PARABOLA

1. Let P be a point on the parabola $y^2 = 4ax$, where $a > 0$. The normal to the parabola at P meets the x-axis at a point Q. The area of the triangle PFQ, where F is the focus of the parabola, is 120. If the slope m of the normal and a are both positive integers, then the pair (a,m) is **[JEE(Advanced) 2023]**

(A) (2, 3) (B) (1, 3) (C) (2, 4) (D) (3, 4)

2. Consider the parabola $y^2 = 4x$. Let S be the focus of the parabola. A pair of tangents drawn to the parabola from the point $P = (-2, 1)$ meet the parabola at P_1 and P_2. Let Q_1 and Q_2 be points on the lines SP_1 and SP_2 respectively such that PQ_1 is perpendicular to SP_1 and PQ_2 is perpendicular to SP_2. Then, which of the following is/are TRUE ? **[JEE(Advanced) 2022]**

(A) $SQ_1 = 2$ (B) $Q_1Q_2 = \frac{3\sqrt{10}}{5}$ (C) $PQ_1 = 3$ (D) $SQ_2 = 1$

3. Let E denote the parabola $y^2 = 8x$. Let $P = (-2, 4)$, and let Q and Q' be two distinct points on E such that the lines PQ and PQ' are tangents to E. Let F be the focus of E. Then which of the following statements is (are) **TRUE**? **[JEE(Advanced) 2021]**

(A) The triangle PFQ is a right-angled triangle

(B) The triangle QPQ' is a right-angled triangle

(C) The distance between P and F is $5\sqrt{2}$

(D) F lies on the line joining Q and Q'

Question Stem for Questions Nos. 4 and 5

Question Stem

Consider the region $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \geq 0 \text{ and } y^2 \leq 4 - x\}$. Let F be the family of all circles that are contained in R and have centers on the x-axis. Let C be the circle that has largest radius among the circles in F. Let (α, β) be a point where the circle C meets the curve $y^2 = 4 - x$.

4. The radius of the circle C is _____. **[JEE(Advanced) 2021]**

5. The value of α is _____. **[JEE(Advanced) 2021]**

6. If a chord, which is not a tangent, of the parabola $y^2 = 16x$ has the equation $2x + y = p$, and midpoint (h, k), then which of the following is(are) possible value(s) of p, h and k ? **[JEE(Advanced) 2017]**

(A) $p = 5, h = 4, k = -3$ (B) $p = -1, h = 1, k = -3$

(C) $p = -2, h = 2, k = -4$ (D) $p = 2, h = 3, k = -4$

7. Let P be the point on the parabola $y^2 = 4x$ which is at the shortest distance from the center S of the circle $x^2 + y^2 - 4x - 16y + 64 = 0$. Let Q be the point on the circle dividing the line segment SP internally. Then-

(A) $SP = 2\sqrt{5}$

(B) $SQ : QP = (\sqrt{5}+1) : 2$

(C) the x-intercept of the normal to the parabola at P is 6

(D) the slope of the tangent to the circle at Q is $\frac{1}{2}$ **[JEE(Advanced) 2016]**

8. If the normals of the parabola $y^2 = 4x$ drawn at the end points of its latus rectum are tangents to the circle $(x-3)^2 + (y+2)^2 = r^2$, then the value of r^2 is **[JEE(Advanced) 2015]**

9. Let the curve C be the mirror image of the parabola $y^2 = 4x$ with respect to the line $x + y + 4 = 0$. If A and B are the points of intersection of C with the line $y = -5$, then the distance between A and B is

[JEE(Advanced) 2015]

10. Let P and Q be distinct points on the parabola $y^2 = 2x$ such that a circle with PQ as diameter passes through the vertex O of the parabola. If P lies in the first quadrant and the area of the triangle ΔOPQ is $3\sqrt{2}$, then which of the following is(are) the coordinates of P ? **[JEE(Advanced) 2015]**

(A) $\left(4, 2\sqrt{2}\right)$ (B) $\left(9, 3\sqrt{2}\right)$ (C) $\left(\frac{1}{4}, \frac{1}{\sqrt{2}}\right)$ (D) $\left(1, \sqrt{2}\right)$

11. The common tangents to the circle $x^2 + y^2 = 2$ and the parabola $y^2 = 8x$ touch the circle at the point P, Q and the parabola at the points R,S. Then the area of the quadrilateral PQRS is -

[JEE(Advanced) 2014]

(A) 3 (B) 6 (C) 9 (D) 15

Paragraph For Questions 12 and 13

Let a,r,s,t be nonzero real numbers. Let $P(at^2, 2at)$, Q, $R(ar^2, 2ar)$ and $S(as^2, 2as)$ be distinct points on the parabola $y^2 = 4ax$. Suppose that PQ is the focal chord and lines QR and PK are parallel, where K is the point (2a, 0).

12. The value of r is- **[JEE(Advanced) 2014]**

(A) $-\frac{1}{t}$ (B) $\frac{t^2+1}{t}$ (C) $\frac{1}{t}$ (D) $\frac{t^2-1}{t}$

13. If st = 1, then the tangent at P and the normal at S to the parabola meet at a point whose ordinate is-

[JEE(Advanced) 2014]

(A) $\frac{\left(t^2+1\right)^2}{2t^3}$ (B) $\frac{a\left(t^2+1\right)^2}{2t^3}$ (C) $\frac{a\left(t^2+1\right)^2}{t^3}$ (D) $\frac{a\left(t^2+2\right)^2}{t^3}$

SOLUTIONS

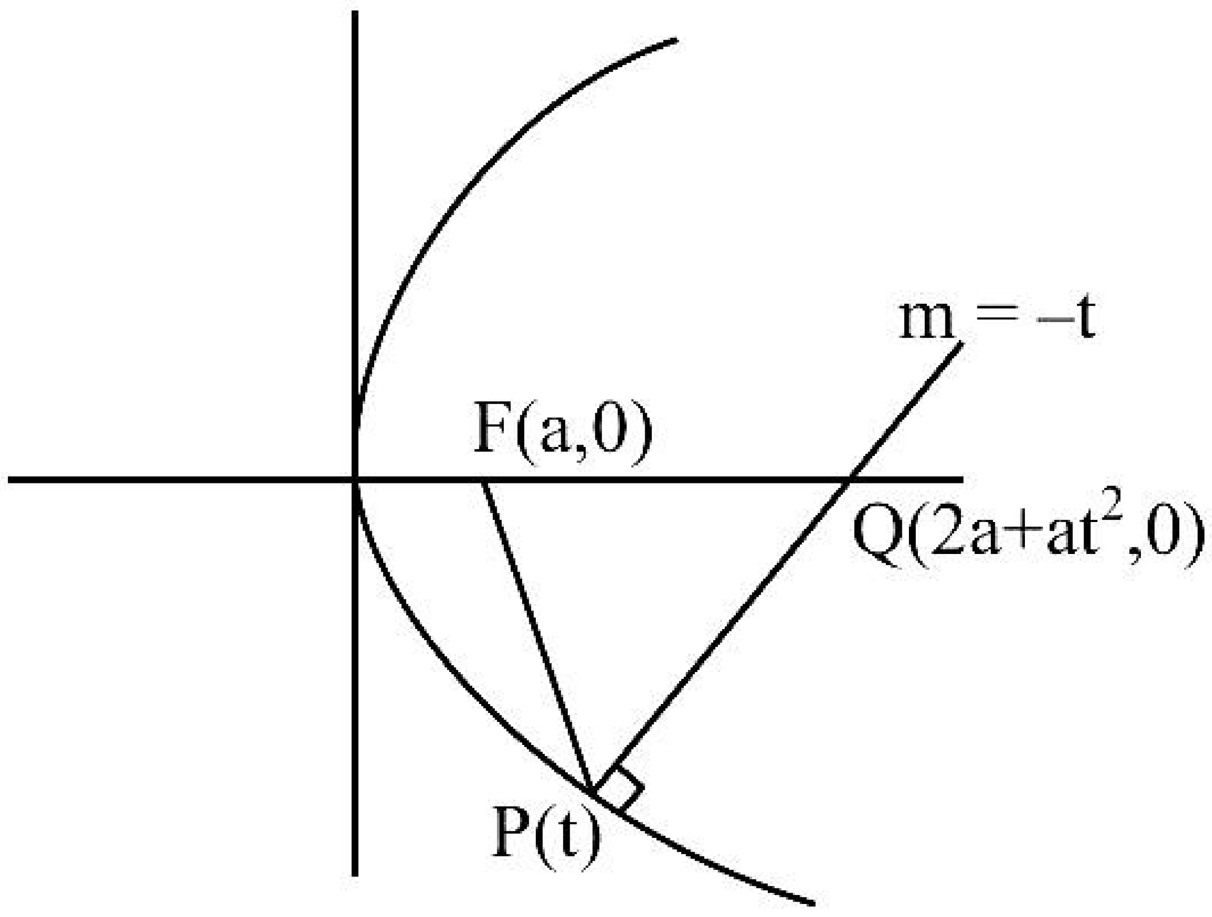
1. Ans. (A)

Sol. Let point P $(at^2, 2at)$

normal at P is $y = -tx + 2at + at^3$

$y = 0,\ x = 2a + at^2$

$Q(2a + at^2, 0)$



Area of $\Delta PFQ = \left|\frac{1}{2}\left(a + at^2\right)\left(2at\right)\right| = 120$

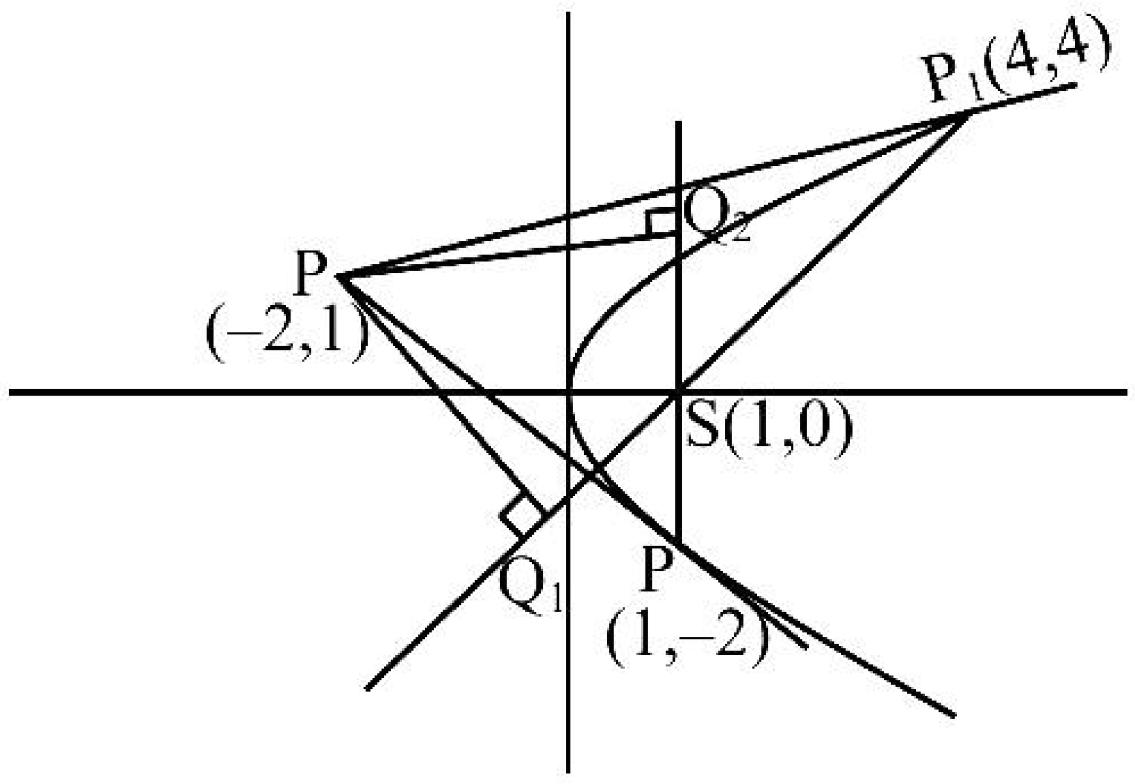
$\because m = -t$

$\because\ a^2\,[1 + m^2]\,m = 120$

(a, m) = (2, 3) will satisfy

2. Ans. (B, C, D)

Sol. Let equation of tangent with slope 'm' be



$T : y = mx + \frac{1}{m}$

T : passes through (–2, 1) so

$1 = -2m + \frac{1}{m}$

$\Rightarrow m = -1$ or $m = \frac{1}{2}$

Points are given by $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

So, one point will be (1, –2) & (4, 4)

Let $P_1(4, 4)$ & $P_2(1, -2)$

$P_1S : 4x - 3y - 4 = 0$

$P_2S : x - 1 = 0$

$PQ_1 = \left|\frac{4(-2) - 3(1) - 4}{5}\right| = 3$

$SP = \sqrt{10}$; $PQ_2 = 3$; $SQ_1 = 1 = SQ_2$

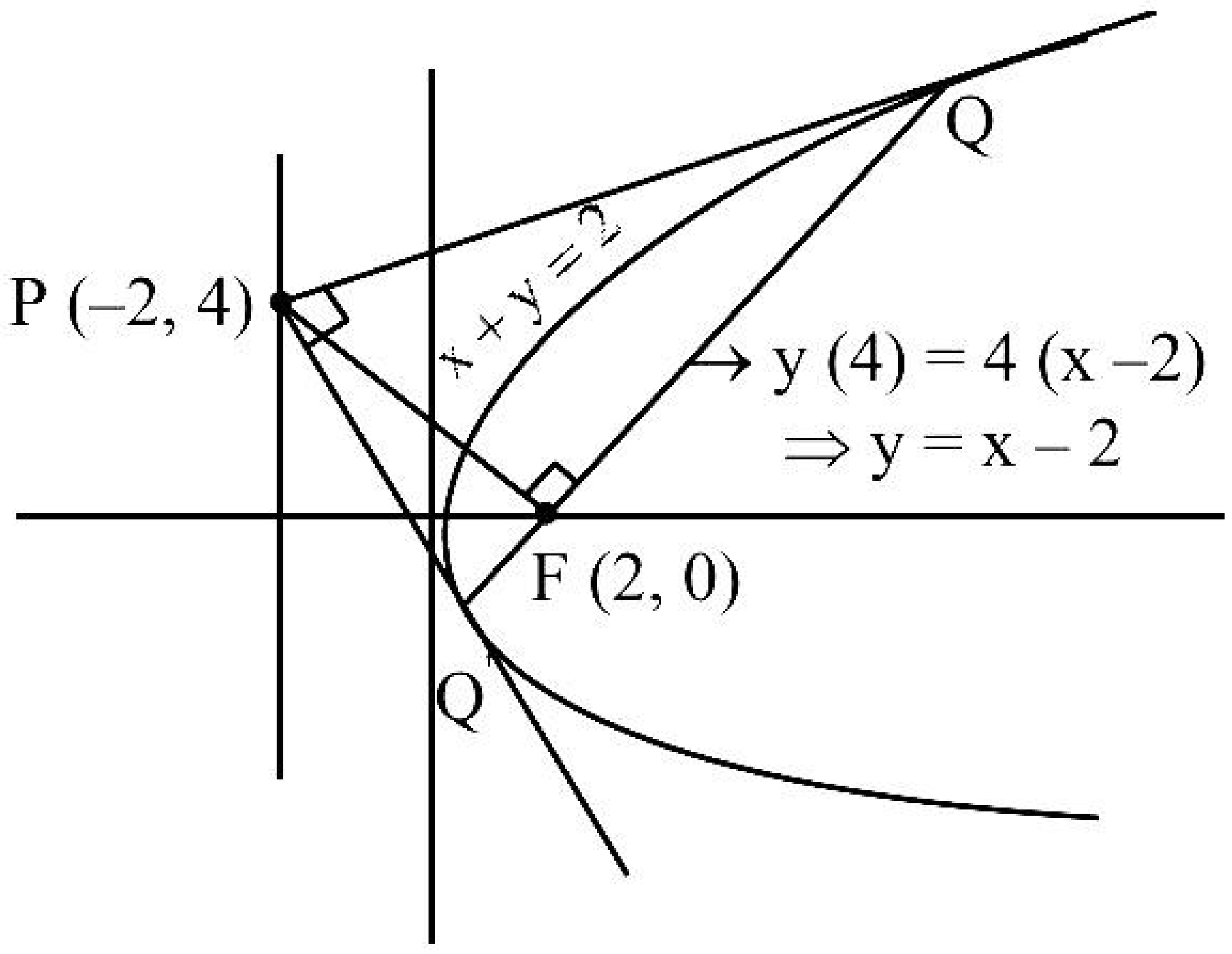
$\frac{1}{2}\left(\frac{Q_1Q_2}{2}\right) \times \sqrt{10} = \frac{1}{2} \times 3 \times 1$

(comparing Areas)

$\Rightarrow Q_1Q_2 = \frac{2 \times 3}{\sqrt{10}} = \frac{3\sqrt{10}}{5}$

3. Ans. (A, B, D)

Sol.



Note that P lies on directrix so triangle PQQ' is right angled, hence QQ' passes through focus F.

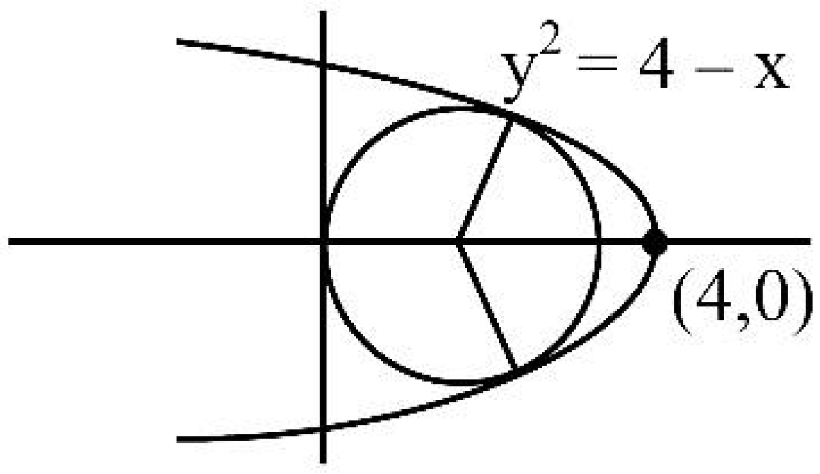
$PF = 4\sqrt{2}$

Equation of QF is $y = x - 2$ & PF is $x + y = 2$

Hence A, B, D.

4. Ans. (1.50)

Sol.



Let the circle be

$x^2 + y^2 + \lambda x = 0$

For point of intersection of circle & parabola

$y^2 = 4 - x$.

$x^2 + 4 - x + \lambda x = 0 \Rightarrow x^2 + x(\lambda - 1) + 4 = 0$

For tangency : $\Delta = 0 \Rightarrow (\lambda - 1)^2 - 16 = 0$

$\Rightarrow \lambda = 5$ (rejected) or $\lambda = -3$

Circle : $x^2 + y^2 - 3x = 0$

Radius $= \frac{3}{2} = 1.5$

5. Ans. (2.00)

Sol. For point of intersection :

$x^2 - 4x + 4 = 0 \Rightarrow x = 2$ so $\alpha = 2$

6. Ans. (D)

Sol. Equation of chord with mid point (h, k) :

$$k.y - 16\left(\frac{x+h}{2}\right) = k^2 - 16h$$

$\Rightarrow \quad 8x - ky + k^2 - 8h = 0$

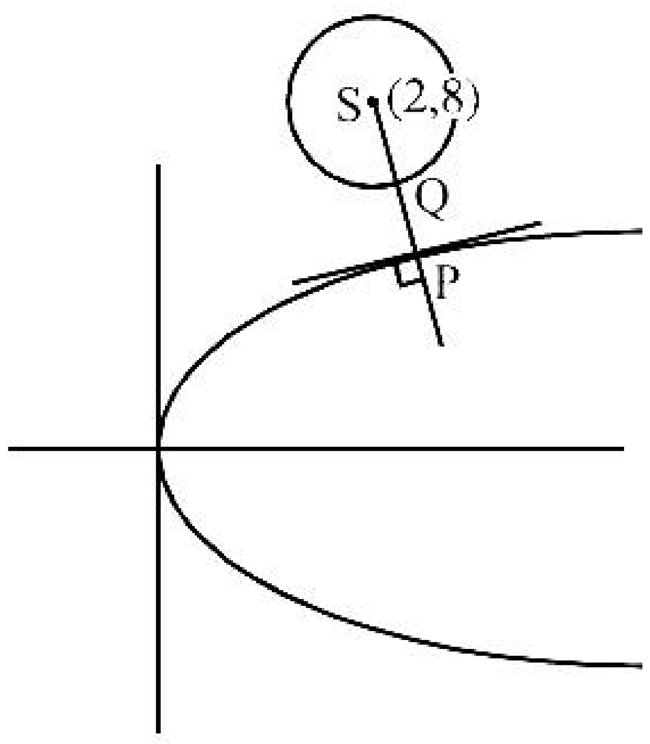
Comparing with $2x + y - p = 0$, we get

$k = -4$; $2h - p = 4$

only (D) satisfies above relation.

7. Ans. (A, C, D)

Sol.



$y^2 = 4x$

point P lies on normal to parabola passing through centre of circle

$y + tx = 2t + t^3$ ....(i)

$8 + 2t = 2t + t^3$

$t = 2$

P(4, 4)

$SP = \sqrt{(4-2)^2 + (4-8)^2}$

$SP = 2\sqrt{5}$

$SQ = 2$

$\Rightarrow \quad PQ = 2\sqrt{5} - 2$

$$\frac{SQ}{QP} = \frac{1}{\sqrt{5}-1} = \frac{\sqrt{5}+1}{4}$$

To find x intercept

put y = 0 in (i)

$\Rightarrow \quad x = 2 + t^2$

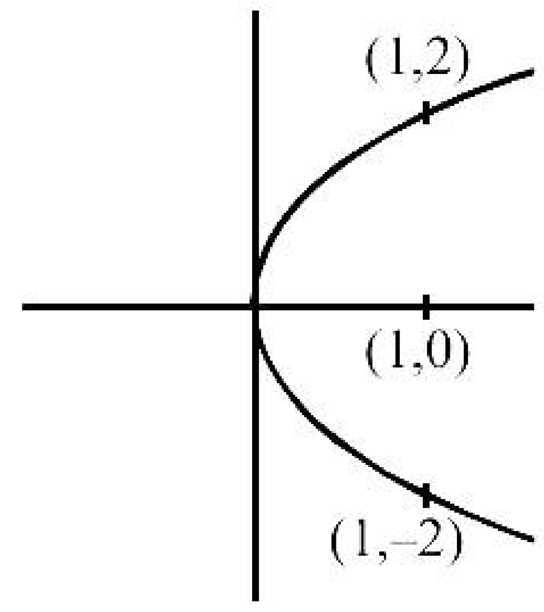
$x = 6$

$\because$ Slope of common normal $= -t = -2$

$\therefore$ Slope of tangent $= \dfrac{1}{2}$

8. Ans. (2)

Sol.



The co-ordinates of latus rectum are (1,2) and (1,–2)

clearly slope of tangent is given by $\dfrac{dy}{dx} = \dfrac{2}{y}$

$\therefore$ At y = 2 slope of normal = –1

and At y = –2 slope of normal = 1

$\therefore$ Equation of normal at (1,2)

$(y - 2) = -1(x - 1) \Rightarrow x + y = 3$

Now, this line is tangent to circle

$(x - 3)^2 + (y + 2)^2 = r^2$

$\therefore$ perpendicular distance from centre to line

= Radius of circle

$$\therefore \frac{|3-2-3|}{\sqrt{2}} = r \Rightarrow r^2 = 2$$

9. Ans. (4)

Sol. Let there be a point $(t^2, 2t)$ on $y^2 = 4x$

Clearly its reflection in $x + y + 4 = 0$ is given by

$$\frac{x-t^2}{1} = \frac{y-2t}{1} = \frac{-2(t^2+2t+4)}{2}$$

$\therefore \quad x = -(2t + 4)$ & $y = -(t^2 + 4)$

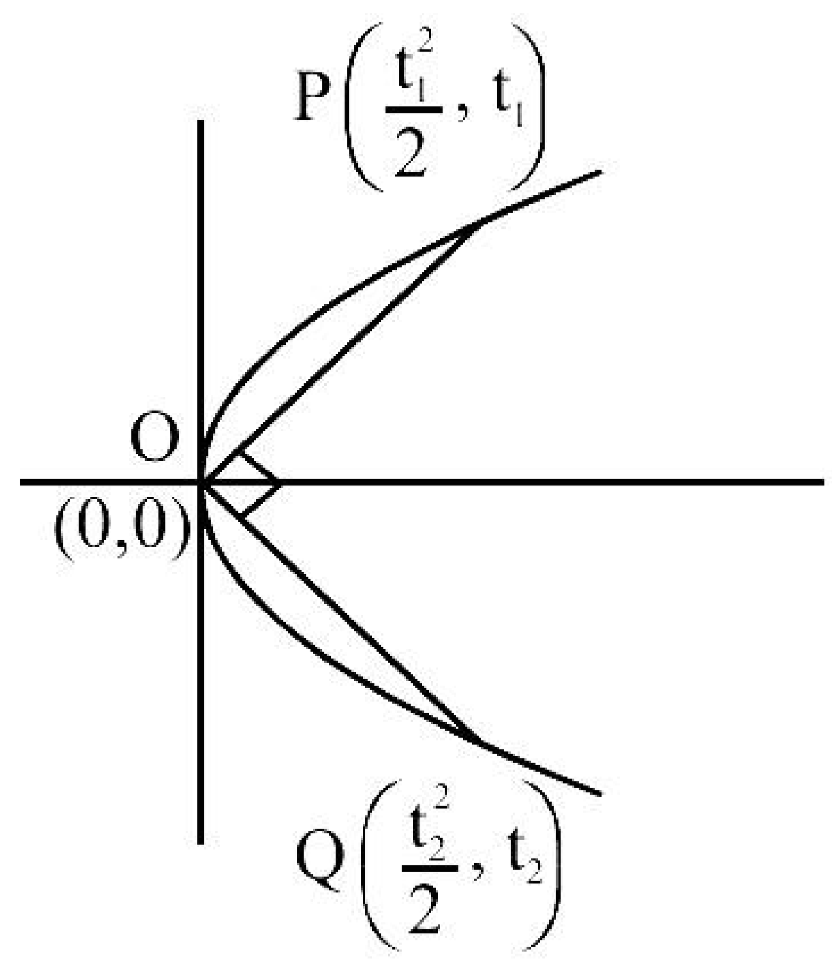
Now, $y = -5 \quad \Rightarrow \quad t = \pm 1$

$\therefore \quad x = -6 \quad$ or $\quad x = -2$

$\therefore \quad$ Distance between A & B = 4

10. Ans. (A, D)

Sol.



$\because \angle POQ = \frac{\pi}{2} \quad \Rightarrow \quad t_1 t_2 = -4$

$$\therefore \quad \left| \frac{1}{2} \begin{vmatrix} 0 & 0 & 1 \\ \frac{t_1^2}{2} & t_1 & 1 \\ \frac{t_2^2}{2} & t_2 & 1 \end{vmatrix} \right| = 3\sqrt{2}$$

$$\Rightarrow \quad \left| \frac{t_1^2 t_2 - t_1 t_2^2}{2} \right| = 6\sqrt{2}$$

$$\Rightarrow \quad |t_1 - t_2| = 3\sqrt{2}$$

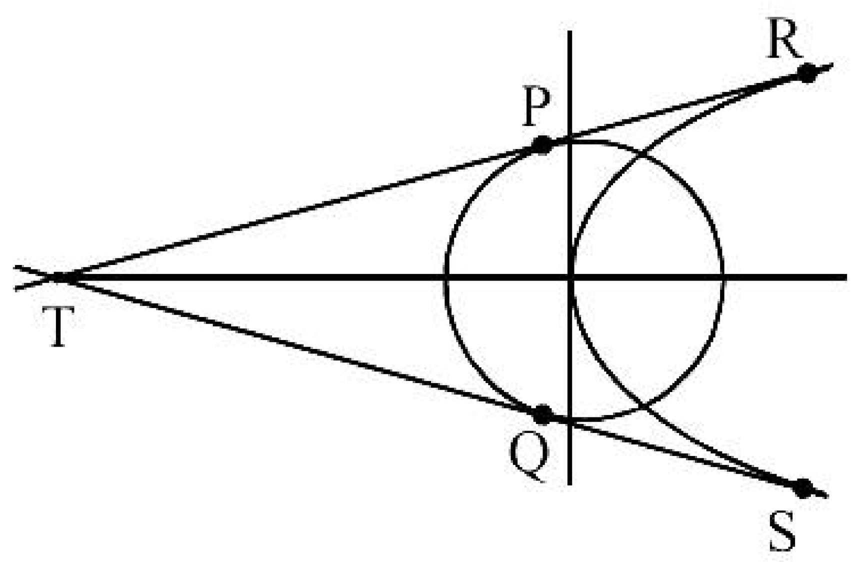
$$\Rightarrow \quad t_1 + \frac{4}{t_1} = 3\sqrt{2} \qquad (\because \quad t_1 > 0)$$

We get $t_1 = 2\sqrt{2}, \sqrt{2}$

$P(4, 2\sqrt{2})$ or $(1, \sqrt{2})$

11. Ans. (D)

Sol.



$$y = mx + \frac{2}{m}$$

$$\frac{\left|0 - 0 + \frac{2}{m}\right|}{\sqrt{1+m^2}} = \sqrt{2} \Rightarrow 2 = m^2(1 + m^2)$$

$\Rightarrow m = \pm 1$

TP : $-x + y = 2$

So $P(-1, 1)$ & $Q(-1, -1)$

& $R\left(\frac{2}{m}, \frac{4}{m}\right) \equiv R(2,4)$ & $S(2,-4)$

So $\Delta = \frac{1}{2} 10.3 = 15$

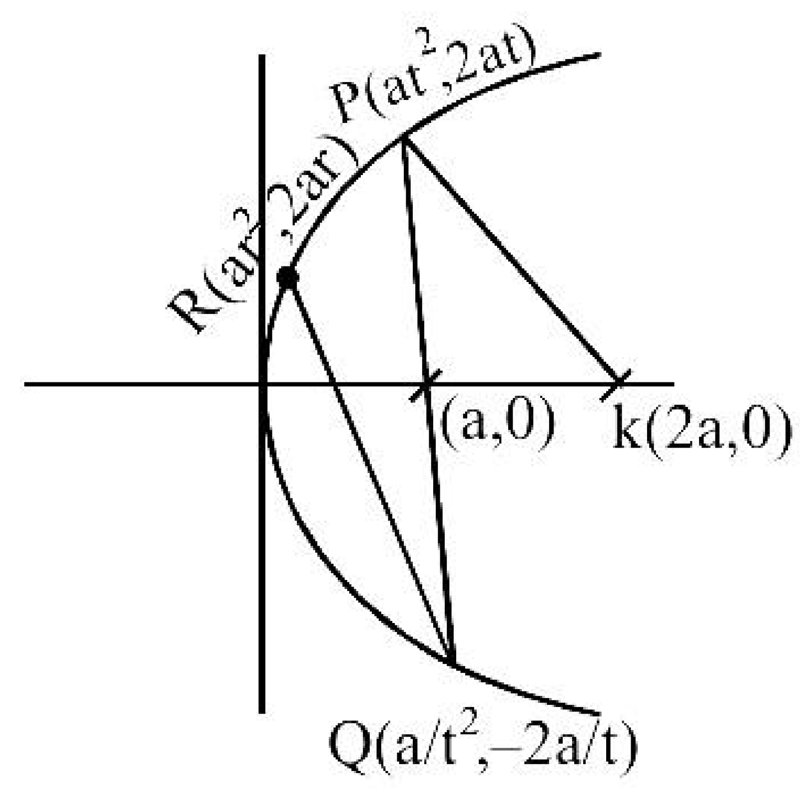
So $P(-1, 1)$ & $Q(-1, -1)$

& $R\left(\frac{2}{m}, \frac{4}{m}\right) \equiv R(2,4)$ & $S(2,-4)$

So $\Delta = \frac{1}{2} 10.3 = 15$

12. Ans. (D)

Sol.



$\because$ PQ is a focal chord

$\therefore$ co-ordinates of point Q are $= \left(\frac{a}{t^2}, -\frac{2a}{t}\right)$

$$m_{QR} = \frac{2a\left(r + \frac{1}{t}\right)}{a\left(r^2 - \frac{1}{t^2}\right)} = \frac{2}{\left(r - \frac{1}{t}\right)}$$

$$m_{PK} = \frac{2at - 0}{a\left(t^2 - 2\right)} = \frac{2t}{t^2 - 2}$$

Given $m_{QR} = m_{PK}$

$$\Rightarrow \frac{2}{r - \frac{1}{t}} = \frac{2t}{t^2 - 2} \Rightarrow r = \frac{t^2 - 2}{t} + \frac{1}{t}$$

$$\Rightarrow r = t - \frac{2}{t} + \frac{1}{t} \Rightarrow r = \frac{t^2 - 1}{t}$$

13. Ans. (B)

Sol. Equation of tangent at point P is

$ty = x + at^2$ ....(i)

Equation of normal at point S is

$$\frac{1}{t}x + y = \frac{2a}{t} + \frac{a}{t^3}$$

$\Rightarrow \quad t^2 x + t^3 y = 2at^2 + a$ ....(ii)

Multiply equation (i) by t^2 and then subtract from equation (ii),

we get,

$$2t^3 y = 2at^2 + at^4 + a$$

$$\Rightarrow 2t^3 y = a(1 + t^4 + 2t^2)$$

$$\Rightarrow \boxed{y = \frac{a(1+t^2)^2}{2t^3}}$$