Chapter 1

Calculus

CHAPTER HIGHLIGHTS

- *Limit of a function*
- *Derivatives*
- *Mean value theorem*

Limit of a Function

Let $y = f(x)$ be a function of x and let 'a' be any real number.

We must first understand what a 'limit' is. A limit is the value, function approaches, as the variable within that function (usually 'x') gets nearer and nearer to a particular value. In other words, when x is very close to a certain number, what is $f(x)$ very close to?

Meaning of '$x \to a$'

Let x be a variable and 'a' be a constant. If x assumes values nearer and nearer to 'a', then we say that 'x tends to a' or 'x approaches a' and is written as '$x \to a$'. By $x \to a$, we mean that $x \neq a$ and x may approach 'a' from left or right, which is explained in the example given below.

Let us look at an example of a limit: What is the limit of the function $f(x) = x^3$ as x approaches 2? The expression 'the limit as x approaches to 2' is written as: $\lim\limits_{x \to 2}$ Let us check out some values of $\lim\limits_{x \to 2}$ as x increases and gets closer to 2, without even exactly getting there.

When $x = 1.9, f(x) = 6.859$
When $x = 1.99, f(x) = 7.88$
When $x = 1.999, f(x) = 7.988$
When $x = 1.9999, f(x) = 7.9988$

As x increases and approaches 2, $f(x)$ gets closer and closer to 8 and since x tends to 2 from left this is called 'left-hand limit' and is written as $\lim\limits_{x \to 2^-}$.

Now, let us see what happens when x is greater than 2.

When $x = 2.1, f(x) = 9.261$
When $x = 2.01, f(x) = 8.12$
When $x = 2.001, f(x) = 8.01$
When $x = 2.0001, f(x) = 8.001$

As x decreases and approaches 2, $f(x)$ still approaches 8. This is called 'right-hand limit' and is written as $\lim\limits_{x \to 2^+}$

$\overrightarrow{x}$ 2 2 $\overleftarrow{x}$

We get the same answer while finding both, left and right hand limits. Hence we write that $\lim\limits_{x \to 2} x^3 = 8$.

Meaning of the Symbol: $\lim\limits_{x \to a} f(x) = l$

Let $f(x)$ be a function of x where x takes values closer and closer to 'a' ($\neq a$), then $f(x)$ will assume values nearer and nearer to l. Hence we say, $f(x)$ tends to the limit 'l' as x tends to a.

The following are some of the simple algebraic rules of limits.

1. $\lim\limits_{x \to a} kf(x) = k \lim\limits_{x \to a} f(x)$
2. $\lim\limits_{x \to a} [f(x) \pm g(x)] = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x)$
3. $\lim\limits_{x \to a} [f(x) \cdot g(x)] = \lim\limits_{x \to a} f(x) \cdot \lim\limits_{x \to a} g(x)$
4. $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{x \to a}{\lim\limits_{x \to a} g(x)} (\lim\limits_{x \to a} g(x) \neq 0)$

NOTES

1. If the left hand limit of a function is not equal to the right hand limit of the function, then the limit does not exist.
2. A limit equal to infinity is not the same as a limit that does not exist.

Continuous Functions

Let $f: A \to B$ be any given function and let $c \in A$. We say f is continuous at c, if given $\in > 0$, there exists $\delta > 0$ such that $|f(x) - f(c)| < \in$ whenever $|x - c| < \delta$

In words, this means that, if x is very close to c in domain, then $f(x)$ is very close to $f(c)$ in range.

Equivalently f is continuous at c. If $\lim_{x \to c} f(x) = f(c)$

We observe

1. $c \in A$, i.e., $f(c)$ must exist
2. $\lim_{x \to c} f(x)$ exists
3. $f(c)$ and $\lim_{x \to c} f(x)$ are equal.

If any of these three conditions fail, then f is discontinuous at $x = c$.

Algebra of Continuous Functions

If f, g be two continuous functions at c, then $f + g$, $f - g$, fg are also continuous at $x = c$.

To solve a problem of continuous functions at a point a, you can take the following approach.

1. Find the value $f(x)$ at $x = a$. If a is in the domain of f, $f(a)$ must exist. If a is not in the domain, then $f(a)$ does not exist. In such a case, f is not continuous at $x = a$.
2. Find $\lim_{x \to a} f(x)$. For this you have to first find $\lim_{x \to \infty} f(x) = l_1(\text{say})$ and $\lim_{x \to a^+} f(x) = l_2(\text{say})$. If $l_1 \neq l_2$ then $\lim_{x \to a} f(x)$ does not exist and so f is not continuous at $x = a$. If $l_1 = l_2$, then $\lim_{x \to a} f(x)$ exists.
3. If $\lim_{x \to a} f(x)$ exists and also $f(a)$ exists.

Then verify whether $\lim_{x \to a} f(x) = f(a)$.

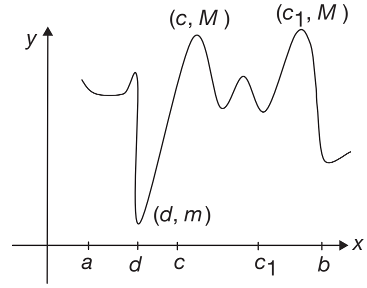
If $\lim_{x \to a} f(x) = f(a)$. Then f is continuous, otherwise it is not continuous at $x = a$.

Problems on continuous functions can be grouped into the following categories.

1. Using $\in$, δ notation.
2. Using existence of right and left hand limits.
3. To find the value of the unknown in $f(x)$ when f is given to be continuous at a point.
4. To find $f(a)$ when f is given to be continuous at $x = a$.

For functions that are continuous on (a, b) the following holds:

f is bounded and attains its bounds at least once on $[a, b]$, i.e., for some $c, d \in [a, b]$,
M = supremum of $f = f(c)$ and m = Infimum of $f = f(d)$



NOTE

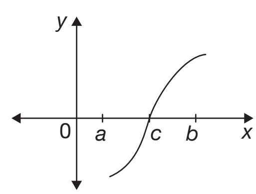
The converse may not be true as $f(x) = \begin{cases} 1; & 0 < x \le 1 \\ -1; & 1 < x \le 2 \end{cases}$ is bounded on $[1, 2]$ but it is not continuous at $x = 1$.

Intermediate-value Theorem

If f is continuous on $[a, b]$ and $f(a) \neq f(b)$ then f takes every value between $f(a)$ and $f(b)$.

Equivalently, if f is continuous on $[a, b]$ and $f(a) < k < f(b)$ or $f(b) < k < f(a)$, then there exists $c \in (a, b)$ such that $f(c) = k$.

Equivalently, If $f(a)$ and $f(b)$ are of opposite signs then there exists $c \in (a, b)$ such that $f(c) = 0$.



$f(a) < 0$ and $f(b) > 0$, clearly $f(c) = 0$.

NOTES

1. If $f(x)$ is continuous in $[a, b]$ then f takes all values between m and M at least once as x moves from a to b, where M = Supremum of f on $[a, b]$ and m = infimum of f on $[a, b]$.
2. If $f(x)$ is continuous in $[a, b]$, then $|f|$ is also continuous on $[a, b]$, where $|f|(x) = |f(x)|\ x \in [a, b]$.
3. Converse may not be true

For instance, $f(x) = \begin{cases} 1; & 0 < x \le 3 \\ -1; & 3 < x \le 5 \end{cases}$

is not continuous at $x = 3$, but $|f|(x) = 1 x \in [0, 5]$, being a constant function is continuous $[0, 5]$.

Inverse-function Theorem

If f is a continuous one-to-one function on $[a, b]$ then f^{-1} is also continuous on $[a, b]$.

Uniform Continuity A function f defined on an interval I is said to be uniformly continuous on I if given $\in > 0$ there exists a $\delta > 0$ such that if x, y are in I and $|x - y| < \delta$ then $|f(x) - f(y)| < \in$.

NOTE

Continuity on $[a, b]$ implies uniform continuity whereas continuity on (a, b) does not mean uniform continuity.

Types of Discontinuity If f is a function defined on an interval I, it is said to have

(TD1) **a removable discontinuity** at $p \in I$, if $\lim_{x\to p} f(x)$ exists, but is not equal to $f(p)$.

(TD2) **a discontinuity of first kind from the left** at p if $\lim_{x\to p^-} f(x)$ exists but is not equal to $f(p)$.

(TD3) **a discontinuity of first kind from the right** at p if $\lim_{x\to p^+} f(x)$ exists but is not equal to $f(p)$.

(TD4) **a discontinuity of first kind** at p if $\lim_{x\to\infty} f(x)$ and $\lim_{x\to p^+} f(x)$ exists but they are unequal.

(TD5) **a discontinuity of second kind from the left** at p if $\lim_{x\to p-} f(x)$ does not exist.

(TD6) **a discontinuity of second kind from the right** at p if $\lim_{x\to p^+} f(x)$ does not exist.

(TD7) **a discontinuity of second kind** at p if neither $\lim_{x\to p^-} f(x)$ nor $\lim_{x\to p^+} f(x)$ exist.

Examples for each type are presented in the following table:

Type	Example	Point of Discontinuity
TD1	$f(x) = \frac{x^2-1}{x-1}, x \neq$ $f(1) = 3$	$x = i$
TD2	$f(x) = x + 3$ for $0 < x < 1$ $f(x) = 5$ for $x \geq 1$	$x = 1$
TD3	$f(x) = x + 3$, for $x > 2$ $f(x) = 8$ for $x \leq 2$	$x = 2$
TD4	$f(x) = \begin{cases} x+3; & x > 2 \\ 7; & x = 2 \\ x-3; & x < 2 \end{cases}$	$x = 2$
TD5	$f(x) = \tan x$ for $x < \pi/2$ $f(x) = 1$, for $x \geq \pi/2$	$x = \frac{\pi}{2}$
TD6	$f(x) = 1$, for $x \leq \pi/2$ $f(x) = \tan x$ for $x > \pi/2$	$x = \frac{\pi}{2}$
TD7	$f(x) = 1/x$ at $x \neq 0$ $f(0)$ $= 3$ at $x = 0$	$x = 0$

NOTES

1. Every differentiable function is continuous, but the converse is not true.
   The example of a function which is continuous but not differentiable at a point $f(x) = |x - 3|$ for $x \in R$ is continuous at $x = 3$, but it is not differentiable at $x = 3$.
2. The function may have a derivative at a point, but the derivative may not be continuous.
   For example the function

$$f(x) = \begin{cases} x^3 \sin\frac{1}{x}; & x \neq 0 \\ 0; & x = 0 \end{cases}$$ has the derivative function

as

$$f'(x) = \begin{cases} 3x^2 \sin\frac{1}{x} - x\cos\frac{1}{x}; & x \neq 0 \\ 0; & x = 0 \end{cases}$$

However $\lim_{x\to 0} f'(x)$ doesn't exist.

SOLVED EXAMPLES

Example 1

Discuss the continuity of the function at $x = 1$ where $f(x)$ is defined by

$$f(x) = \frac{3x-2}{x} \text{ for } 0 < x \leq 1$$
$$= \frac{\sin(x-1)}{(x-1)} \text{ for } x > 1$$

Solution

Consider the left and right handed limits

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1} \frac{3x-2}{x} = 1$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1} \frac{\sin(x-1)}{x-1}$$
$$= \lim_{(x-1)\to 0} \frac{\sin(x-1)}{(x-1)} = 1 \text{ and } f(1)$$
$$= \frac{3(1)-2}{1} = 1$$

$$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$$

$\therefore f$ is continuous at $x = 1$.

Example 2

If $f(x) = \dfrac{(2^x-1)^2}{(\sin 2x)\log(1+x)}$ for $x \neq 0$ and $f(x) = \log 2$ for $x = 0$, discuss the continuity at $x = 0$.

Solution

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{(2^x-1)}{(\sin 2x)\log(1+x)}$$

$$= \lim_{x\to 0} \frac{\left(\frac{2^x-1}{x}\right)^2}{\frac{\sin 2x}{2x}(2)\frac{\log(1+x)}{x}}$$

$$= \lim_{x\to 0} \frac{\left(\frac{2^x-1}{x}\right)^2}{2\left(\frac{\sin 2x}{2x}\right)\log(1+x)^{\frac{1}{x}}}$$

$$= \frac{1}{2}\frac{\lim_{x\to 0}\left(\frac{2^x-1}{x}\right)^2}{\left(\lim_{x\to 0}\frac{\sin 2x}{2x}\right)\left(\log \lim_{x\to 0}(1+x)^{\frac{1}{x}}\right)}$$

$$= \frac{1}{2}(\log 2)^2.$$

But given $f(x) = 2 \log 2$ at $x = 0$

$\therefore \lim_{x\to 0} f(x) \neq f(0)$

$\therefore f(x)$ is not continuous at $x = 0$.

Example 3

Find the value of k if

$$f(x) = \frac{2x^3-5x^2+4x+11}{x+1}, \text{ for } x \neq -1$$

And $f(-1) = k$ is continuous at $x = -1$.

Solution

Given $f(x)$ is continuous at $x = -1$

$$\Rightarrow \lim_{x\to -1} f(x) = f(-1) = k.$$

$$\Rightarrow \lim_{x\to -1} f(x) \lim_{x\to -1}\left[\frac{2x^3-5x^2+4x+11}{x+1}\right]$$

$$= \lim_{x\to -1}\frac{(x+1)(2x^2-7x+11)}{x+1}$$

$$= 2(-1)^2 - 7(-1) + 11$$

$$= 2 + 7 + 11 = 20$$

$$\therefore k = 20$$

Example 4

If $f(x) = \frac{x-4}{|x-4|} + a$, for $x < 4$, $= a + b$ for $x = 4$, $= \frac{x-4}{|x-4|} + b$, for $x > 4$

And $f(x)$ is continuous at $x = 4$, then find the values of a and b.

Solution

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-}\frac{x-4}{|x-4|} + a$$

$$= \lim_{x\to 4^-}\frac{(x-4)}{-(x-4)} + a = -1 + a$$

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+}\frac{x-4}{|x-4|} + b$$

$$= \lim_{x\to 4^+}\frac{x-4}{(x-4)} + b = 1 + b$$

Since given $f(x)$ is continuous at $x = 4$

$$\lim_{x\to 4^-} f(x) = f(4) = \lim_{x\to 4^+} f(x)$$

$\Rightarrow -1 + a = a + b = 1 + b \Rightarrow a = 1, b = -1$

Example 5

Examine the continuity of the given function at origin where,

$$f(x) = \begin{cases} \frac{xe^{\frac{1}{x}}}{1+e^{\frac{1}{x}}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Solution

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\frac{xe^{\frac{1}{x}}}{1+e^{\frac{1}{x}}} = 0$$

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0^+}\frac{x}{e^{-1/x}+1} = 0$$

Then,

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = \lim_{x\to 0} f(x) = 0$$

Thus the function is continuous at the origin.

Derivatives

In this section we will look at the simplistic form of the definition of a derivative, the derivatives of certain standard functions and application of derivatives.

For a function $f(x)$, the ratio $\frac{[f(a+h)-f(a)]}{h}$ is the rate of change of $f(x)$ in the interval $[a, (a + h)]$.

The limit of this ratio as h tends to zero is called the derivative of $f(x)$. This is represented as $f'(x)$, i.e.,

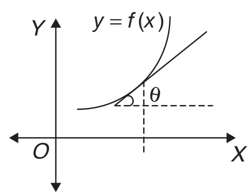
$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = f'(x)$$

The derivative $f'(x)$ is also represented as $\frac{d\{f(x)\}}{dx}$ or $\frac{d}{dx}\{f(x)\}$

Hence, if $y = f(x)$, i.e., y is a function of x, then $\frac{dy}{dx}$ is the derivative of y with respect to x.

NOTES

1. $\frac{dy}{dx}$ is the rate of change of y with respect to x.
2. If the function y can be represented as a general curve, and a tangent is drawn at any point where the tangent makes an angle θ with the horizontal (as shown in the figure), then $\frac{dy}{dx} = \tan\theta$, In other words, derivative of a function at a given point is the slope of the curve at that point, i.e., tans of the angle, the tangent drawn to the curve at that point, makes with the horizontal.



Standard Results

If $f(x)$ and $g(x)$ are two functions of x and k is a constant, then

1. $\frac{d}{dx}(c) = 0$ (c is a constant)

2. $\frac{d}{dx}k \cdot f(x) = k\frac{d}{dx}f(x)$ (k is a constant)

3. $\frac{d}{dx}(f(x) \pm g(x))$

$= \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$

Product Rule

4. $\frac{d}{dx}\{f(x)\cdot g(x)\} = f'(x)\cdot g(x) + f(x)\cdot g'(x)$

Quotient Rule

5. $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{(g(x))^2}$

Chain Rule

6. If $y = f(u)$ and $u = g(x)$ be two functions, then $\frac{dy}{dx} = \left(\frac{dy}{du}\right)\times\left(\frac{du}{dx}\right)$

Derivatives of Some Important Functions

1. (a) $\frac{d}{dx}(x^n) = n\cdot x^{n-1}$

 (b) $\frac{d}{dx}\left[\frac{1}{x^n}\right] = \frac{-n}{x^{n+1}}$

 (c) $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}};\ x \neq 0$

2. $\frac{d}{dx}[ax^n + b] = an \cdot x^{n-1}$
3. $\frac{d}{dx}[ax + b]^n = n\,a\,(ax + b)^{n-1}$
4. $\frac{d}{dx}[e^{ax}] = a \cdot e^{ax}$
5. $\frac{d}{dx}[\log x] = \frac{1}{x};\ x > 0$
6. $\frac{d}{dx}[a^x] = a^x \log a;\ a > 0$
7. (a) $\frac{d}{dx}[\sin x] = \cos x$

 (b) $\frac{d}{dx}[\cos x] = -\sin x$

 (c) $\frac{d}{dx}[\tan x] = \sec^2 x$

 (d) $\frac{d}{dx}[\cot x] = -\mathrm{cosec}^2 x$

 (e) $\frac{d}{dx}[\sec x] = \sec x \cdot \tan x$

 (f) $\frac{d}{dx}[\mathrm{cosec}\, x] = -\mathrm{cosec}\, x \cdot \cot x$

Inverse Rule

If $y = f(x)$ and its inverse $x = f^{-1}(y)$ is also defined, then $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$.

Second Derivative

If $y = f(x)$, then the derivative of derivative of y is called as second derivative of y and is represented by $\frac{d^2y}{dx^2}$.

$\frac{d^2y}{dx^2} = f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right)$ where $\frac{dy}{dx}$ is the first derivative of y.

8. (a) $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$

 (b) $\frac{d}{dx}\mathrm{cosec}^{-1} x = \frac{-1}{|x|\sqrt{x^2-1}}$

 (c) $\frac{d}{dx}\cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$

 (d) $\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$

(e) $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$

(f) $\frac{d}{dx}\cot^{-1}x = \frac{-1}{1+x^2}$

9. (a) $\frac{d}{dx}\sinh x = \cosh x$

(b) $\frac{d}{dx}\cosh x = \sinh x$

(c) $\frac{d}{dx}\tanh x = \text{sech}^2 x$

(d) $\frac{d}{dx}\coth x = -\text{cosech}^2 x$

(e) $\frac{d}{dx}\text{sech}\, x = -\text{sech}\, x \tanh x$

(f) $\frac{d}{dx}\text{cosech} x = -\text{cosech} x \coth x$

10. (a) $\frac{d}{dx}\sinh^{-1}x = \frac{1}{\sqrt{1+x^2}}$

(b) $\frac{d}{dx}\cosh^{-1}x = \frac{1}{\sqrt{x^2-1}}$

(c) $\frac{d}{dx}\tanh^{-1}x = \frac{1}{1-x^2}$

(d) $\frac{d}{dx}\coth^{-1}x = \frac{-1}{x^2-1}$

(e) $\frac{d}{dx}\text{sech}^{-1}x = \frac{-1}{x\sqrt{1-x^2}}$

(f) $\frac{d}{dx}\text{cosech}^{-1}x = \frac{-1}{x\sqrt{x^2+1}}$

Successive Differentiation

If f is differentiable function of x and the derivative f' is also a differentiable function of x, then f'' is called the second derivative of f. Similarly 3rd, 4th ... nth derivative of f may be defined and are denoted by f''', f'''', ... f^n or $y_3, y_4 \dots y_n$.

11. The n^{th} derivatives of some special functions:

(a) $\frac{d^n}{dx^n}x^n = n\,!$

(b) $\frac{d^n}{dx^n}x^m = \frac{m!}{(m-n)}x^{m-n}$ s(m being a positive integer more than n)

(c) $\frac{d^n}{dx^n}e^{ax} = a^n e^{ax}$

(d) $\frac{d^n}{dx^n}\left(\frac{1}{x-a}\right) = \frac{(-1)^n n!}{(x+a)^{n+1}}; x \neq -a$

(e) $\frac{d^n}{dx^n}\log(x+a) = \frac{(-1)^{n-1}(n-1)!}{(x+a)}$; $(x+a) > 0$

(f) $\frac{d^n}{dx^n}\sin(ax+b) = a^n \sin\left(\frac{n\pi}{2}+ax+b\right)$

(g) $\frac{d^n}{dx^n}\cos(ax+b) = a^n \cos\left(\frac{n\pi}{2}+ax+b\right)$

(h) $\frac{d^n}{dx^n}(e^{ax}\sin bx)$

$= (a^2+b^2)^{n/2} e^{ax} \sin\left(bx + n\tan^{-1}\frac{b}{a}\right)$

(i) $\frac{d^n}{dx^n}(e^{ax}\cos bx)$

$= (a^2+b^2)^{n/2} e^{ax} \cos\left(bx + n\tan^{-1}\frac{b}{a}\right)$

(j) $\frac{d^n}{dx^n}\left(\frac{1}{x^2+a^2}\right) = \frac{(-1)^n n}{a^{n+2}}\sin^{n+1}\theta \sin(n+1)\theta$

where $\theta = \tan^{-1}\left(\frac{x}{a}\right)$

(k) $\frac{d^n}{dx^n}(\tan^{-1}x) = (-1)^{n-1}(n-1)!\ \sin^n\theta \cdot \sin n\theta$

where $\theta = \cot^{-1}x$.

Application of Derivatives

Errors in Measurement

Problems relating to errors in measurement can be solved using the concept of derivatives. For example, if we know the error in measurement of the radius of a sphere, we can find out the consequent error in the measurement of the volume of the sphere. Without going into further details of theory, we can say dx = error in measurement of x and dy = consequent error in measurement of y, Where $y = f(x)$. Hence, we can rewrite $\frac{dy}{dx} = f'(x)$ as $dy = f'(x)\cdot dx$. Thus, if we know the function $y = f(x)$ and dx, error in measurement of x, we can find out dy, the error in measurement of y.

NOTES

1. An error is taken to be positive when the measured value is greater than the actual value and negative when it is less.
2. Percentage error in y is given by $\left(\frac{dy}{y}\right)\times 100$.

Rate of Change

While defining the derivative, we have seen that derivative is the 'rate of change'. This can be applied to motion of bodies to determine their velocity and acceleration.

Velocity If we have s, the distance covered by a body expressed as a function of t, i.e., $s = f(t)$, then rate of change of s is called velocity (v). $v = \frac{ds}{dt} = f'(t)$.

Acceleration Rate of change of velocity is defined as acceleration. Since $v = f'(t)$ itself is a function of t, we can write $v = f'(t)$.

$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$, i.e., acceleration is the second derivative of the function $s = f(t)$.

Maxima and Minima

A function takes a maximum value or a minimum value when the slope of the tangent of the curve at that point is zero, i.e., when the first derivative of the function is zero. If $y = f(x)$, then y is maximum or minimum at the point $x = x_1$ if $\left(\frac{dy}{dx}\right)_{x=x_1} = 0$.

Thus we can find the value of x_1 by equating $\frac{dy}{dx} = 0$.

As mentioned above that y can have a maximum or a minimum value at $x = x_1$. Whether y is a maximum value or minimum is governed by the sign of the second derivative. The function y has a minimum value if the second derivative is positive. In other words, y is maximum at $x = x_1$ if $\frac{d^2y}{dx^2} < 0$ at $x = x_1 \cdot y$ is minimum at $x = x_1$ if $\frac{d^2y}{dx^2} > 0$ at $x = x_1 \cdot \left(\frac{dy}{dx}\right)_{x=x_1} = 0.$ in both the cases discussed above.

The above discussion can be summerized as follows:

1. If $f'(c) = 0$ and $f''(c)$ is negative, then $f(x)$ is maximum for $x = c$
2. If $f'(c) = 0$ and $f''(c)$ is positive, then $f(x)$ is minimum for $x = c$
3. If $f'(c) = f''(c) = \cdots = f^{r-1}(c) = 0$ and $f^r(c) \neq 0$, then
   (a) If r is even, then $f(x)$ is maximum or minimum for $x = c$ according as $f^r(c)$ is negative or positive.
   (b) If r is odd, then there is neither maximum nor a minimum for $f(x)$ at $x = c$.

Mean Value Theorems

Rolle's Theorem Let f be a function defined on $[a, b]$ such that

1. f is continuous on $[a, b]$;
2. f is differentiable on (a, b) and
3. $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$

Lagrange's Mean Value Theorem Let f be a function defined on $[a, b]$ such that

1. f is continuous on $[a, b]$,
2. f is differentiable on (a, b) then there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Another Form If f is defined on $[a, a + h]$ such that

1. f is continuous on $[a, a + h]$.
2. f is differentiable on $(a, a + h)$ then there exists atleast one $\theta \in (0, 1)$ such that $f(a + h) = f(a) + hf'(a + \theta h)$.

Meaning of the sign of the derivative

SIGN OF $f'(x)$ on [a, b]	Meaning
$f'(x) \geq 0$	f is non-decreasing
$f'(x) > 0$	f is increasing
$f'(x) < 0$	f is non-increasing
$f'(x) < 0$	f is decreasing
$f'(x) = 0$	f is constant

Example: The function f, defined on R by $f(x) = x^3 - 15x^2 + 75x - 125$ is non-decreasing in every interval as $f'(x) = 3(x^2 - 10x + 15) = 3(x - 5)^2 \geq 0$

Thus f is non-decreasing on R.

Cauchy's Mean Value Theorem Let f and g be two functions defined on $[a, b]$ such that

1. f and g are continuous on $[a, b]$
2. f and g are differentiable on (a, b)
3. $g'(x) \neq 0$ for any $x \in (a, b)$ then there exists at least one real number $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Taylor's Theorem

Let f be a real-valued function defined on $[a, a + h]$ such that

1. f^{n-1} is continuous on $[a, a + h]$
2. f^{n-1} is derivable on $(a, a + h)$, then there exists a number $\theta \in (0, 1)$ such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots$$

$$+ \frac{h^{n-1}}{(n-1)}f^{n-1}(a) + R_n.$$

Where

$$R_n = \frac{h^n f^n(a + \theta h)}{n!}$$

(Lagranges' form of remainder)

$$R_n = \frac{h^n (1-\theta)^{n-1} f^n(a+\theta h)}{(n-1)!}$$

(Cauchy's form of remainder)

Maclaurin's Theorem Let $f \cdot [0, x] \to R$ such that

1. f^{n-1} is continuous on $[0, x]$,
2. f^{n-1} is derivable on $(0, x)$

Then there exists a real number $\theta \in (0, 1)$ such that

$$f(x) = f(0) + xf'(0) + \frac{x^2}{n!}f''(0) + \cdots$$

$$+ \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + R_n.$$

Where

$$R_n = \frac{x^n}{n!}f^n(\theta x)$$

(Lagranges form of remainder)

$$R_n = \frac{x^n(1-\theta)^{n-1}f^n(\theta x)}{(n-1)!}$$

(Cauchy's form of remainder)

Maclaurin's Series Let $f(x)$ be a function which posses derivatives of all orders in the interval $[0, x]$, then

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)}f^{(n-1)}(0)$$

$$+ \frac{x^n}{n!}f^n(0) + \cdots \text{ is known as}$$

Maclaurin's infinite series.

Series expansions of some standard functions

1. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$
2. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$
3. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$
4. $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + \cdots$
5. $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots$
6. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} - \cdots + \frac{(-1)^{n-1}x^n}{n}$
7. $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \cdots$
8. $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \cdots$
9. $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$
10. $(1-x)^{-\frac{1}{2}} = 1 + \frac{x}{2} + \frac{1\cdot 3}{2\cdot 3}x^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot x^3 + \cdots$
11. $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^{n-1}}{(2n-1)}x^{2n-1} + \cdots$
12. $\sin^{-1}x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5} + \cdots$

Example 6

For the function $f(x) = x(x^2 - 1)$ test for the applicability of Rolle's theorem in the interval $[-1, 1]$ and hence find c such that $-1 < c < 1$.

Solution

Given $f(x) = x(x^2 - 1)$

1. f is continuous in $[-1, 1]$
2. f is differentiable in $(-1, 1)$
3. $f(-1) = f(1) = 0$

$\therefore$ $f(x)$ satisfies the hypothesis of Rolle's theorems

$\therefore$ We can find a number c such that $f'(c) = 0$, i.e., $f'(x) = 3x^2 - 1$

$$f'(c) = 0 \Rightarrow 3c^2 - 1 = 0 \Rightarrow c = \pm\sqrt{\frac{1}{3}}$$

$$\Rightarrow c = \sqrt{\frac{1}{3}}$$

Example 7

If $f(x) = 2x^2 + 3x + 4$, then find the value of θ in the mean value theorem.

Solution

$f(a) = 2a^2 + 3a + 4$

$f(a + h) = 2(a^2 + 2ah + h^2) + 3a + 3h + 4$

$f(a + h) - f(a) = 4ah + 2h^2 + 3h = 2(2ah + h^2) + 3h$

$$\frac{f(a+h) - f(a)}{h} = 2(2a + h) + 3$$

$$= 4\left(a + \frac{h}{2}\right) + 3 \quad (1)$$

Now $f'(x) = 4x + 3$, $f^1(a + \theta h)$

$= 4a + 4h\theta + 3$ (2)

Comparing Eqs. (1) and (2) we have $4\left(a + \frac{h}{2}\right) + 3$

$$= 4a + 4h\theta + 3 \Rightarrow a + h\theta = a + \frac{h}{2}$$

$$\Rightarrow \theta = \frac{1}{2}$$

Partial Differentiation

Let u be a function of two variables x and y. Let us assume the functional relation as $u = f(x, y)$. Here x alone or y alone or both x and y simultaneously may be varied and in each case a change in the value of u will result. Generally the change in the value of u will be different in each of these three cases. Since x and y are independent, x may be supposed to vary when y remains constant or the reverse.

The derivative of u wrt x when x varies and y remains constant is called the partial derivative of u wrt x and is denoted by $\frac{\partial u}{\partial x}$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right), \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right).$$

Total Differential Co-efficient

If u be a continuous function of x and y and if x and y receive small increments Δx and Δy, u will receive, in turn, a small increment Δu. This Δu is called total increment of u.

$\Delta u = f(x + \Delta x, y + \Delta y) - f(x, y)$

In the differential form, this can be written as

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy.$$

du is called the total differential of u. If $u = f(x, y, z)$ then

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \cdot \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$$

Implicit Function

If the relation between x and y be given in the form $f(x, y) = c$ where c is a constant, then the total differential co-efficient wrt x is zero.

Homogeneous Functions

Let us consider the function $f(x, y) = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \cdots + a_n y^n$. In this expression the sum of the indices of the variable x and y in each term is n. Such an expression is called a homogeneous function of degree n.

Euler's Theorem

If $f(x, y)$ is a homogeneous function of degree n, then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$.

This is known as Euler's theorem on homogeneous function.

Maxima and Minima for Function of Two Variables

A function $f(x, y)$ is said to have a local maximum at a point (a, b), if $f(a + h, b + k) \le f(a, b)$ for all small values of h and k, i.e., $f(x, y)$ has a local maximum at (a, b), if $f(a, b)$ has a highest value in a neighbourhood of (a, b).

Similarly, $f(x, y)$ is said to have a local minimum at a point (a, b), if $f(x, y)$ has least value at (a, b) in a neighbourhood of (a, b).

Procedure to Obtain Maxima and Minima

Let $f(x, y)$ be a function of two variables for which we need to find maxima and minima.

1. Find $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$

2. Take $f_x = 0$ and $f_y = 0$ and solve them as simultaneous equations to get pairs of values for x and y, which are called stationary points.

3. Find $r = f_{xx} = \frac{\partial^2 f}{\partial x^2}, \quad s = f_{xy} = \frac{\partial^2 f}{\partial x \partial y}$ and

$t = f_{yy} = \frac{\partial^2 f}{\partial y^2}$ and find $rt - s^2$.

4. At a stationary point, say (a, b)

(a) If $rt - s^2 > 0$, then (a, b) is called an extreme point of $f(x, y)$ at which $f(x, y)$ has either maximum or minimum which can be found as follows.

Case 1: If $r < 0$, then $f(x, y)$ has a local maximum at $a, b)$

Case 2: If $r > 0$, then $f(x, y)$ has a local minimum at (a, b).

(b) If $rt - s^2 < 0$, then (a, b) is called as saddle point of $f(x, y)$ where $f(x, y)$ has neither maximum nor minimum at (a, b).

Example 8

Find the stationary points of the function $f(x, y) = x^2 y + 3xy - 7$ and classify them into extreme and saddle points.

Solution

Given $f(x, y) = x^2 y + 3xy - 7$

$\therefore f_x = \frac{\partial f}{\partial x} = 2xy + 3y$ and $f_y = \frac{\partial f}{\partial y} = x^2 + 3x$

Now $f_x = 0 \Rightarrow 2xy + 3y = 0$ and $f_y = 0$

$\Rightarrow x^2 + 3x = 0$

$\Rightarrow \quad y = 0$ and $x = \frac{-3}{2}$; $x(x+3)x = 0$ and $x = -3$

But for $x = \frac{3}{2}$, $f_y \ne 0$

$\therefore$ The stationary points of $f(x, y)$ are $(0, 0)$ and $(-3, 0)$

Now $r = f_{xx} = 2y$; $s = f_{xy} = 2x + 3$ and $t = f_{yy} = 0$

And $rt - s^2 = 2y \times 0 - (2x + 3)^2 = -(2x + 3)^2$

$\therefore rt - s^2 < 0$ at $(0, 0)$ as well as $(-3, 0)$

Hence the two stationary points $(0, 0)$ and $(-3, 0)$ are saddle points where $f(x, y)$ has neither maximum nor minimum.

Example 9

Find the maximum value of the function $f(x, y, z) = z - 2x^2 - 3y^2$ where $3xy - z + 7 = 0$.

Solution

Given $f(x, y, z) = z - 2x^2 - 3y^2$ (1)

Where $3xy - z + 7 = 0$ (2)

$\Rightarrow z = 3xy + 7$ (3)

Substituting the value of z in (1), we have $f = 3xy + 7 - 2x^2 - 3y^2$

$$\therefore f_x = \frac{\partial f}{\partial x} = 3y - 4x \text{ and } f_y = \frac{\partial f}{\partial y} = 3x - 6y$$

$f_x = 0 \Rightarrow 3y - 4x = 0$ and $f_y = 0 \Rightarrow 3x - 6y = 0$

$f_x = 0$ and $f_y = 0$ only when $x = 0$ and $y = 0$

$\therefore$ The stationary point is (0, 0)

Now $r = f_{xx} = \dfrac{\partial^2 f}{\partial x^2} = -4;\ s = f_{xy} = \dfrac{\partial^2 f}{\partial x \partial y} = 3$ and

$$t = f_{yy} = \frac{\partial^2 f}{\partial y^2} = -6$$

$\therefore rt - s^2 = (-4)(-6) - 3^2 = 24 - 9 = 15 > 0$ and $r = -4 < 0$

$\therefore f$ has a maximum value at (0, 0)

For $x = 0, y = 0$, from (3), $z = 3 \times 0 \times 0 + 7 \Rightarrow z = 7$

$\therefore$ The maximum value exists for $f(x, y, z)$ at (0, 0, 7) and that maximum value is $f(x, y, z)_{at\ (0,0,7)} = 7 - 2 \times 0^2 - 3 \times 0^2 = 7$.

Indefinite Integrals

If $f(x)$ and $g(x)$ are two functions of x such that $g'(x) = f(x)$, then the integral of $f(x)$ is $g(x)$. Further, $g(x)$ is called the antiderivative of $f(x)$.

The process of computing an integral of a function is called Integration and the function to be integrated is called integrand.

An integral of a function is not unique. If $g(x)$ is any one integral of $f(x)$, then $g(x) + c$ is also its integral, where C is any constant termed as constant of integration.

Some Standard Formulae

1. $\int x^n dx = \dfrac{x^{n+1}}{n+1} + c \ (n \neq -1)$
2. $\int (ax+b)^n dx = \dfrac{(ax+b)^{n+1}}{(n+1)a} + c \ (n \neq -1)$
3. $\int \dfrac{1}{x} dx = \log x + c$
4. $\int \dfrac{1}{ax+b} dx = \dfrac{\log(ax+b)}{a} + c$
5. $\int a^x dx = \dfrac{a^x}{\log a} + c$
6. $\int e^x dx = e^x + c$
7. $\int \sin x \, dx = -\cos x + c$
8. $\int \cos x \, dx = \sin x + c$
9. $\int \sec^2 x \, dx = \tan x + c$
10. $\int \operatorname{cosec}^2 x \, dx = -\cot x + c$
11. $\int \sec x \tan x \, dx = \sec x + c$
12. $\int \operatorname{cosec} x \cot x \, dx = \operatorname{cosec} x + c$
13. $\int \tan x \, dx = \log(\sec x) + c$
14. $\int \cot x \, dx = \log(\sin x) + c$
15. $\int \sec x \, dx = \log(\sec x + \tan x) + c$

    $= \log \tan\left[\dfrac{\pi}{4} + \dfrac{x}{2}\right] + c$
16. $\int \operatorname{cosec} x \, dx = \log(\operatorname{cosec} x + \cot x) + c$

    $= \log \tan \dfrac{x}{2} + c$
17. $\int \dfrac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$ or $-\cos^{-1} x + c$
18. $\int \dfrac{1}{1+x^2} dx = \tan^{-1} x + c$ or $-\cot^{-1} x + c$
19. $\int \dfrac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + c$ or $-\operatorname{cosec}^{-1} x + c$
20. $\int \sinh x \, dx = \cosh x + c$
21. $\int \cosh x \, dx = \sinh x + c$
22. $\int \operatorname{sech}^2 x \, dx = \tanh x + c$
23. $\int \operatorname{cosech}^2 x \, dx = -\coth x + c$
24. $\int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x + c$
25. $\int \operatorname{sech} x \coth x \, dx = -\operatorname{cosech} x + c$
26. $\int K f(x) dx = K \int f(x) dx + c$
27. $\int (f(x) \pm g(x)) \, dx = \int f(x) \, dx \pm \int g(x) \, dx + c$
28. $\int \dfrac{f'(x)}{f(x)} dx = \log[f(x)] + c$
29. $\int f(x)^n \cdot f'(x) dx = \dfrac{[f(x)]^{n+1}}{n+1} + c$

30. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}+c$

31. $\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\frac{x}{a}+c$ or

$\log|x+\sqrt{a^2+x^2}|+c$

32. $\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a}+c$ or

$\log|x+\sqrt{x^2+a^2}|+c$

33. $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$

34. $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c$

35. $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+c$

36. $\int \sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2+x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+c$

37. $\int \sqrt{a^2+x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin h^{-1}\frac{x}{a}+c$

38. $\int \sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\cos h^{-1}\frac{x}{a}+c$

39. $\int \log x\,dx = x(\log x-1) = x\log\left(\frac{x}{e}\right)+c$

40. $\int e^x[f(x)+f'(x)]\,dx = e^x f(x)+c$

Definite Integrals

The difference in the values of an integral of a function $f(x)$ for two assigned values say a, b of the independent variable x, is called the Definite Integral of $f(x)$ over the interval $[a, b]$ and is denoted by $\int_a^b f(x)dx$.

The number 'a' is called the lower limit and the number 'b' is the upper limit of integration.

Fundamental Theorem of Integral Calculus

If $f(x)$ is a function of x continuous in $[a, b]$, then $\int_a^b f(x)dx = g(b)-g(a)$ where $g(x)$ is a function such that $\frac{d}{dx}g(x) = f(x)$.

Properties of definite integrals

1. If $f(x)$ is a continuous function of x over $[a, b]$, and c belongs to $[a,b]$, then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.
2. If $f(x)$ is continuous function of x over $[a, b]$, then $\int_a^b Kf(x)dx = K\int_a^b f(x)dx$.
3. If $f(x)$ is continuous function of x over $[a, b]$, then $\int_b^a f(x)dx = -\int_a^b f(x)dx$.
4. If $f(x)$ is continuous in some neighbourhood of a, then $\int_a^a f(x)dx = 0$.
5. If $f(x)$ and $g(x)$ are continuous in $[a, b]$, then $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$.
6. $\int_a^b f(x)dx = \int_a^b f(z)dz = \int_a^b f(t)dt$
7. $\int_0^a f(x)dx = \int_0^a f(a-x)dx$
8. $\int_{-a}^a f(x) = 0$, if $f(x)$ is odd
9. $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$ if $f(x)$ is even
10. $\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$, if $f(2a-x) = f(x)$

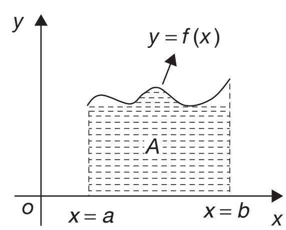
    $= 0$ if $f(2a-x) = -f(x)$
11. $\int_0^{na} f(x)dx = n\int_0^a f(x)dx$, if $f(a+x) = f(x)$

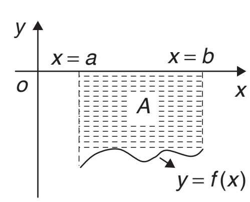
Applications of Integration

Area as a Definite Integral

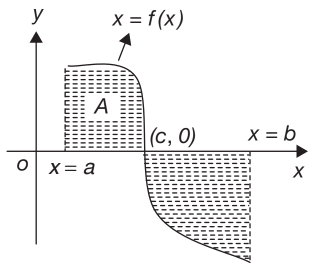
1. The area enclosed by a curve $y = f(x)$, the lines $x = a$ and $x = b$ and the x-axis is given by:

$$A = \int_a^b |f(x)|dx = \begin{cases} \int_a^b f(x)dx, & \text{if } f(x)\ge 0,\ a\le x\le b \\ -\int_a^b f(x)dx, & \text{if } f(x)\le 0,\ a\le x\le b \end{cases}$$

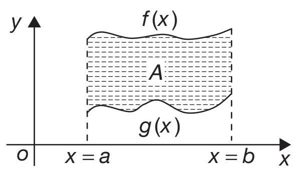


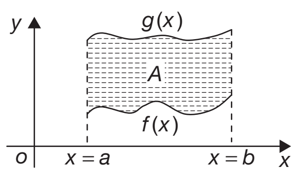


2. Similarly, the area enclosed by the curve $x = g(y)$, the lines $y = c$ and $y = d$ and the y-axis is $A = \int_c^d |g(y)|dy$
3. When $f(x)\ge 0$ for $a\le x\le c$ and $f(x)\le 0$ for $c\le x\le b$, then the area enclosed by the curve $y = f(x)$, the lines $x = a$ and $x = b$ and the x-axis is $A = \int_a^c f(x)\,dx - \int_c^b f(x)\,dx$



4. The area enclosed by the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$ is given by,





$$A = \int_a^b |f(x) - g(x)|dx = \begin{cases} \int_a^b (f(x) - g(x))dx, \text{ if } f(x) \geq g(x), \\ \quad a \leq x \leq b \\ \int_a^b (g(x) - f(x))dx, \text{ if } f(x) \leq g(x); \\ \quad a \leq x \leq b \end{cases}$$

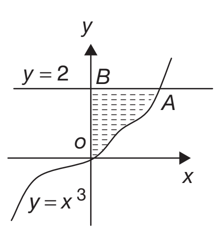
Example 10

Find the area enclosed by the curve $y = x^3$, the line $y = 2$ and the y-axis in first quadrant?

Solution

The area bounded by $y = x^3$, $y = 2$ and the y-axis is the area OAB as shown in the figure.

So, the region OAB is bounded by the curve $x = y^{\frac{1}{3}}$, the lines $y = 0$ and $y = 2$ and the y-axis and $x = y^{\frac{1}{3}} \geq 0,\ y \in [0,\ 2]$



∴ The required area

$$= \left[\int_{y=0}^{2} y^{\frac{1}{3}} dy = \frac{3}{4} y^{\frac{4}{3}}\right]_0^2$$

$$= \frac{3}{4} \times 2^{\frac{4}{3}}$$

$$= \frac{3}{2^{\frac{2}{3}}}$$

$$= \frac{3}{\sqrt[3]{4}}$$

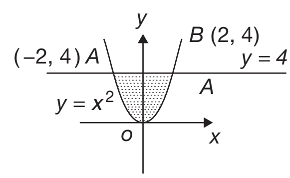
Example 11

Find the area enclosed by the curve $y = x^2$ and line $y = 4$?

Solution

The area enclosed by the curve $y = x^2$ and the line $y = 4$ is the region OAB.

∴ The region OAB is bounded by line $y = 4$ and the curve $y = x^2$ from $x = -2$ to $x = 2$ and $4 \geq x^2$ for all $x \in [-2, 2]$



∴ The required area $= \int_{x=-2}^{2} (4 - x^2)\, dx$

$$= 2\int_0^2 (4 - x^2)dx \ (\because \quad 4 - x^2 \text{as even})$$

$$= 2\left[4x - \frac{x^3}{3}\right]_0^2 = \frac{32}{3}$$

Rectification

The process of determining the length of arcs of plane curves is called Rectification. The length of the arc can be calculated by any one of the methods given below.

Cartesian Equations Let $y = f(x)$ be a function of x. The length of arc between the points with x-coordinates 'a' and 'b' is given by $S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$, provided $\frac{dy}{dx}$ is continuous on $[a, b]$.

NOTE

If the equation of the curve is given in the form $x = f(y)$, then the length of the arc between the points with y-coordinates 'c' and 'd' is given by

$$S = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy \text{ provided } \frac{dx}{dy} \text{ is continuous on } [c, d]$$

Parametric Equations Let $x = f(t)$ and $y = g(t)$ be parametric functions of 't'. The length of the arc between the points $\{f(t_1), g(t_1)\}$ and $\{f(t_2), g(t_2)\}$ is given by

$\int_{t_1}^{t_2}\sqrt{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]}\,dt$ provided $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are both continuous on $[t_1, t_2]$.

Polar Equations Let $r = f(\theta)$ be a function of θ, the length of the arc between the points $\{f(\theta_1), \theta_1\}$ and $\{f(\theta_2), \theta_2\}$ is given by $S = \int_{\theta_1}^{\theta_2}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$ provided $\frac{dr}{d\theta}$ is continuous along the arc.

If the equation of the curve is given in the form $\theta = f(r)$, then the length of the arc between the points $(r_1, f(r_1))$, $(r_2, f(r_2))$ is given by

$S = \int_{r_1}^{r_2}\sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}\,dr$ provided $\frac{d\theta}{dr}$ is continuous along the arc.

Theorems on Integration

1. If f is a continuous function on $[a, b]$ then there exists $c \in (a, b)$ such that $\int_a^b f(x)\,dx = f(c)(b-a)$
2. If $f, g \in R\,[a, b]$ and g keeps the same sign on $[a, b]$ then there exists $\mu \in R$ lying between the infimum and the suprimum of f such that $\int_a^b f(x)g(x) = \mu\int_a^b g(x)dx$

NOTE

This is called the first mean value theorem.

3. If $f, g \in R\,[a, b]$, g is positive and decreasing on $[a, b]$ Then there exists $\mu \in [a, b]$ such that

$$\int_a^b f(x)g(x)\,dx = g(a)\int_a^\mu f(x)dx$$

NOTE

This is known as Bonnet mean value theorem.

4. If $f, g \in R\,[a, b]$ and is monotonic on $[a, b]$ then there exists $\mu \in (a, \text{b})$ such that $\int_a^b f(x)g(x)\,dx = g(x)\int_a^\mu f(x)\,dx + g(x)\int_\mu^b f(x)\,dx$

NOTE

This is known as second mean value theorem or weierstrars theorem.

Example 12

Prove that these exists $\mu \in \left(0, \frac{\pi}{2}\right)$ such that

$$\int_0^{\frac{\pi}{2}} x\cos x\,dx = \mu$$

Solution

Take $f(x) = x$ and $g(x) = \cos x$

$\therefore$ f is continuous on $\left[0, \frac{\pi}{2}\right]$ and g is integrable on $\left[0, \frac{\pi}{2}\right]$ also

$$g(x) \geq 0 \text{ in } \left[0, \frac{\pi}{2}\right]$$

$\therefore$ By first mean value theorem,

$$\int_0^{\frac{\pi}{2}} x\cos dx = \mu\int_0^{\pi}\cos x\,dx = \mu$$

$\therefore$ There exists is $\mu \in \left(0, \frac{\pi}{2}\right)$ such that $\int_0^{\frac{\pi}{2}}\cos x\,dx = \mu$

Example 13

Verify second mean value theorem for $f(x) = x^2$ and $g(x) = x^2$ on $[-1, 1]$.

Solution

Given $f(x) = x^2$ and $g(x) = x^2$ on $[-1, 1]$ both f and g are continuous and integrable on $[-1, 1]$ but g is a decreasing function on $[-1, 0]$ and increasing function on $[0, 1]$ $\therefore$ g is not monotonic.

$$\therefore \int_{-1}^1 f(x)g(x)dx = \int_{-1}^1 x^2\cdot x^2 dx$$

$$= \left[\frac{x^5}{5}\right]_{-1}^1 = \frac{1}{5}+\frac{1}{5} = \frac{2}{5} \quad (1)$$

But by second mean value theorem,

$$\int_a^b f(x)g(x)\,dx = g(a)\int_a^\mu f(x)dx + g(b)\int_\mu^b f(x)\,dx$$

$$\therefore \int_{-1}^1 x^4 dx = g(-1)\int_{-1}^\mu x^2 dx + g(1)\int_\mu^1 x^2 dx$$

$$= \int_{-1}^\mu x^2 dx + \int_\mu^1 x^2 dx = \int x^2 dx = \frac{2}{3} \quad (2)$$

Since (1) and (2) are not equal the mean value theorem does not hold.

Improper Integrals

Consider definite integral $\int_a^b f(x)dx$ (1)

If $f(x)$ is a function defined in a finite interval $[a, b]$ and $f(x)$ is continuous for all x which belongs to $[a, b]$

Then (1) is called proper integral.

If $f(x)$ is violated, at least one of these conditions then the integral is known as improper integral. These improper integrals are classified into three kinds.

Improper Integral of the First Kind In a definite integral if one or both limits of integration are infinite then it is an improper integral of first kind.

1. $\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx.$

   (Singularity at upper limit)

2. $\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx.$

3. $\int_{-\infty}^\infty f(x)dx = \lim_{\substack{b\to\infty \\ a\to-\infty}}\int_a^b f(x)dx.$ Or

4. $\int_{-\infty}^{\infty} f(x)dx = \lim_{a\to-\infty}\int_a^0 f(x)dx + \lim_{b\to\infty}\int_0^b f(x)dx.$

Or $= \lim_{r\to\infty}\int_{-r}^{r} f(x)dx.$

Convergent: If the limits of the above integral exists or finite then the integral is said to be converge.

Divergent: If the limits do not exist then they are said to be Divergent.

NOTES

1. Geometrically for $f(x) \geq 0$, the improper integral $\int_a^\infty f(x)dx$ denotes the area of an unbounded region lying between the curve $y = f(x)$ the ordinate $x = a$ and x-axis.
2. Let $f(x)$ and $g(x)$ be non-negative functions and $0 \leq f(x) \leq g(x)$ for $x \geq a$. If $\int_a^\infty g(x)dx$ converges then $\int_a^\infty f(x)dx$ is also converges and $\int_a^\infty f(x)dx \leq \int_a^\infty g(x)dx.$

   Similarly let $0 \leq g(x) \leq f(x)$. If $\int_a^\infty g(x)dx$ diverges then $\int_a^\infty f(x)dx$ also diverges.

   That is the convergent or divergent of an improper integral by comparing it with a simple integral.

Improper Integral of the Second Kind

Consider $$\int_a^b f(x)dx \qquad (1)$$

If both the limits of Eq. (1) are finite and $f(x)$ is undefined or discontinuous at a point in between a and b, then Eq. (1) is known as Improper integral of second kind.

This can be evaluated as follows.

Let $f(x)$ be undefined at a point c which belongs to (a, b) then

$$\int_a^b f(x)dx = \lim_{\in\to 0}\int_a^{c-\in} f(x)dx + \lim_{\in\to 0}\int_{c+\in}^b f(x)dx.$$

If these limits exist then it is convergent otherwise it is divergent.

Improper Integral of Third Kind If the limits of the integral are infinite or $f(x)$ may be discontinuous or both then the improper integral is known as third kind.

NOTES

1. $\int_1^\infty \frac{1}{x^p}dx$ is convergent when $p > 1$ and it is divergent when $p \leq 1$. This result is used in comparison test for testing the convergence or divergence of the integral of first kind.
2. $\int_a^c \frac{1}{(x-c)^p}dx$ is convergent for $p < 1$ and is divergent for $p \leq 1$. This is used for convergence or divergence of an improper integral of second kind.

Example 14

Examine $\int_1^\infty \frac{dx}{x^p}$ for convergence/divergence.

Solution

Consider $\int_1^k \frac{dx}{x^p} = \int_1^k x^{-p}dx = \left[\frac{x^{-p+1}}{-p+1}\right]_1^k$ if $p \neq 1$

And $\Rightarrow$ $[\log x]_1^k$ if $p = 1$

Case 1: If $p = 1$, $\int_1^k \frac{dx}{x} = \log k - \log 1 = \log k \to \infty$ when $k \to \infty$ it does not tend to a finite limit.

$\therefore$ It is divergent.

Case 2: If $p \neq 1 \int_1^k \frac{dx}{x^p} = \frac{1}{1-p}[k^{1-p}]$ it converges

If $p > 1$ and diverges if $p \leq 1$.

Multiple integrals

Double Integrals: Integration of $f(x, y)$ over a region R in xy-plane is called a double integral.

$$\iint_R f(x, y)dR = \int_{x=x_1}^{x_2}\int_{y=y_1}^{y_2} f(x, y)dxdy$$

Order of Integration in a Double Integral Order of integration depends on the nature of limits of the variables.

Case 1: If the limits of y are function of x, say $y_1 = f_1(x)$ and $y_2 = f_2(x)$ and the limits of x are constants, say $x_1 = a$ and $x_2 = b$, where a and b are constants, then integrate wrt y first treating x as constant and then integrate wrt x.

$$\text{That is, } \iint_R f(x, y)dR = \int_{x_1=a}^{x_2=b}\left(\int_{y_1 = f_1(x)}^{y_2 = f_2(x)} f(x, y)dy\right)dx$$

Case 2: If the limits of x are function of y, say $x_1 = g_1(y)$ and $x_2 = g_2(y)$ and the limits of y are constants, say $y_1 = c$ and $y_2 = d$, then integrate wrt x first treating y as constant and then integrate wrt y.

$$\text{That is, } \iint_R f(x, y)dR = \int_{y_1=c}^{y_2=d}\left(\int_{x_1 = g_1(y)}^{x_2 = g_2(y)} f(x, y)dx\right)dy$$

Case 3: If both the variables x and y have constant limits, then one can follow any order of integration.

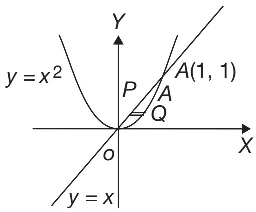
Change of Order of Integration Evaluation of some of the double integrals can be made simple by changing the order of integration. In change of order of integration, we take the limits of the variables for the given region of integration in such a way that the order of integration reverses.

Example 15

Evaluate $\int_{x=0}^{1}\int_{y=x^2}^{x}\frac{x}{y}e^{-\frac{x^2}{y}}\,dy\,dx$

Solution

Let $I=\int_{x=0}^{1}\int_{y=x^2}^{x}\frac{x}{y}e^{-\frac{x^2}{y}}\,dy\,dx$ (1)



Evaluation of this integral can be made simple by changing the order of integration.

From the limits of x and y given, the region of integration is the region bounded by the line $y = x$ and the parabola $y = x^2$ as shown in figure.

Now by changing the order of integration, we first integrate wrt x, along the horizontal strip PQ from $P(x = y)$ to $Q(x=\sqrt{y})$ and then

We integrate wrt y from $0(y = 0)$ to $A(y = 1)$

$$\therefore\ I=\int_{y=0}^{1}\left(\int_{x=y}^{\sqrt{y}}\frac{x}{y}e^{-\frac{x^2}{y}}dx\right)dy \quad (2)$$

Put $\frac{x^2}{y}=t \Rightarrow \frac{2x}{y}dx=dt$

$$\Rightarrow \frac{x}{y}dx=\frac{1}{2}dt$$

$$x=y \Rightarrow t=\frac{y^2}{y}=y \text{ and } x=\sqrt{y} \Rightarrow t\frac{(\sqrt{y})^2}{y}=1$$

$\therefore$ Eq. (2) Becomes

$$I=\int_{y=0}^{1}\left(\int_{t=y}^{1}e^{-1}\frac{1}{2}dt\right)dy$$

$$=\int_{y=0}^{1}(-e^{-t})\Big]_{t=y}^{1}dy=\int_{y=0}^{1}[-e^{-1}+e^{-y}]dy$$

$$=-ye^{-1}-e^{-y}]_0^1$$

$$=(-e^{-1}-e^{-1})-(0-e^{-0})$$

$$=1-2e^{-1}=\frac{e-2}{e}$$

Triple Integrals Integration of a function $f(x, y, z)$ over a 3-dimensional region V is called the triple integral.

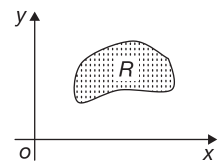
$$\iint_V\int f(x, yx, z)dv=\int_{x=x_1}^{x_2}\int_{y=y_1}^{y_2}\int_{z=z_1}^{z_2}f(x, y, z)dxdydz$$

Like double integrals, in triple integrals also the order of integration depends on the nature of the limits of the variables.

Applications of Double and Triple Integrals

1. Area of the region R in xy-plane is given by

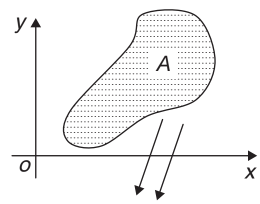
Area of $R=\int_R\int dxdy$



2. Volume of the solid of revolution:

(a) The volume of the solid of revolution obtained by revolving the area A about x-axis is

Volume $=V=\int_A\int 2\pi y\,dx\,dy$

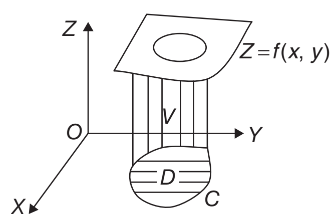


(b) The volume of the solid of revolution obtained by revolving the area A about y-axis is

Volume $=V=\iint_A 2\pi x\,dx\,dy$

(c) Volume under the surface as a double integral: The volume V of the solid under the surface $z = f(x, y)$ and above the xy-plane with the projection of $z = f(x, y)$ on xy plane as its base is

Volume $=\int\int_D f(x, y)dx\,dy$



(d) Volumes as a triple integral: The volume of the 3-dimensional region V is given by $\int_V\iint dx\,dy\,dz$

Example 16

Find the volume under the surface $x + 2y + z = 4$ and above the circle $x^2 + y^2 = 4$ in the xy-plane.

Solution

Given surface is $x + 2y + z = 4$

$\Rightarrow z = 4 - x - 2y$ (1)

Let D be the region bounded by the circle $x^2 + y^2 = 4$ in xy-plane

∴ In D, y varies from $y = -\sqrt{4-x^2}$ to $y = \sqrt{4-x^2}$ and x varies from $x = -2$ to $x = +2$.

∴ The volume under the surface $x + 2y + z = 4$ and above the circle $x^2 + y^2 = 4$ in xy-plane is

$$V = \int_D \int z dx dy = \int_{x=-2}^{2} \int_{y=-2\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - x - 2y) dx dy \qquad (2)$$

Evaluation of this double integral can be made simple by changing it into polar coordinates.

In polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$ and

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

∴ $J = r$. Also, in the circle $x^2 + y^2 = 4$, r varies from $r = 0$ to $r = 2$ and θ varies from $\theta = 0$ to $\theta = 2\pi$

∴ From (2),

$$V = \iint_D (4 - x - 2y) dx\, dy$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} (4 - r\cos\theta - 2r\sin\theta) |J| dr d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} (4 - r\cos\theta - 2r\sin\theta) r dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \left(\int_{r=0}^{2} (4r - r^2\cos\theta - 2r^2\sin\theta) dr \right) d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[2r^2 - \frac{r^3}{3}\cos\theta - \frac{2r^3}{3}\sin\theta \right]_{r=0}^{2} d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[8 - \frac{8}{3}\cos\theta - \frac{16}{3}\sin\theta \right] d\theta$$

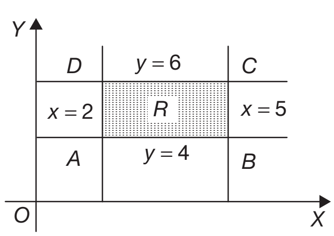
$$= 8\theta - \frac{8}{3}\sin\theta + \frac{16}{3}\cos\theta \Big]_{\theta=0}^{2\pi} = 16\pi$$

Example 17

Find the volume generated by the revolution of the rectangle formed by the lines $x = 2$, $x = 5$, $y = 4$ and $y = 6$ about x-axis.

Solution

The volume of the solid generated by revolving the rectangle $ABCD$ about x-axis $= V = \int_R \int 2\pi y dx dy$



$$= \int_{x=2}^{5} \int_{y=4}^{6} 2\pi y dx dy = \left(\int_{x=2}^{5} dx \right) \left(\int_{y=4}^{6} 2\pi y\, dy \right)$$

$$= \left(x \right]_{x=2}^{5} \left) \left(\pi y^2 \right]_{y=4}^{6} \right) = 3 \times 20\pi = 60\pi$$

Change of Variables Evaluation of some of the double (or) triple integrals can be made simple by changing the variables.

1. **In a double integral:** Let a double integral $\int_{R_{xy}} \iint f(x, y) dx dy$ in x and y is to be converted into the variables u and ϑ where $x = \phi(u, \vartheta)$ and $y = \Psi(u, \vartheta)$. Then

$$\int_{R_{xy}} \int f(x, y) dx dy = \int_{R^1_{u\vartheta}} \int f(\phi(u, \vartheta), \psi(u, \vartheta)) |J| du d\vartheta$$

Where $J = \frac{\partial(x, y)}{\partial(u, \vartheta)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial \vartheta} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial \vartheta} \end{vmatrix}$

Is the Jacobian of x and y wrt u and ϑ and $R'_{u\vartheta}$ is the region of integration in u, ϑ-plane corresponding to R_{xy} in xy-plane.

2. **In a triple integral:** Let a triple integral $\int_{R_{xyz}} \iint f(x, y, z)\, dx dy dz$ in x, y and z is to be converted into the variables u, ϑ and w, where $x = \phi(u, \vartheta, \omega)$, $y = \Psi(u, \vartheta, \omega)$ and $z = h(u, \vartheta, \omega)$

Then $\int_{R_{xyz}} \iint f(x, y, z) dx dy dz = \int_{R^1_{u\vartheta\omega}} \iint f(\phi(u, \vartheta, \omega), \psi(u, \vartheta, \omega), h(u, \vartheta, \omega)) |J| du d\vartheta d\omega$

where $J = \frac{\partial(x, y, z)}{\partial(u, \vartheta, \omega)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial \vartheta} & \frac{\partial x}{\partial \omega} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial \vartheta} & \frac{\partial y}{\partial \omega} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial \vartheta} & \frac{\partial z}{\partial \omega} \end{vmatrix}$ is the

jacobian of x, y and z wrt u, ϑ and ω and $R'_{u\vartheta\omega}$ is the region of integration in u, ϑ, ω, coordinate system corresponding to the region R_{xyz} in xyz co-ordinate system.

Vector Calculus

If $\bar{r}$ is the position vector of a point P, having co-ordinates (x, y, z), then $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, where $\bar{i}, \bar{j}, \bar{k}$ are unit vectors along OX, OY, OZ respectively, and

$$|\bar{r}| = |x\bar{i} + y\bar{j} + z\bar{k}| = \sqrt{x^2 + y^2 + z^2}.$$

Given any vector $\bar{v} = a\bar{i} + b\bar{j} + c\bar{k}$ its direction ratios are a, b, c and its direction cosines are given by:

$$l = \frac{a}{|\bar{v}|},\ m = \frac{b}{|\bar{v}|},\ n = \frac{c}{|\bar{v}|} \text{ and } l^2 + m^2 + n^2 = 1$$

Linear Combinations

A vector $\overline{r}$ is said to be a linear combination of the vectors $\overline{a}, \overline{b}, \overline{c} \ldots$ etc. if there exist scalars $x, y, z, \ldots$ such that

$\overline{r} = x\overline{a} + y\overline{b} + z\overline{c} + \cdots$

Test of Collinearity

Three points A, B, C with position vectors $\overline{a}, \overline{b}, \overline{c}$ respectively are collinear if *f* there exist scalars x, y, z not all zero such that $x\overline{a} + y\overline{b} + z\overline{c} = 0$, where $x + y + z = 0$

Test of Coplanarity

Four points A, B, C and D with position vectors $\overline{a}, \overline{b}, \overline{c}, \overline{d}$ are coplanar if there exist scalars x, y, z and u (not all zero) such that $x\overline{a} + y\overline{b} + z\overline{c} + u\overline{d} = 0$, where $x + y + z + u = 0$

Linear Dependence and Independence

A system of vectors $\overline{a}, \overline{b}, \overline{c} \ldots$ is said to be linearly independent (L.I.) if $x\overline{a} + y\overline{b} + z\overline{c} + \cdots = 0$

$\Rightarrow x = y = z \cdots = 0$

If $\overline{a}, \overline{b}, \overline{c} \ldots$ is a system of vectors which is not LI, then they are linearly dependent (L.D) and for such system of vectors there exist scalars $x, y, z \cdots$ (not all zeros) such that

$x\overline{a} + y\overline{b} + z\overline{c} + \cdots = 0$

NOTE

Every non-zero vector is LI.
Every pair of non-zero non-collinear vectors is LI.
Every pair of collinear vectors is LD.
Three non-coplanar vectors are LI.
Three coplanar vectors are LD.

Multiplication of Vectors

Scalar or Dot Product If $\overline{a}$ and $\overline{b}$ are two non-zero vectors and θ is the angle between them $(0 \le \theta \le \pi)$, then their dot or scalar product is given by $\overline{a}\cdot\overline{b} = |\overline{a}||\overline{b}|\cos\theta \cdot \overline{a}\cdot\overline{b}$ is a scalar.

NOTES

1. If one or both of $\overline{a}, \overline{b}$, are $\overline{0}$, then $\overline{a}\cdot\overline{b} = 0$
2. $\overline{a}\cdot\overline{b} = |\overline{a}|\cdot$(scalar component of $\overline{b}$ along $\overline{a}$)
   $= |\overline{b}|$ (scalar component of $\overline{a}$ along $\overline{b}$),
3. $\overline{a}\cdot\overline{b} = \overline{b}\cdot\overline{a}$
4. If $\overline{a}, \overline{b}, \overline{c}$ are any three vectors, then $\overline{a}\cdot(\overline{b}+\overline{c}) = \overline{a}\cdot\overline{b} + \overline{a}\cdot\overline{c}$
5. Two non-zero vectors $\overline{a}$ and $\overline{b}$ are perpendicular if $\overline{a}\cdot\overline{b} = 0$
6. $\overline{i}\cdot\overline{j} = \overline{j}\cdot\overline{i} = \overline{j}\cdot\overline{k} = \overline{k}\cdot\overline{j} = \overline{j}\cdot\overline{k} = \overline{k}\cdot\overline{i} = 0$
7. $\overline{a}\cdot\overline{b}$ is positive, negative or zero according as $0 \le \theta < 90°$, $90° < \theta \le 180°$ or $\theta = 90°$
8. The square of a vector is the square of its modulus, i.e., $(\overline{a})^2 = |\overline{a}|^2$
   $\overline{i}^2 = \overline{k}^2 = \overline{j}^2 = 1$
9. m is a scalar, then
   $m(\overline{a}\cdot\overline{b}) = (m\overline{a})\cdot\overline{b} = \overline{a}\cdot(m\overline{b})$
10. If $\overline{a} = a_1\overline{i} + a_2\overline{j} + a_3\overline{k}$ and $\overline{b} = b_1\overline{i} + b_2\overline{j} + b_3\overline{k}$, then $\overline{a}\cdot\overline{b} = a_1b_1 + a_2b_2 + a_3b_3$ and angle between the vectors is
    $$\cos\theta = \frac{\overline{a}\cdot\overline{b}}{|\overline{a}||\overline{b}|} = \frac{a_1\cdot b_1 + a_2\cdot b_2 + a_3\cdot b_3}{\sqrt{a_1^2 + a_2^2 + a_3^3}\cdot\sqrt{b_1^2 + b_2^2 + b_3^2}}$$
11. Work done $= \overline{F}\cdot\overline{S}$

Vector or Cross Product

$\overline{a}\times\overline{b} = |\overline{a}||\overline{b}|\sin\theta\cdot\hat{n}$ where θ $(0 \le \theta \le 180)$ is the angle between $\overline{a}$ and $\overline{b}$, and $\hat{n}$ is a unit vector such that it is perpendicular to both $\overline{a}$ and $\overline{b}$.

$\overline{a}, \overline{b}$ and $\hat{n}$ (in the same order) are in the right handed orientation (i.e., the rotation of a right handed screw from $\overline{a}$ to $\overline{b}$ advances it in the direction of $\hat{n}$).

NOTES

1. $\overline{a}\times\overline{b} \ne \overline{b}\times\overline{a}$ but $\overline{a}\times\overline{b} = -\overline{b}\times\overline{a}$
2. If $\overline{a}$ and $\overline{b}$ are parallel, then $\overline{a}\times\overline{b} = \overline{0}$
3. $\overline{i}\times\overline{j} = \overline{k}, \overline{j}\times\overline{k} = \overline{i}, \overline{k}\times\overline{i} = \overline{j}$ and
   $\overline{j}\times\overline{i} = -\overline{k}, \overline{k}\times\overline{j} = -\overline{i}, \overline{i}\times\overline{k} = -\overline{j}$
   $\overline{i}\times\overline{i} = \overline{j}\times\overline{j} = \overline{k}\times\overline{k} = 0$? [In particular $\overline{a}\times\overline{a} = 0$]
4. The angle between two vectors: $\sin\theta = \dfrac{|\overline{a}\times\overline{b}|}{|\overline{a}||\overline{b}|}$
5. A unit vector perpendicular to the plane of $\overline{a}$ and $\overline{b}$ is given by $\hat{n}$ where $\hat{n} = \dfrac{\overline{a}\times\overline{b}}{|\overline{a}\times\overline{b}|}$
6. Area of parallelogram whose adjacent sides are $\overline{a}$ and $\overline{b}$ is given by $|\overline{a}\times\overline{b}|$
7. When the diagonals are given, the vector area of parallelogram $ABCD$ is $\frac{1}{2}(\overline{AB}\times\overline{AC})$
8. The vector area of the triangle $ABC = \frac{1}{2}(\overline{AB}\times\overline{AC})$
9. If $\overline{a} = a_1\overline{i} + a_2\overline{j} + a_3\overline{k}$ and $\overline{b} = b_1\overline{i} + b_2\overline{j} + b_3\overline{k}$,
   $$\text{Then } \overline{a}\times\overline{b} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$
10. Vector product is distributive with respect to vector addition $\overline{a}\times(\overline{b}+\overline{c}) = \overline{a}\times\overline{b} + \overline{a}\times\overline{c}$

Triple Products

Scalar Triple Product The Scalar triple product of three vectors $\bar{a}$, $\bar{b}$, $\bar{c}$ is $(\bar{a}\times\bar{b})\cdot\bar{c}$ denoted by $[\bar{a}\bar{b}\bar{c}]$

The Scalar triple product of orthonormal right handed vector triad $\bar{i}$, $\bar{j}$, $\bar{k}$ is equal to unity

That is, $[\bar{i}\ \bar{j}\ \bar{k}] = [\bar{j}\ \bar{k}\ \bar{i}] = [\bar{k}\ \bar{i}\ \bar{j}] = 1$.

1. The volume of a parallelepiped having $\bar{a}$, $\bar{b}$, $\bar{c}$ as co-terminus edges $= [\bar{a}\bar{b}\bar{c}]$.
2. If three vectors are coplanar then $[\bar{a}\bar{b}\bar{c}] = 0$
3. If two of the three vectors are equal, then their scalar triple product is zero, i.e., $[\bar{a}\ \bar{b}\ \bar{c}] = 0$
4. If $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$, $\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$,

   $\bar{c} = c_1\bar{i} + c_2\bar{j} + c_3\bar{k}$, then $\left[\bar{a}\bar{b}\bar{c}\right] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$
5. The volume of a tetrahedron with co-terminus edges $\bar{a}$, $\bar{b}$, $\bar{c}$ is $\frac{1}{6}[\bar{a}\bar{b}\bar{c}]$ cubic units.
6. $[\bar{a}\bar{b}\bar{c}] = (\bar{a}\times\bar{b})\cdot\bar{c} = \bar{a}\cdot(\bar{b}\times\bar{c})$

Vector Triple Product If $\bar{a}$, $\bar{b}$, $\bar{c}$ are three vectors, then the triple product $\bar{a}\times(\bar{b}\times\bar{c})$ is called the vector triple product.

If $\bar{a}$, $\bar{b}$, $\bar{c}$ are any three vectors, then $\bar{a}\times(\bar{b}\times\bar{c}) = (\bar{a}\cdot\bar{c})\bar{b} - (\bar{a}\cdot\bar{b})\bar{c}$

Vector Variable

A variable of the form $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is called a vector variable and x, y, z are scalar variables.

Scalar Function If t is a scalar variable on a range $a \le t \le b$ and a function f defined as $f = f(t)$ for $t \in [a, b]$ is called a scalar function of t.

Example: $f(t) = 9t^3 + 4t^2 + 7$,
$f(t) = \sin t + 5\cos t + e^t$, etc.

Vector Function If t is a scalar variable defined on a domain $[a, b]$, and a function $\bar{F}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ is called a vector function of the scalar variable t.

NOTE

t is generally taken as 'time'.

Differentiation If $\bar{F}(t)$ is a continuous single valued vector function of the variable t, then the derivative of $\bar{F}(t)$ is defined as $\frac{d\bar{F}}{dt} = \lim_{\Delta t\to 0}\frac{F(t+\Delta t) - F(t)}{\Delta t}$ where Δt is a small increment in t.

One can also look at second and higher order derivatives in a similar way.

Differentiation Formula

1. The derivative of a constant vector with respect to any scalar variable is 0.
2. $\frac{d}{dt}[\bar{F}(t) \pm \bar{G}(t)] = \frac{d\bar{F}}{dt} \pm \frac{d\bar{G}}{dt}$.
3. $\frac{d}{dt}[s(t)\bar{F}(t)] = s(t)\cdot\frac{d\bar{F}}{dt} + \frac{ds}{dt}\cdot\bar{F}$
4. Chain rule: $\frac{d\bar{F}}{dt} = \frac{d\bar{F}}{du}\times\frac{du}{dt}$, where $\bar{F} = \bar{F}(u)$ and u is a function of t.
5. Dot and cross products:

   $$\frac{d}{dt}(\bar{F}\cdot\bar{G}) = \bar{F}\cdot\frac{d\bar{G}}{dt} + \frac{d\bar{F}}{dt}\cdot\bar{G},$$

   $$\frac{d}{dt}(\bar{F}\times\bar{G}) = \bar{F}\times\frac{d\bar{G}}{dt} + \frac{d\bar{F}}{dt}\times\bar{G}.$$
6. Partial derivatives: If $\bar{F}$ is vector function dependent on x, y and z, say $\bar{F} = \bar{F}(x, y, z)$, then partial derivative of $\bar{F}$ with respect to x is defined as $\frac{\partial\bar{F}}{\partial x} = \lim_{\Delta x\to 0}\frac{\bar{F}(x+\Delta x, y, z) - \bar{F}(x, y, z)}{\Delta x}$.

Likewise, one can also define $\frac{\partial\bar{F}}{\partial y}$ and $\frac{\partial\bar{F}}{\partial z}$.

It is also possible to define higher order partial derivatives as:

$$\frac{\partial^2\bar{F}}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial\bar{F}}{\partial x}\right),\quad \frac{\partial^2\bar{F}}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial\bar{F}}{\partial y}\right).$$

$$\frac{\partial^2\bar{F}}{\partial x\partial z} = \frac{\partial}{\partial x}\left(\frac{\partial\bar{F}}{\partial z}\right),\ \text{etc}$$

Differential Vectors

1. If $\bar{G} = \bar{G}(x, y, z)$ then

   $$d\bar{G} = \frac{\partial\bar{G}}{\partial x}dx + \frac{\partial\bar{G}}{\partial y}dy + \frac{\partial\bar{G}}{\partial z}dz$$
2. If $\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$, then

   $$d\bar{F} = dF_1\hat{i} + dF_2\hat{j} + dF_3\hat{k}$$
3. $d(\bar{F}\cdot\bar{G}) = \bar{F}\cdot d\bar{G} + d\bar{F}\cdot\bar{G}$
4. $d(\bar{F}\times\bar{G}) = \bar{F}\times d\bar{G} + d\bar{F}\times\bar{G}$

Vector Differential Operators ∇ is to be read as del or nabla

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$ is called Laplacian.

Gradient of a Scalar Function

If $\phi(x, y, z)$ is a scalar function, then $\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}$ is known as the gradient of ϕ and is denoted by grad ϕ. One can also write the gradient of ϕ using the ∇ operator as grad

$$\phi = \hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z} = \nabla\phi$$

Now $\nabla\phi$ denotes a vector field.

NOTES

1. If ϕ is a constant, then $\nabla\phi = \bar{0}$
2. If a vector $\bar{G}(x, y, z)$ is defined at all points in a region we say $\bar{G}$ is a vector field. A vector field is said to be irrotational if $\bar{G}$ = grad ϕ for some scalar function ϕ.
3. Gradient can be used in finding directional derivative. (An example is discussed in worked examples section)
4. $\nabla\phi$ also gives the normal to the surface $\phi(x, y, z) = C$.
5. If $\nabla^2\phi = 0$, the function is called the harmonic function.
6. The directional derivative of $\phi(x, y, z)$ in the direction of a vector $\bar{a}$ is $\nabla\phi\cdot\hat{n}$, where $\hat{n} = \frac{\bar{a}}{|\bar{a}|}$.

Divergence of Vector

$\bar{F}(x, y, z)$ be a vector field which is differentiable at each point (x, y, z) in some region of space, i.e., $\bar{F}$ is differentiable vector field. The scalar product of the vector operator ∇ and $\bar{F}$ gives a scalar which is termed as divergence.

$$\nabla\cdot\bar{F} = \hat{i}\cdot\frac{\partial\bar{F}}{\partial x}+\hat{j}\cdot\frac{\partial\bar{F}}{\partial y}+k\cdot\frac{\partial\bar{F}}{\partial z}$$

NOTE

If div $(\bar{F})$ or $\nabla\cdot\bar{F} = 0$, then $\bar{F}$ is called 'solenoidal'

Curl of a Vector

Let $\bar{F}(x, y, z)$ is a vector field defined for all (x, y, z) in a certain region of space and is differentiable, i.e., $\bar{F}$ is a differentiable vector field. The cross product of the vector operator ∇ with the vector $\bar{F}$ is termed as curl $\bar{F}$.

$$\text{curl } F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}; \ \bar{F} = F_1\hat{i}+F_2\hat{j}+F_3\hat{k}$$

NOTE

If curl $\bar{F} = \bar{0}$, then $\bar{F}$ is said to be irrotational.

Standard Results

1. div $(\phi\bar{F}) = \phi$ div $\bar{F} + \bar{F}\cdot$grad ϕ or $\nabla\cdot\phi\bar{F} = \phi\nabla\cdot\bar{F} + \bar{F}\cdot\nabla\phi$
2. curl $(\phi\bar{F}) = \nabla\phi\times\bar{F} + \phi$ curl $\bar{F}$
3. div $(\bar{F}\times\bar{G}) = \bar{F}\cdot$ curl $\bar{G} - \bar{G}\cdot$ curl $\bar{F}$
4. $\nabla\cdot\nabla\phi$ = div (grad ϕ) or $\nabla\cdot\nabla\phi = \nabla^2\phi$
5. curl (grad ϕ) = $\bar{0}$ or $\nabla\times(\nabla\phi) = \bar{0}$, i.e., curl of a gradient equals $\bar{0}$.
6. div (curl $\bar{F}$) = 0 or $\nabla\cdot(\nabla\times\bar{F}) = 0$
7. curl (curl $\bar{F}$) = grad (div $\bar{F}$) $- \nabla^2\bar{F}$ (or) $\nabla\times(\nabla\times\bar{F}) = \nabla(\nabla\cdot\bar{F}) - \nabla^2\bar{F}$

Integration

Line Integral

Let $\bar{F}(x, y, z)$ be a vector function defined on a region of space and let C be curve in that region, then the integral $\int_c \bar{F}\cdot d\bar{r}$ is called the line integral.

For Riemann Integration,

$\int_{x=a}^{x=b} f dx$ the limits of integration are along the line segment joining $(a, 0)$, $(b, 0)$, where $a < b$.

Here instead of line, we integrate along the curve C.

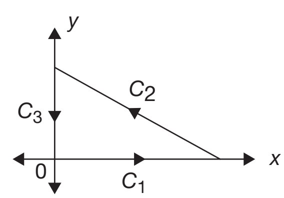
Circulation

The line integral around a closed curve C denoted by $\oint F\cdot dr$ is called circulation of F around C.

Example 18

Evaluate $\int_C \bar{F}\cdot d\hat{r}$, where $\bar{F} = xy\hat{i} + y^2\hat{j}$ along the triangle $x = 0$, $y = 0$ and $x + y = 1$ in the first quadrant.

Solution



$$\int_c F\cdot dr = \int_{C_1}(xydx + y^2dy) + \int_{C_2}(xydx + y^2dy) + \int_{C_3}(xydx + y^2dy)$$

C_1	C_2	C_3
$y = 0$	$y = 1 - x$	$x = 0$
$0 < x < 1$	$1 < x < 0$	$dx = 0$
$dy = 0$	$dy = -dx$	$1 < y < 0$

$$= \int_{x=0}^{1}[x(0)dx + 0 + \int_{x=1}^{0} x(1-x)dx + (1-x)^2(-dx) + \int_1^0 y^2dy$$

$$= \int_1^0 (x - x^2 - 1 - x^2 + 2x)dx + \int_1^0 y^2dy$$

$$= \int_1^0 (-2x^2 + 3x - 1)dx - \int_0^1 y^2dy$$

$$= \left(\frac{2}{3} - \frac{3}{2} + 1\right) - \frac{1}{3} = \frac{-1}{6}$$

Surface Integral Let S be a closed surface, then the normal surface integral $\int_s FN\,ds$ is called the flux of F over S.

Cartesian Form Let $F(r) = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$, where, F_1, F_2, F_3, are continuous and differentiable functions of x, y, z. If $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ be the direction cosines of the unit normal N, then

$$N = \hat{i}\cos\alpha + \hat{j}\cos\beta + \hat{k}\cos\gamma.$$

$$\therefore \int_S F\cdot N\,ds = \int_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)ds$$

But then $ds\cos\alpha$, $ds\cos\beta$ and $ds\cos\gamma$ are the projections of ds on yz, zx and xy planes. If dx, dy, dz are the differentials along the areas then

$ds\cos\alpha = dy\,dz$; $ds\cos\beta = dz\,dx$; $ds\cos\gamma = dx\,dy$.

$$\therefore \int_S F\cdot N\,ds = \iint_S (F_1 dy\,dz + F_2 dz\,dx + F_3 dx\,dy)$$

NOTE

If R_1 is the projection of S on xy-plane, then

$$\int_S F\cdot Nds = \int_{R_1}\int F\cdot N\frac{dxdy}{\cos\gamma}$$

$$= \iint_S F\cdot N\cdot\frac{dxdy}{|N\cdot\hat{k}|}(|N\cdot\hat{k}| = \cos\gamma)$$

Equivalently,

$$\iint_S F\cdot Nds = \iint_{R_2} F\cdot N\frac{dydz}{|N\cdot\hat{i}|} = \iint_{R_3} F\cdot N\frac{dzdx}{|N\cdot\hat{j}|}$$

Volume Integral

$$\int_{x_1}^{x^2}\int_{y_1}^{y^2}\int_{z_1}^{z^2} f(x, y, z)dz\,dy\,dx$$

$$= \int_{x_1}^{x_2}\left[\int_{y_1}^{y_2}\left[\int_{z_1}^{z_2} f(x, y, z)\,dz\right]dy\right]dx$$

Gauss' Divergence Theorem

If $\bar{F}$ is continuously differentiable vector function in the region bounded by a surface S, then $\iint_S \bar{F}\cdot\bar{N}\,ds = \iiint_V \text{div}\,\bar{F}dv$ where $\bar{N}$ is the unit normal to the surface.

Green's Theorem If P and Q are scalar point functions, possessing continuous derivatives of the first order, in a region S of the xy plane bounded by a closed curve C then $\int_C Pdx + Qdy = \iint_S\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy.$

Stoke's Theorem If S is an open surface bounded by a closed curve C and $\bar{F}$ is a continuously differentiable vector point function, then $\int_C \bar{F}\cdot d\bar{r} = \int_S \text{curl}\,\bar{F}\cdot Nds$, where N is unit outward drawn normal at any point on the surface.

Example 19

If $A = x^3\hat{i} + x^2\hat{j} + x\hat{k}$ and $B = -x\hat{i} + x^2\hat{j} + x^3\hat{k}$, then find the values of

(i) $\frac{d}{dx}(A\cdot B)$ and (ii) $\frac{d}{dx}(A\times B)$.

Solution

(i) $$\frac{d}{dx}(A\cdot B) = A\cdot\frac{d}{dx}(B) + B\cdot\frac{d}{dx}(A)$$

$$= (x^3\hat{i} + x^2\hat{j} + x\hat{k})\cdot\frac{d}{dx}(-x\hat{i} + x^2\hat{j} + x^3\hat{k})$$

$$+(-x\hat{i} + x^2\hat{j} + x^3\hat{k})\cdot\frac{d}{dx}(x^3\hat{i} + x^2\hat{j} + x\hat{k})$$

$$= (x^3\hat{i} + x^2\hat{j} + x\hat{k})\cdot(-\hat{i} + 2x\hat{j} + 3x^2\hat{k})$$

$$+(-x\hat{i} + x^2\hat{j} + x^3\hat{k})\cdot(3x^2\hat{i} + 2x\hat{j} + \hat{k})$$

$$= -x^3 + 2x^3 + 3x^3 - 3x^3 + 2x^3 + x^3 = 4x^3.$$

(ii) $$\frac{d}{dx}(A\times B) = A\times\frac{dB}{dx} + \frac{dA}{dx}\times B$$

$$= (x^3\hat{i} + x^2\hat{j} + x\hat{k})\times\frac{d}{dx}(-x\hat{i} + x^2\hat{j} + x^3\hat{k})$$

$$+\frac{d}{dx}(x^3\hat{i} + x^2\hat{j} + x\hat{k})\times(-x\hat{i} + x^2\hat{j} + x^3\hat{k})$$

$$= (x^3\hat{i} + x^2\hat{j} + x\hat{k})\times(-\hat{i} + 2x\hat{j} + 3x^2\hat{k})$$

$$+(3x^2\hat{i} + 2x\hat{j} + \hat{k})\times(-x\hat{i} + x^2\hat{j} + x^3\hat{k})$$

$$= \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ x^3 & x^2 & x\\ -1 & 2x & 3x^2\end{vmatrix} + \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3x^2 & 2x & 1\\ -x & x^2 & x^3\end{vmatrix}$$

$$= \hat{i}(5x^4 - 3x^2) - \hat{j}(6x^5 + 2x) + \hat{k}(5x^4 + 3x^2)$$

Example 20

If $f = x^3 - 6xy^2 - 9xyz$ is a scalar function, then find $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial x\partial y}$.

Solution

$f = x^3 - 6xy^2 - 9xyz$

$$\therefore \frac{\partial f}{\partial x} = 3x^2 - 6y^2 - 9yz$$

$$\therefore \frac{\partial^2 f}{\partial x^2} = 6x - 0 = 6x$$

$$\frac{\partial f}{\partial y} = -12xy - 9xz$$

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(-12xy - 9xz)$$

$$= -12y - 9z.$$

Example 21

If $\phi \equiv x^3 + y^3 + z^3 - 3xyz$, then find the value of grad ϕ at (2, 1, 1).

Solution

$$\text{Grad } \phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z} \text{ (by definition)}$$

$$= \hat{i}\frac{\partial}{\partial x}(x^3 + y^3 + z^3 - 3xyz) + \hat{j}\frac{\partial}{\partial y}(x^3 + y^3 + z^3 - 3xyz)$$

$$+ \hat{k}\frac{\partial}{\partial z}(x^3 + y^3 + z^3 - 3xyz).$$

$$= 3[\hat{i}(x^2 - yz) + \hat{j}(y^2 - xz) + \hat{k}(z^2 - xy)]$$

$\therefore$ grad ϕ at (2, 1, 1)

$$= 3[\hat{i}(4-1) + \hat{j}(1-2) + \hat{k}(1-2)]$$

$$= 9\hat{i} - 3\hat{j} - 3\hat{k}.$$

Example 22

If $\bar{P} = x^2 y\hat{i} - x^3\hat{j} + xyz^2\hat{k}$, then find div p and curl p.

Solution

(i) div $\bar{P} = \nabla \cdot \bar{P}$

$$= \frac{\partial}{\partial x}(x^2 y) - \frac{\partial}{\partial y}(-x^3) + \frac{\partial}{\partial z}(xyz^2)$$

$$= 2xy - 0 + 2xyz = 2xy(1 + z)$$

(ii) curl $\bar{P} = \nabla \times \bar{P} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & -x^3 & xyz^2 \end{vmatrix}$

$$= \hat{i}\left(\frac{\partial}{\partial y}(xyz)^2 - \frac{\partial}{\partial z}(-x^3)\right) - \left(\hat{j}\frac{\partial}{\partial x}(xyz^2) - \frac{\partial}{\partial Z}(x^2 y)\right)$$

$$+ \hat{k}\left(\frac{\partial}{\partial x}(-x^3) - \frac{\partial}{\partial y}(x^2 y)\right)$$

$$= xz^2\hat{i} - yz^2\hat{j} + \hat{k}(-3x^2 - x^2)$$

$$= xz^2\hat{i} - yz^2\hat{j} - 4x^2\hat{k}.$$

Example 23

Find the value of r if,

$p = xy^2\hat{i} + xyz^2\hat{j} + (r-2)xyz^3\hat{k}$ is solenoidal at (1, – 1, 1).

Solution

p is solenoidal $\Rightarrow$ div $p = 0 \Rightarrow \nabla \cdot p = 0$

$$\Rightarrow \frac{\partial p_1}{\partial x} + \frac{\partial p_2}{\partial y} + \frac{\partial p_3}{\partial z} = 0$$

$$\Rightarrow \frac{\partial}{\partial x}xy^2 + \frac{\partial}{\partial y}(xyz^2) + \frac{\partial}{\partial z}[(r-2)xyz^3] = 0$$

$\Rightarrow y^2 + xz^2 + (r-2)\,3xyz^2 = 0$ at (1, −1, 1), div $p = 0$

$\Rightarrow (-1)^2 + (1)^2 + (r-2)\,3\,(1)(-1)(1)^2 = 0$

$$\Rightarrow 1 + 1 - 3r + 6 = 0 \Rightarrow r = \frac{8}{3}.$$

Example 24

Find the value of a, if $P = (y^2 + 2xz)\hat{i} + (z^2 + 2xy)\hat{j} + (x^2 + ayz)\hat{k}$ is irrotational.

Solution

The vector P is irrotational

$\Rightarrow$ curl $P = \bar{0} \Rightarrow \nabla \times P = \bar{0}$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 + 2xz & z^2 + 2xy & ayz + x^2 \end{vmatrix} = \bar{0}$$

$$\Rightarrow \hat{i}\left(\frac{\partial}{\partial y}(x^2 + ayz) - \frac{\partial}{\partial z}(2xy + z^2)\right)$$

$$-\hat{j}\left(\frac{\partial}{\partial x}(x^2 + ayz) - \frac{\partial}{\partial z}(y^2 + 2xz)\right)$$

$$+\hat{k}\left(\frac{\partial}{\partial x}(z^2 + 2xy) - \frac{\partial}{\partial y}(y^2 + 2xz)\right) = \bar{0}$$

$$\Rightarrow \hat{i}(az - 2z) + \hat{j}(2x - 2x) + \hat{k}(2y - 2y) = \bar{0}$$

$$\Rightarrow \hat{i}\,z(a-2) = 0 = 0\,\hat{i} \Rightarrow z(a-2) = 0$$

$$\Rightarrow a - 2 = 0 \Rightarrow a = 2$$

Example 25

Find the angle between the surfaces $xy^2 z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at (1, −2, 1).

Solution

Let $f = xy^2 z - 3x - z^2 = 0$ and

$g = 3x^2 - y^2 + 2z - 1 = 0$.

$\therefore$ grad $f = \hat{i}(y^2 z - 3) + \hat{j}(2xyz) + \hat{k}(xy^2 - 2z)$

grad $g = \hat{i}(6x) + \hat{j}(-2y) + \hat{k}(2)$

But, angle between two surfaces at a point is equal to angle between the normals to the surfaces at that point.

$\therefore$ Let $\bar{n}_1$ = grad f at (1, – 2, 1) and $\bar{n}_2$ = grad g at (1, – 2, 1) respectively

$\therefore$ $\bar{n}_1$ = (grad f) at (1, – 2, 1)

$$= \hat{i}[(-2)^2 1 - 3] + \hat{j}[2(1)(-2)1]$$

$$+ \hat{k}[1(-2)^2 - 2(1)] = \hat{i} - 4\hat{j} + 2\hat{k}$$

$\bar{n}_2$ = (grad g) at (1, – 2, 1)

$$= \hat{i}[6\,(A)] + \hat{j}[-2(-2)] + \hat{k}(2) = 6\hat{i} + 4\hat{j} + 2\hat{k}$$

Let the angle between the normals $\bar{n}_1$ and $\bar{n}_2$ be θ.

So, $\bar{n}_1 \cdot \bar{n}_2 = |n_1||n_2|\cos\theta \;\Rightarrow\; 6 - 16 + 4$

$$= (\sqrt{1+16+4})(\sqrt{36+16+4})\cos\theta$$

$$\therefore \cos\theta = \left|\frac{-6}{\sqrt{21}\sqrt{56}}\right| = \left|\frac{-3}{7\sqrt{6}}\right| = \frac{3}{7\sqrt{6}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{3}{7\sqrt{6}}\right)$$

Example 26

If $\bar{F} = (x^2 + y^2)\hat{i} - 2xy\hat{j}$ evaluate $\oint \bar{F}\cdot d\bar{r}$ along the straight line C from (0, 0, 0) to (1, 2, 3).

Solution

The equation of the line joining (0, 0, 0) and (1, 2, 3) is $\frac{x}{1} = \frac{y}{2} = \frac{z}{3} = (t)$.

Then along the line C, $x = t$, $y = 2t$, $z = 3t$.

$$\therefore \quad \bar{r} = x\hat{i} + y\hat{j} + z\hat{k} = t\hat{i} + 2t\hat{j} + 3t\hat{k}$$

$$d\bar{r} = \hat{i} + 2\hat{j} + 3\hat{k}$$

Given $\bar{F} = (x^2 + y^2)\hat{i} - 2xy\hat{j}$

And along C, $\bar{F} = [t^2 + (2t)^2]\hat{i} - 2t(2t)\hat{j} = 5t^2\hat{i} - 4t^2\hat{j}$

$\therefore \; \bar{F}\cdot d\bar{r} = (5t^2 - 8t^2 + 0)\,dt = -3t^2\,dt$

at (0, 0, 0), $t = 0$ and at (1, 2, 3), $t = 1$.

$$\therefore \int_c \bar{F}\cdot d\bar{r} = \int_{t=0}^{1} -3t^2 dt = \left(\frac{-3t^3}{3}\right)_0^1 = -1.$$

Example 27

If $F = 3x\hat{i} - z^2\hat{k}$, , evaluate $\oint F\cdot dr$, where the curve C is the rectangle in the xz bounded by $z = 0$, $z = 2$, $x = 0$, $x = 3$.

Solution

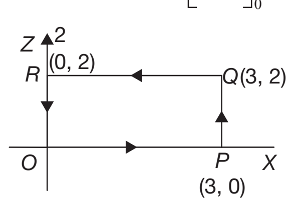
Since the integration takes place in xz-plane ($y = 0$)

$$\therefore \int_0 F\cdot dr = \int_0 f_1 dx + f_2 dz = \int_0 3x\,dx - z^2 dz$$

$$\oint_C \bar{F}\cdot d\bar{r} = \int_{OP} \bar{F}\cdot d\bar{r} + \int_{PQ} \bar{F}\cdot d\bar{r} + \int_{QR} \bar{F}\cdot d\bar{r} + \int_{RO} \bar{F}\cdot d\bar{r}$$

(i) Along OP:
z = 0, $dz = 0$ and x varies from 0 to 3

$$\int F\cdot dr = \int_0^3 3x\,dx = \left[\frac{3x^2}{2}\right]_0^3 = \frac{27}{2}$$



(ii) Along PQ:
$x = 3$, $dx = 0$ and z changes from 0 to 2.

$$\therefore \int_{PQ} \bar{F}\cdot dr = \int_0^2 -z^2 dz = \left[\frac{-Z^3}{3}\right]_0^2 = \frac{-8}{3}$$

(3) Along QR:
$y = 2$, $dy = 0$ and x changes from 3 to 0

$$\therefore \int_{QR} F\cdot dr = \int_3^0 3x\,dx = \left(3\frac{x^2}{2}\right)_3^0 = \frac{-27}{2}$$

(4) Along RO:
$x = 0$,
$dx = 0$ and y varies from 2 to 0.

$$\therefore \int_{RO} F\cdot dr = \int_2^0 -z^2 dz = -\left(\frac{z^3}{3}\right)_2^0 = \frac{8}{3}$$

$$\text{Thus } \oint_C F\cdot dr = \frac{27}{2} - \frac{8}{3} - \frac{27}{2} + \frac{8}{3} = 0$$

Example 28

Evaluate by Green's theorem $(\oint)\,(xy + y^2)\,dx + x^2\,dy$, where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.

Solution

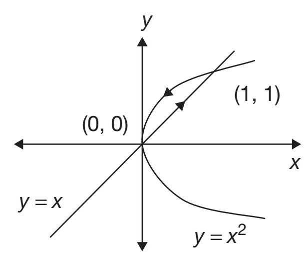
Here $P = xy + y^2 \quad \therefore \quad \frac{\partial P}{\partial y} = x + 2y$

$$Q = x^2 \quad \therefore \quad \frac{\partial Q}{\partial x} = 2x$$

Hence by Green's theorem,

$$\int_C (xy + y^2)dx + x^2 dy = \iint_S (2x - x - 2y)dx\,dy$$

$$= \iint_S (x - 2y)dx\,dy = \int_{x-0}^{1}\left[\int_{y-x}^{x}(x - 2y)dy\right]dx$$



$$= \int_{x=0}^{1}[xy - y^2]_{y=x}^{x^2} dx = \int_{x=0}^{1}(x^3 - x^4)dx$$

$$= \left(\frac{x^4}{4} - \frac{x^5}{5}\right)_0^1 = \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$$

Example 29

By applying Gauss theorem, evaluate $\iint_S (x^3 dy\,dz + x^2 y dz\,dx + x^2 z dx\,dz)$, where S is the closed surface consisting of the cylinder $x^2 + y^2 = a^2$ and the circular discs $z = 0$ and $z = b$.

Solution

We have

$F_1 = x^3;\ F_2 = x^2y;\ F^3 = x^2z$

$$\therefore\ \frac{\partial F_1}{\partial x} = 3x^2,\ \frac{\partial F_2}{\partial y} = x^2,\ \frac{\partial F_3}{\partial z} = x^2$$

$$\therefore\ \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 3x^2 + x^2 + x^2 = 5x^2$$

∴ Using Gauss theorem,

$$\int\int_S F_1 dy\,dz + F_2 dz\,dx + F_3 dx\,dy$$

$$= \int\int_V \int\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)dx\,dy\,dz$$

$$\therefore\ \int\int_S x^3 dy\,dz + x^2 ydz\,dx + x^2 zdx\,dy$$

$$= \int\int_V \int 5x^2 dx\,dy\,dz = 20\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\int_{z=0}^{b} x^2 dx\,dy\,dz$$

$$= 20\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} x^2 b\,dx\,dy.$$

$$= 20b\int_0^a x^2\sqrt{a^2-x^2}\,dx$$

[Let $x = a\sin\theta$; $dx = a\cos\theta\,d\theta$

Upper limit: $x = a \Rightarrow a\sin\theta = a \Rightarrow \theta = \frac{\pi}{2}$

Lower limit: $x = 0 \Rightarrow a\sin\theta = 0 \Rightarrow \theta = 0$]

$$= 20b\int_0^{\frac{\pi}{2}} a^2\sin^2\theta\sqrt{a^2(1-\sin^2\theta)}\,a\cos\theta d\theta$$

$$= 20a^4 b\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta d\theta$$

$$= 20a^4 b\int_0^{\frac{\pi}{2}} \frac{1}{4}\sin^2 2\theta\,d\theta$$

$$= 5a^4 b\int_0^{\frac{\pi}{2}}\left(\frac{1-\cos 4\theta}{2}\right)d\theta$$

$$= \frac{5a^4 b}{2}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{5a^4 b}{2}\left[\frac{\pi}{2} - 0\right] = \frac{5\pi}{4}a^4 b$$

Example 30

Evaluate $\int_c F\cdot dr$ by Stokes theorem,

If $F = (x^2+y^2)\hat{i} - 2xy\hat{j}$, where c is the rectangle formed by the lines $x = \pm a$, $y = 0$ and $y = b$.

Solution

$\bar{F} = (x^2+y^2)\hat{i} - 2xyj$

By Stoke's theorem,

$$\int(\nabla\times\bar{F})\cdot\bar{N}\,ds = \int_c \bar{F}\cdot d\bar{r}$$

$$\nabla\times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix} = -4y\hat{k}$$

$$\therefore\ \int(\nabla\times\bar{F})\cdot(N\cdot k)ds$$

$$= \int(-4ky)\cdot N\,ds = \int -4y\,(N\cdot k)\,ds\ \int_R\int -4y\,dx\,dy$$

Since $N\cdot k\,ds = dx\,dy$

And R is the region bounded by the rectangle.

$$= \int_{x=-a}^{a}\int_{y=0}^{b}(-4y)dy\,dx = \int_{-a}^{a}\left[\frac{-4y^2}{2}\right]_0^b dx$$

$$= -2\int_{-a}^{a}(b^2-0)dx = -2b^2[x]_{-a}^{a} = -4ab^2.$$

Exercises

1. $\lim_{x\to\infty}\{3x - \sqrt{9x^2 - x}\} =$ ______.
   (A) $\frac{1}{6}$ (B) 3
   (C) 6 (D) None of these

2. $\lim_{x\to 0}\left[\frac{24\cos x - 24 + 12x^2 - x^4}{24x^6}\right] =$
   (A) $\frac{1}{720}$ (B) $\frac{-1}{120}$
   (C) $\frac{1}{120}$ (D) $\frac{-1}{720}$

3. Evaluate $\lim_{x\to 2.7}(x - [x])$, where $[x]$ is the greatest integer less than equal to x.
   (A) -0.3 (B) 0.7
   (C) 4.7 (D) 2

4. Evaluate $\lim_{x\to 0}\frac{1}{x^{189}}$.
   (A) 0 (B) ∞
   (C) $-\infty$ (D) None of these

5. $\lim_{x\to 0}\left(\frac{2^x + 3^x}{2}\right)^{1/x} =$
   (A) 1 (B) $\sqrt{3}$
   (C) $\sqrt{6}$ (D) $\sqrt{2}$

6. $\lim_{x\to 2} |x-2| + [x-2] =$
(A) 0.
(B) only left limit exists.
(C) only right limit exists.
(D) limit does not exist.

7. Let the function $f(x) = [x]$. Where $[x]$ is the greatest integer less than or equal to x. Which of the following is/are true?
(A) $f(x)$ has jump discontinuity at all $x \in Z$.
(B) $f(x)$ has removable discontinuity at all $x \in Z$.
(C) $f(x)$ is continuous at all irrational values.
(D) both (A) and (C).

8. $f(x) = \begin{cases} 5x-4 & 0 < x \le \\ 4x^2-3x & 1 < x < 2 \end{cases}$ at $x = 1$
(A) Left hand continuous at $x = 1$.
(B) Right hand continuous at $x = 1$.
(C) continuous at $x = 1$.
(D) None of these

9. The function $f(x) = \dfrac{x \sin x}{(x^2+2)}$ is
(A) continuous for all x.
(B) discontinuous for all x.
(C) constant function.
(D) discontinuous only at $x = \pm 2$.

10. Check the continuity of the following function

$$f(x) = \begin{cases} \dfrac{\sin^2 ax}{x^2}, & \text{when } x \neq 0 \\ a^2 & \text{when } x=0 \end{cases} \text{ at } x = 0$$

(A) continuous at $x = 0$
(B) discontinuous at $x = 0$
(C) discontinuous of first kind
(D) None of these

11. If $f(x) = \begin{cases} 7 & x < 5, \\ ax+b & 5 < x < 7, \\ 11 & x > 7 \end{cases}$ is c continuous on R

then the values of a and b are
(A) $a = 2, b = 3$ (B) $a = -2, b = 3$
(C) $a = 3, b = -2$ (D) $a = 2, b = -3$

12. Let $f(x) = \max(1-x, x^2-1)$. Then f is
(A) not continuous at $x = 1, -2$.
(B) continuous and differentiable everywhere.
(C) not differentiable at $x = -2, 1$.
(D) continuous but not differentiable at $x = 1, -1$.

13. Consider the function $f(x) = \dfrac{1}{x-1} + \dfrac{1}{3-x}$ defined in the interval [1, 3]
P. f is continuous on [1, 3]
Q. f is differentiable on (1, 3)
R. there exists $c \in (1, 3)$ such that $f'(c) = 0$ which of the above statements are true?
(A) P, Q only (B) Q, R only
(C) P, R only (D) P, Q, R

14. A function $f: R \to R$ is such that $f(x+y) = f(x) \cdot f(y)$ for all x, y in R and $f(x) \neq 0$ for any x in R. If $f(x)$ is differentiable and $f'(0) = 2$, then
(A) $f'(x) = 2f(x)$ (B) $f(x) = 2f'(x)$
(C) $f(x) = f'(x)$ (D) $f'(x) = -f(x)$

15. Which of the following statement(s) is/are true?
(A) $y = x^2$ has a minimum value at $x = 0$
(B) $y = |x-3|$ has a minimum value at $x = 3$
(C) The maximum value of the function $y = \dfrac{1}{1+x^2}$ is 1
(D) All of these

16. The maximum and minimum values of $f(x) = 3\sin^2 x + 4\cos^2 x$ is
(A) $\{-4, -3\}$ (B) $\{7, 3\}$
(C) $\{4, -3\}$ (D) $\{4, 3\}$

17. If the function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$, where $a > 0$, attains its maximum and minimum at $x = p$ and $x = q$ respectively such that $p^2 = q$, then the value of 'a' is
(A) 2 (B) $\dfrac{1}{4}$
(C) $\dfrac{1}{8}$ (D) 4

Direction for questions 18 and 19:

The sum of the hypotenuse and one side of a right angled triangle is given as a units.

18. When the area is maximum the ratio of the side and the hypotenuse is ______.
(A) 2 : 1 (B) 1 : 3
(C) 1 : 2 (D) 2 : 3

19. When the area is maximum, find the angle between the hypotenuse and the other side is ______.
(A) 60° (B) 30°
(C) 45° (D) None of these

20. Consider $f(x) = |x^2 - 3|$, $0 \le x \le \sqrt{6}$ and $g(x) = \begin{cases} 3^x, & 0 \le x \le 1 \\ 4-x, & 1 < x \le 3 \end{cases}$. Then Rolle's theorem can be applied in the respective intervals
(A) to both $f(x)$ and $g(x)$.
(B) only to $f(x)$.
(C) only to $g(x)$.
(D) neither to $f(x)$ nor to $g(x)$.

21. If the function $f(x) = px^2 + qx^2 + rx + s$ on [0, 1], satisfies the mean value theorem, then the value of c in the interval (0, 1) is
(A) $\dfrac{1}{2}$ (B) $\dfrac{1}{3}$
(C) $\dfrac{2}{3}$ (D) $\dfrac{2}{3\sqrt{3}}$

22. $f(x)=\dfrac{x^2}{x+1}$ increases in

(A) $(-2, 0)$
(B) $[-4, -2]$
(C) $(-\infty, -2]\cup[0, \infty)$
(D) $(-\infty, -2)\cup(0, \infty)$

23. Let $f(x) = e^{ax}$ and $g(x) = e^{-ax}$ be two functions defined in $[p, q]$, If the functions satisfies Cauchy mean value theorem then the value of 'c' is _______.

(A) $p+q$ (B) $\dfrac{p+q}{2}$
(C) $2(p+q)$ (D) None of these

24. If $x = \cos(z + y^2)$, then $\dfrac{\partial z}{\partial y} =$

(A) 1 (B) y
(C) $2y$ (D) $-2y$

25. If $u=\left[\dfrac{\sqrt[4]{x}+\sqrt[4]{y}}{\sqrt[6]{x}+\sqrt[6]{y}}\right]^6$, then $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial x}=$

(A) $\dfrac{u}{2}$ (B) $\dfrac{4}{u}$
(C) $4u$ (D) $6u$

26. The stationary points of the function $f(x, y) = x^3 + y^4 - 27x + 32y + 100$ is/are

(A) $(3, 2), (3, -2)$
(B) $(-3, 2), (-3, -2)$
(C) $(3, 2), (-3, -2)$
(D) $(3, -2), (-3, -2)$

27. For the function $f(x, y) = 2x^2 + 4y^2 + 4xy + 2x + 10y + 7$.

(A) Local maximum exists, but no local minimum.
(B) Local minimum exists, but no local maximum.
(C) Neither local minimum nor local maximum exists.
(D) Both local minimum and local maximum exists.

28. For the function xyz, if $x + y + z = 3$, then the local maximum occurs for xyz at the point _______.

(A) $\left(4, \dfrac{1}{2}, \dfrac{1}{2}\right)$
(B) $(5, -1, -1)$
(C) $(1, 1, 1)$
(D) $(7, -3, -1)$

29. The ratio of the dimensions of a rectangular box of volume 64 cubic units and open at the top that requires least material for its construction is

(A) 2 : 2 : 1 (B) 2 : 4 : 5
(C) 2 : 3 : 4 (D) 1 : 2 : 3

30. Which of the following function/s is/are integrable but not continuous on (0, 10)?

(A) $f(x) = [x]$ (greatest integer function)
(B) $f(x) = |x - 3|$
(C) $f(x) = |x - 5| + |x - 2|$
(D) $f(x) = x^2 + 5x + 9$

31. $\int \sec^3 x \; dx =$ _______.

(A) $\dfrac{\sec x \tan x}{3}+\log\,(\sec\, x+\text{an}\; x)$

(B) $\dfrac{\sec^2 x \tan x}{3}+\dfrac{1}{3}\log\tan\left(\dfrac{\pi}{4}+x\right)$

(C) $\dfrac{\sec x \tan x}{2}+\dfrac{1}{2}\log\tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)$

(D) None of these

32. $\displaystyle\int_0^{\pi/2} \sin^4 x \cos^6 x dx =$ _______.

(A) $\dfrac{3\pi}{128}$ (B) $\dfrac{2\pi}{425}$
(C) $\dfrac{3\pi}{2560}$ (D) $\dfrac{3\pi}{512}$

33. Area bounded by the curve $y = -3x^2$, $x = 2$ and the two coordinate axes is _______ sq units

(A) 2 (B) 3
(C) 6 (D) 8

34. The volume of the solid obtained by revolving the area bounded by the parabola $y^2 = x - 4$, x-axis and the lines $x = 4$ and $x = 7$, about x-axis is _______ cubic unit

(A) $\dfrac{9}{2}\pi$ (B) $\dfrac{11}{2}\pi$
(C) $\dfrac{13}{2}\pi$ (D) $\dfrac{15}{2}\pi$

35. The length of arc of the curve $y = \ln(\cos x)$ from $x = 0$ to $x=\dfrac{\pi}{4}$ is _____.

(A) $\ln(1+\sqrt{2})$ (B) $\ln(\sqrt{2}-1)$
(C) $\ln(2+\sqrt{3})$ (D) $\ln(2-\sqrt{3})$

36. Evaluate $\displaystyle\int_0^{\pi/4}\int_0^{\pi/4}(3\cos\theta+4\sin\theta)d\theta d\phi$ _______.

(A) $\left(\dfrac{\sqrt{2}-1}{\sqrt{2}}\right)\pi$ (B) $\dfrac{(4\sqrt{2}-1)\pi}{4\sqrt{2}}$
(C) $\dfrac{(4\sqrt{2}-1)\pi}{\sqrt{2}}$ (D) $\dfrac{(4\sqrt{2}-1)\pi}{4\sqrt{2}}$

37. Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{dxdy}{\sqrt{1-x^2-y^2}}$

(A) $\frac{\pi}{4}$ (B) 0

(C) $\frac{\pi}{2}$ (D) 1

38. By changing the order of integration, the integral $\int_2^\infty \int_0^{x-2} f(x,y)dxdy$ becomes _______.

(A) $\int_0^\infty \int_2^{y+2} f(x,y)dxdy$ (B) $\int_0^\infty \int_{y+2}^\infty f(x,y)\, dxdy$

(C) $\int_0^\infty \int_0^\infty f(x,y)dxdy$ (D) $\int_1^\infty \int_{y+2}^1 f(x,y)dxdy$

39. By changing the variables in the double integral $\iint_R \frac{dxdy}{xy}$, where $x = e^{u+v}$ and $y = uv$, it changes to $\iint_R \phi(uv)dudv$ then $\phi(u, v)$ is

(A) $\frac{R}{(e^{u+v})(uv)}$ (B) $\frac{e^{u+v}}{uv}$

(C) $\frac{1}{v}+\frac{1}{u}$ (D) $\frac{1}{v}-\frac{1}{u}$

40. By changing the variables from x, y to u, v where $x = u + 2v$ and $y = 4u + 3v$, the given integral $\iint_R f(x,y)dxdy$ changes to $\iint_R f(u+2v, 4u+3v)\psi(u,v)dud\,v$ then $\Psi(u, v)$ is _______.

(A) 5 (B) −5

(C) $\frac{1}{5}$ (D) $-\frac{1}{5}$

41. The area bounded by the circle $x^2 + y^2 = 6$ and the parabola $y = x^2$ is given by:

(A) $\int_{x=-2}^{2} \int_{y=\sqrt{x}}^{\sqrt{x^2-6}} dydx$

(B) $\int_{x=-\sqrt{2}}^{\sqrt{2}} \int_{y=\sqrt{x}}^{\sqrt{6-x^2}} dydx$

(C) $\int_{x=-2}^{2} \int_{y=\sqrt{x}}^{\sqrt{x^2-6}} (x^2+y^2)dydx$

(D) $\int_{x=-\sqrt{2}}^{\sqrt{2}} \int_{y=\sqrt{x}}^{\sqrt{6-x^2}} (y-x^2)dxdy$

42. The volume of the solid bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 4$ is _______ cubic units.

(A) $\frac{32}{3}$ (B) $\frac{64}{3}$

(C) 32 (D) 64

43. The acute angle between the vectors $3i + j + 2k$ and $i - j + k$ is θ, then the value of $\cos\theta$ is

(A) $\frac{8}{21}$ (B) $\frac{8}{\sqrt{21}}$

(C) $21\sqrt{8}$ (D) $\sqrt{\frac{8}{21}}$

44. If $\bar{r}$ is the position vector of a particle which passes along the curve $x = 3 \sin 4t$, $y = 3 \cos 4t$, and $z = 5t$ ($t > 0$). The magnitude of its velocity and acceleration respectively are

(A) 13, 45 (B) 12, 48

(C) 13, 48 (D) 12, 45

45. $\bar{f}(t)$ be a vector function and $\bar{f} \times \frac{d\bar{f}}{dt} = 0$ implies

(A) f is a vector function with constant magnitude.

(B) f is a vector function both in direction and magnitude.

(C) f is a vector function of constant direction.

(D) Either A or C.

46. The directional derivative of $f = x^3 y + y^3 z + z^3 x$ in the direction of $i + 2j + 2k$ at (0, 1, −1) is

(A) $\frac{5}{3}$ (B) $\frac{4}{3}$

(C) $\frac{-4}{3}$ (D) $\frac{-5}{3}$

47. If $\bar{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$ and $|r| = r$, then $\nabla r^n =$

(A) $n(n-1)r^{n-1}\,\bar{r}$ (B) $n(n-2)r^{n-2}\,\bar{r}$

(C) $n \cdot r^{n-2} \times \bar{r}$ (D) $n(n-1)\bar{r}$

Direction for questions 48 and 49:

Two equations $f = x y^2 z - 2y + z^2$ and $g = x^2 + yz - x - 2$ represents two surfaces

48. Find normal vector to 'g' at (1, −1, 2)

(A) $i + 2j + 2k$ (B) $i + 2j - k$

(C) $2i - j - k$ (D) $i - j - 2k$

49. The acute angle between the surfaces f and g at (1, −1, 2) is

(A) $\cos^{-1}\left(\frac{15}{\sqrt{390}}\right)$ (B) $\cos^{-1}\left(\sqrt{\frac{15}{390}}\right)$

(C) 60° (D) 30°

50. The magnitude of maximum directional derivative of $\phi = 2xy^2 - xyz + y^2 z$ in the direction from the point (1, – 1, 1) is

(A) 62 (B) $\sqrt{52}$

(C) $\sqrt{62}$ (D) $\sqrt{56}$

51. The directional derivative of a scalar point function is a function of

(A) only direction (B) only position

(C) either A or B (D) both A and B

52. The values of div $\bar{r}$ and curl $\bar{r}$ respectively when $\bar{r} = 2x\hat{i} - y\hat{j} + 3z\bar{k}$ is

(A) $4; \hat{i}$ (B) $0, \bar{0}$

(C) $4, 4\bar{k}$ (D) $4, \bar{0}$

53. The necessary and sufficient condition that the force field $\bar{F}(x, y, z)$ is conservative is

(A) (curl $\bar{F}$) = $-\bar{F}$ (B) div $\bar{F} = \bar{0}$

(C) curl $\bar{F} = \bar{F}$ (D) curl $\bar{F} = \bar{0}$

54. Which of the following is/are true?

(A) $\nabla(\bar{r} \times \bar{a}) = 0$

(B) Grad $(\bar{r} \cdot \bar{a}) = \bar{a}$

(C) $\nabla \times (\bar{r} \times \bar{a}) = -2\bar{a}$

(D) All of these

55. Compute the value of div$(\nabla\phi \times \nabla f)$.

(A) ∇f curl $(\nabla\phi)$ (B) $\nabla\phi$ curl (∇f)

(C) curl $(\nabla\phi \times \nabla f)$ (D) 0

56. For what value of p the vector $f = (2x + 3y)i + (z + 2y)j + (x - pz)k$ is solenoidal?

(A) 4 (B) –4

(C) 2 (D) 0

57. For what values of p, q and r the vector $\bar{f} = (x + ry - z)i + (3x - y + qz)j + (px + y - z)k$ is irrotational?

(A) $p = 1, q = -1, r = 3$

(B) $p = -1, q = 1, r = 3$

(C) $p = -1, q = 1, r = -3$

(D) $p = 1, q = 1, r = -3$

58. If $\nabla\phi = yz\hat{i} + zx\hat{j} + xy\hat{k}$, then $\phi(x, y, z) =$

(A) $xyz + f(y, z)$; $f \neq$ constant

(B) $xyz + g(x, z)$; $g \neq$ constant

(C) $xyz + h(x, y)$; $h \neq$ constant

(D) $xyz + k$; k is a constant

59. If $\bar{F} = (5xy - 6x^2)\hat{i} + (2y - 4x)\hat{j}$, compute the line integral $\int_C \bar{F}\, d\bar{r}$ where $C \equiv y = x^3$ in the xy-plane joining (1, 1) and (2, 8).

(A) 35 (B) –32

(C) 12 (D) 18

60. Compute $\int_S x^2 y^2 ds$ around the circle $x = \cos t$ and $y = \sin t$.

(A) $\frac{\pi}{4}$ (B) 0

(C) $\frac{\pi}{2}$ (D) π

61. If $\bar{F} = y^2\hat{i} - 2xy\hat{j}$, compute the circulation $\int_C \bar{F}\, d\bar{r}$ where C is the rectangle bounded by $y = 0$, $y = 1$, $x = 0$ and $x = 2$.

(A) 3 (B) 4

(C) –4 (5) –3

62. A particle in the force field $\bar{F} = 2x^2 i + (y - 3xz)j + 2z k$ is moving along a space curve defined by $x = 2t$, $y = t^2$, $z = 3t^2 - 2$. Find the work done by $\bar{F}$ in moving a particle along the straight line from $A(0, 0, 0)$ to $B(2, 1, 1)$.

(A) $\frac{107}{30}$ (B) $\frac{121}{30}$

(C) $\frac{113}{30}$ (D) $\frac{109}{30}$

63. Evaluate $\oint_C (x^2y dx + xy^2 dy)$ using greens theorem where C is the triangle with vertices (0, 0), (2, 0) and (2, 1).

(A) $\frac{11}{24}$ (B) $\frac{11}{12}$

(C) $\frac{-11}{6}$ (D) $\frac{11}{4}$

64. Find the area of the region in the first quadrant bounded by the curves $y = 4x$, $y = \frac{1}{x}$ and $y = \frac{x}{4}$ using green's theorem.

(A) log 2 (B) $\frac{1}{2}$ log 2

(C) log 4 (D) log 16

65. Evaluate $\int\int_s F\, nds$ where $F = 2xz\hat{i} - yz\hat{j} + yx\bar{k}$ where S is the cube bounded by $x = 0$, $x = 3$, $y = 0$, $y = 3$ and $z = 0$, $z = 3$.

(A) $\frac{27}{2}$ (B) $\frac{81}{4}$

(C) $\frac{27}{4}$ (D) $\frac{81}{2}$

66. For the force field $\bar{F} = x^2 i + xyj$ in the square region in the xy-plane bounded by the lines x = 0, $y = 0$, $x = 2$, $y = 2$. Using stokes theorem, find the value of $\int_C F, dr$.

(A) 4 (B) 6

(C) 8 (D) 2

67. Evaluate the volume integral $\int_V div\, \bar{N}\, dv$, where N is the outward drawn normal to the surface described by $x^2 + (y-5)^2 + (z-8)^2 = 12$.

(A) 8π (B) 12π
(C) 48π (D) 24π

68. If S is a closed surface and n is unit normal to the surface 'S' then $\iint_S \bar{r}\, nds =$ ______.

(A) 4V (B) 3V
(C) 2V (D) V

69. $\int_{1x}^{\infty} \frac{1}{1.0001} dx =$ ______.

(A) 1000 (B) 100000
(C) 10000 (D) 1000000

70. $\int_0^3 \frac{1}{(x-2)^{4/5}} dx =$ ______.

(A) $5 - 2^{1/5}$ (B) $5 + 2^{1/5}$
(C) $5(1-2)^{1/5}$ (D) $5\left[1+2^{1/2}\right]$

Previous Years' Questions

1. Evaluate $\int_0^{\infty} \frac{\sin t}{t} dt$ **[GATE, 2007]**

(A) π (B) $\frac{\pi}{2}$
(B) $\frac{\pi}{4}$ (D) $\frac{\pi}{8}$

2. A velocity vector is given as $\bar{v} = 5xy\bar{i} + 2y^2\bar{j} + 3yz^2\bar{k}$. The divergence of the this velocity vector at (1, 1, 1) is **[GATE, 2007]**

(A) 9 (B) 10
(C) 14 (D) 15

3. The value of $\int_0^3\int_0^x (6-x-y)dx\,dy$ is **[GATE, 2008]**

(A) 13.5 (B) 27.0
(C) 40.5 (D) 54.0

4. The inner (dot) product of two vectors $\bar{P}$ and $\bar{Q}$ is zero. The angle (degrees) between the two vectors is **[GATE, 2008]**

(A) 0 (B) 30
(C) 90 (D) 120

5. For a scalar function $f(x, y, z) = x^2 + 3y^2 + 2z^2$, the gradient at the point $P(1, 2, -1)$ is **[GATE, 2009]**

(A) $2\vec{i} + 6\vec{j} + 4\vec{k}$ (B) $2\vec{i} + 12\vec{j} - 4\vec{k}$
(C) $2\vec{i} + 12\vec{j} + 4\vec{k}$ (D) $\sqrt{56}$

6. The $\lim_{x\to 0} \frac{\sin\left[\frac{2}{3}\right]x}{x}$ is **[GATE, 2010]**

(A) $\frac{2}{3}$ (B) 1
(C) $\frac{3}{2}$ (D) ∞

7. Given function
$F(x, y) = 4x^2 + 6y^2 - 8x - 4y + 8$. The optimal value of $f(x, y)$ **[GATE, 2010]**

(A) is a minimum equal to $\frac{10}{3}$
(B) is a maximum equal to $\frac{10}{3}$
(C) is a minimum equal to $\frac{8}{3}$
(D) is a maximum equal to $\frac{8}{3}$

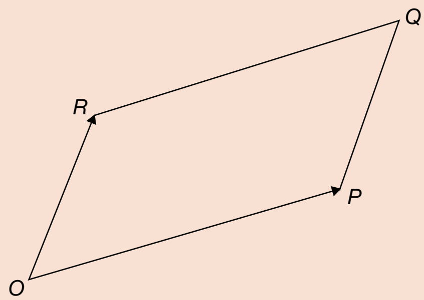
8. What is the value of the definite integral, $\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}} dx$? **[GATE, 2011]**

(A) 0 (B) $\frac{a}{2}$
(C) a (D) $2a$

9. If $\bar{a}$ and $\bar{b}$ are two arbitrary vectors with magnitudes a and b, respectively, $\left|\bar{a}\times\bar{b}\right|^2$ will be equal to **[GATE, 2011]**

(A) $a^2b^2 - (\vec{a}\cdot\vec{b})^2$
(B) $ab - \vec{a}\cdot\vec{b}$
(C) $a^2b^2 + (\vec{a}\cdot\vec{b})^2$
(D) $ab + \vec{a}\cdot\vec{b}$

10. For the parallelogram $OPQR$ shown in the sketch, $\overrightarrow{OP} = a\bar{i} + bj$ and $\overrightarrow{OR} = ci + dj$. The area of the parallelogram is



[GATE, 2012]

(A) $ad - bc$
(B) $ac + bd$
(C) $ad + bc$
(D) $ab - cd$

11. There is no value of x that can simultaneously satisfy both the given equations. Therefore, find the least square error solution to the two equations, i.e., find the value of x that minimizes the sum of squares of the errors in the two equations
$2x = 3$
$4x = 1$ **[GATE, 2013]**

12. The solution for $\int_0^{\pi/6} \cos^4 3\theta \sin^3 6\theta d\theta$ is **[GATE, 2013]**

(A) 0 (B) $\frac{1}{15}$
(C) 1 (D) $\frac{8}{3}$

13. $\lim_{x\to\infty}\left(\frac{x+\sin x}{x}\right)$ equals is **[GATE, 2014]**

(A) $-\infty$ (B) 0
(C) 1 (D) ∞

14. The expression $\lim_{x\to 0}\frac{x^\alpha - 1}{\alpha}$ is equal to **[GATE, 2014]**

(A) $\ln x$ (B) 0
(C) $x\ln x$ (D) ∞

15. With reference to the conventional cartesian (x, y) coordinate system, the vertices of a triangles have the following coordinates: $(x_1, y_1) = (1, 0)$: $(x_2, y_2) = (2, 2)$: and (x_3, y_3), = (4, 3). The area of the triangle is equal to **[GATE, 2014]**

(A) $\frac{3}{2}$ (B) $\frac{3}{4}$
(C) $\frac{4}{5}$ (D) $\frac{5}{2}$

16. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{2x}$ is equal to **[GATE, 2015]**

(A) e^{-2} (B) e
(C) 1 (D) e^2

17. While minimizing the function $f(x)$, necessary and sufficient conditions for a point, x_0 to be a minima are: **[GATE, 2015]**

(A) $f'(x_0) > 0$ and $f''(x_0) = 0$
(B) $f'(x_0) < 0$ and $f''(x_0) = 0$
(C) $f'(x_0) = 0$ and $f''(x_0) < 0$
(D) $f'(x_0) = 0$ and $f''(x_0) > 0$

18. The directional derivative of the field $u(x, y, z) = x^2 - 3yz$ in the direction for the vector $(\hat{i} + \hat{j} - 2\hat{k})$ at point (2, –1, 4) is _______. **[GATE, 2015]**

19. The optimum value of the function $f(x) = x^2 - 4x + 2$ is **[GATE, 2016]**

(A) 2 (maximum) (B) 2 (minimum)
(C) –2 (maximum) (D) –2 (minimum)

20. The quadratic approximation of $f(x) = x^3 - 3x^2 - 5$ at the point $x = 0$ is **[GATE, 2016]**

(A) $3x^2 - 6x - 5$ (B) $-3x^2 - 5$
(C) $-3x^2 + 6x - 5$ (D) $3x^2 - 5$

21. What is the value of $\lim_{\substack{x\to 0\\ y\to 0}}\frac{xy}{x^2+y^2}$? **[GATE, 2016]**

(A) 1 (B) –1
(C) 0 (D) Limit does not exist

22. The area between the parabola $x^2 = 8y$ and the straight line $y = 8$ is _______. **[GATE, 2016]**

23. The area of the region bounded by the parabola $y = x^2 + 1$ and the straight line $x + y = 3$ is **[GATE, 2016]**

(A) $\frac{59}{6}$ (B) $\frac{9}{2}$
(C) $\frac{10}{3}$ (D) $\frac{7}{6}$

24. The angle of intersection of the curves $x^2 = 4y$ and $y^2 = 4x$ at point (0, 0) is **[GATE, 2016]**

(A) 0° (B) 30°
(C) 45° (D) 90°

25. The value of $\int_0^\infty \frac{1}{1+x^2}dx + \int_0^\infty \frac{\sin x}{x}dx$ is **[GATE, 2016]**

(A) $\frac{\pi}{2}$ (B) π
(C) $\frac{3\pi}{2}$ (D) 1

Answer Keys

Exercises

1. A	**2.** D	**3.** B	**4.** D	**5.** C	**6.** D	**7.** D	**8.** C	**9.** A	**10.** A
11. D	**12.** C	**13.** B	**14.** A	**15.** D	**16.** D	**17.** A	**18.** C	**19.** B	**20.** D
21. A	**22.** D	**23.** B	**24.** D	**25.** A	**26.** D	**27.** B	**28.** C	**29.** A	**30.** A
31. C	**32.** D	**33.** D	**34.** A	**35.** A	**36.** B	**37.** C	**38.** B	**39.** D	**40.** A
41. B	**42.** A	**43.** D	**44.** C	**45.** C	**46.** D	**47.** C	**48.** B	**49.** A	**50.** C
51. D	**52.** D	**53.** D	**54.** D	**55.** D	**56.** A	**57.** B	**58.** D	**59.** A	**60.** A
61. C	**62.** D	**63.** C	**64.** C	**65.** D	**66.** A	**67.** C	**68.** B	**69.** C	**70.** D

Previous Years' Questions

1. B	**2.** D	**3.** A	**4.** C	**5.** B	**6.** A	**7.** A	**8.** B	**9.** A	**10.** A
11. 0.875	**12.** B	**13.** C	**14.** A	**15.** A	**16.** D	**17.** D	**18.** −5.72 to −5.70		**19.** D
20. B	**21.** D	**22.** 85.33	**23.** B	**24.** D	**25.** B				