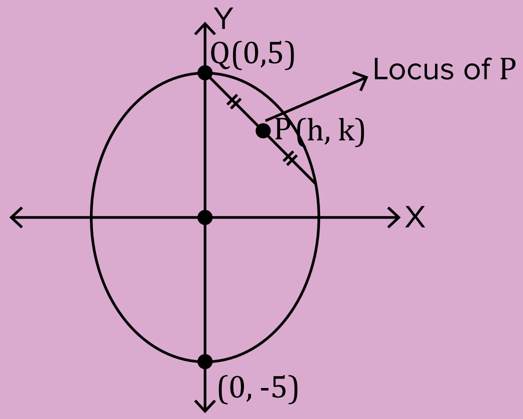
⇒ The major axis of the ellipse is the y-axis.

We need to figure out the locus of the middle point of chord passing through Q(0, 5).



Step 2

To understand this better, consider a chord passing through $Q(0, 5)$. Assume the middle point of the chord as $P(h, k)$. Therefore, we just need to have the locus of this point P.



Step 3

If the middle point is known for any chord of an ellipse, then the equation can be easily written as:

$T = S_1$

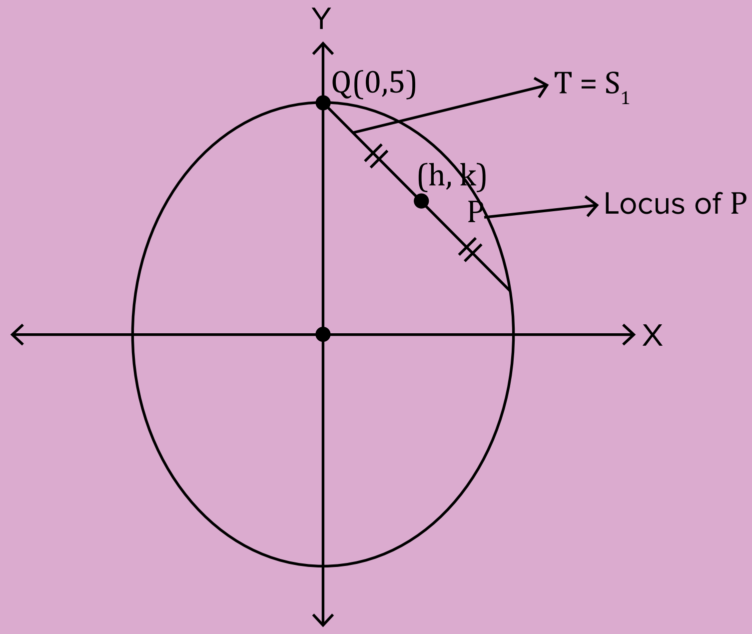
S: $\frac{x^2}{16} + \frac{y^2}{25} - 1$

T: $\frac{xh}{16} + \frac{yk}{25} - 1$

S_1: $\frac{h^2}{16} + \frac{k^2}{25} - 1$

Therefore, the equation of chord with middle point $P(h, k)$ is:

$$\frac{xh}{16} + \frac{yk}{25} - 1 = \frac{h^2}{16} + \frac{k^2}{25} - 1$$



Step 4

This chord passes through the point $Q(0, 5)$. Substituting 0 in place of x and 5 in place of y, we get,

$$\frac{(0)h}{16} + \frac{(5)k}{25} - 1 = \frac{h^2}{16} + \frac{k^2}{25} - 1$$

$$0 + \frac{k}{5} = \frac{h^2}{16} + \frac{k^2}{25}$$

Now, to get the required locus of point P, replace h with x and k with y.

$$\frac{y}{5} = \frac{x^2}{16} + \frac{y^2}{25}$$

Upon simplifying, we have,

$25x^2 + 16y^2 - 80y = 0$

To confirm that this is an ellipse, we need to refer to the condition for the second degree equation for a conic.

$25x^2 + 16y^2 + 0xy + 0x - 80y + 0 = 0$

For this to be an ellipse, $h^2 < ab$

I.e., $0 < (25)(16)$

Yes! This is in fact an ellipse.

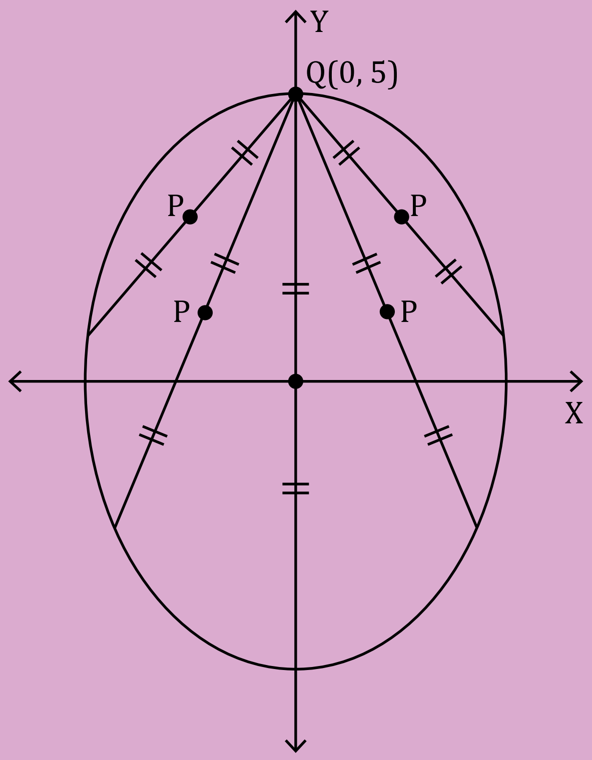
However, we need to find its centre.

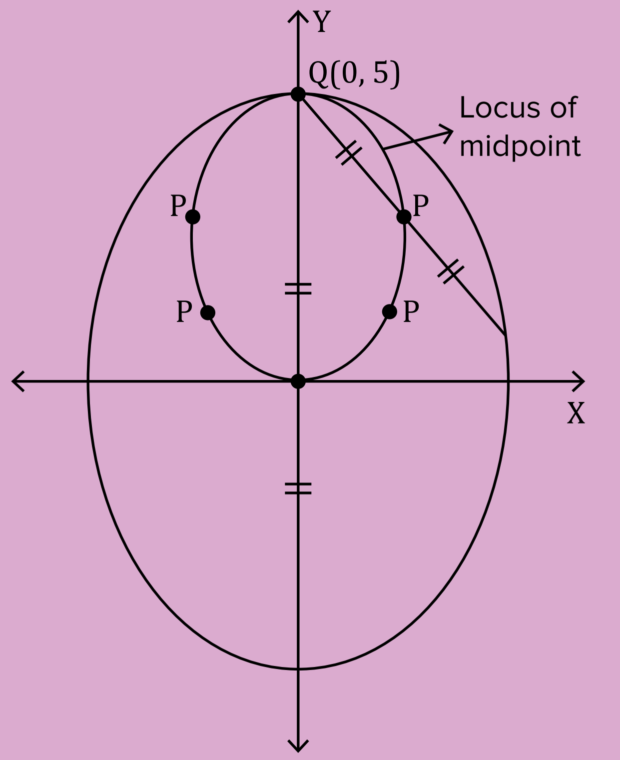
$25x^2 + 16y^2 - 80y = 0$

$$25x^2 + 16\left(y^2 - 5y + \frac{25}{4}\right) = 100$$

$$\frac{x^2}{4} + \frac{\left(y - \frac{5}{2}\right)^2}{\frac{25}{4}} = 1$$

This is an ellipse having centre $\left(0, \frac{5}{2}\right)$



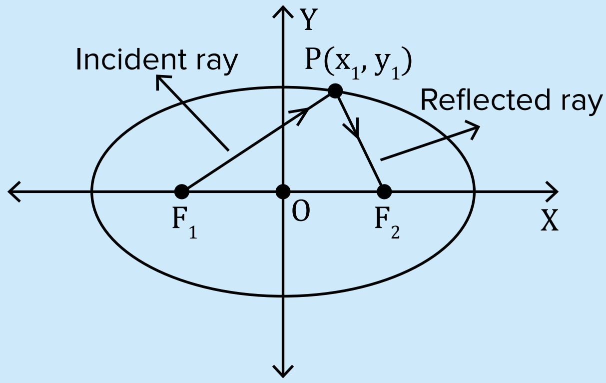


Geometrical Properties of Ellipse

1. The incident ray from focus F_1 passes through other focus F_2 after the reflection by the ellipse at point P.

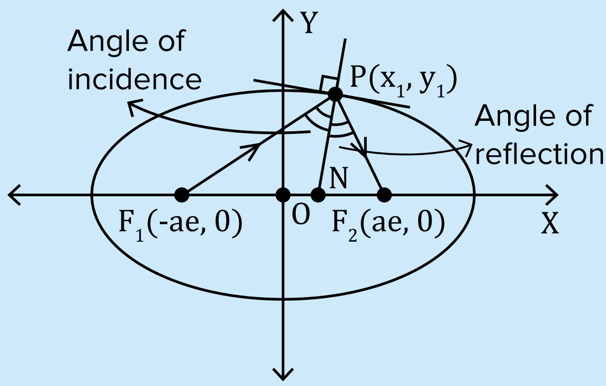
Proof

Let the ray of light be emanating from F_1.
After meeting the elliptical curve at a point, say $P(x_1, y_1)$, it gets reflected and we have to prove that it passes through F_2.



Consider the normal to the ellipse at point P. Let us say that it is intersecting the x-axis at N.
To prove the property, we have to show that the angle of incidence is equal to the angle of reflection.
I.e; $\angle F_1PN = \angle NPF_2$
Mathematically,
In ΔPF_1F_2, PN is the internal angle bisector of $\angle P$.
To show this, we have to prove the angle bisector theorem, i.e.,

$$\frac{(F_1N)}{(F_2N)} = \frac{(F_1P)}{(F_2P)}$$



Consider the standard horizontal ellipse,

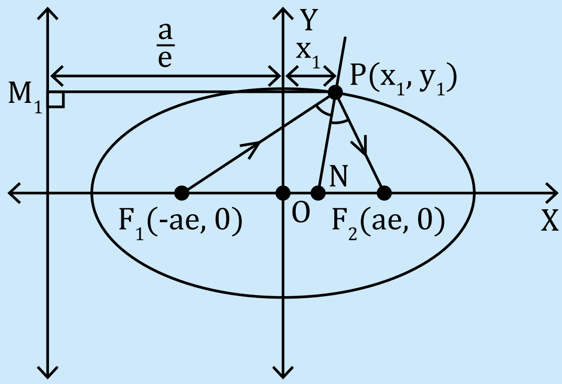
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$

The coordinates of foci are given by $(\pm ae, 0)$.

The focal distance of P from focus F_1 is equal to e times of PM_1.

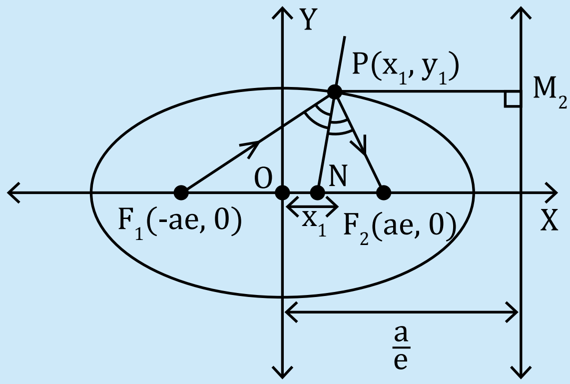
I.e., $PF_1 = ePM_1$

$\Rightarrow PF_1 = e\left[\frac{a}{e} + x_1\right] = a + ex_1$



Now, the focal distance of P from focus F_2 is equal to e times of PM_2.

I.e., $PF_2 = e[(\frac{a}{e} - x_1)] = a - ex_1$



The normal drawn at point P intersects the x-axis at N.

Equation of the normal NP in point form is given by,

$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$

At point N, $y = 0$

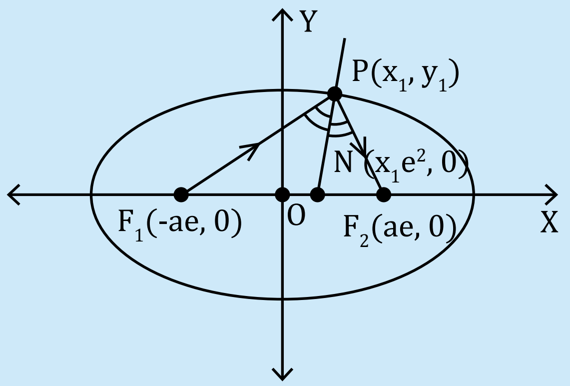
$\Rightarrow \frac{a^2x}{x_1} = a^2 - b^2$

$\Rightarrow x = \frac{(a^2 - b^2)x_1}{a^2}$

$\Rightarrow x = x_1\left(1 - \frac{b^2}{a^2}\right)$

$\Rightarrow x = x_1 e^2$

Therefore, $N(x_1e^2, 0)$.



We have the coordinates of points F_1, F_2, and N as:

$F_1(-ae, 0))$, $N(x_1e^2, 0)$, $F_2(ae, 0)$

$\Rightarrow F_1N = ae + x_1e^2 = e(a + ex_1)$

Recollecting the fact that $F_1P = (a + ex_1)$

$\Rightarrow F_1N = eF_1P$ .....(i)

And

$F_2N = ae - x_1e^2 = e(a - ex_1)$

$\Rightarrow F_2N = e\,F_2P$ ....(ii)

Dividing equations (i) and (ii), we get,

$\Rightarrow \frac{(F_1N)}{(F_2N)} = \frac{(F_1P)}{(F_2P)}$ $\Rightarrow$ PN bisects $\angle F_1PF_2$.

∴ The incident ray from focus F_1 passes through other focus F_2 after the reflection by the ellipse at point P.

Hence proved.

Solve

A ray emanating from point $(0, 6)$ is incident on the ellipse $25x^2 + 16y^2 = 1600$ at point P with ordinate 5. After the reflection, the ray cuts the y-axis at S'. Find the length of PS'.

(a) 7 (b) 13 (c) 5 (d) None of these

Solution

Step 1

Given, ellipse:

$25x^2 + 16y^2 = 1600$

Dividing both the sides by 1600, we get,

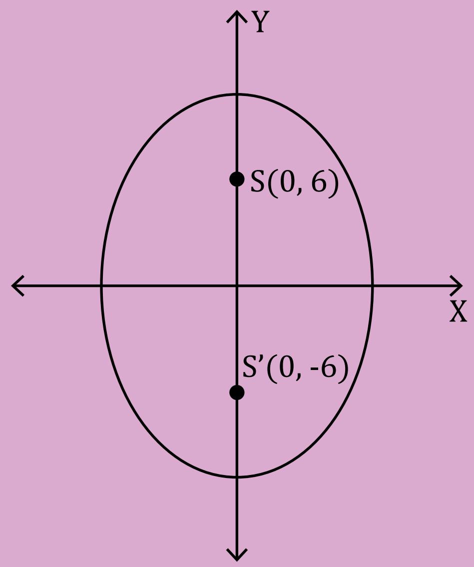
$\frac{x^2}{64} + \frac{y^2}{100} = 1$

$a^2 = 64, b^2 = 100$

$\Rightarrow a = 8, b = 10$

$\therefore b > a$

Therefore, the ellipse has its centre at the origin and the major axis along the y-axis. The foci lie along the y-axis.



Step 2

Let us name the foci as S and S'. The coordinates of the foci for the standard vertical ellipse are given by $(0, \pm be)$,

where $e = \sqrt{\left[1 - \left(\frac{a^2}{b^2}\right)\right]}$

$e = \sqrt{\left[1 - \left(\frac{64}{100}\right)\right]} = \sqrt{\frac{36}{100}} = \frac{6}{10}$

$\Rightarrow$ Foci $\equiv (0, \pm be) = \left(0, \pm 10\left(\frac{6}{10}\right)\right) = (0, \pm 6)$

Hence, the given points S and S' are the two foci.

Step 3

P is the point with ordinate 5. It can be written as $(x_1, 5)$.

As P lies on the ellipse, it satisfies the equation of the ellipse.

$\Rightarrow \frac{(x_1)^2}{64} + \frac{5^2}{100} = 1 \Rightarrow \frac{(x_1)^2}{64} + \frac{1}{4} = 1$

$\Rightarrow \frac{(x_1)^2}{64} = \frac{3}{4} \Rightarrow (x_1)^2 = 48$

$\Rightarrow x_1 = \pm 4\sqrt{3}$

∴ The coordinates of P are $(\pm 4\sqrt{3}, 5)$.
The incident ray from focus S passes through other focus S' after the reflection by the ellipse at point P.
By distance formula,

$d = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2]}$

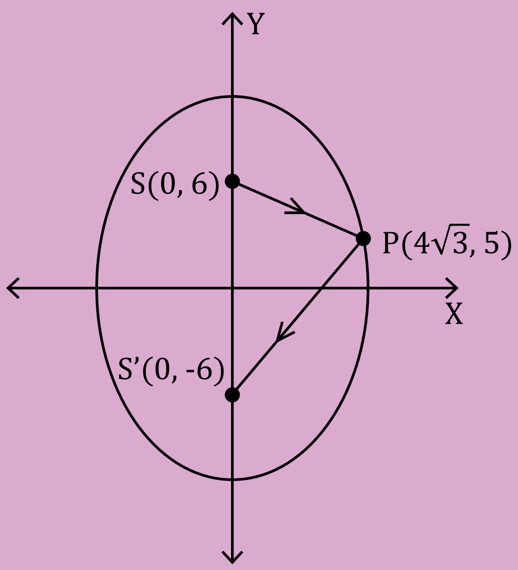
$\Rightarrow PS' = \sqrt{[(0 - 4\sqrt{3})^2 + (-6 - 5)^2]}$

$\Rightarrow PS' = \sqrt{[48 + 121]}$

$\Rightarrow PS' = \sqrt{169}$

$\Rightarrow PS' = 13$ units

∴ Option (b) is the correct answer



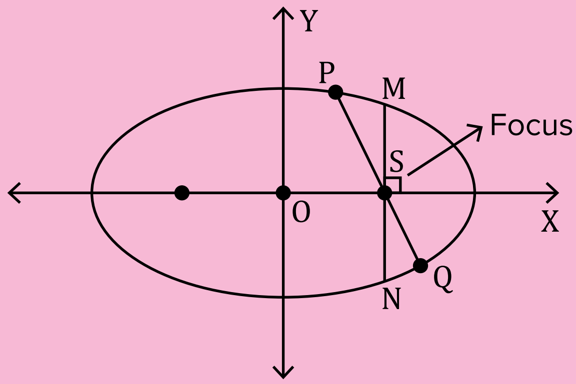
2. **If PSQ is a focal chord of the ellipse $S = 0$, then $\frac{1}{SP} + \frac{1}{SQ} = \frac{2a}{b^2}$ or the semi-latus rectum is the H.M. of SP and SQ.**

Proof

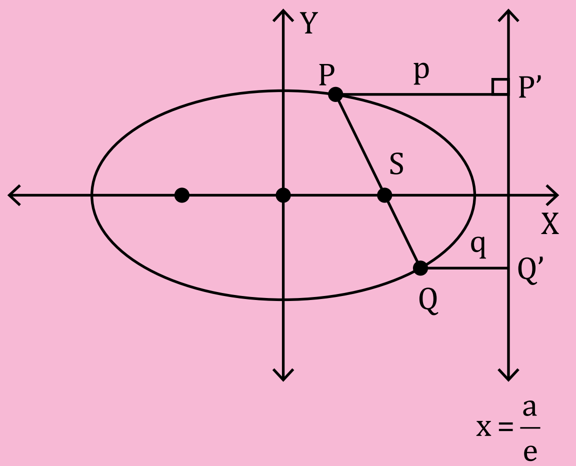
Given, an ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

A focal chord PQ is drawn that passes through focus S.
This focus S divides the chord PQ into two line segments SP and SQ.
We need to prove that the semi-latus rectum SM is the H.M of SP and SQ.

I.e., $\frac{1}{SP} + \frac{1}{SQ} = \frac{2a}{b^2}$



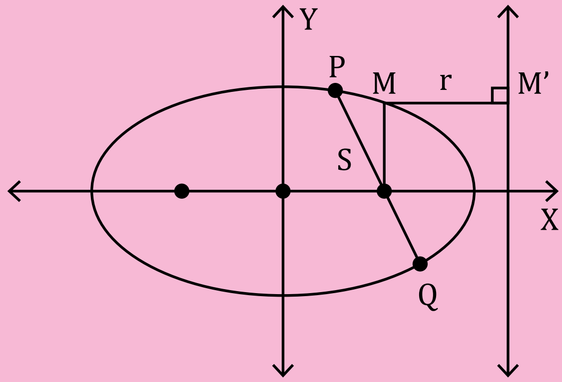
To establish this relation, let us make use of the basic conic condition.
Let us say that the foot of the perpendicular from P onto the directrix is P'.
$PS = ePP'$ or $PS = e \times p$
Similarly, $SQ = eQQ'$ or $SQ = e \times q$



The semi-latus rectum (SM) that is $\frac{b^2}{a}$ can be written as eMM'or $e \times r$.

$SM = e \times r$

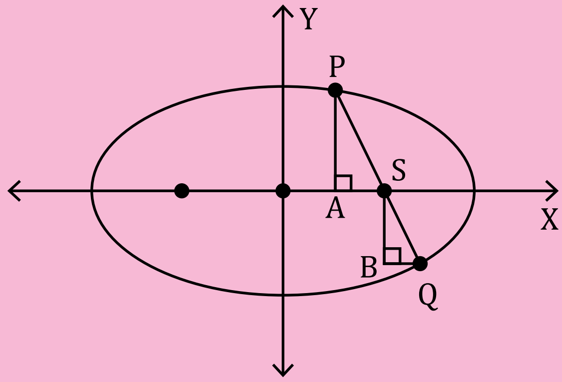
Where M' is the foot of perpendicular from M to the directrix.



As the lengths of the segments SP, SQ, and SM are known, we can proceed to prove the property. However, we need a bit of mathematical manipulation.

I.e., drop a perpendicular from P onto the x-axis so that a right-angled triangle, PAS, is formed.

Similarly, form a right-angled triangle SBQ by dropping a vertical line from S and joining a horizontal line from Q.



Now, these two triangles PAS and SBQ are similar triangles and therefore, the ratio of the corresponding sides will be equal.

I.e., $\frac{AS}{PS} = \frac{BQ}{SQ}$

$\Rightarrow \frac{p - r}{ep} = \frac{r - q}{eq}$..

Upon simplifying, we get,

$$r = \frac{2pq}{p + q}$$

Multiplying p, q, and r by e, we get,

$$(e \times r) = \frac{2(e \times p)(e \times q)}{(e \times p) + (e \times q)}$$

We already know,

$SM = e \times r, SP = e \times p, SQ = e \times q$

Therefore,

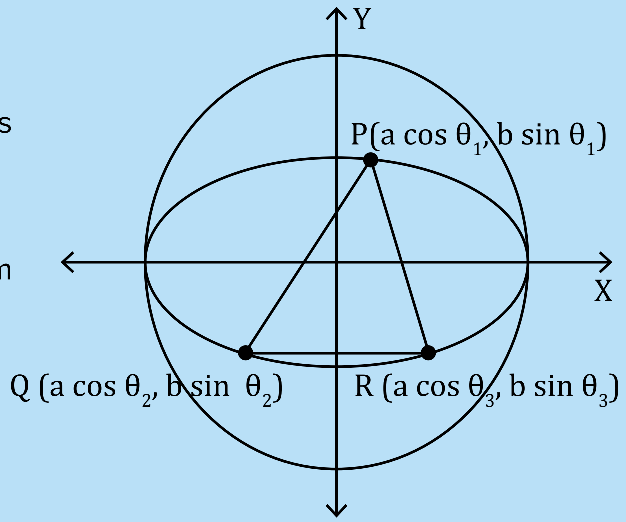
$$SM = \frac{2SP \cdot SQ}{SP + SQ}$$

SM, the semi-latus rectum, is the harmonic mean of SP and SQ.

Hence proved.

3. The ratio of area of any triangle inscribed in an ellipse to the area of a triangle formed by corresponding points on the auxiliary circle is equal to the ratio of semi-minor axis to the semi-major axis.

Proof



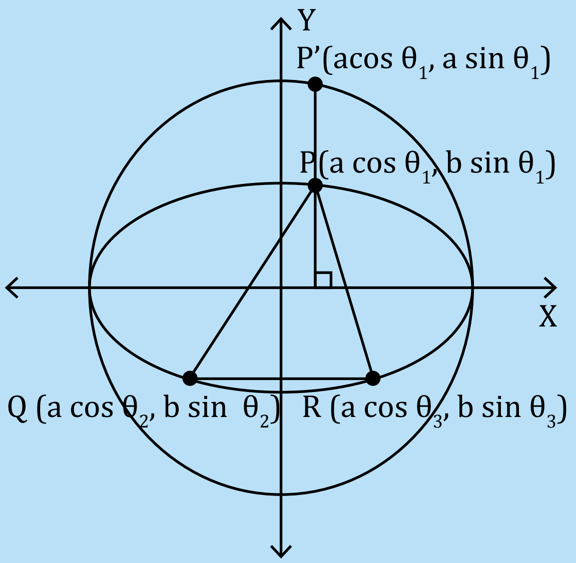
Consider a standard ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with the x-axis as the major axis and its auxiliary circle as $x^2 + y^2 = a^2$

Let us randomly pick three points on the ellipse to form a triangle PQR and let the area be represented as Δ_1

To determine the area, we need the vertices.
Say,
$P \equiv (a\cos\theta_1, b\sin\theta_1)$
$Q \equiv (a\cos\theta_2, b\sin\theta_2)$
$R \equiv (a\cos\theta_3, b\sin\theta_3)$

Now,

$$\Delta_1 = \frac{1}{2}\begin{vmatrix} a\cos\theta_1 & b\sin\theta_1 & 1 \\ a\cos\theta_2 & b\sin\theta_2 & 1 \\ a\cos\theta_3 & b\sin\theta_3 & 1 \end{vmatrix}$$



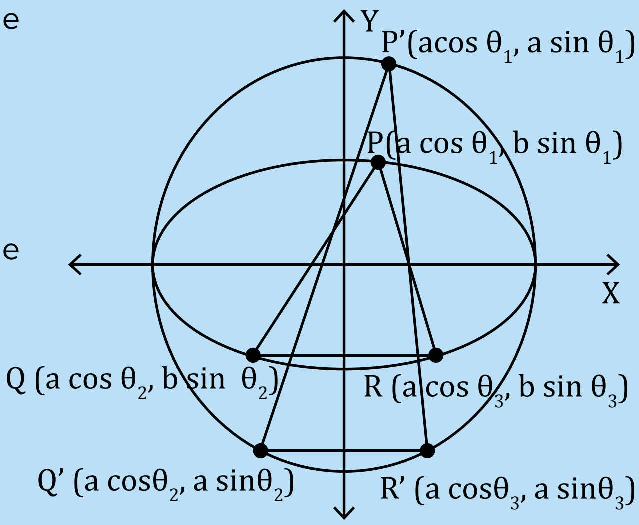
Now, to have the triangle formed by corresponding points on the auxiliary circle, we need those points on the auxiliary circle corresponding to the points P, Q, and R.
In order to do that, let us drop a perpendicular from P onto the x-axis and extend it vertically to meet the auxiliary circle at P'.
Therefore, $P' \equiv (a\cos\theta_1, a\sin\theta_1)$

Now, simultaneously drop the perpendiculars from Q and R onto the x-axis, and extend them to meet the auxiliary circle at Q' and R'.

$Q' \equiv (a\cos\theta_2, a\sin\theta_2)$
$R' \equiv (a\cos\theta_3, a\sin\theta_3)$

Now, joining the points P', Q', and R' gives us the required triangle and let us call its area as Δ_2.

$$\Delta_2 = \frac{1}{2}\begin{vmatrix} a\cos\theta_1 & a\sin\theta_1 & 1 \\ a\cos\theta_2 & a\sin\theta_2 & 1 \\ a\cos\theta_3 & a\sin\theta_3 & 1 \end{vmatrix}$$



Using the properties of determinants,

$$\Delta_1 = \frac{1}{2}\begin{vmatrix} a\cos\theta_1 & b\sin\theta_1 & 1 \\ a\cos\theta_2 & b\sin\theta_2 & 1 \\ a\cos\theta_3 & b\sin\theta_3 & 1 \end{vmatrix} = \frac{ab}{2}\begin{vmatrix} \cos\theta_1 & \sin\theta_1 & 1 \\ \cos\theta_2 & \sin\theta_2 & 1 \\ \cos\theta_3 & \sin\theta_3 & 1 \end{vmatrix}$$

$$\Delta_2 = \frac{1}{2}\begin{vmatrix} a\cos\theta_1 & a\sin\theta_1 & 1 \\ a\cos\theta_2 & a\sin\theta_2 & 1 \\ a\cos\theta_3 & a\sin\theta_3 & 1 \end{vmatrix} = \frac{a^2}{2}\begin{vmatrix} \cos\theta_1 & \sin\theta_1 & 1 \\ \cos\theta_2 & \sin\theta_2 & 1 \\ \cos\theta_3 & \sin\theta_3 & 1 \end{vmatrix}$$

Therefore, $\frac{\Delta_1}{\Delta_2} = \frac{b}{a}$

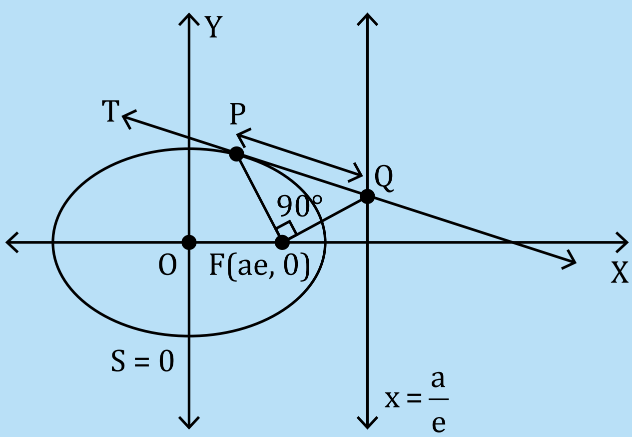
Hence proved.

4. The portion of the tangent to an ellipse between the point of contact and the directrix subtends a right angle at the corresponding focus, i.e., ∠PFQ = 90°

Consider a standard ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Let P(a cosθ, b sinθ) be the point at which the tangent is drawn and let it intersect the directrix at a point Q.
The corresponding focus is F(ae, 0).

We need to prove that the angle PFQ is 90°.



Equation of tangent at P,

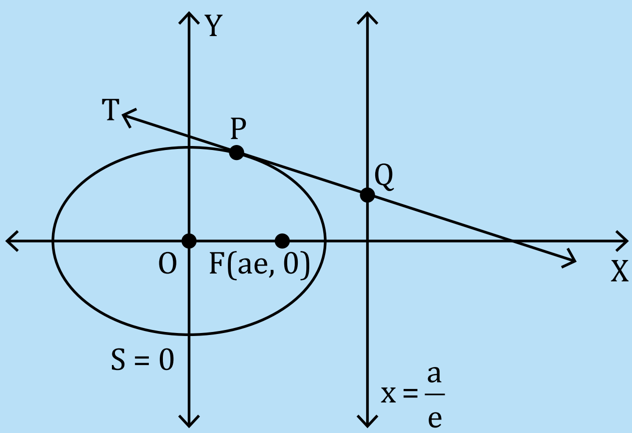
$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$$

Q is the point of intersection of the tangent and the directrix, $x = \frac{a}{e}$

Therefore,

$$\frac{\frac{a}{e}\cos\theta}{a} + \frac{y\sin\theta}{b} = 1 \Rightarrow y = \frac{b(e - \cos\theta)}{e\sin\theta}$$

$$Q \equiv \left(\frac{a}{e}, \frac{b(e - \cos\theta)}{e\sin\theta}\right)$$



We know the coordinates of the points P(a cosθ, b sinθ), F(ae, 0).

$$Q \equiv \left(\frac{a}{e}, \frac{b(e-\cos\theta)}{e\sin\theta}\right)$$

Slope of PF,

$$m_{PF} = \frac{b\sin\theta}{a(\cos\theta - e)}$$

Slope of FQ,

$$m_{FQ} = \frac{\frac{b(e-\cos\theta)}{e\sin\theta} - 0}{\frac{a}{e} - ae} = \frac{b(e-\cos\theta)}{a(1-e^2)\sin\theta}$$

Using the relation $b^2 = a^2(1 - e^2)$, we get the product of the slopes of PF and FQ as -1.

$$m_{PF} \cdot m_{FQ} = \frac{b\sin\theta}{a(\cos\theta - e)} \cdot \frac{b(e-\cos\theta)}{a(1-e^2)\sin\theta} = -1$$

Therefore, the portion of the tangent to an ellipse between the point of contact and the directrix subtends a right angle at the corresponding focus.

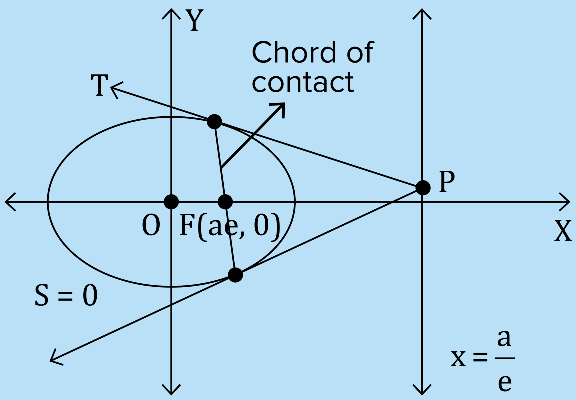
Hence proved.

5. Chord of contact of any point on the directrix passes through the corresponding focus.

Consider a standard ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Let us take a random point P on the directrix $x = \frac{a}{e}$. As P lies on the directrix, its abscissa will be $\frac{a}{e}$, therefore, $P \equiv \left(\frac{a}{e}, y_1\right)$. Chord of contact of any point P can be written using the equation T = 0

$$\frac{x.\frac{a}{e}}{a^2} + \frac{yy_1}{b^2} - 1 = 0$$



$$\Rightarrow \frac{x}{ae} + \frac{yy_1}{b^2} = 1$$

This is the chord of contact at any point on the directrix. We now have to check whether it passes through the corresponding focus F which is (ae, 0).

Upon substituting F(ae, 0) in the equation, we get,

$$\frac{ae}{ae} + \frac{y(0)}{b^2} = 1$$

$$\Rightarrow 1 = 1$$

Hence proved

Consider an ellipse E with its centre as C(1, 3), focus at S(6, 3), which passes through the point P(4, 7),

(i) Find the product of the lengths of the perpendicular segments from the foci on tangent at point P.

(a) 20 (b) 45 (c) 40 (d) Cannot be determined

(ii) If the normal at a variable point on the ellipse E meets its axes in Q and R, then the locus of the midpoint of QR is a conic with an eccentricity (e') is.

(a) $\frac{3}{\sqrt{10}}$ (b) $\frac{\sqrt{5}}{3}$ (c) $\frac{3}{\sqrt{5}}$ (d) $\frac{\sqrt{10}}{3}$

Solution

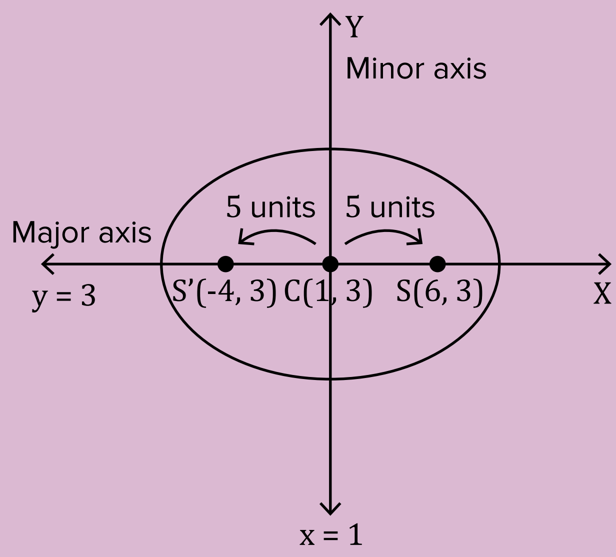
(i)

Given, centre C(1, 3), focus at S(6, 3)

Major axis is the line joining centre and foci. This gives us the equation of major axis as y = 3

Equation of the minor axis which is perpendicular to the major axis is x = 1

The centre of the ellipse is the midpoint of the foci. The distance between focus S(6, 3) and centre C (1, 3) is 5 units along the major axis. Focus S' will be at a distance 5 units to the left of C along the major axis. So, coordinates of focus S' will be (-4, 3).

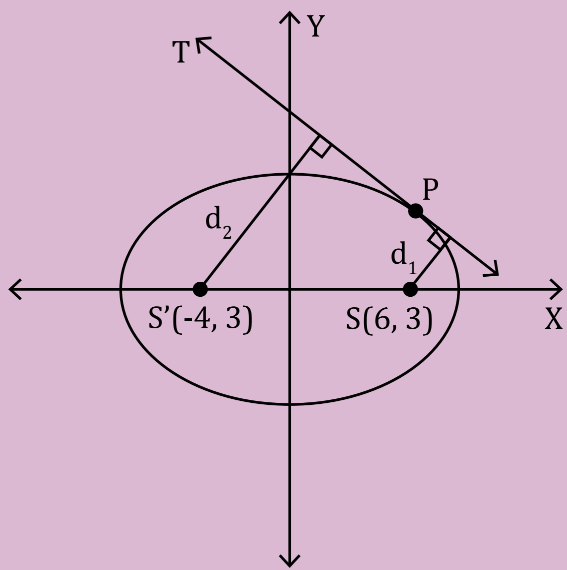


Let d_1 and d_2 be the perpendiculars from the foci S and S' to the tangent at P(4, 7).

Recall the property which says that the product of the lengths of the perpendicular segments from the foci onto the tangent at any point is equal to the square of the semi-minor axis.

I.e., $d_1.d_2 = b^2$

Now, everything depends upon the value of b, the length of the semi-minor axis.



To figure out the value of b, we need to know the values of a and e.
Utilising the basic property of ellipse, the sum of the focal distances of any point on an ellipse is the length of the major axis.
Therefore, PS + PS' = 2a

$$PS' = \sqrt{\left((4+4)^2 + (7-3)^2\right)} = 4\sqrt{5}$$

$$PS = \sqrt{\left((4-6)^2 + (7-3)^2\right)} = 2\sqrt{5}$$

$$PS + PS' = 6\sqrt{5} = 2a$$

$$a = 3\sqrt{5}$$

Also, the distance between the two foci in any ellipse is 2ae.
We have, SS' = 10

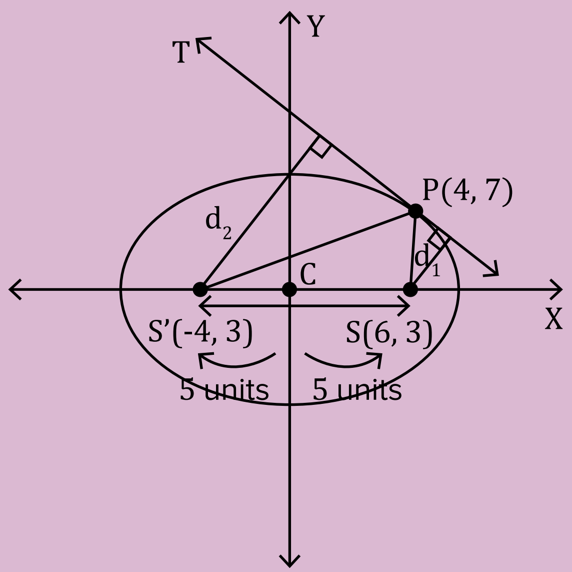
$$\Rightarrow 2ae = 10 \Rightarrow ae = 5 \Rightarrow e = \frac{\sqrt{5}}{3}$$

Also, $b^2 = a^2(1 - e^2)$

$$b^2 = \left(3\sqrt{5}\right)^2 \cdot \left(1 - \left(\frac{\sqrt{5}}{3}\right)^2\right) \Rightarrow b^2 = 45\left(1 - \frac{5}{9}\right)$$

$$b^2 = 45\left(\frac{4}{9}\right) \Rightarrow b^2 = 20$$

Hence, option(a) is the correct answer

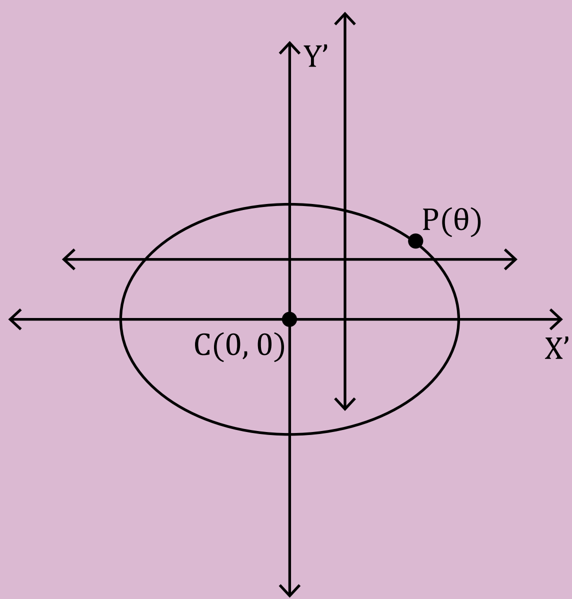


(ii)
Given, an ellipse with centre C ≡ (1, 3),
Foci S ≡ (6, 3), S' ≡ (-4, 3)
We also know, $a^2 = 45$, $b^2 = 20$
Equation of the ellipse can be written as:

$$\frac{(x-1)^2}{45} + \frac{(y-3)^2}{20} = 1$$

Next step is to have the equation of normal at any point P on the ellipse so that points Q and R can be found.
To ease the process, let us translate the ellipse so that its centre is the origin and the major axis is the x-axis.
Let $x - 1 = x'$ and $y - 3 = y'$
So, the ellipse is transformed to

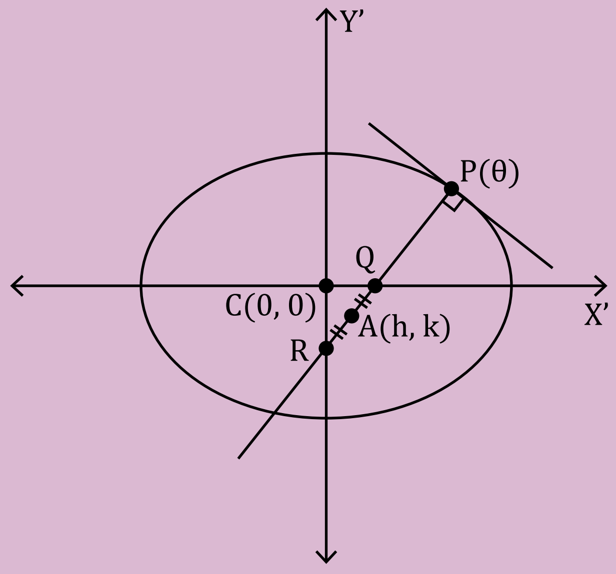
$$\frac{(x')^2}{45} + \frac{(y')^2}{20} = 1$$



Equation of normal at any point $P(\theta)$

$$\frac{ax'}{\cos\theta}-\frac{by'}{\sin\theta}=a^2-b^2$$

Q is the point where the normal intersects the x-axis.



$$\frac{ax'}{\cos\theta}-\frac{b(0)}{\sin\theta}=a^2-b^2$$

$$x'=\frac{(a^2-b^2)\cos\theta}{a}$$

$$Q \text{ is } \left(\frac{(a^2-b^2)\cos\theta}{a},0\right)$$

R is the point where the normal intersects the y - axis.

$$\frac{a(0)}{\cos\theta}-\frac{by'}{\sin\theta}=a^2-b^2$$

$$y'=-\frac{(a^2-b^2)\sin\theta}{b}$$

$$R \text{ is } \left(0,-\frac{(a^2-b^2)\sin\theta}{b}\right)$$

So, the midpoint of QR is:

$$\left(\frac{(a^2-b^2)\cos\theta}{2a},-\frac{(a^2-b^2)\sin\theta}{2b}\right)$$

Let the locus of the midpoint of QR be $A=(h,k)$

So, $h=\frac{(a^2-b^2)\cos\theta}{2a}$,

$k=-\frac{(a^2-b^2)\sin\theta}{2b}$,

$\cos\theta=\frac{2ah}{(a^2-b^2)}$, and

$\sin\theta=-\frac{2bk}{(a^2-b^2)}$

Also, $(\cos\theta)^2+(\sin\theta)^2=1$

$$\left(\frac{2ah}{(a^2-b^2)}\right)^2+\left(-\frac{2bk}{(a^2-b^2)}\right)^2=1$$

$$\left(\frac{4a^2h^2}{(a^2-b^2)^2}\right)+\left(\frac{4b^2k^2}{(a^2-b^2)^2}\right)=1$$

Replacing (h,k) with (x',y'), we get,

$$\left(\frac{4a^2(x')^2}{(a^2-b^2)^2}\right)+\left(\frac{4b^2(y')^2}{(a^2-b^2)^2}\right)=1$$

$$\left(\frac{(x')^2}{\frac{(a^2-b^2)^2}{4a^2}}\right)+\left(\frac{(y')^2}{\frac{(a^2-b^2)^2}{4b^2}}\right)=1$$

Locus is an ellipse with equation

$$\left(\frac{(x')^2}{\frac{(a^2-b^2)^2}{4a^2}}\right)+\left(\frac{(y')^2}{\frac{(a^2-b^2)^2}{4b^2}}\right)=1$$

Let $A^2=\frac{(a^2-b^2)^2}{4a^2}$ and $B^2=\frac{(a^2-b^2)^2}{4b^2}$

So, the equation of ellipse becomes

$$\left(\frac{(x')^2}{A^2}\right)+\left(\frac{(y')^2}{B^2}\right)=1$$

Since $a^2 = 45$ and $b^2 = 20$,

$a^2 > b^2$

$\frac{1}{a^2}<\frac{1}{b^2}$

$\frac{1}{4a^2}<\frac{1}{4b^2}$

$\frac{(a^2-b^2)^2}{4a^2}<\frac{(a^2-b^2)^2}{4b^2}$

$A^2 < B^2$

Equation of ellipse is

$$\left(\frac{(x')^2}{A^2}\right)+\left(\frac{(y')^2}{B^2}\right)=1\ (B>A)$$

This is a vertical ellipse.

So, for eccentricity e',

$A^2=B^2\left(1-(e')^2\right)$

$e'=\sqrt{\left(1-\frac{A^2}{B^2}\right)}$

$e'=\sqrt{1-\frac{\frac{(a^2-b^2)^2}{4a^2}}{\frac{(a^2-b^2)^2}{4b^2}}}$

$e'=\sqrt{\left(1-\frac{4b^2}{4a^2}\right)}$

$e'=\sqrt{\left(1-\frac{b^2}{a^2}\right)}=e\,(\text{Orginal ellipse eccenticity})=\frac{\sqrt{5}}{3}$

So, option (b) is correct.

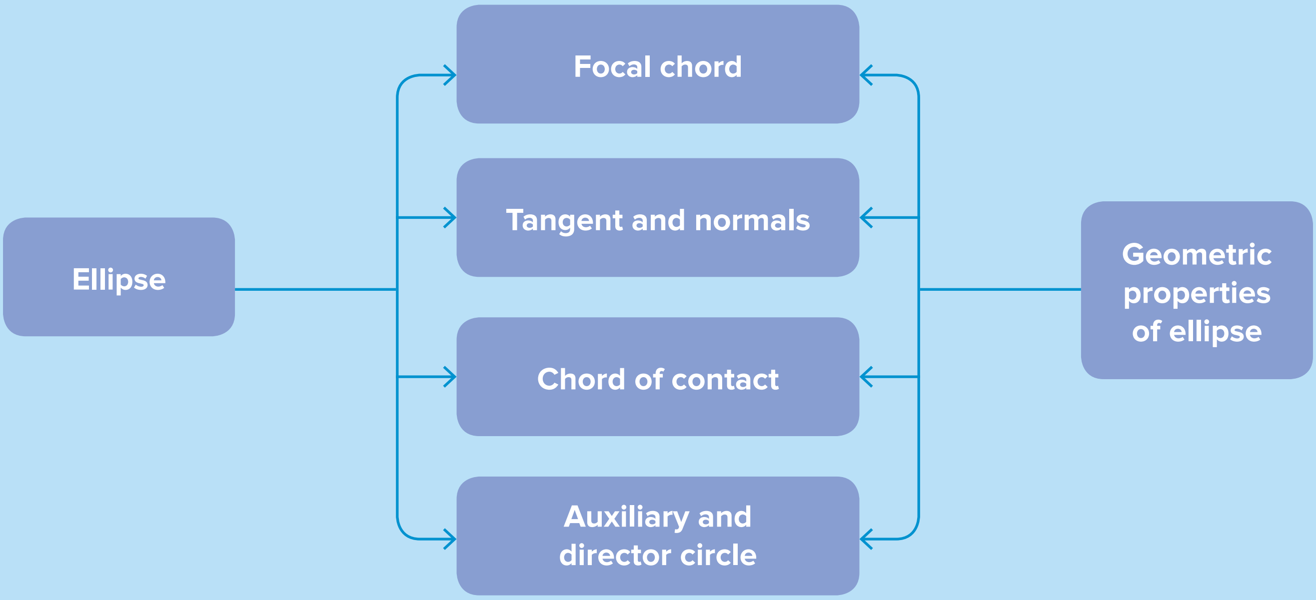
Summary sheet

Key Result

Geometrical properties of ellipse

- The incident ray from one focus S passes through the other focus S' after the reflection by the ellipse.
- If PSQ is a focal chord of the ellipse S = 0, then $\frac{1}{SP}+\frac{1}{SQ}=\frac{2a}{b^2}$ or the semi-latus rectum is the H.M. of SP and SQ.
- The ratio of area of any triangle inscribed in an ellipse to the area of a triangle formed by corresponding points on the auxiliary circle is equal to the ratio of semi-minor axis to the semi-major axis.
- The portion of the tangent to an ellipse between the point of contact and the directrix subtends a right angle at the corresponding focus.
- Chord of contact of any point on the directrix passes through the corresponding focus.

Mind map



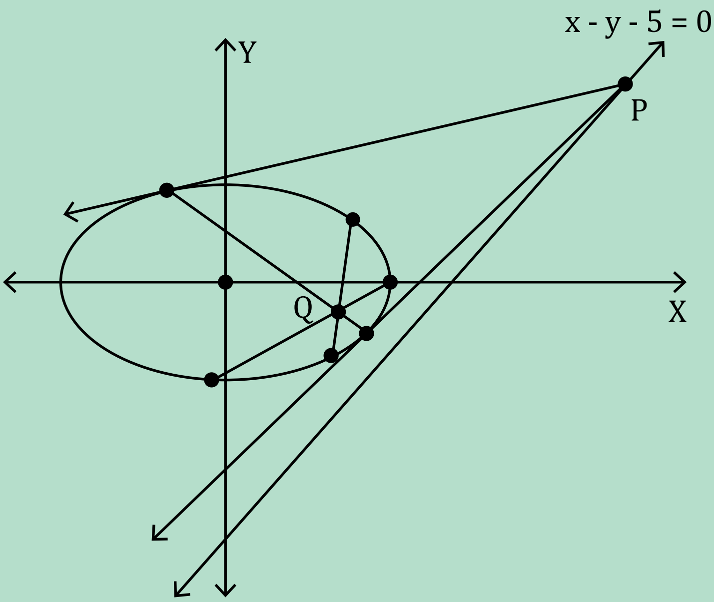
Self-Assessment

1. If the normal at the point $P\left(2, \frac{3\sqrt{3}}{2}\right)$ meets the major axis of ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ at Q and S and S' are the foci of the given ellipse, then find $SQ : S'Q$.

Answers

Concept Check 1

Step 1:
Given, ellipse: S: $x^2 + 4y^2 - 4 = 0$
Line: $x - y - 5 = 0$
This line is a non-intersecting line to the given ellipse. Hence, any point on this line will lie exterior to the ellipse. Therefore, from any point P on this given line, a pair of tangents can be drawn and respective chord of contact can be obtained.
The question says, all of these chords of contact are intersecting at a single point, let that point be Q.



Step 2:

Let $P(\lambda, \lambda - 5)$ be any point on line $x - y - 5 = 0$, $\lambda \in \mathbb{R}$

Equation of chord of contact of P is T = 0

$$\frac{x\lambda}{4} + \frac{y(\lambda - 5)}{1} = 1$$

$$\lambda x + 4\lambda y - 20y - 4 = 0$$

$$\underbrace{(-20y - 4)}_{L_1} + \lambda \underbrace{(x + 4y)}_{L_2} = 0.$$

$L_1 + \lambda L_2 = 0$

Point Q is the point of intersection of L_1 and L_2.

I.e., $(-20y - 4) = 0$

$(x + 4y) = 0$

Upon solving,

$$\Rightarrow x = \frac{4}{5} \text{ and } y = -\frac{1}{5}$$

Therefore, the point of intersection is

$$Q\left(\frac{4}{5}, -\frac{1}{5}\right).$$

Self-Assessment 1

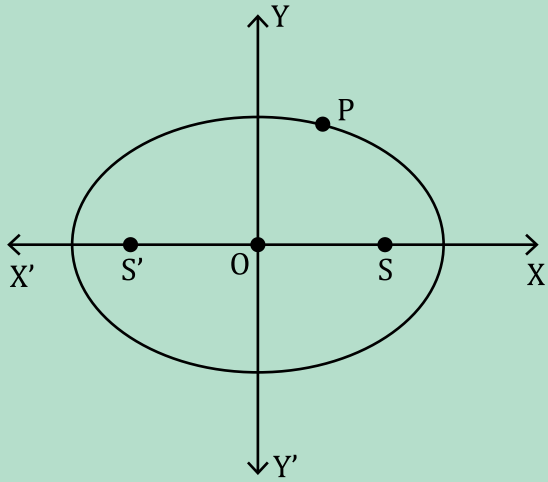
Given an ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ which is standard horizontal with $a^2 = 16$ & $b^2 = 9$ And $a^2(1 - e^2) = b^2$

$$\Rightarrow 16(1 - e^2) = 9 \Rightarrow e = \frac{\sqrt{7}}{4}$$

Therefore the foci S and S' are $(\pm ae, 0)$

$$S \equiv \left(4 \times \frac{\sqrt{7}}{4}, 0\right) = \left(\sqrt{7}, 0\right)$$

$$S' \equiv \left(-\sqrt{7}, 0\right)$$



A normal is drawn at point $P\left(2, \frac{3\sqrt{3}}{2}\right)$ and this intersects the x axis at Q.

To find out the ratio of SQ and S'Q we just need to make use of the property that this normal bisects the angle S'PS.

Therefore $\frac{SP}{S'P} = \frac{SQ}{S'Q} = \frac{8 - \sqrt{7}}{8 + \sqrt{7}}$