# UNIT - 7 : PERMUTATION AND COMBINATION [JEE – MAIN CRASH COURSE]

## Factorial Notation

The product of first $n$ natural numbers is denoted by $n!$ and is read as "factorial $n$". Thus

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-1) \cdot n$$
$$= n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$$

***Notes:***

- $n! = (n-1)! \cdot n = n \cdot (n-1)!$

  Similarly, $(n-1)! = (n-1) \cdot (n-2)!$

  Thus $n! = n \cdot (n-1)!$
  $= n \cdot (n-1) \cdot (n-2)!$
  $= n \cdot (n-1) \cdot (n-2) \cdot (n-3)!$

  and so on

- If $n$ and $r$ are positive integers, then

$$\frac{n!}{r!} = \frac{1 \cdot 2 \cdot 3 \cdots n}{1 \cdot 2 \cdot 3 \cdots r}$$
$$= \frac{1 \cdot 2 \cdot 3 \cdots (r-1) \cdot r \cdot (r+1) \cdot (r+2) \cdots (n-1) \cdot n}{1 \cdot 2 \cdot 3 \cdots (r-1) \cdot r}$$
$$= (r+1) \cdot (r+2) \cdots (n-1) \cdot n$$
$$= n \cdot (n-1) \cdot (n-2) \cdots (r+1)$$

## Exponent of Prime in $n!$

Let $p$ be a given prime and $n$ any positive integer, then maximum power of $p$ present in $n!$ is $\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \cdots$ where $[\cdot]$ denotes the greatest integer function.

The above formula does not work for composite numbers. For example, if we have to find the maximum power of 6 present in 32! then the answer is not $\left[\frac{31}{6}\right] + \left[\frac{32}{6^2}\right] + \cdots = 5$, as 5 is the number of integral multiples of 6 in 1, 2, ..., 32; and 6 can be obtained on multiplying 2 by 3 also. Hence for the required number, we find the maximum powers of 2 and 3 (say $r$ and $s$) present in 32!. Using the above formula $r = 31$ and $s = 14$. Hence 2 and 3 will be combined (to form 6) 14 times. Thus maximum power of 6 present is 32! is 14.

## Fundamental Principle of Counting

**Multiplication rule** If a work $A$ can be done in $m$ ways and another work $B$ can be done in $n$ ways and $C$ is a work which is done only when both $A$ and $B$ are done.

**Addition rule** If a work $A$ can be done in $m$ ways and another work $B$ can be done in $n$ ways and $C$ is a work which is done only when either $A$ or $B$ is done, then number of ways of doing the work $C$ is $(m + n)$.

## Permutation

Each of the different arrangements which can be made by taking some or all of a number of given things or objects at a time is called a permutation. In permutation order of appearance of things is taken into account.

${}^nP_r$ = number of permutations of $r$ things out of $n$ different things.

= numbers of ways of filling up $r$ vacant places with $n$ different things. (in each place exactly one object is put) $(n > r)$

= numbers of ways of filling up $n$ vacant places with $r$ different things. (in each place exactly one object is put) $(n > r)$

For example, the following six arrangements can be made with three distinct objects taking two at a time *ab*, *ba*, *bc*, *cb*, *ac*, *ca*. Each of these arrangements is called a permutation.

### Different cases of permutations

- The number of permutations of $n$ different things taken $r$ at a time: ${}^nP_r = \dfrac{n!}{(n-r)!}$.
- The number of permutations of $n$ different things taken all at a time $= n!$.
- The number of permutations of things taken all at a time when $p$ of them are similar and are of one type, $q$ of them are similar and are of second type, $r$ of them are similar and are of third type and rest are all different: $\dfrac{n!}{p!\,q!\,r!}$.
- The number of permutations of $n$ different things taken $r$ at a time when each thing can be repeated $r$ times: $n^r$.

### Permutations under restrictions

#### *When particular objects are never together (gap method)*

**Illustration:** Find the number of ways five girls and five boys can be arranged in row if no two boys are together.

**Sol.** In the question there is no any condition for arranging the girls. Now five girls can be arranged in 5! ways.

$$\times\, G \times G \times G \times G \times G \times$$

When girls are arranged, six gaps are generated as shown in diagram with "×".

Now boys must occupy the places with "×" marked so that no two boys are together.

Now five boys can be arranged in these six gaps in ${}^6P_5$ ways. Hence, total number of arrangement is $5! \times {}^6P_5$.

#### *When particular objects are always together*

**Illustration:** Find the number of ways in which six boys and six girls be seated in a row so that all the girls sit together and all the boys sit together.

**Sol.** $(b_1\, b_2\, b_3\, b_4\, b_5\, b_6)$ $(g_1\, g_2\, g_3\, g_4\, g_5\, g_6)$

Considering boys and girls as two units, there permutations $= 2! \cdot 6! \cdot 6! = 2 \cdot (6!)^2$.

### Circular permutations

- The total number of circular arrangements of $n$ persons is $\dfrac{n!}{n} = (n-1)!$.
- The total number of different arrangements of $n$ different beads in circle is $\dfrac{1}{2}(n-1)!$.

## Combination

Each of the different groups or selections which can be made by taking some or all of a number of given things or objects at a time is called a combination. In combination, order of appearance of things is not taken into account.

### Different cases of combinations (selections)

| Case | Number of ways |
|---|---|
| Number of combinations of $n$ different things taking $r$ at a time $(r < n)$ or Number of ways $r$ objects can be selected from $n$ different objects | ${}^nC_r = \dfrac{n!}{r!(n-r)!} = \dfrac{{}^nP_r}{r!}$ |
| Number of combinations of $n$ different things taken $r$ at a time when $p$ particular things are always included | ${}^{n-p}C_{r-p}$ |
| Number of combinations of $n$ different things taken $r$ at a time when $p$ particular things are always to be excluded | ${}^{n-p}C_r$ |
| The total number of combinations of $n$ different things taken one or more at a time or Number of ways in which at least one object can be selected from $n$ different objects | ${}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = ({}^nC_0 + {}^nC_1 \cdots + {}^nC_n) - {}^nC_0 = 2^n - 1$ |
| Number of ways of selecting at least one thing from each of three groups of size $p$, $q$, and $r$. Each group containing different objects | $(2^p-1)(2^q-1)(2^r-1)$ |
| Number of ways of selecting $r$ things out of $n$ identical things $(r \le n)$ | 1 |
| The number of ways of selecting at least one thing out of $n$ identical things | $n$ |
| Number of ways of selecting zero or more things out of $n$ identical things | $n+1$ |
| Number of ways of selecting at least one thing out of $p$ identical things of one type, $q$ identical things of another type, $r$ identical things of the third type | $(p+1)(q+1)(r+1)-1$ |

| Case | Number of ways |
|---|---|
| Number of ways of selecting at least one thing of each type out of $p$ identical things of one type, $q$ identical things of another type, $r$ identical things of the third type | $pqr$ |
| Number of ways of selecting exactly $r$ objects from two groups one containing $n$ different objects and other containing $m$ different objects | ${}^mC_r\,{}^nC_0 + {}^mC_{r-1}\,{}^nC_1 + \cdots + {}^mC_0\,{}^nC_r$ when $r \le m, n$; ${}^mC_m\,{}^nC_{r-m} + {}^mC_{m-1}\,{}^nC_{r-(m-1)} + \cdots + {}^mC_{r-n}\,{}^nC_n$ when $m, n \le r \le m + n$ |
| Number of ways of selecting exactly $r$ objects from two groups one containing $n$ different objects and other containing $m$ identical objects | ${}^mC_r + {}^mC_{r-1} + \cdots + {}^mC_0$, when $r \le m, n$; ${}^mC_m + {}^mC_{m-1} + \cdots + {}^mC_{r-n}$, when $m, n \le r \le m + n$ |

## Number of Divisors

Number of divisors of the number $n = a^\alpha b^\beta c^\gamma \ldots$ are $(\alpha + 1)(\beta + 1)(\gamma + 1)\ldots$

where $a, b, c, \ldots$ are distinct prime numbers.

As number of divisors is equivalent to number of ways of selecting zero or more objects from $(a, a, a, \ldots \alpha$ times), $(b, b, b \ldots \beta$ times), $(c, c, c, \ldots \gamma$ times), …

Also number of divisors of $n$ can been seen as number of different terms in the expansion of

$$(a^0 + a^1 + a^2 + \cdots + a^\alpha) \times (b^0 + b^1 + b^2 + \cdots + b^\beta)$$

$\times\ (c^0 + c^1 + c^2 + \cdots + c^\gamma) \ldots$ which also gives sum of divisors which is $\dfrac{a^\alpha - 1}{a - 1} \times \dfrac{b^\beta - 1}{b - 1} \times \dfrac{c^\gamma - 1}{c - 1} \cdots$

## Combination Based on Geometry

1. Number of diagonals in convex polygon of $n$ sides: For diagonal we must join any two vertices but not adjacent vertices. Hence number of diagonals $= {}^nC_2 - n$.
2. Number of rectangles in chess board: There are nine vertical lines and nine horizontal lines on the chess board. For rectangle we require two horizontal lines and two vertical lines, which can be selected in ${}^9C_2 \times {}^9C_2$ ways.
3. Number of lines joining $n$ points on the plane when $r$ points are collinear $(n \ge r)$: Number of lines is equivalent to number of ways two points can be selected which are ${}^nC_2$ but of this ${}^rC_2$ selection gives the same line, which must be considered once. Hence number of lines $= {}^nC_2 - {}^rC_2 + 1$.

## Division and Distribution

| Case | Number of ways |
|---|---|
| Division of $m + n$ distinct objects into two groups of the size $m$ and $n$. $(m \ne n)$ This is equivalent to number of ways of selecting $m$ objects from $m + n$ distinct objects for the first group | ${}^{m+n}C_m = \dfrac{(m+n)!}{m!\,n!}$ |
| Division of $m + n + p$ distinct objects into three groups of the size $m$, $n$, and $p$. $(m \ne n \ne p)$ This is equivalent to number of ways of selecting $m$ objects from $m + n + p$ distinct objects for the first group and then selecting $n$ objects from remaining $n + p$ objects for the second group | ${}^{m+n+p}C_m\,{}^{n+p}C_n = \dfrac{(m+n+p)!}{m!\,n!\,p!}$ |
| Distribution of $m + n$ distinct objects between two persons if one gets $m$ and the other gets $n$ objects This is equivalent to the first forming groups of size $m$ and $n$ and then distributing these two groups between two persons | $\dfrac{(m+n)!}{m!\,n!}2!$ |
| Distribution of $m + n + p$ objects among three persons if they get $m$, $n$, and $p$ objects This is equivalent to the first forming groups of size $m$, $n$, and $p$ and then distributing these three groups among three persons | $\dfrac{(m+n+p)!}{m!\,n!\,p!}3!$ |
| Division of $x_1 + x_2 + x_3 \cdots + x_n$ into $n$ groups of the size $x_1, x_2, x_3, \cdots, x_n$ $(x_1 \ne x_2 \cdots \ne x_n)$ | $\dfrac{(x_1 + x_2 \ldots x_n)!}{x_1!\,x_2! \ldots x_n!}$ |
| Distribution ways of the above $n$ groups among $n$ persons | $\dfrac{(x_1 + x_2 \ldots x_n)!}{x_1!\,x_2! \ldots x_n!}n!$ |
| Division of $2n$ objects into two groups of equal size (**Note:** Here group size is same) | $\dfrac{\frac{(2n)!}{n!\,n!}}{2!} = \dfrac{(2n)!}{n!\,n!\,2!}$ |
| Distribution ways of this two groups between two persons | $\dfrac{(2n)!}{n!\,n!\,2!}2! = \dfrac{(2n)!}{n!\,n!}$ |

| Case | Number of ways |
|---|---|
| Division of $3n$ objects into three groups of equal size $n$ | $\dfrac{\frac{(3n)!}{n!\,n!\,n!}}{3!} = \dfrac{(3n)!}{n!\,n!\,n!\,3!}$ |
| Distribution ways of this three groups among three persons | $\dfrac{(3n)!}{n!\,n!\,n!\,3!}3! = \dfrac{(3n)!}{n!\,n!\,n!}$ |
| Distribution of $n$ distinct objects in $r$ different boxes if in any box any number of objects are placed (empty boxes are allowed)<br>Here each object has $r$ possibilities. Hence number of ways $= r \times r \times r \times r \cdots n$ times $= r^n$ | $r^n$ |
| Distribution of $n$ distinct objects into $r$ different boxes if empty boxes are not allowed or in each box at least one object is put. (This can be derived from the principle of inclusion and exclusion) | $r^n - {}^rC_1(r-1)^n + {}^rC_2(r-2)r^n - {}^rC_3(r-3)^n \cdots + (-1)^{r-1}\,{}^rC_{r-1}1$ |

| Case | Number of ways |
|---|---|
| Distribution of $n$ identical objects in $r$ different boxes if empty boxes are not allowed<br>or<br>Number of positive integral solutions of the equation $x_1 + x_2 + \cdots + x_r = n$ | ${}^{n-1}C_{r-1}$ |
| Distribution of $n$ identical objects in $r$ different boxes if empty boxes are allowed<br>or<br>Number of non-negative integral solutions of the equation $x_1 + x_2 + \cdots + x_r = n$ | ${}^{n+r-1}C_{r-1}$ |

## Derangement

If there are $n$ letters and $n$ corresponding envelopes. In how many ways letters can be placed in the envelopes (one letter in each envelope) so that no letter is placed in correct envelope. The number of ways are

$$n!\left[1 - \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{(-1)^n}{n!}\right]$$