CHAPTER : 9 SEQUENCES AND SERIES

Sequences and Series

Definitions

Sequence is a function whose domain is the set of N natural numbers.

Real sequence: A sequence whose range is a subset of R is called a real sequence

Series: If $a_1, a_2, a_3, \ldots a_n$, is a sequence, then the expression $a_1+a_2+a_3+\ldots a_n+\ldots$ is a series. A series is finite or infinite according to the no. of terms in the corresponding sequence as finite or infinite.

Progression: Those sequences whose terms follow certain patterns are called progression.

Arithmetic Progression (A.P.)

An A.P. is a sequence in which terms increase or decrease regularly by the fixed number (same constant). This fixed number is called 'common difference' of the A.P. If 'a' is the first term & 'd' is the common difference and 'l' is the last term of A.P., then general term or the n[th] term of the A.P. is given by $a_n = a + (n-1)d$ from starting and $a_n = l-(n-1)\,d$ from the end.

The sum S_n of the first n terms of an A.P. is given by

$$S_n = \frac{n}{2}[2a+(n-1)d] = \frac{n}{2}[a+l]$$

Arithmetic Mean (AM)

If three numbers are in A.P., then the middle term is called AM between the other two, so if a, b, c, are in A.P., b is AM of a and c.

AM for any 'n' +ve numbers $a_1, a_2, a_3, \ldots, a_n$ is

$$AM = \frac{a_1+a_2+a_3+\ldots+a_n}{n}$$

n-Arithmetic Mean between Two Numbers: If a, b are any two given numbers & a, $A_1, A_2, .., A_n$, b are in A.P. then $A_1, A_2, \ldots, A_n$ are n AM's between a & b.

$$A_1 = a + \frac{b-a}{n+1}, A_2 = a + \frac{2(b-a)}{n+1} \ldots A_n = a + \frac{n(b-a)}{n+1}$$

or $A_1 = a + d$, $A_2 = a+2d$, …, $A_n = a + nd$,

where $d = \frac{b-a}{n+1}$

Sum upto terms of some Special Sequences

- Sum of first 'n' natural numbers

$$\sum_{k=1}^{n} k = 1+2+3+\ldots+n = \frac{n(n+1)}{2}$$

- Sum of squares of first n natural numbers

$$\sum_{k=1}^{n} k^2 = 1^2+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

- Sum of cubes of first n natural numbers

$$\sum_{k=1}^{n} k^3 = 1^3+2^3+3^3+\ldots+n^3 = \left(\frac{n(n+1)}{2}\right)^2 = \left(\sum_{k=1}^{n} k\right)^2$$

- Sum of first 'n' odd natural numbers

$$\sum_{k=1}^{n}(2k-1) = 1+3+5+\ldots+(2n-1) = n^2$$

Relationship between AM and GM

Let A and G be the AM and GM of two given positive real numbers a & b, respectively. Then,

$$A = \frac{a+b}{2} \text{ and } G = \sqrt{ab}$$

Thus, we have , $A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2}$

$$= \frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{2} \geq 0 \quad \text{.............. (1)}$$

From (1), we obtain the relationship A ≥ G

Properties of AM & GM between two quantities

- If A.M.: G.M. of two positive numbers a and b is m : n, then a : b = $\left(m+\sqrt{m^2-n^2}\right):\left(m-\sqrt{m^2-n^2}\right)$
- If A and G be the AM & GM between two positive numbers, then the numbers are $A \pm \sqrt{A^2-G^2}$

Arithmetico Geometric Series

In a series,where each term is formed by multiplying the corresponding term of an AP & GP is called Arithmetico-Geometric Series.

Eg: $1+3x+5x^2+7x^3+\ldots$

Here, 1, 3, 5, … are in AP and 1, x, x^2, … are in GP.

Geometric Mean (GM)

- If a, b, c are in G.P., b is the GM between a & c, $b^2 = ac$, therefore $b = \sqrt{ac}$; a > 0, c > 0
- If a, b are two given numbers & a, $G_1, G_2, \ldots, G_n$, b are in G.P., then $G_1, G_2, G_3 \ldots G_n$ are n GMs between a & b.

$$G_1 = a\cdot\left(\frac{b}{a}\right)^{\frac{1}{n+1}} G_2 = a\cdot\left(\frac{b}{a}\right)^{\frac{2}{n+1}} \ldots\ldots G_n = a\cdot\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

or

$$G_1 = ar,\ G_2 = ar^2,\ \ldots\ G_n = ar^n$$

where $r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

Geometric Progression (G.P.)

A sequence is said to be a geometric progression or G.P., if the ratio of any term to its preceding term is same throughout.This constant factor is called **'Common ratio'.** Usually we denote the first term of a G.P. by 'a' and its common ratio by 'r'. The gereral or n[th] term of G.P. is given by $a_n = ar^{n-1}$

The sum S_n of the first n terms of G.P. is given by

$$S_n = \frac{a(r^n-1)}{r-1} \text{ or } \frac{a(1-r^n)}{1-r} \text{ if } r\neq 1$$

Sum of infinite terms of G.P. is given by

$$S_n = \frac{a}{1-r};\ |r|<1$$