Relations & Functions-1

1. **(b)** Put $x = y = 1$, $(f(1))^2 = 3f(1) - 2$

$\Rightarrow f(1) = 1$ or 2

Let $f(1) = 1$, then put $y = 1$

$f(x).f(1) = f(x) + f(1) + f(x) - 2$

$\Rightarrow f(x) = 1$ constant function $\therefore f(1) \neq 1$,

hence $f(1) = 2$

2. **(a)** Given

$f(x) = \sqrt{x^{14} - x^{11} + x^6 - x^3 + x^2 + 1}$

for f (x) to be defined,

$x^{14} - x^{11} + x^6 - x^3 + x^2 + 1 \geq 0$

Case 1 : x ≥ 1

$x^{14} - x^{11} + x^6 - x^3 + x^2 + 1$

$= (x^{14} - x^{11}) + (x^6 - x^3) + (x^2 + 1) > 0$

Case 2 : $0 \leq x \leq 1$

$x^{14} - x^{11} + x^6 - x^3 + x^2 + 1$

$= x^{14} - \{(x^{11} - x^6) + (x^3 - x^2)\} + 1 > 0$

$\{\because x^{11} - x^6 \leq 0, x^3 - x^2 \leq 0\}$

Case 3 : x < 0

$x^{14} - x^{11} + x^6 - x^3 + x^2 + 1 > 0$

$(\because x^{11} < 0, x^3 < 0, \; x^{14}, x^6, x^2 > 0)$

Thus, for all real x,

$x^{14} - x^{11} + x^6 - x^3 + x^2 + 1 \geq 0$

Hence, the domain of $f(x) = R = (-\infty, \infty)$

3. **(a)** Given $f(x) = \cos(\log x)$

$\therefore f(xy) = \cos(\log xy)$

$f(xy) = \cos[\log x + \log y]$ ....(1)

And $f\left(\frac{x}{y}\right) = \cos\left(\log\frac{x}{y}\right)$

$f\left(\frac{x}{y}\right) = \cos(\log x - \log y)$ ....(2)

Adding (1) and (2), we get

$f(xy) + f\left(\frac{x}{y}\right)$

$= \cos(\log x + \log y) + \cos(\log x - \log y)$

$= 2\cos(\log x).\cos(\log y)$

$\Rightarrow f(xy) + f\left(\frac{x}{y}\right) = 2f(x).f(y)$

Then the value of $f(x)f(y)$

$-\frac{1}{2}\left\{f\left(\frac{x}{y}\right) + f(xy)\right\}$

$= f(x)\,f(y) - \frac{1}{2}.2\{f(x).f(y)\} = 0$

4. **(c)** We must $x^4 - 21x^2 \ge 0$ and

$10 - \sqrt{x^4 - 21x^2} \ge 0$

$\Rightarrow \quad x^2\left(x^2 - 21\right) \ge 0$ ...(1)

and $100 \ge x^4 - 21x^2$ ...(2)

Eq. (1) gives $x = 0$ or $x \le -\sqrt{21}$ or $x \ge \sqrt{21}$

Eq. (2) $\Rightarrow x^4 - 21x^2 - 100 \le 0$

$\Rightarrow \quad \left(x^2 - 25\right)\left(x^2 + 4\right) \le 0$

$\Rightarrow \quad x^2 - 25 \le 0$ (as $x^2 + 4 > 0$ always)

$\Rightarrow \quad -5 \le x \le 5$

Domain is given by

$\left[-5, -\sqrt{21}\right] \cup \left[\sqrt{21}, 5\right] \cup \{0\}$

5. **(b)** $3f(x) + 2f\left(\frac{x+59}{x-1}\right) = 10x + 30$

For $x = 7$, $3f(7) + 2f(11) = 70 + 30 = 100$

For $x = 11$, $3f(11) + 2f(7) = 140$

$\frac{f(7)}{-20} = \frac{f(11)}{-220} = \frac{-1}{9-4}$ or $f(7) = 4$

6. **(a)** $af(x+1) + bf\left(\frac{1}{x+1}\right) = (x+1) - 1$ ...(1)

Replacing $x + 1$ by $\frac{1}{x+1}$, we get

$af\left(\frac{1}{x+1}\right) + bf(x+1) = \frac{1}{x+1} - 1$ ...(2)

Eq. (1) × a – Eq. (2) × b

$\Rightarrow \quad (a^2 - b^2)f(x+1) = a(x+1) - a - \frac{b}{x+1} + b$

Putting $x = 1$, $(a^2 - b^2)f(2) = 2a - a - \frac{b}{2} + b$

$= a + \frac{b}{2} = \frac{2a+b}{2}$

7. **(d)** $f(x) = \sqrt{\cos(\sin x)} + \sqrt{\log_x\{x\}}$

Domain $\cos(\sin x) \ge 0$ $\{x\} > 0,\ x > 0,\ x \ne 1,$ $\log_x\{x\} \ge 0$

(i) $\cos(\sin x) \ge 0$ for all x, $x \in R\,[-1, 1]$

(ii) $\{x\} > 0,\ x \notin$ Int. (iii) $x > 0,\ x \in (0, \infty)$

(iv) $x \ne 1$

(v) $\log_x\{x\} \ge 0 \Rightarrow 1 > f(x) \ge 0$,

so $1 > x \ge 0$ $\log_x f(x)$ is positive $x \in [0, 1)$

$\Rightarrow x \in (0, 1)$

8. **(c)** Given that,

$f(x) = \frac{1}{2} - \tan\left(\frac{\pi x}{2}\right), -1 < x < 1$

Here, domain of $f(x) = (-1, 1)$ and

$g(x) = \sqrt{3 + 4x - 4x^2} = \sqrt{-(2x-3)(2x+1)}$

$\Rightarrow (2x-3)(2x+1) \le 0 \Rightarrow -\frac{1}{2} \le x \le \frac{3}{2}$

$\therefore$ Domain of $g(x) = \left[-\frac{1}{2}, \frac{3}{2}\right]$

Hence, domain of $(f + g)$ = intersection of their domains $= \left[-\frac{1}{2}, 1\right)$.

9. **(b)** We have f(x)

$= \log_4[\log_5\{\log_3(18x - x^2 - 77)\}]$

Since, $\log_a x$ is defined for all $x > 0$, f(x) is defined if

$\log_5\{\log_3(18x - x^2 - 77)\} > 0$ and $18x - x^2 - 77 > 0$

or $\log_3(18x - x^2 - 77) > 5^0$ and $x^2 - 18x + 77 < 0$

or $\log_3(18x - x^2 - 77) > 1$ and $(x - 11)(x - 7) < 0$

or $18x - x^2 - 77 > 3^1$ and $7 < x < 11$

or $18x - x^2 - 80 > 0$ and $7 < x < 11$

or $x^2 - 18x + 80 < 0$ and $7 < x < 11$

or $(x-10)(x-8)<0$ and $7<x<11$
or $8<x<10$ and $7<x<11$ or $8<x<10$
or $x \in (8, 10)$
Hence, the domain of f(x) is (8, 10).

10. **(a)** Clearly, $g(x)=\frac{1}{2}(a^x+a^{-x})$ and

$h(x)=\frac{1}{2}(a^x-a^{-x})$. Now $g(x+y)+g(x-y)$

$$=\frac{1}{2}(a^{x+y}+a^{-(x+y)})+\frac{1}{2}(a^{x-y}+a^{-x+y})$$

$$=\frac{1}{2}(a^x a^y+a^x a^{-y}+a^{-x}a^y+a^{-x}a^{-y})$$

$$=\frac{1}{2}(a^x(a^y+a^{-y})+a^{-x}(a^y+a^{-y}))$$

$$=2\left(\frac{1}{2}(a^x+a^{-x})\right)\left(\frac{1}{2}(a^y+a^{-y})\right)=2g(x)g(y)$$

11. **(b)** Given $f(\lambda+x)=f(\lambda-x)$ ...(1)
$f(2\lambda+x)=-f(2\lambda-x)$ ...(2)
for $\lambda>0$
Replacing x by $\lambda-x$ in $\lambda-x$ in (1), we get
$f(2\lambda-x)=f(x)$ ...(3)
$\therefore$ From (2) and (3), $f(x)=-f(2\lambda+x)$
$\Rightarrow f(x)=-[-f(2\lambda+2\lambda+x)]$
$\Rightarrow f(x)=f(x+4\lambda)$ ...(4)
$\Rightarrow f(x)$ is periodic with period 4λ.
Further from (3), replacing x by $-x$, we get
$f(2\lambda+x)=f(-x)$ ...(5)
From (2), (3) and (5), we have
$f(-x)=f(2\lambda+x)=-f(2\lambda-x)=-f(x)$
i.e. $f(-x)=-f(x)$
$\Rightarrow f(x)$ is odd function
Thus, f is odd and periodic function.

12. **(d)** $f(x)=\alpha x^3-\beta x-(\tan x)\operatorname{sgn} x$

Since $f(-x)=f(x)$

$\Rightarrow -\alpha x^3+\beta x-\tan x \operatorname{sgn} x$

$=\alpha x^3-\beta x-(\tan x)(\operatorname{sgn} x)$

$\Rightarrow 2\left(\alpha x^2-\beta\right)x=0\ \forall\ x\in R \Rightarrow \alpha=0$ and $\beta=0$

$\therefore [a]^2-5[a]+4=0$

and $6\{a\}^2-5\{a\}+1=0$
$\Rightarrow ([a]-1)([a]-4)=0$ and
$(3\{x\}-1)(2\{x\}-1)=0$

$\Rightarrow [a]=1, 4$ and $\{a\}=\frac{1}{3}, \frac{1}{2}$

$\therefore a=1+\frac{1}{3}, 1+\frac{1}{2}, 4+\frac{1}{3}, 4+\frac{1}{2}$

Sum of all possible values of $a=\frac{35}{3}$

13. **(c)** $f(x)=\cos nx \,.\, \sin\left(\frac{5x}{n}\right)$;

Period of $\cos nx=\frac{2\pi}{|n|}$

Period of $\sin\frac{5x}{n}=\frac{2\pi}{\left|\frac{5}{n}\right|}=\frac{2|n|\pi}{5}$

$\therefore$ Period of $f(x)=$ L.C.M. $\left(\frac{2\pi}{|n|}, \frac{2|n|\pi}{5}\right)=2\pi$ (given)

$\Rightarrow$ L.C.M. $\left(\frac{1}{|n|}, \frac{|n|}{5}\right)=1=\frac{\text{L.C.M.}(1,|n|)}{\text{H.C.F.}(|n|,5)}=1$

$\Rightarrow \frac{|n|}{\text{H.C.F.}(|n|,5)}=1 \Rightarrow$ H.C.F. $(|n|,5)=|n|$
If g.c.d. $(|n|, 5)=1 \Rightarrow |n|=1 \Rightarrow n=1$
If g.c.d. $(|n|, 5)\neq 1 \Rightarrow |n|=5m; m\in\mathbb{N}$
$\Rightarrow$ g.c.d. $(5m, 5)=1$
$\Rightarrow |n|=5 \Rightarrow n=\pm 5 \therefore n\in\{\pm 1, \pm 5\}$

14. **(d)** $f: \mathbb{R}\to\mathbb{R}, g; \mathbb{R}\to\mathbb{R}$
We know that min. $\{f_1(x), f_2(x)\}$
$$=\frac{(f_1(x)+f_2(x))-|f_1(x)-f_2(x)|}{2}$$
$\therefore$ min $\{f(x)-g(x), 0\}$
$$=\frac{(f(x)-g(x)+0)-|f(x)-g(x)-0|}{2}$$
$$=\frac{(f(x)-g(x))-|f(x)-g(x)|}{2}$$

15. **(b)** $y=f(e^x)+f(|\ln|\,|x|)$ Domain $f(x)=(0,1)$
$\Rightarrow 0<e^x<1 \Rightarrow x<0$ ...(1)
and $0<\ln|x|<1 \Rightarrow 1<|x|<e$
$\Rightarrow x\in\{-e,-1\}\cup(1,e)$ ... (2)
Taking intersection $x\in(-e,-1)$

16. **(b)** $f\left(x+\frac{1}{2}\right)=f(x); f(2)=5;$

$f\left(\frac{9}{4}\right)=2, f(-3)-f\left(\frac{1}{4}\right)=?$

$\because\ f(x)$ is periodic with period $\frac{1}{2}$

$\Rightarrow\ f(x) = f\left(x + \frac{n}{2}\right)\ \forall\ x \in \mathbb{N}$

$\Rightarrow\ f(-3) = f\left(-3 + \frac{10}{2}\right) = f(2) = 5$

and $f\left(\frac{1}{4}\right) = f\left(\frac{1}{4} + 4\left(\frac{1}{2}\right)\right) = f\left(\frac{9}{4}\right) = 2$

$\therefore\ f(-3) - f\left(\frac{1}{4}\right) = 5 - 2 = 3$

17. (c) $f(x) = \begin{cases} 2x+3 & x \le 1 \\ a^2x+1 & x > 1 \end{cases}$

For $x \le 1$; $f(x) \le 5$
So for range of f(x) to be R.
$\Rightarrow a^2 + 1 \le 5$ and $a \ne 0 \quad \Rightarrow a \in [-2, 2]$
Hence, $a = \{-2, -1, 1, 2\}$

18. (b) $g(\sqrt{2}f(x) + 3)$

$= \log_{\sqrt{5}}(\sqrt{2}\ (\sin x - \cos x) + 3)$

We know that

$-\sqrt{2} \le (\sin x - \cos x) \le \sqrt{2}\ \forall\ x \in \mathbb{R}$

$\left[\because -\sqrt{a^2 + b^2} \le a \sin x + b \cos x \le \sqrt{a^2 + b^2}\right]$

$\Rightarrow\ -2 \le \sqrt{2}(\sin x - \cos x) \le 2$

$\Rightarrow\ 1 \le \sqrt{2}(\sin x - \cos x) + 3 \le 5$

$\Rightarrow\ 0 \le \log_{\sqrt{5}}(\sqrt{2}\ (\sin x - \cos x) + 3) \le 2$

($\because\ \log_a x$ is increasing for $a > 1$)

Hence, range of $g(\sqrt{2}f(x) + 3)$ is [0, 2].

19. (d) $-g(2, f(x)) = -\log_2\left(1 + \frac{1}{4\sqrt{x}}\right)$

$\Rightarrow\ -g(2, f(x)) - 1 = -\log_2\left(1 + \frac{1}{4\sqrt{x}}\right) - 1$

$\therefore\ g\left(\frac{1}{2}, -g(2, f(x)) - 1\right)$

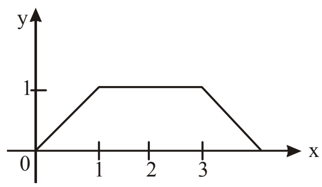
$= \log_{1/2}\left(-\log_2\left(1 + \frac{1}{4\sqrt{x}}\right) - 1\right)$

$\Rightarrow\ \log_2\left(1 + \frac{1}{4\sqrt{x}}\right) + 1 < 0$

$\Rightarrow\ 0 < 1 + \frac{1}{4\sqrt{x}} < 2^{-1}$

$\Rightarrow\ -1 < \frac{1}{4\sqrt{x}} < -\frac{1}{2}\ \Rightarrow x \in \phi$ (null set)

20. (d)



$0 \le f(x) \le 1 \quad \Rightarrow\ 0 \le 7\,f(x) \le 7$
$\Rightarrow\ -1 \le \sin(7\,f(x)) \le 1$

21. (5.5) Here $\phi(-x) = \frac{1}{1+e^x}$

So, $\phi(x) + \phi(-x) = \frac{1}{1+e^{-x}} + \frac{1}{1+e^x}$

$= \frac{e^x}{e^x+1} + \frac{1}{1+e^x} = \frac{e^x+1}{e^x+1} = 1$

$\therefore S = \{\phi(5) + \phi(-5)\} + \ldots + \{\phi(1) + \phi(-1)\} + \phi(0)$

$= 1 + 1 + 1 + 1 + 1 + \phi(0) = 5 + \frac{1}{1+e^0} = 5 + \frac{1}{2} = \frac{11}{2}$

22. (2) $f(x+p) = 1 + \{1 - 3f(x) + 3f^2(x) - f^3(x)\}^{1/3}$
$\Rightarrow\ f(x+p) = 1 + (1 - f(x)) = 2 - f(x)$
$\Rightarrow\ f(x+p) = 2 - [2 - f(x-p)]$
$\Rightarrow f(x+p) = f(x-p)$
$\Rightarrow\ f(x) = f(x+2p)$
$\Rightarrow$ Period of $f(x) = 2p = \lambda p$ (given)
$\Rightarrow\ \lambda = 2$

23. (9) Given $f(x+2) = f(x) + f(2)$
Put $x = -1$. Then $f(1) = f(-1) + f(2)$
or $f(1) = -f(1) + f(2)$ [as f(x) is an odd function]
or $f(2) = 2f(1) = 6$
Now, put $x = 1$
We have $f(3) = f(1) + f(2) = 3 + 6 = 9$

24. (6) Given $f(x, y) = f(2x + 2y, 2y - 2x)\ \forall\ x, y \in \mathbb{R}$,
$f(x) = f(2^x, 0)$ and $f(x)$ is periodic with period k.
$\Rightarrow\ f(x) = f(2^x, 0) = f(2.2^x + 2(0), 2(0) - 2.2^x)$
$= f(2^{x+1}, 2^{x+1})$
$= f(2.2^{x+1} - 2.2^{x+1}, -2.2^{x+1} - 2.2^{x+1})$
$= f(0, -2^{x+3})$
$= f(2.(-2^{x+3}), -2.2^{x+3}) = f(-2^{x+4}, -2^{x+4})$
$= f(-2^{x+6}, 0)$
$= f(-2^{x+7}, 2^{x+7}) = f(0, 2^{x+9})$
$= f(2^{x+10}, 20^{x+10}) = f(2^{x+12}, 0) = f(x+12)$
$\Rightarrow\ f(x)$ is periodic with period $12 \Rightarrow\ k = 12$.

25. **(6)** $f(x)=\left[\frac{x}{15}\right]\left[-\frac{15}{x}\right] \qquad x \in (0, 90)$

$0 \le x < 15 \qquad f(x)=0$

$15 \le x < 30 \qquad f(x)=-1$

$30 \le x < 45 \qquad f(x)=-2$

$45 \le x < 60 \qquad f(x)=-3$

$60 \le x < 75 \qquad f(x)=-4$

$75 \le x < 90 \qquad f(x)=-5$

Total integers in range $f(x)=\{0,-1,-2,-3,-4,-5\}$