BINOMIAL THEOREM [JEE ADVANCED PREVIOUS YEAR SOLVED PAPER]

JEE ADVANCED

Single Correct Answer Type

1. Given positive integers $r > 1$, $n > 2$ and that the coefficient of $(3r)^{th}$ and $(r+2)^{th}$ terms in the binomial expansion of $(1+x)^{2n}$ are equal. Then
   a. $n = 2r$ b. $n = 2r + 1$
   c. $n = 3r$ d. none of these
   (IIT-JEE 1983)

2. The coefficient of x^4 in $(x/2 - 3/x^2)^{10}$ is
   a. $\frac{405}{256}$ b. $\frac{504}{259}$ c. $\frac{450}{263}$ d. none of these
   (IIT-JEE 1983

3. If C_r stands for ${}^nC_r$, then the sum of the series
   $$\frac{2\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}{n!}[C_0^2 - 2C_1^2 + 3C_2^2 - \cdots + (-1)^n (n+1)C_n^2],$$
   where n is an even positive integer is equal to
   a. 0 b. $(-1)^{n/2}(n+1)$
   c. $(-1)^n(n+2)$ d. $(-1)^n n$ **(IIT-JEE 1986)**

4. If $a_n = \sum_{r=0}^{n} \frac{1}{{}^nC_r}$, then $\sum_{r=0}^{n} \frac{r}{{}^nC_r}$ equals
   a. $(n-1)a_n$ b. na_n
   c. $(1/2)na_n$ d. none of these
   (IIT-JEE 1988)

5. The expression $\left(x + (x^3 - 1)^{\frac{1}{2}}\right)^5 + \left(x - (x^3 - 1)^{\frac{1}{2}}\right)^5$ is a polynomial of degree
   a. 5 b. 6 c. 7 d. 8
   (IIT-JEE 1992)

6. For $2 \le r \le n$, $\binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2} =$
   a. $\binom{n+1}{r-1}$ b. $2\binom{n+1}{r+1}$ c. $2\binom{n+2}{r}$ d. $\binom{n+2}{r}$
   (IIT-JEE 2000)

7. In the binomial expansion of $(a-b)^n$ $n \ge 5$, the sum of the 5th and 6th terms is zero. Then a/b equals
   a. $(n-5)/6$ b. $(n-4)/5$ c. $n/(n-4)$ d. $6/(n-5)$
   (IIT-JEE 2001)

8. The sum $\sum_{i=0}^{m}\binom{10}{i}\binom{20}{m-i}$, where $\binom{p}{q} = 0$ if $p < q$ is maximum when m is
   a. 5 b. 10 c. 15 d. 20
   (IIT-JEE 2002)

9. The coefficient of t^{24} in $(1+t^2)^{12}(1+t^{12})(1+t^{24})$ is
   a. ${}^{12}C_6 + 3$ b. ${}^{12}C_6 + 1$ c. ${}^{12}C_6$ d. ${}^{12}C_6 + 2$
   (IIT-JEE 2003)

10. If ${}^{n-1}C_r = (k^2 - 3)\,{}^nC_{r+1}$, then $k \in$
    a. $(-\infty, -2]$ b. $[2, \infty)$
    c. $[-\sqrt{3}, \sqrt{3}]$ d. $(\sqrt{3}, 2]$ **(IIT-JEE 2004)**

11. The value of
    $$\binom{30}{0}\binom{30}{10} - \binom{30}{1}\binom{30}{11} + \binom{30}{2}\binom{30}{12} - \cdots + \binom{30}{20}\binom{30}{30},$$
    where $\binom{n}{r} = {}^nC_r$ is
    a. $\binom{30}{10}$ b. $\binom{30}{15}$ c. $\binom{60}{30}$ d. $\binom{31}{10}$
    (IIT-JEE 2005)

12. For $r = 0, 1, \ldots, 10$, let A_r, B_r and C_r denote, respectively, the coefficients of x^r in the expansions of $(1+x)^{10}$, $(1+x)^{20}$ and $(1+x)^{30}$. Then $\sum_{r=1}^{10} A_r(B_{10}B_r - C_{10}A_r)$ is equal to
    a. $B_{10} - C_{10}$ b. $A_{10}(B_{10}{}^2 - C_{10}A_{10})$
    c. 0 d. $C_{10} - B_{10}$
    (IIT-JEE 2010)

13. Coefficient of x^{11} in the expansion of $(1+x^2)^4(1+x^3)^7(1+x^4)^{12}$ is
    a. 1051 b. 1106 c. 1113 d. 1120
    (JEE Advanced 2014)

Integer Answer Type

1. The coefficients of three consecutive terms of $(1+x)^{n+5}$ are in the ratio 5 : 10 : 14. Then $n =$
   (JEE Advanced 2013)

2. The coefficient of x^9 in the expansion of $(1+x)(1+x^2)(1+x^3)\ldots(1+x^{100})$ is
   (JEE Advanced 2015)

Fill in the Blanks Type

1. The larger of $99^{50} + 100^{50}$ and 101^{50} is ______.
   (IIT-JEE 1982)

2. The sum of the coefficients of the polynomial $(1 + x - 3x^2)^{2163}$ is ______. **(IIT-JEE 1982)**

3. If $(1+ax)^n = 1 + 8x + 24x^2 + \cdots$, then $a =$ ______ and $n =$ ______. **(IIT-JEE 1983)**

4. The sum of the rational terms in the expansion of $(\sqrt{2} + 3^{1/5})^{10}$ is ______. **(IIT-JEE 1997)**

Subjective Type

1. Given that $C_1 + 2C_2x + 3C_3 x^2 + \cdots + 2n\ C_{2n}\ x^{2n-1} = 2n(1+x)^{2n-1}$, where $C_r = (2n)!/[r!(2n-r)!]$; $r = 0, 1, 2, \ldots, 2n$, then prove that $C_1^2 - 2C_2^2 + 3C_3^2 - \cdots - 2nC_{2n}^2 = (-1)^n nC_n$. **(IIT-JEE 1979)**

2. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$, then show that the sum of the products of the coefficient taken two at a time, represented by $\sum\sum_{0\le i<j\le n} {}^nC_i\, {}^nC_j$ is equal to $2^{2n-1} - \dfrac{(2n)!}{2(n!)^2}$. **(IIT-JEE 1983)**

3. Given,
$$s_n = 1 + q + q^2 + \cdots + q^n,\ S_n = 1 + \frac{q+1}{2} + \left(\frac{q+1}{2}\right)^2 + \cdots + \left(\frac{q+1}{2}\right)^n,\ q \ne 1$$
Prove that ${}^{n+1}C_1 + {}^{n+1}C_2\, s_1 + {}^{n+1}C_3\, s_2 + \cdots + {}^{n+1}C_{n+1}\, s_n = 2^n S_n$. **(IIT-JEE 1984)**

4. Prove that
$$\sum_{r=0}^{n}(-1)^r\, {}^nC_r\left[\frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + \frac{15^r}{2^{4r}} + \cdots \text{up to } m \text{ terms}\right] = \frac{2^{mn}-1}{2^{mn}(2^n-1)}$$
(IIT-JEE 1985)

5. Let $R = (5\sqrt{5} + 11)^{2n+1}$ and $f = R - [R]$ where [ ] denotes the greatest integer function, prove that $Rf = 4^{2n+1}$. **(IIT-JEE 1988)**

6. Prove that $C_0 - 2^2 C_1 + 3^2 C_2 - 4^2 C_3 + \cdots + (-1)^n (n+1)^2 \times C_n = 0$ where $C_r = {}^nC_r$. **(IIT-JEE 1989)**

7. If $\sum_{r=0}^{2n} a_r(x-2)^r = \sum_{r=0}^{2n} b_r(x-3)^r$ and $a_k = 1$ for all $k \ge n$, then show that $b_n = {}^{2n+1}C_{n+1}$. **(IIT-JEE 1992)**

8. Prove that $\sum_{r=1}^{k}(-3)^{r-1}\, {}^{3n}C_{2r-1} = 0$, where $k = 3n/2$ and n is an even integer. **(IIT-JEE 1993)**

9. Let n be a positive integer and $(1+x+x^2)^n = a_0 + a_1x + \cdots + a_{2n}x^{2n}$. Show that $a_0^2 - a_1^2 + a_2^2 + \cdots + a_{2n}^2 = a_n$. **(IIT-JEE 1994)**

10. Prove that
$$\frac{3!}{2(n+3)} = \sum_{r=0}^{n}(-1)^r\left(\frac{{}^nC_r}{{}^{r+3}C_r}\right)$$
(IIT-JEE 1997)

11. For any positive integer m, n (with $n \ge m$), let $\binom{n}{m} = {}^nC_m$. Prove that
$$\binom{n}{m} + \binom{n-1}{m} + \binom{n-2}{m} + \cdots + \binom{m}{m} = \binom{n+1}{m+1}.$$
Hence or otherwise, prove that
$$\binom{n}{m} + 2\binom{n-1}{m} + 3\binom{n-2}{m} + \cdots + (n-m+1)\binom{m}{m} = \binom{n+2}{m+2}$$
(IIT-JEE 2000)

12. Prove that $(25)^{n+1} - 24n + 5735$ is divisible by $(24)^2$ for all $n = 1, 2, \ldots$ **(IIT-JEE 2002)**

13. If n and k are positive integers, show that
$$2^k\binom{n}{0}\binom{n}{k} - 2^{k-1}\binom{n}{1}\binom{n-1}{k-1} + 2^{k-2}\binom{n}{2}\binom{n-2}{k-2} - \cdots + (-1)^k\binom{n}{k}\binom{n-k}{0} = \binom{n}{k}$$
where $\binom{n}{k}$ stands for ${}^nC_k$. **(IIT-JEE 2003)**

Answer Key

JEE Advanced

Single Correct Answer Type

1. a.	2. a.	3. c.	4. c.	5. c.
6. d.	7. b.	8. c.	9. d.	10. d.
11. a.	12. d.	13. c.		

Integer Answer Type

1. (6) 2. (8)

Fill in the Blanks Type

1. 101^{50}	2. -1
3. $a = 2, n = 4$	4. 41

Subjective Type

1. $(-1)^n\, n\, C_n$	3. $2^n S_n$	4. $\dfrac{2^{mn}-1}{2^{mn}(2^n-1)}$
5. 4^{2n+1}	6. 0	7. ${}^{2n+1}C_{n+1}$
8. 0	9. a_n	10. $\dfrac{3!}{2(n+3)}$
11. ${}^{n+2}C_{m+2}$	12. 24^2	13. ${}^nC_k$

Hints and Solution

JEE Advanced

Single Correct Answer Type

1. a. Given that r and n are +ve integers such that $r > 1$, $n > 2$.
Also, in the expansion of $(1 + x)^{2n}$,
Coefficient of $3r$th term = Coefficient of $(r + 2)$th term

$\Rightarrow \quad {}^{2n}C_{3r-1} = {}^{2n}C_{r+1}$

or $3r - 1 = r + 1$ or $3r - 1 + r + 1 = 2n$

[Using ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$]

$\Rightarrow$ $r = 1$ or $2r = n$.

But $r > 1$

$\therefore$ $n = 2r$

2. a. $\left(\frac{x}{2} - \frac{3}{x^2}\right)^{10}$

General term in this expansion is

$$T_{r+1} = {}^{10}C_r\left(\frac{x}{2}\right)^{10-r}\left(\frac{-3}{x^2}\right)^r = {}^{10}C_r\, x^{10-3r}\frac{(-1)^r 3^r}{2^{10-r}}$$

For coefficient of x^4, we should have $r = 2$.

Therefore, coefficient of x^4 is ${}^{10}C_2\ \frac{(-1)^2 3^2}{2^8} = \frac{405}{256}$

3. c. Since n is even, let $n = 2m$. Then,

$$\text{L.H.S.} = S = \frac{2\times m!m!}{(2m)!}\left[C_0^2 - 2C_1^2 + 3C_2^2 + \cdots + (-1)^{2m}\ 2m + (2m+1)C_{2m}^2\right] \quad (1)$$

or $$S = \frac{2\times m!m!}{(2m)!}\left[(2m+1)C_0^2 - 2mC_1^2 + (2m-1)C_2^2 + \cdots + C_m^{\ 2}\right] \quad (2) \text{ (Using } C_r = C_{n-r})$$

Adding (1) and (2), we get

$$2S = 2\frac{m!m!}{(2m)!}(2m+2)\left[C_0^2 - C_1^2 + C_2^2 + \cdots + C_{2m}^2\right]$$

We know that $C_0^2 - C_1^2 + C_2^2 - \cdots + C_n^2 = (-1)^{n/2}\,{}^nC_{n/2}$

if n is even, we get

$$S = 2\frac{m!\,m!}{(2m)!}(m+1)\left[(-1)^m\,{}^{2m}C_m\right]$$

$$= 2\left(\frac{n}{2}+1\right)(-1)^{n/2}$$

$$= (-1)^{n/2}(n+2)$$

4. c. Let

$$b = \sum_{r=0}^{n}\frac{r}{{}^nC_r} \quad (1)$$

$$= \sum_{r=0}^{n}\frac{n-r}{{}^nC_{n-r}}$$ (we can replace r by $n - r$ or writing series in reverse order)

$$\therefore\quad b = \sum_{r=0}^{n}\frac{n-r}{{}^nC_r} \quad (2)$$

Adding (1) and (2), we have

$$2b = \sum_{r=0}^{n}\frac{r}{{}^nC_r} + \sum_{r=0}^{n}\frac{n-r}{{}^nC_r}$$

$$= n\sum_{r=0}^{n}\frac{1}{{}^nC_r}$$

$$= na_n$$

or $$b = \frac{n}{2}a_n$$

5. c. The given expression is $\left(x + \sqrt{x^3 - 1}\right)^5 + \left(x - \sqrt{x^3 - 1}\right)^5$.

We know that

$$(x+a)^n + (x-a)^n = 2\left[{}^nC_0 x^n + {}^nC_2 x^{n-2}a^2 + {}^nC_4 x^{n-4}a^4 + \cdots\right]$$

Therefore the given expression is equal to $2[{}^5C_0x^5 + {}^5C_2x^3(x^3 - 1) + {}^5C_4x(x^3 - 1)^2]$.

Maximum power of x involved here is 7, also only +ve integral powers of x are involved; therefore, the given expression is a polynomial of degree 7.

6. d. We know that $\binom{n}{r} + 2\binom{n}{r-1} = \binom{n+1}{r}$

$$\binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2}$$

$$= \left[\binom{n}{r} + \binom{n}{r-1}\right] + \left[\binom{n}{r-1} + \binom{n}{r-2}\right]$$

$$= \binom{n+1}{r} + \binom{n+1}{r-1} = \binom{n+2}{r} \quad [\because\ {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r]$$

7. b. In the binomial expansion,

$(a - b)^n$, $n \ge 5$

$T_5 + T_6 = 0$

$\Rightarrow$ ${}^nC_4\, a^{n-4} b^4 - {}^nC_5\, a^{n-5} b^5 = 0$

or $\frac{{}^nC_4}{{}^nC_5}\frac{a}{b} = 1$ or $\frac{4+1}{n-4}\frac{a}{b} = 1$ $\left[\text{Using } \frac{{}^nC_r}{{}^nC_{r+1}} = \frac{r+1}{n-r}\right]$

or $\frac{a}{b} = \frac{n-4}{5}$

8. c. $\sum_{i=0}^{m}{}^{10}C_i\,{}^{20}C_{m-i} = {}^{10}C_0\,{}^{20}C_m + {}^{10}C_1\,{}^{20}C_{m-1} + {}^{10}C_2\,{}^{20}C_{m-2} + \cdots + {}^{10}C_m\,{}^{20}C_0$

= Coefficient of x^m in the expansion of product $(1 + x)^{10}(x + 1)^{20}$

= Coefficient of x^m in the expansion of $(1+x)^{30}$

$= {}^{30}C_m$

Hence, the maximum value ${}^{30}C_m$ is ${}^{30}C_{15}$.

9. d. $(1 + t^2)^{12}(1 + t^{12})(1 + t^{24})$

$= (1 + t^{12} + t^{24} + t^{36})(1 + t^2)^{12}$

$\therefore$ Coefficient of t^{24}

$= 1 \times$ coefficient of t^{24} in $(1 + t^2)^{12} + 1 \times$ coefficient of t^{12} in $(1 + t^2)^{12} + 1 \times$ constant term in $(1 + t^2)^{12}$

$= {}^{12}C_{12} + {}^{12}C_6 + {}^{12}C_0 = 1 + {}^{12}C_6 + 1 = {}^{12}C_6 + 2$

10. d. ${}^{n-1}C_r = {}^nC_{r+1}(k^2 - 3)$

$\Rightarrow$ $k^2 - 3 = \frac{{}^{n-1}C_r}{{}^nC_{r+1}} = \frac{r+1}{n}$

Now,

$0 \le r \le n - 1$

or $1 \le r + 1 \le n$

or $\frac{1}{n} \le \frac{r+1}{n} \le 1$

or $\frac{1}{n} \le k^2 - 3 \le 1$

or $3 + \frac{1}{n} \le k^2 \le 4$ or $\sqrt{3 + \frac{1}{n}} \le k \le 2$

When $n \to \infty$, we have

$$\sqrt{3} < k \le 2$$

$\Rightarrow \quad k \in (\sqrt{3}, 2]$

11. a. Given series is

$${}^{30}C_0\,{}^{30}C_{10} - {}^{30}C_1\,{}^{30}C_{11} + {}^{30}C_2\,{}^{30}C_{12} - \cdots + {}^{30}C_{20}\,{}^{30}C_{30}$$

which is

$${}^{30}C_0\,{}^{30}C_{20} - {}^{30}C_1\,{}^{30}C_{19} + {}^{30}C_2\,{}^{30}C_{18} - \cdots + {}^{30}C_{20}\,{}^{30}C_0$$

= Coefficient of x^{20} in the expansion of $(x+1)^{30}(1-x)^{30}$

= Coefficient of x^{20} in the expansion of $(1-x^2)^{30}$

$= {}^{30}C_{10}$

12. d. A_r, B_r and C_r denote, respectively, the coefficients of x^r in the expansions of $(1+x)^{10}$, $(1+x)^{20}$ and $(1+x)^{30}$.

$\therefore \quad A_r = {}^{10}C_r,\ B_r = {}^{20}C_r,\ C_r = {}^{30}C_r$

$$\therefore \quad \sum_{r=1}^{10} A_r(B_{10}B_r - C_{10}A_r)$$

$$= B_{10}\sum_{r=1}^{10} A_rB_r - C_{10}\sum_{r=1}^{10}(A_r)^2$$

$$= B_{10}\sum_{r=1}^{10} {}^{10}C_r\,{}^{20}C_r - C_{10}\sum_{r=1}^{10}({}^{10}C_r)^2$$

$$= B_{10}\sum_{r=1}^{10} {}^{10}C_r\,{}^{20}C_{20-r} - C_{10}\sum_{r=1}^{10}({}^{10}C_r)^2$$

$$= B_{10}\left[\left(\sum_{r=0}^{10} {}^{10}C_r\,{}^{20}C_{20-r}\right) - 1\right] - C_{10}\left[\left(\sum_{r=0}^{10}({}^{10}C_r)^2\right) - 1\right]$$

$$= B_{10}\left[{}^{30}C_{20} - 1\right] - C_{10}\left[{}^{20}C_{10} - 1\right]$$

$$\left(\because\ {}^nC_0^2 + {}^nC_1^2 + {}^nC_2^2 + \ldots + {}^nC_n^2 = {}^{2n}C_n\right)$$

$$= \left[B_{10}\,{}^{30}C_{20} - C_{10}\,{}^{20}C_{10}\right] + \left[C_{10} - B_{10}\right]$$

$$= \left[{}^{20}C_{10}\,{}^{30}C_{20} - {}^{30}C_{10}\,{}^{20}C_{10}\right] + \left[C_{10} - B_{10}\right]$$

$$= C_{10} - B_{10}$$

13. c. $2x_1 + 3x_2 + 4x_3 = 11$

Possibilities are (0, 1, 2); (1, 3, 0); (2, 1, 1); (4, 1, 0).

$\therefore$ Required coefficients

$= ({}^4C_0 \times {}^7C_1 \times {}^{12}C_2) + ({}^4C_1 \times {}^7C_3 \times {}^{12}C_0) + ({}^4C_2 \times {}^7C_1 \times {}^{12}C_1) + ({}^4C_4 \times {}^7C_1 \times 1)$

$= (1 \times 7 \times 66) + (4 \times 35 \times 1) + (6 \times 7 \times 12) + (1 \times 7)$

$= 462 + 140 + 504 + 7 = 1113$

Integer Answer Type

1. (6) Let T_{r-1}, T_r, T_{r+1} are three consecutive terms of $(1+x)^{n+5}$

$T_{r-1} = {}^{n+5}C_{r-2}\,(x)^{r-2}$, $T_r = {}^{n+5}C_{r-1}\,x^{r-1}$, $T_{r+1} = {}^{n+5}C_r x^r$,

where, ${}^{n+5}C_{r-2} : {}^{n+5}C_{r-1} : {}^{n+5}C_r = 5 : 10 : 14$.

So $$\frac{{}^{n+5}C_{r-2}}{5} = \frac{{}^{n+5}C_{r-1}}{10} = \frac{{}^{n+5}C_r}{14}$$

Comparing first two results we have

$$\frac{{}^{n+5}C_{r-1}}{{}^{n+5}C_{r-2}} = r \quad \text{or} \quad \frac{(n+5)-(r-1)+1}{r-1} = r$$

or $\quad n - 3r = -9 \qquad (1)$

Comparing last two results we have

$$\frac{{}^{n+5}C_r}{{}^{n+5}C_{r-1}} = \frac{7}{5} \quad \text{or} \quad \frac{n+5-r+1}{r} = \frac{7}{5}$$

or $\quad 5n - 12r = -30 \qquad (2)$

From equation (1) and (2), $n = 6$

2. (8) x^9 can be formed in 8 ways.

i.e. $x^9, x^{1+8}, x^{2+7}, x^{3+6}, x^{4+5}, x^{1+2+6}, x^{1+3+5}, x^{2+3+4}$ and coefficient in each case is 1.

$\therefore$ Coefficient of $x^9 = 1 + 1 + 1 + \ldots$ 8 times

$= 8$

Fill in the Blanks Type

1. We have

$$101^{50} = (100+1)^{50} = 100^{50} + 50 \times 100^{49} + \frac{50\times 49}{2\times 1}100^{48} + \cdots \qquad (1)$$

$$99^{50} = (100-1)^{50} = 100^{50} - 50 \times 100^{49} + \frac{50 \times 49}{2 \times 1}100^{48} - \cdots \qquad (2)$$

Subtracting (2) from (1), we get

$$101^{50} - 99^{50} = 100^{50} + 2\,\frac{50 \times 49 \times 48}{1 \times 2 \times 3}100^{47} + \cdots > 100^{50}$$

Hence, $101^{50} > 100^{50} + 99^{50}$.

2. If we put $x = 1$ in the expansion of $(1 + x - 3x^2)^{2163} = A_0 + A_1x + A_2x^2 + \cdots$ we will get the sum of coefficients of the given polynomial, which is equal to -1.

3. $(1+ax)^n = 1 + 8x + 24x^2 + \cdots$

$$\Rightarrow \quad 1 + nxa + \frac{n(n-1)}{2!}a^2x^2 + \cdots = 1 + 8x + 24x^2 + \cdots$$

Comparing like powers of x, we get

$nax = 8x$ or $na = 8 \qquad (1)$

$$\frac{n(n-1)a^2}{2} = 24 \text{ or } n(n-1)a^2 = 48 \qquad (2)$$

Solving (1) and (2), $n = 4$, $a = 2$.

4. Let T_{r+1} be the general term in the expansion of $\left(\sqrt{2} + 3^{1/5}\right)^{10}$

$\therefore \quad T_{r+1} = {}^{10}C_r\,(\sqrt{2})^{10-r}(3^{1/5})^r \quad (0 \le r \le 10)$

$$= \frac{10!}{r!(10-r)!}\,2^{5-r/2}\,3^{r/5}$$

T_{r+1} will be rational if $2^{5-r/2}$ and $3^{r/5}$ are rational numbers. Hence,

$5 - r/2$ and $r/5$ are integers. So, $r = 0$ and $r = 10$. Therefore, T_1 and T_{11} are rational terms. Now, sum of T_1 and T_{11} is ${}^{10}C_0\,2^{5-0} \times 3^0 + {}^{10}C_{10}\,2^{5-5} \times 3^2 = 32 + 9 = 41$.

Subjective Type

1. Given that

$$C_1 + 2C_2x + 3C_3x^2 + \cdots + 2n\,C_{2n}x^{2n-1} = 2n(1+x)^{2n-1}$$

where

$$C_r = \frac{2n!}{r!(2n-r)!} \qquad (1)$$

Integrating both sides with respect to x, under the limits 0 to x, we get

$$[C_1x + C_2x^2 + C_3x^3 + \cdots + C_{2n}x^{2n}]_0^x = \left[(1+x)^{2n}\right]_0^x$$

or $\quad C_1x + C_2x^2 + C_3x^3 + \cdots + C_{2n}x^{2n} = (1+x)^{2n} - 1$

or $\quad C_0 + C_1x + C_2x^2 + C_3x^3 + \cdots + C_{2n}x^{2n} = (1+x)^{2n} \qquad (2)$

Replacing x by $-1/x$, we get

$$C_0 - \frac{C_1}{x} + \frac{C_2}{x^2} - \frac{C_3}{x^3} + \cdots + (-1)^{2n}\frac{C_{2n}}{x^{2n}} = \left(1 - \frac{1}{x}\right)^{2n}$$

or $\quad C_0x^{2n} - C_1x^{2n-1} + C_2x^{2n-2} - C_3x^{2n-3} + \cdots + C_{2n} = (x-1)^{2n} \quad (3)$

Multiplying Eqs. (1) and (3) and equating the coefficient of x^{2n-1} on both sides, we get

$$-C_1^2 + 2C_2^2 - 3C_3^2 + \cdots + 2n\,C_{2n}^2$$

$= \text{Coefficient of } x^{2n-1} \text{ in } 2n\,(x^2-1)^{2n-1}(x-1)$

$= 2n[\text{coefficient of } x^{2n-2} \text{ in } (x^2-1)^{2n-1} - \text{coefficient of } x^{2n-1} \text{ in } (x^2-1)^{2n-1}]$

$= 2n[{}^{2n-1}C_{n-1}(-1)^{n-1} - 0]$

$= (-1)^{n-1}\, 2n\,{}^{2n-1}C_{n-1}$

$\Rightarrow \quad C_1^2 - 2C_2^2 + 3C_3^2 + \cdots + 2nC_{2n}^2$

$= (-1)^n\, 2n\,{}^{2n-1}C_{n-1}$

$= (-1)^n\, n \times \left(\frac{2n}{n} \times {}^{2n-1}C_{n-1}\right)$

$= (-1)^n\, n\, {}^{2n}C_n$

$= (-1)^n\, n\, C_n \quad (\because\ {}^{2n}C_n = C_n)$

Alternative Method:

Consider expansion

$(1+x)^{2n} = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots + 2n\,{}^{2n}C_{2n}x^{2n}$

Now ${}^{2n}C_1{}^2 - 2\,{}^{2n}C_2{}^2 + 3\,{}^{2n}C_3{}^2 - \ldots - 2n\,{}^{2n}C_{2n}{}^2$

$= \sum_{r=1}^{2n}(-1)^{r-1}\, r\,({}^{2n}C_r)^2$

$= \sum_{r=1}^{2n}(-1)^{r-1}\, r\,.{}^{2n}C_r\,{}^{2n}C_{2n-r}$

$= -2n\sum_{r=1}^{2n}(-1)^{r-1}\,{}^{2n-1}C_{r-1}\,{}^{2n}C_{2n-r}$

$= -2n({}^{2n-1}C_0\,{}^{2n}C_{2n-1} - {}^{2n-1}C_1\,{}^{2n}C_{2n-2} - \ldots - {}^{2n-1}C_{2n-1}\,{}^{2n}C_0)$

$= -2n \times (\text{coefficient of } x^{2n-1} \text{ in } (1+x)^{2n}(1-x)^{2n-1})$

$= -2n \times (\text{coefficient of } x^{2n-1} \text{ in } (1+x)(1-x^2)^{2n-1})$

$= -2n \times (\text{coefficient of } x^{2n-1} \text{ in } (1-x^2)^{2n-1} + \text{coefficient of } x^{2n-1} \text{ in } x(1-x^2)^{2n-1})$

$= -2n \times (0 + \text{coefficient of } x^{2n-2} \text{ in } (1-x^2)^{2n-1})$

$= -2n \times (-1)^{n-1}\,{}^{2n-1}C_{n-1}$

$= (-1)^n\, n\,.\left(\frac{2n}{n}.\,{}^{2n-1}C_{n-1}\right)$

$= (-1)^n\, n\,.\,{}^{2n}C_n$

2. We have $S = \sum\sum_{0\le i<j\le n} {}^nC_i\,.\,{}^nC_j$

Here, i and j are dependent.

Now, consider sum $\sum_{i=0}^{n}\sum_{j=0}^{n} {}^nC_i{}^nC_j$, where i and j are independent.

This sum contain terms for $i<j$, $i>j$ and $i=j$.

Because of symmetry of terms sum of terms for $i<j$ and $i>j$ is same.

So required sum

$$S = \frac{\left(\sum_{i=0}^{n}\sum_{j=0}^{n} {}^nC_i{}^nC_j\right) - \sum_{i=0}^{n}({}^nC_i)^2}{2}$$

$$= \frac{\left(\sum_{i=0}^{n} {}^nC_i\right)\left(\sum_{j=0}^{n} {}^nC_j\right) - \sum_{i=0}^{n}({}^nC_i)^2}{2}$$

$$= \frac{2^n2^n - \sum_{i=0}^{n}({}^nC_i)^2}{2}$$

Now $\sum_{i=0}^{n}({}^nC_i)^2$

$= {}^nC_0{}^2 + {}^nC_1{}^2 + \ldots + {}^nC_n{}^2$

$= ({}^nC_0\,.\,{}^nC_n + {}^nC_1\,.\,{}^nC_{n-1} + {}^nC_2\,.\,{}^nC_{n-2} + \ldots + {}^nC_n\,.\,{}^nC_0)$

$= \text{Coefficient of } x^n \text{ in } (1+x)^n(1+x)^n$

$= \text{Coefficient of } x^n \text{ in } (1+x)^{2n}$

$= {}^{2n}C_n$

$\therefore \quad S = \dfrac{4^n - {}^{2n}C_n}{2}$

3. s_n is geometric progression, hence

$\therefore \quad s_n = \dfrac{q^{n+1}-1}{q-1},\ q \ne 1$

$$S_n = \frac{\left(\frac{q+1}{2}\right)^{n+1} - 1}{\left(\frac{q+1}{2}\right) - 1} = \frac{(q+1)^{n+1} - 2^{n+1}}{2^n(q-1)} \qquad (1)$$

Consider

${}^{(n+1)}C_1 + {}^{(n+1)}C_2s_1 + {}^{(n+1)}C_3s_3 + \cdots + {}^{(n+1)}C_{n+1}s_n$

$$= {}^{(n+1)}C_1\left(\frac{q-1}{q-1}\right) + {}^{(n+1)}C_2\,\frac{q^2-1}{q-1} + \cdots + {}^{(n+1)}C_{n+1}\,\frac{q^{n+1}-1}{q-1}$$

$$= \left(\frac{1}{q-1}\right)\left[\left\{{}^{(n+1)}C_1q + {}^{(n+1)}C_2q^2 + \cdots + {}^{(n+1)}C_{n+1}q^{n+1}\right\} - \left\{{}^{(n+1)}C_1 + {}^{(n+1)}C_2 + \cdots + {}^{(n+1)}C_{n+1}\right\}\right]$$

$$= \left(\frac{1}{q-1}\right)\left[\left\{(1+q)^{n+1} - 1\right\} - \left\{2^{n+1} - 1\right\}\right] \qquad (2)$$

$$= \frac{(1+q)^{n+1} - 2^{n+1}}{q-1}$$

Thus,

$${}^{(n+1)}C_1 + {}^{(n+1)}C_2s_1 + \cdots + {}^{(n+1)}C_{n+1}s_n = \frac{(1+q)^{n+1} - 2^{n+1}}{q-1}$$

But from (1), we have

$${}^{n+1}C_1 + {}^{(n+1)}C_2 s_1 + \cdots + {}^{(n+1)}C_{n+1} s_{n+1} = 2^n S_n$$

4. $\sum_{r=0}^{n} (-1)^r \times {}^nC_r \left(\frac{1}{2^r} + \left(\frac{3}{4}\right)^r + \left(\frac{7}{8}\right)^r + \cdots m \text{ terms} \right)$

$$= \left(1-\frac{1}{2}\right)^n + \left(1-\frac{3}{4}\right)^n + \left(1-\frac{7}{8}\right)^n + \left(1-\frac{15}{16}\right)^n + \cdots m \text{ terms}$$

$$= \frac{1}{2^n} + \frac{1}{2^{2n}} + \frac{1}{2^{3n}} + \frac{1}{2^{4n}} + \cdots m \text{ terms}$$

$$= \frac{\frac{1}{2^n}\left[1-\left(\frac{1}{2^n}\right)^m\right]}{1-\frac{1}{2^n}} = \frac{2^{mn}-1}{2^{mn}(2^n-1)}$$

5. Here $f = R - [R]$ is the fractional part of R. Thus, if I is the integral part of R, then

$$R = I + f = (5\sqrt{5} + 11)^{2n+1}, \text{ and } 0 < f < 1$$

Let $f' = (5\sqrt{5} - 11)^{2n+1}$. Then $0 < f' < 1$ (as $5\sqrt{5} - 11 < 1$)

Now, $I + f - f' = (5\sqrt{5} + 11)^{2n+1} - (5\sqrt{5} - 11)^{2n+1}$

$= 2[{}^{2n+1}C_1 (5\sqrt{5})^{2n} \times 11 + {}^{2n+1}C_3 (5\sqrt{5})^{2n-2} \times 11^3 + \cdots]$

= an even integer (1)

$\Rightarrow$ $f - f'$ must also be an integer

$\Rightarrow$ $f - f' = 0$, ($\because$ $0 < f < 1, 0 < f' < 1$, $\therefore$ $-1 < f - f' < 1$)

$\Rightarrow$ $f = f'$

$\therefore$ $Rf = Rf' = (5\sqrt{5} + 11)^{2n+1}(5\sqrt{5} - 11)^{2n+1}$

$= (125 - 121)^{2n+1} = 4^{2n+1}$

6. $S = C_0 - 2^2C_1 + 3^2C_2 - \cdots + (-1)^n (n+1)^2 C_n$

$T_r = (-1)^r r^2 \, {}^nC_r$

$= (-1)^r r(r\,{}^nC_r)$

$= (-1)^r r(n\,{}^{n-1}C_{r-1})$

$= n(-1)^r ((r-1)+1)({}^{n-1}C_{r-1})$

$= n(-1)^r ((r-1)\,{}^{n-1}C_{r-1} + {}^{n-1}C_{r-1})$

$= n(-1)^r ((n-1)\,{}^{n-2}C_{r-2} + {}^{n-1}C_{r-1})$

$= n(n-1)\,{}^{n-2}C_{r-2}(-1)^{r-2} - n\,{}^{n-1}C_{r-1}(-1)^{r-1}$

$\Rightarrow$ $S = \sum_{r=0}^{n} T_r$

$= n(n-1)(1-1)^{n-2} - n(1-1)^{n-1}$

$= 0$

7. Given that

$$\sum_{r=0}^{2n} a_r (x-2)^r = \sum_{r=0}^{2n} b_r (x-3)^r \quad (1)$$

and

$a_k = 1, \forall k \ge n$

In Eq. (1), put $x - 3 = y$ or $x - 3 = y$ or $x - 2 = y + 1$

So, we get

$$\sum_{r=0}^{2n} a_r (1+y)^r = \sum_{r=0}^{2n} b_r (y)^r$$

$\Rightarrow$ $a_0 + a_1(1+y) + \cdots + a_{n-1}(1+y)^{n-1}$

$+ (1+y)^n + (1+y)^{n+1} + \cdots + (1+y)^{2n} = \sum_{r=0}^{2n} b_r y^r$

[Using $a_k = 1, \forall k \ge n$]

Equating the coefficients of y^n on both the sides, we get

$${}^nC_n + {}^{n+1}C_n + {}^{n+2}C_n + \cdots + {}^{2n}C_n = b_n$$

or $\left({}^{n+1}C_{n+1} + {}^{n+1}C_n\right) + {}^{n+2}C_n + \cdots + {}^{2n}C_n = b_n$

[Using ${}^nC_n = {}^{n+1}C_{n+1} = 1$]

or $b_n = {}^{n+2}C_{n+1} + {}^{n+2}C_n + \cdots + {}^{2n}C_n$ [Using ${}^mC_r + {}^mC_{r-1} = {}^{m+1}C_r$]

Combining the terms in similar way, we get

$b_n = {}^{2n}C_{n+1} + {}^{2n}C_n$

$= {}^{2n+1}C_{n+1}$

8. $S = \sum_{r=1}^{k} (-3)^{r-1} \, {}^{3n}C_{2r-1}$, $k = \frac{3n}{2}$ and n is even or $n = 2m, n \in N$

$\Rightarrow k = \frac{3(2m)}{2} = 3m$

$\Rightarrow S = \sum_{r=1}^{3m} (-3)^{r-1} \times {}^{6m}C_{2r-1} = {}^{6m}C_1 - 3\,{}^{6m}C_3 + 3^2\,{}^{6m}C_5 - \cdots (-3)^{3m-1}\,{}^{6m}C_{6m-1}$

$= \frac{1}{\sqrt{3}}\left[\sqrt{3}\,{}^{6m}C_1 - (\sqrt{3})^3\,{}^{6m}C_3 + (\sqrt{3})^5\,{}^{6m}C_5 - \cdots + (-1)^{3m-1}(\sqrt{3})^{6m-1}\,{}^{6m}C_{6m-1}\right]$

There is an alternate sign series with odd binomial coefficients. Hence, we should replace x by $\sqrt{3}i$ in $(1+x)^{6m}$. Therefore,

$(1+\sqrt{3}i)^{6m} = {}^{6m}C_0 + {}^{6m}C_1(\sqrt{3}i) + {}^{6m}C_2(\sqrt{3}i)^2 + {}^{6m}C_3(\sqrt{3}i)^3 + \cdots + {}^{6m}C_{6m}(\sqrt{3}i)^{6m}$

$\Rightarrow$ $\sqrt{3} \times {}^{6m}C_1 - (\sqrt{3})^3\,{}^{6m}C_3 + (\sqrt{3})^5\,{}^{6m}C_5 + \cdots$

= Imaginary part in $(1+\sqrt{3}i)^{6m}$

$= \text{Im}\left[2^{6m}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^{6m}\right]$

$= \text{Im}\left[2^{6m}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]^{6m}$

$= \text{Im}\left[2^{6m}(\cos 2m\pi + i\sin 2m\pi)\right] = \text{Im}[2^{6m}] = 0$

$\Rightarrow$ $S = 0$

9. $a_0^2 - a_1^2 + a_2^2 - a_3^2 + \cdots$ suggests that we have to multiply two expansions

$(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \cdots + a_{2n}x^{2n}$ (1)

Replacing x by $-1/x$, we get

$$\left(1 - \frac{1}{x} + \frac{1}{x^2}\right)^n = a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} - \cdots + \frac{a_{2n}}{x^{2n}}$$

$\Rightarrow$ $(1 - x + x^2)^n = a_0x^{2n} - a_1x^{2n-1} + a_2x^{2n-2} - \cdots + a_{2n}$ (2)

Clearly,

$a_0^2 - a_1^2 + a_2^2 \cdots + a_{2n}^2$ is the coefficient of x^{2n} in $(1+x+x^2)^n(1-x+x^2)^n$

$\Rightarrow$ $a_0^2 - a_1^2 + a_2^2 - \cdots + a_{2n}^2$

= Coefficient of x^{2n} in $(1 + x^2 + x^4)^n$

In $(1+x^2+x^4)^n$, replace x^2 by y, then the coefficient of y^n in $(1+y+y^2)^n$ is a_n. Hence,

$$a_0^2 - a_1^2 + a_2^2 - \cdots + a_{2n}^2 = a_n$$

10. $\sum_{r=0}^{n}(-1)^r\left(\frac{^nC_r}{^{r+3}C_3}\right)$

$$= \sum_{r=0}^{n}(-1)^r \frac{n!}{(n-r)!r!}\frac{3!r!}{(r+3)!}$$

$$= 3!\sum_{r=0}^{n}(-1)^r \frac{n!}{(n-r)!(r+3)!}$$

$$= \frac{3!}{(n+1)(n+2)(n+3)}\sum_{r=0}^{n}(-1)^r\, {}^{n+3}C_{r+3}$$

$$= -\frac{3!}{(n+1)(n+2)(n+3)}\sum_{r=0}^{n}(-1)^{r+3}\, {}^{n+3}C_{r+3}$$

$$= -\frac{3!}{(n+1)(n+2)(n+3)}\left[-{}^{n+3}C_3 + {}^{n+3}C_4 - \cdots (-1)^{n+3}\,{}^{n+3}C_{n+3}\right]$$

$$= -\frac{3!}{(n+1)(n+2)(n+3)}\left[({}^{n+3}C_0 - {}^{n+3}C_1 + {}^{n+3}C_2 - {}^{n+3}C_3 + \cdots + (-1)^{n+3}\,{}^{n+3}C_{n+3}) - ({}^{n+3}C_0 - {}^{n+3}C_1 + {}^{n+3}C_2)\right]$$

$$= -\frac{3!}{(n+1)(n+2)(n+3)}\left[(1-1)^{n+3} - (1-(n+3) - \frac{(n+3)(n+2)}{2}\right]$$

$$= \frac{3!}{(n+1)(n+2)(n+3)}\left[(1-n-3+\frac{(n+3)(n+2)}{2}\right]$$

$$= \frac{3!}{(n+1)(n+2)(n+3)}\frac{(n^2+3n+2)}{2} = \frac{3!}{2(n+3)}$$

11. We know that the coefficient of x^r in the binomial expansion of $(1+x)^n$ is ^nC_r.

$\therefore$ $^nC_m + {}^{n-1}C_m + {}^{n-2}C_m + \cdots + {}^mC_m$

= Coefficient of x^m in the expansion of $[(1+x)^n + (1+x)^{n-1} + (1+x)^{n-2} + \cdots + (1+x)^m]$

= Coefficient of x^m in $[(1+x)^m + (1+x)^{m-1} + (1+x)^{m-2} + \cdots + (1+x)^n]$ (writing in reserve order)

= Coefficient of x^m $\left[(1+x)^m \frac{\{(1+x)^{n-m+1}-1\}}{1+x-1}\right]$

[sum of G.P.]

= Coefficient of x^m in $\frac{\left[(1+x)^{n+1} - (1+x)^m\right]}{x}$

= Coefficient of x^{m+1} in $[(1+x)^{n+1} - (1+x)^m]$

$= {}^{n+1}C_{m+1} - 0$

$= {}^{n+1}C_{m+1}$

Now, we have to prove

$$^nC_m + 2\,{}^{n-1}C_m + 3\,{}^{n-2}C_m + \cdots + (n-m+1)\,{}^mC_m = {}^{n+2}C_{m+2}$$

L.H.S. = coefficient of x^m in

$[(+x)^n + 2(1+x)^{n-1} + 3(1+x)^{n-2} + \ldots + (n-m+1)x^m]$

Let us consider

$$S = (1+x)^n + 2(1+x)^{n-1} + 3(1+x)^{n-2} + \cdots + (n-m+1)\times(1+x)^m \quad (1)$$

$$(1+x)S = (1+x)^{n+1} + 2(1+x)^n + 3(1+x)^{n-1} + \cdots + (n-m+1)(1+x)^{m+1} \quad (2)$$

Subtracting (1) from (2), we get

$$xS = (1+x)^{n+1} + (1+x)^n + (1+x)^{n-1} + (1+x)^{n-2} + \cdots + (1+x)^m + (n-m+1)(1+x)^m$$

$$= \frac{(1+x)^{n+1}\left[1-\left(\frac{1}{1+x}\right)^{n+1-m}\right]}{1-\frac{1}{1+x}} + (n-m+1)(1+x)^m$$

$$= \frac{(1+x)^{n+1}\left[(1+x)^{n+1-m}-1\right]}{x(1+x)^{n-m}} + (n-m+1)(1+x)^m$$

$$= \frac{(1+x)^{m+1}\left[(1+x)^{n+1-m}-1\right]}{x} + (n-m+1)(1+x)^m$$

$$\Rightarrow \quad S = \frac{(1+x)^{n+2} - (1+x)^{m+1}}{x^2} + \frac{(n-m+1)(1+x)^m}{x}$$

Now,

$^nC_m + 2\times{}^{n-1}C_m + 3\times{}^{n-2}C_m + \cdots + (n-m+1)\,{}^mC_m$

= Coefficient of x^m is S

= Coefficient of x^m in

$$\left[\frac{(1+x)^{n+2} - (1+x)^{m+1}}{x^2} + \frac{(n-m+1)(1+x)^m}{x}\right]$$

= Coefficient of x^{m+2} in $[(1+x)^{n+2} - (1+x)^{m+1}]$

$= {}^{n+2}C_{m+2}$

12. $(25)^{n+1} - 24n + 5735$

$= (1+24)^{n+1} - 24n + 5735$

$= {}^{n+1}C_0 + {}^{n+1}C_1\,24 + {}^{n+1}C_2 24^2 + \cdots - 24n + 5735$

$= 1 + 24(n+1) + {}^{n+1}C_2 24^2 + \cdots + {}^{n+1}C_{n+1}24^{n+1} - 24n + 5735$

$= 5760 + 24^2({}^{n+1}C_2 + \cdots + {}^{n+1}C_{n+1}24^{n-1})$

$= 24^2[10 + ({}^{n+1}C_2 + \cdots + {}^{n+1}C_{n+1}24^{n-1})]$

which is divisible by 24^2.

13. $S = 2^k\binom{n}{0}\binom{n}{k} - 2^{k-1}\binom{n}{1}\binom{n-1}{k-1} + 2^{k-2}\binom{n}{2}\binom{n-2}{k-2} - \cdots + (-1)^k\binom{n}{k}\binom{n-k}{0}$

$$= 2^k\,{}^nC_0\,{}^nC_k - 2^{k-1}\,{}^nC_1\,{}^{n-1}C_{k-1} + \cdots + (-1)^k\,{}^nC_k\,{}^{n-k}C_0$$

$$= \sum_{r=0}^{k}(-1)^r 2^{k-r}\,{}^nC_r\,{}^{n-r}C_{k-r}$$

$$= \sum_{r=0}^{k}(-1)^r 2^{k-r}\frac{n!}{(n-r)!r!}\frac{(n-r)!}{(k-r)!(n-k)!}$$

$$= \frac{n!}{k!(n-k)!}\sum_{r=0}^{k}(-1)^r 2^{k-r}\frac{k!}{r!(k-r)!}$$

$$= {}^nC_k\sum_{r=0}^{k}(-1)^r 2^{k-r}\,{}^kC_r$$

$$= {}^nC_k\,[{}^kC_0 2^k - {}^kC_1 2^{k-1} + {}^kC_2 2^{k-2} - \cdots + (-1)^k\,{}^kC_k]$$

$$= {}^nC_k\,(2-1)^k = {}^nC_k$$