Session 2

Area Bounded by Two or More Curves

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Area bounded by the curves $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$.

Let the curves $y = f(x)$ and $y = g(x)$ be represented by AB and CD, respectively. We assume that the two curves do not intersect each other in the interval $[a, b]$.

Thus, shaded area = Area of curvilinear trapezoid $APQB$ − Area of curvilinear trapezoid $CPQD$

$$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b \{f(x) - g(x)\}\,dx$$

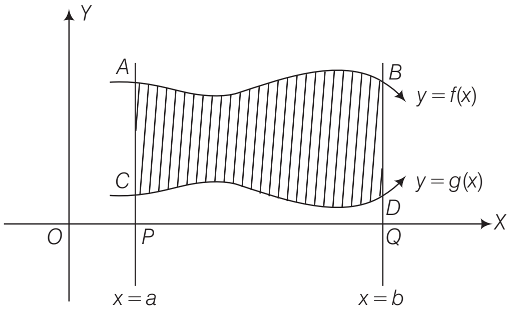


Figure 3.32

Now, consider the case when $f(x)$ and $g(x)$ intersect each other in the interval $[a, b]$.

First of all we should find the intersection point of $y = f(x)$ and $y = g(x)$. For that we solve $f(x) = g(x)$. Let the root is $x = c$. (We consider only one intersection point to illustrate the phenomenon).

Thus, required (shaded) area

$$= \int_a^c \{f(x) - g(x)\}dx + \int_c^b \{g(x) - f(x)\}\,dx$$

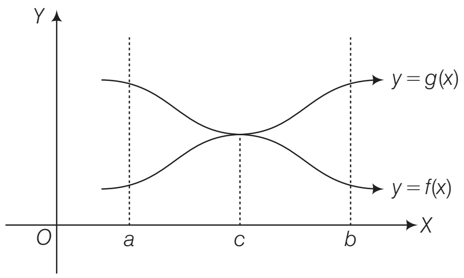


Figure 3.33

If confusion arises in such case evaluate $\int_a^b |f(x) - g(x)|\,dx$ which gives the required area.

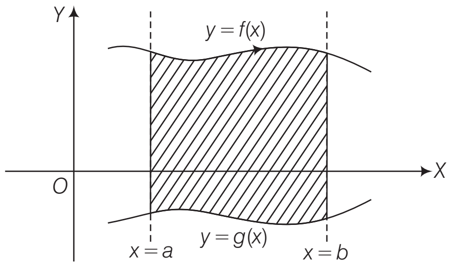


Figure 3.34

Area between two curves $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$ is always given by $\int_a^b \{f(x) - g(x)\}\,dx$ provided $f(x) > g(x)$ in $[a, b]$; the position of the graph is immaterial. As shown in Fig. 3.34, Fig. 3.35, Fig. 3.36.

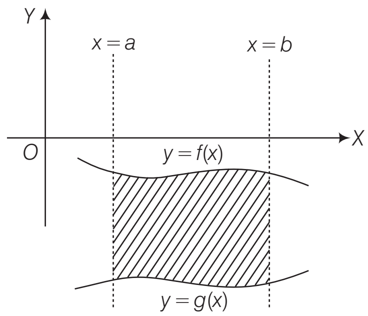


Figure 3.35

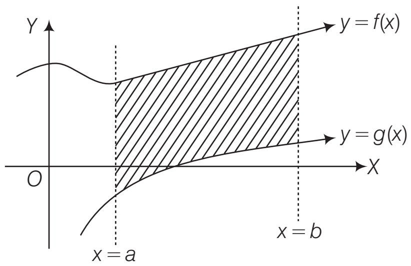
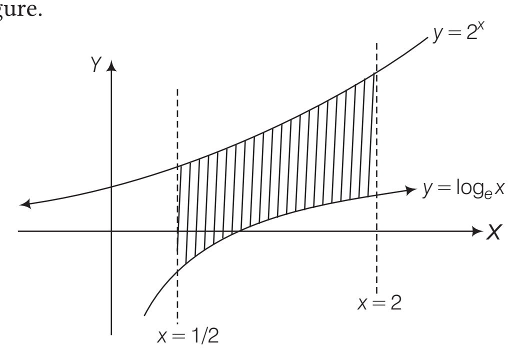


Figure 3.36

Example 17 Sketch the curves and identify the region bounded by $x = 1/2$, $x = 2$, $y = \log_e x$ and $y = 2^x$. Find the area of this region. ***[IIT JEE 1991]***

Sol. The required area is the shaded portion in the following figure.



In the region $\frac{1}{2} \le x \le 2$; the curve $y = 2^x$ lies above as compared to $y = \log_e x$.

Hence, required area $= \int_{1/2}^{2} (2^x - \log x)\, dx$

$$= \left[\frac{2^x}{\log 2} - (x \log x - x) \right]_{1/2}^{2}$$

$$= \left(\frac{4 - \sqrt{2}}{\log 2} - \frac{5}{2} \log 2 + \frac{3}{2} \right) \text{sq units}$$

Example 18 Find the area given by $x + y \le 6, x^2 + y^2 \le 6y$ and $y^2 \le 8x$.

Sol. Let us consider the curves

$$P \equiv y^2 - 8x = 0 \quad \text{...(i)}$$

$$C \equiv x^2 + y^2 = 6y$$

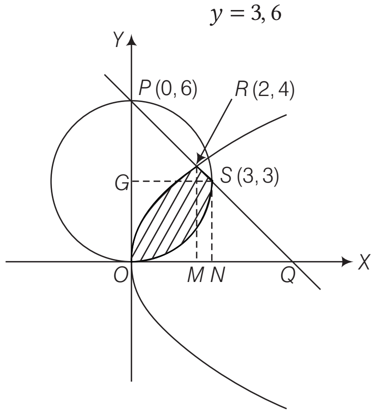
i.e. $$x^2 + (y-3)^2 - 9 = 0 \quad \text{...(ii)}$$

and $$S \equiv x + y - 6 = 0 \quad \text{...(iii)}$$

The intersection points of the curves (ii) and (iii) are given by

$$(6 - y)^2 + y^2 - 6y = 0$$

i.e. $$y = 3, 6$$



Therefore, the points are (0, 6) and (3, 3). The intersection points of the curves (i) and (iii) are given by

$$y^2 = 8(6 - y), \quad \text{i.e. } y = 4, -12$$

Therefore, the point of intersection in 1st quadrant is (2, 4).

Now, we know that

$C \le 0$ denotes the region, inside the circle $C = 0$.

$P \le 0$ denotes the region, inside the parabola $P = 0$.

$S \le 0$ denotes the region, which is negative side of the line $S = 0$.

$\therefore$ Required area = Area of curvilinear $\Delta OMRO$ + Area of trapezium $MNSR$ − Area of curvilinear $\Delta ONSO$

$$= \int_0^2 \sqrt{8x}\, dx + \frac{1}{2}(MR + NS) \cdot MN$$

− (Area of square $ONSG$ − Area of sector $OSGO$)

$$= \int_0^2 \sqrt{8x}\, dx + \frac{1}{2}(4 + 3) \cdot 1 - \left(3^2 - \frac{\pi \cdot 3^2}{4} \right)$$

$$= \left(\frac{9\pi}{4} - \frac{1}{6} \right) \text{sq units}$$

Example 19 Find the area of the region $\{(x, y) : 0 \le y \le x^2 + 1, 0 \le y \le x + 1, 0 \le x \le 2\}$.

Sol. Let $R = \{(x, y) : 0 \le y \le x^2 + 1, 0 \le y \le x + 1, 0 \le x \le 2\}$

$$= \{(x, y) : 0 \le y \le x^2 + 1\} \cap \{(x, y) : 0 \le y \le x + 1\} \cap \{(x, y) : 0 \le x \le 2\}$$

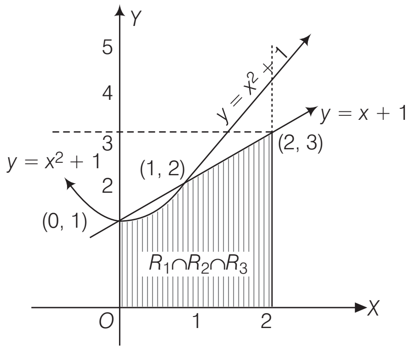
$$= R_1 \cap R_2 \cap R_3$$

where, $R_1 = \{(x, y) : 0 \le y \le x^2 + 1\}$

$R_2 = \{(x, y) : 0 \le y \le x + 1\}$

and $R_3 = \{(x, y) : 0 \le x \le 2\}$

Thus, the sketch of R_1, R_2 and R_3 are



From the above figure,

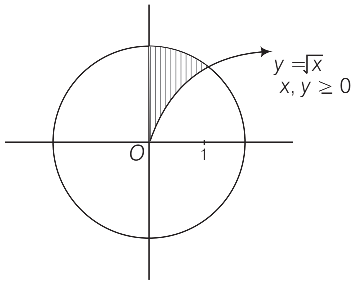
$$\text{Required area} = \int_0^1 (x^2 + 1)\, dx + \int_1^2 (x + 1)\, dx$$

$$= \left(\frac{x^3}{3} + x \right)_0^1 + \left(\frac{x^2}{2} + x \right)_1^2 = \frac{23}{6} \text{ sq units}$$

Example 20 The area common to the region determined by $y \ge \sqrt{x}$ and $x^2 + y^2 < 2$ has the value

(a) π sq units
(b) $(2\pi - 1)$ sq units
(c) $\left(\frac{\pi}{4} - \frac{1}{6} \right)$ sq units
(d) None of these

Sol. The region formed by $y \geq \sqrt{x}$ is the outer region of the parabola $y^2 = x$, when $y \geq 0$ and $x \geq 0$ and $x^2 + y^2 < 2$ is the region inner to circle $x^2 + y^2 = 2$ shown as in figure.



Now, to find the point of intersection put $y^2 = x$ in $x^2 + y^2 = 2$.

$$\Rightarrow \quad x^2 + x - 2 = 0$$

$$\Rightarrow \quad (x+2)(x-1) = 0$$

$$\Rightarrow \quad x = 1, \text{ as } x \geq 0$$

$\therefore$ Required area $= \int_0^1 (\sqrt{2-x^2} - \sqrt{x})\, dx$

$$= \left[\frac{x\sqrt{2-x^2}}{2} + \sin^{-1}\frac{x}{\sqrt{2}} - \frac{2}{\sqrt{3}}x^{3/2}\right]_0^1$$

$$= \frac{1}{2} + \frac{\pi}{4} - \frac{2}{3} = \left(\frac{\pi}{4} - \frac{1}{6}\right) \text{ sq units}$$

Hence, (c) is the correct answer.

Example 21 Find the area of the region enclosed by the curve $5x^2 + 6xy + 2y^2 + 7x + 6y + 6 = 0$.

Sol. Comparing $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get $a = 5, b = 2, h = 3, g = 7/2, f = 3$ and $c = 6$

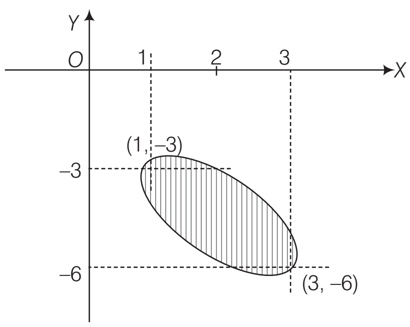
$$\Rightarrow \quad h^2 - ab = -1 < 0$$

So, the above equation represents an ellipse.

$$\therefore \quad 2y^2 + 6(1+x)y + (5x^2 + 7x + 6) = 0$$

$$\Rightarrow \quad y = \frac{-3(1+x) \pm \sqrt{(3-x)(x-1)}}{2}$$

Clearly, the values of y are real for all $x \in [1, 3]$. Thus, the graph is as shown below



Thus, required area

$$= \left| \int_1^3 \left(\frac{-3(1+x) - \sqrt{(3-x)(x-1)}}{2}\right) - \left(\frac{-3(1+x) + \sqrt{(3-x)(x-1)}}{2}\right) dx \right|$$

$$= \left| -\int_1^3 \sqrt{(3-x)(x-1)}\, dx \right| = \left| \int_1^3 \sqrt{1^2 - (x-2)^2}\, dx \right|$$

$$= \left| -\left\{\frac{1}{2}(x-2)\sqrt{-x^2 + 4x - 3} + \frac{1}{2}\sin^{-1}\left(\frac{x-2}{1}\right)\right\}_1^3 \right|$$

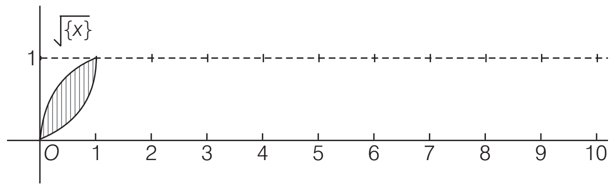
$$= \frac{\pi}{2} \text{ sq units}$$

Example 22 ***If*** $f(x) = \begin{cases} \sqrt{\{x\}}, & x \notin Z \\ 1, & x \in Z \end{cases}$ ***and*** $g(x) = \{x\}^2$ ***(where, {.} denotes fractional part of*** x***), then the area bounded by*** $f(x)$ ***and*** $g(x)$ ***for*** $x \in [0, 10]$ ***is***

(a) $\frac{5}{3}$ sq units (b) 5 sq units

(c) $\frac{10}{3}$ sq units (d) None of these

Sol. As, $f(x) = \begin{cases} \sqrt{\{x\}}, & x \notin z \\ 1, & x \in z \end{cases}$ and $g(x) = \{x\}^2$, where both $f(x)$ and $g(x)$ are periodic with period '1' shown as



Thus, required area $= 10\int_0^1 [\sqrt{\{x\}} - \{x\}^2]\, dx$

$$= 10\int_0^1 [(x)^{1/2} - x^2]\, dx$$

$$= 10\left[\frac{x^{3/2}}{3/2} - \frac{x^3}{3}\right]_0^1$$

$$= 10\left(\frac{2}{3} - \frac{1}{3}\right) = \frac{10}{3} \text{ sq units}$$

Hence, (c) is the correct answer.

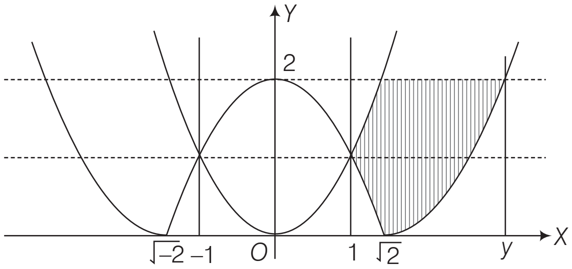
Example 23 Find the area of the region bounded by the curves $y = x^2$, $y = |2 - x^2|$ and $y = 2$, which lies to the right of the line $x = 1$. ***[IIT JEE 2002]***

Sol. The region bounded by given curves on the right side of $x = 1$ is shown as

Required area $= \int_1^{\sqrt{2}} \{x^2 - (2 - x^2)\}\, dx + \int_{\sqrt{2}}^2 \{4 - x^2\}\, dx$

$$= \int_1^{\sqrt{2}} (2x^2 - 2)\, dx + \int_{\sqrt{2}}^{2} (4 - x^2)\, dx$$

$$= \left(2 \cdot \frac{x^3}{3} - 2x\right)_1^{\sqrt{2}} + \left(4x - \frac{x^3}{3}\right)_{\sqrt{2}}^{2}$$



$$= \left(\frac{4\sqrt{2}}{3} - 2\sqrt{2}\right) - \left(\frac{2}{3} - 2\right) + \left(8 - \frac{8}{3}\right) - \left(4\sqrt{2} - \frac{2\sqrt{2}}{3}\right)$$

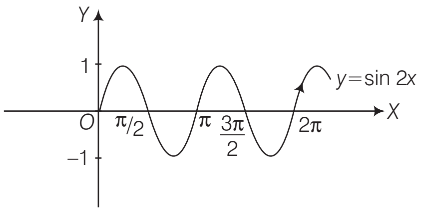
$$= \left(-4\sqrt{2} + \frac{20}{3}\right) = \left(\frac{20 - 12\sqrt{2}}{3}\right) \text{ sq units}$$

Example 24 The area enclosed by the curve $|y| = \sin 2x$, when $x \in [0, 2\pi]$ is

(a) 1 sq unit (b) 2 sq units

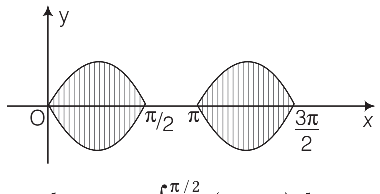
(c) 3 sq units (d) 4 sq units

Sol. As, we know $y = \sin 2x$ could be plotted as



Thus, $|y| = \sin 2x$ is whenever positive, y can have both positive and negative values, i.e. the curve is symmetric about the axes.

$\sin 2x$ is positive only in $0 \le x \le \frac{\pi}{2}$ and $\pi \le x \le \frac{3\pi}{2}$ · Thus, the curve consists of two loops one in $\left[0, \frac{\pi}{2}\right]$ and another in $\left[\pi, \frac{3\pi}{2}\right]$.



Thus, required area $= 4 \int_0^{\pi/2} (\sin 2x)\, dx$

$$= 4\left(-\frac{\cos 2x}{2}\right)_0^{\pi/2} = -2(\cos \pi - \cos 0)$$

$$= -2(-1 - 1) = 4 \text{ sq units}$$

Hence, (d) is the correct answer.

Example 25 Let $f(x) = x^2$, $g(x) = \cos x$ and $\alpha, \beta\ (\alpha < \beta)$ be the roots of the equation $18x^2 - 9\pi x + \pi^2 = 0$. Then, the area bounded by the curves $y = fog(x)$, the ordinates $x = \alpha$, $x = \beta$ and the X-axis is

(a) $\frac{1}{2}(\pi - 3)$ sq units (b) $\frac{\pi}{3}$ sq units

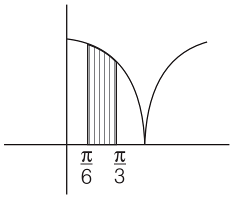
(c) $\frac{\pi}{4}$ sq units (d) $\frac{\pi}{12}$ sq units

Sol. Here, $y = fog(x) = f\{g(x)\} = (\cos x)^2 = \cos^2 x$

Also, $18x^2 - 9\pi x + \pi^2 = 0$

$\Rightarrow$ $(3x - \pi)(6x - \pi) = 0$

$\Rightarrow$ $x = \frac{\pi}{6}, \frac{\pi}{3}$ (as α, β)



$\therefore$ Required area of curve

$$= \int_{\pi/6}^{\pi/3} \cos^2 x\, dx = \frac{1}{2}\int_{\pi/6}^{\pi/3} (1 + \cos 2x)\, dx$$

$$= \frac{1}{2}\left\{x + \frac{\sin 2x}{2}\right\}_{\pi/6}^{\pi/3} = \frac{1}{2}\left\{\left(\frac{\pi}{3} - \frac{\pi}{6}\right) + \frac{1}{2}\left(\sin\frac{2\pi}{3} - \sin\frac{2\pi}{6}\right)\right\}$$

$$= \frac{1}{2}\left\{\frac{\pi}{6} + \frac{1}{2}\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right)\right\} = \frac{\pi}{12}$$

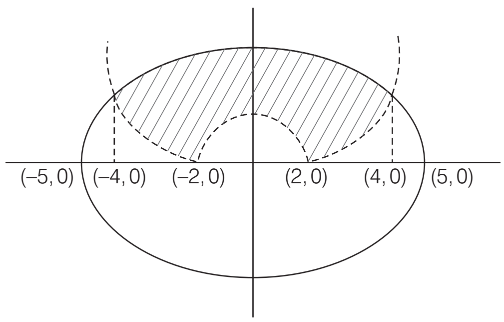
Hence, (d) is the correct answer.

Example 26 ***Find the area bounded by the curves $x^2 + y^2 = 25$, $4y = |4 - x^2|$ and $x = 0$ above the X-axis.***

Sol. The 1st curve is a circle of radius 5 with centre at (0, 0).

The 2nd curve is $y = \left|\frac{4 - x^2}{4}\right| = \left|1 - \frac{x^2}{4}\right|$

which can be traced easily by graph transformation.



When the two curves intersect each other, then

$$x^2 + \left(1 - \frac{x^2}{4}\right)^2 = 25 \Rightarrow x = \pm 4$$

Hence, required area $= 2\int_0^4 \left(\sqrt{25 - x^2} - \left|1 - \frac{x^2}{4}\right|\right) dx$

$$= 2\left[\int_0^4 \sqrt{25 - x^2}\,dx - \int_0^2 \left(1 - \frac{x^2}{4}\right) dx + \int_2^4 \left(1 - \frac{x^2}{4}\right) dx\right]$$

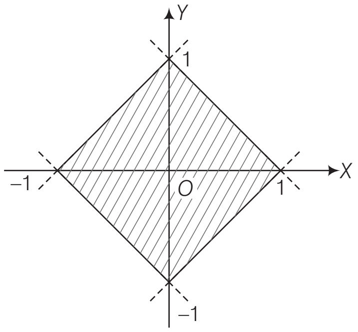
$$= 2\left[6 + \frac{25}{2}\sin^{-1}\left(\frac{4}{5}\right) - \frac{4}{3} - \frac{8}{3}\right] = \left\{25\sin^{-1}\left(\frac{4}{5}\right) + 4\right\}$$

Example 27 ***Find the area enclosed by*** $|x| + |y| = 1$.

Sol. From the given equation, we have

$$|y| = 1 - |x| \,|\, 0 \qquad [\because |y| \,|\, 0]$$

$$\Rightarrow \qquad -1 \le x \le 1$$



Therefore, the curve exists for $x \in [-1, 1]$ only

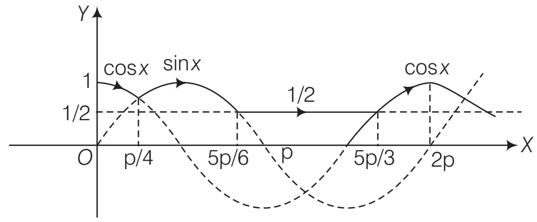
and for $-1 \le x \le 1$; $\quad y = \pm(1 - |x|)$ i.e. $y = \begin{cases} |x| - 1 \\ -(|x| - 1) \end{cases}$

Thus, the required graph is as given in figure.

$\therefore$ Required area $= (\sqrt{2})^2 = 2$ sq units

Example 28 Let $f(x) = \max\left\{\sin x, \cos x, \frac{1}{2}\right\}$, then determine the area of region bounded by the curves $y = f(x)$, X-axis, Y-axis and $x = 2\pi$.

Sol. We have, $f(x) = \max\left\{\sin x, \cos x, \frac{1}{2}\right\}$. Graphically, $f(x)$ could be drawn as



Here, the graph is plotted between 0 to 2π and between the points of intersection the maximum portion is included, thus the shaded part is required area

		Interval	Value of $f(x)$
i.e.	for	$0 \le x < \pi/4$	$\cos x$
	for	$\pi/4 \le x < 5\pi/6$	$\sin x$
	for	$5\pi/6 \le x < 5\pi/3$	$1/2$
	for	$5\pi/3 \le x < 2\pi$	$\cos x$

Hence, required area

$$I = \int_0^{\pi/4} \cos x\,dx + \int_{\pi/4}^{5\pi/6} \sin x\,dx + \int_{5\pi/6}^{5\pi/3} \frac{1}{2}\,dx + \int_{5\pi/3}^{2\pi} \cos x\,dx$$

$$= (\sin x)_0^{\pi/4} - (\cos x)_{\pi/4}^{5\pi/6} + \frac{1}{2}(x)_{5\pi/6}^{5\pi/3} + (\sin x)_{5\pi/3}^{2\pi}$$

$$= \left(\frac{1}{\sqrt{2}} - 0\right) - \left(-\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\right) + \frac{1}{2}\left(\frac{5\pi}{3} - \frac{5\pi}{6}\right) + \left(0 + \frac{\sqrt{3}}{2}\right)$$

$$= \left(\frac{5\pi}{12} + \sqrt{2} + \sqrt{3}\right) \text{ sq units}$$

Exercise for Session 2

1. The area of the region bounded by $y^2 = 2x + 1$ and $x - y - 1 = 0$ is

(a) 2/3 (b) 4/3 (c) 8/3 (d) 16/3

2. The area bounded by the curve $y = 2x - x^2$ and the straight line $y = x$ is given by

(a) 9/2 (b) 43/6 (c) 35/6 (d) None of these

3. The area bounded by the curve $y = x|x|$, X-axis and the ordinates $x = -1$, $x = 1$ is given by

(a) 0 (b) 1/3 (c) 2/3 (d) None of these

4. Area of the region bounded by the curves $y = 2^x$, $y = 2x - x^2$, $x = 0$ and $x = 2$ is given by

(a) $\frac{3}{\log 2} - \frac{4}{3}$ (b) $\frac{3}{\log 2} + \frac{4}{3}$ (c) $3\log 2 - \frac{4}{3}$ (d) None of these

5. The area of the figure bounded by the cuves $y = e^x$, $y = e^{-x}$ and the straight line $x = 1$ is

(a) $e + \frac{1}{e}$ (b) $e - \frac{1}{e}$ (c) $e + \frac{1}{e} - 2$ (d) None of these

6. Area of the region bounded by the curve $y^2 = 4x$, Y-axis and the line $y = 3$ is

(a) 2 (b) 9/4 (c) $6\sqrt{3}$ (d) None of these

7. The area of the figure bounded by $y = \sin x$, $y = \cos x$ is the first quadrant is

(a) $2(\sqrt{2} - 1)$ (b) $\sqrt{3} + 1$ (c) $2(\sqrt{3} - 1)$ (d) None of these

8. The area bounded by the curves $y = xe^x$, $y = xe^{-x}$ and the line $x = 1$ is

(a) $\frac{2}{e}$ (b) $1 - \frac{2}{e}$ (c) $\frac{1}{e}$ (d) $1 - \frac{1}{e}$

9. The areas of the figure into which the curve $y^2 = 6x$ divides the circle $x^2 + y^2 = 16$ are in the ratio

(a) $\frac{2}{3}$ (b) $\frac{4\pi - \sqrt{3}}{8\pi + \sqrt{3}}$ (c) $\frac{4\pi + \sqrt{3}}{8\pi - \sqrt{3}}$ (d) None of these

10. The area bounded by the Y-axis, $y = \cos x$ and $y = \sin x$, $0 \geq x \leq \pi/2$ is

(a) $2(\sqrt{2} - 1)$ (b) $\sqrt{2} - 1$ (c) $(\sqrt{2} + 1)$ (d) $\sqrt{2}$

11. The area bounded by the curve $y = \frac{3}{|x|}$ and $y + |2 - x| = 2$ is

(a) $\frac{4 - \log 27}{3}$ (b) $2 - \log 3$ (c) $2 + \log 3$ (d) None of these

12. The area bounded by the curves $y = x^2 + 2$ and $y = 2|x| - \cos + x$ is

(a) 2/3 (b) 8/3 (c) 4/3 (d) 1/3

13. The are bounded by the curve $y^2 = 4x$ and the circle $x^2 + y^2 - 2x - 3 = 0$ is

(a) $2\pi + \frac{8}{3}$ (b) $4\pi + \frac{8}{3}$ (c) $\pi + \frac{8}{3}$ (d) $\pi - \frac{8}{3}$

14. A point P moves inside a triangle formed by $A(0, 0)$, $B\left(1, \frac{1}{\sqrt{3}}\right)$, $C(2, 0)$ such that min $\{PA, PB, PC\} = 1$, then the area bounded by the curve traced by P, is

(a) $3\sqrt{3} - \frac{3\pi}{2}$ (b) $\sqrt{3} + \frac{\pi}{2}$ (c) $\sqrt{3} - \frac{\pi}{2}$ (d) $3\sqrt{3} + \frac{3\pi}{2}$

15. The graph of $y^2 + 2xy + 40|x| = 400$ divides the plane into regions. The area of the bounded region is

(a) 400 (b) 800 (c) 600 (c) None of these

16. The area of the region defined by $||x| - |y|| \leq 1$ and $x^2 + y^2 \leq 1$ in the xy- plane is

(a) π (b) 2π (c) 3π (d) 1

17. The area of the region defined by $1 \leq |x - 2| + |y + 1| \leq 2$ is

(a) 2 (b) 4 (c) 6 (d) None of these

18. The area of the region enclosed by the curve $|y| = -(1 - |x|)^2 + 5$, is

(a) $\frac{8}{3}(7 + 5\sqrt{5})$ sq units (b) $\frac{2}{3}(7 + 5\sqrt{5})$ sq units (c) $\frac{2}{3}(5\sqrt{5} - 7)$ sq units (d) None of these

19. The area bounded by the curve $f(x) = ||\tan x + \cot x| - |\tan x - \cot x||$ between the lines $x = 0$, $x = \frac{\pi}{2}$ and the X-axis is

(a) $\log 4$ (b) $\log\sqrt{2}$ (c) $2\log 2$ (d) $\sqrt{2}\log 2$

20. If $f(x) = \max\left\{\sin x, \cos x, \frac{1}{2}\right\}$, then the area of the region bounded by the curves $y = f(x)$, X-axis, Y-axis and $x = \frac{5\pi}{3}$ is

(a) $\left(\sqrt{2} - \sqrt{3} + \frac{5\pi}{12}\right)$ sq units (b) $\left(\sqrt{2} + \sqrt{3} + \frac{5\pi}{2}\right)$ sq units

(c) $\left(\sqrt{2} + \sqrt{3} + \frac{5\pi}{2}\right)$ sq units (d) None of these

Answers

Exercise for Session 2

1. (d)	**2.** (a)	**3.** (c)	**4.** (d)	**5.** (a)
6. (b)	**7.** (a)	**8.** (a)	**9.** (c)	**10.** (b)
11. (d)	**12.** (b)	**13.** (a)	**14.** (c)	**15.** (b)
16. (a)	**17.** (c)	**18.** (a)	**19.** (a)	**20.** (b)