Chapter 2

Ordinary Differential Equations

CHAPTER HIGHLIGHTS

- *Introduction*
- *Differential equations*
- *Laplace transforms*

Introduction

Familiarity with various methods used in evaluating indefinite integrals or finding anti-derivatives of functions [or, in other words, evaluating $\int f(x)\,dx$] is a pre-requisite.

Differential Equations

An equation involving derivatives of a dependent variable with respect to one or more independent variables is called a differential equation. The equation may also contain the variables and/or their functions and constants. If there is only one independent variable, the corresponding equation is called an ordinary differential equation. If the number of independent variables is more than one, the corresponding equation is called a partial differential equation.

Examples:

1. $\dfrac{dy}{dx} = x^4 + e^{-x} + y$
2. $x^2 \dfrac{d^2y}{dx^2} + 3\left(\dfrac{dy}{dx}\right)^2 + 3y^4x = \sin x + 6$
3. $\dfrac{dy}{dx} + 5y = x^3 - \tan x$
4. $\dfrac{d^2y}{dx^2} + 4y = 0$
5. $\left(\dfrac{d^3y}{dx^3}\right)^2 + 5\left(\dfrac{dy}{dx}\right)^4 + e^{2xy} = 6$
6. $\dfrac{d^3y}{dx^3} + 8\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + 9y = 16x^2$
7. $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 8u$
8. $\dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial x^2} = 10$
9. $\dfrac{\partial^2 u}{\partial y^2} = 25\dfrac{\partial^2 u}{\partial x^2}$
10. $\dfrac{\partial^4 u}{\partial x^4} + 6\dfrac{\partial^2 u}{\partial x^2 \partial y^2} + \dfrac{\partial^4 u}{\partial y^4} = e^{3xy}$

We note that in the given examples, Eqs. (1) to (6) are ordinary differential equations while Eqs. (7) to (10) are partial differential equations. We refer to these examples later on in next chapter.

Certain Geometrical Results may also be Expressed as Differential Equations

Illustration 1 Consider a family of parallel lines. All these lines have the same slope. If k represents the slope, we may interpret the family of parallel lines as curves having the same slope. As $\dfrac{dy}{dx}$ represents the slope of the tangent to a curve at any point (x, y), we may say that the differential equation $\dfrac{dy}{dx} = k$ represents a family of parallel lines.

Illustration 2 The differential equation $y\frac{dy}{dx} = k$ (a constant) may be said to represent the family of curves having the length of subnormal equal k at every point (x, y) on the curve. (We may note that the family of curves is the family of parabolas). Our study is confined to ordinary differential equations. In what follows, differential equation means ordinary differential equations.

Order of a Differential Equation

It is defined as the order of the highest derivative present in the equation. Examples (1), (3) are of first order; (2), (4) are of second order and (5), (6) are of third.

Degree of a Differential Equation

The degree of a differential equation is defined as the degree of the highest order derivative present in the equation. (It is assumed that the various order differential co-efficients or derivatives present in the equation are made free from fractional powers).

Examples (1), (2), (3), (4), (6) are of first degree while Example (5) is of second degree.

Consider the differential equation,

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{5/2} = 4\frac{d^3y}{dx^3}.$$

Taking the square on both sides (to free it from fractional powers), the differential equation is

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^5 = 16\left(\frac{d^3y}{dx^3}\right)^2.$$

This is a third order second degree differential equation.

Linear Differential Equation

If, in a differential equation, the dependent variable and the derivatives appear only in the first degree and there is no term involving products of the above or containing functions of the dependent variable, it is called **linear differential equation.**

1. $\frac{dy}{dx} + Py = Q$ (where P and Q are functions of only x) is an example of a first order linear differential equation.
2. $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$, where P, Q, R are functions of only x; $\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = f(x)$, where a_1, a_2 are constants and $f(x)$ is a function of x are examples of second order linear differential equations.

Similarly, we can have nth order linear differential equation.

$$P_0\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots + P_{n-1}\frac{dy}{dx} + P_ny = Q$$

where $P_0, P_1, P_2, \ldots, P_n, Q$ are functions of x or constants. If an equation is not linear, it is called a non-linear differential equation. In examples, 1, 3, 4, 6 are linear differential equations, while examples 2 and 5 are non-linear differential equations.

Solution of a Differential Equation

A function $y = f(x)$ or $F(x, y) = 0$ is called a solution of a given differential equation if it is defined and differentiable (as many times as the order of the given differential equation) throughout the interval where the equation is valid, and is such that the equation becomes an identity when $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots$ are replaced by $f(x), f'(x), f''(x), \ldots$ respectively.

[In the case of $F(x, y) = 0$ one has to get $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots$ by successive differentiation of $F(x, y) = 0$ with respect to x].

Examples:

1. $y = e^{7x}$ is a solution of $\frac{dy}{dx} = 7y$, since on substitution of $y = e^{7x}$, both left and right sides of the differential equation become identical. We find that $y = e^{7x}, 3e^{7x}, \frac{-1}{2}e^{7x}$ or, in general, $y = Ce^{7x}$, where C is an arbitrary constant represents solutions of $\frac{dy}{dx} = 7y$.
2. $y^2 - x^2 = 4$ is a solution of the differential equation $\frac{dy}{dx} = \frac{x}{y}$. Also, $y^2 - x^2 = 5$, $y^2 - x^2 = -10$, … or, in general, $y^2 - x^2 = C$ where C is an arbitrary constant represents solutions of $\frac{dy}{dx} = \frac{x}{y}$.

In both the above examples, we could represent the solutions of the differential equations which involve an arbitrary constant denoted by C. We now define the general solution of a first order differential equation.

The ***general solution*** of a first order differential equation is a relation between x and y involving one arbitrary constant such that the differential equation is satisfied by this relation or, the general solution of a first order differential equation is a one parameter family of curves where the parameter is the arbitrary constant. By assigning particular values to the arbitrary constant, we generate ***particular solutions*** of the equation.

In Example (1) $y = Ce^{7x}$ represents the general solution of the differential equation $\frac{dy}{dx} = 7y$ and the solutions

$y = e^{7x}, y = 3e^{7x}, \ldots$ are its particular solutions. The general solution represents a family of exponential curves.

In Example (2) $y^2 - x^2 = C$ represents the general solution of the differential equation $\frac{dy}{dx} = \frac{x}{y}$ and the solutions $y^2 - x^2 = 4$, $y^2 - x^2 = 5$, … are its particular solutions. The general solution in this case represents a family of rectangular hyperbolas.

3. $y = 2e^{-3x} + 5e^{6x}$ is a solution of the second order differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 18y = 0.$
(which can be verified by actual substitution). Also, $y = 4e^{-3x} - 10e^{6x}$, $e^{-3x} + e^{6x}, \ldots$ or, in general, $y = Ae^{-3x} + Be^{6x}$ where A and B are arbitrary constants represents solution of $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 18y = 0.$
4. $y = 2 \cos 4x + 3 \sin 4x$ or, in general, $y = A \cos 4x + B \sin 4x$ where A and B are arbitrary constants represents solutions of $\frac{d^2y}{dx^2} + 16y = 0.$

In Example (3), the general solution is $y = Ae^{-3x} + Be^{6x}$ and in Example (4), the general solution is $y = A \cos 4x + B \sin 4x$.

By assigning particular values to the arbitrary constants one can generate particular solutions.

From Examples (3) and (4), we infer that the general solution of a second order differential equation is a relation between x and y involving two arbitrary constants such that the differential equation is satisfied by this relation or the general solution of a second order differential equation is a two-parameter family of curves where the parameters are the arbitrary constants.

To sum up, the general solution of an nth order differential equation is a relation between x and y involving n arbitrary constants, such that the differential equation is satisfied by this relation or the general solution of an nth order differential equation is an n-parameter family of curves where the parameters are the arbitrary constants. For the first and second order differential equations, we have

First Order Equation

One parameter family of curves:

Representation: Relation between x and y involving one arbitrary constant, say C.

Eliminate: Eliminate C to obtain a DE representing the given curve.

Second Order Equation

Two-parameter family of curves:

Representation: Relation between x and y involving two arbitrary constants, say A and B

Elimination: Eliminate A and B to obtain a DE representing the two-parameter family of curves.

We shall work out a few examples to illustrate the formation of differential equations.

SOLVED EXAMPLES

Example 1

Form the differential equation representing the one-parameter family of curves

$$x^3 - Ay = 0.$$

Solution

Given, $x^3 - Ay = 0$ (1)

$$Ay = x^3$$

$$A\frac{dy}{dx} = 3x^2 \quad \Rightarrow \quad A = \frac{3x^2}{\frac{dy}{dx}} \tag{2}$$

Substituting A in the Eq. (1),
we have

$$x^3 - \frac{3x^2}{\frac{dy}{dx}} \cdot y = 0 \quad \Rightarrow \quad x\frac{dy}{dx} - 3y = 0.$$

Example 2

Obtain the differential equation of all the circles in the first quadrant, which touch the co-ordinate axes.

Solution

The equation of any circle in the first quadrant, which touches the co-ordinate axes may be represented as $(x - h)^2 + (y - h)^2 = h^2$.

Differentiating with respect to x,

$$2(x-h) + 2(y-h)\frac{dy}{dx} = 0$$

or
$$h = \frac{x + y\frac{dy}{dx}}{\left(1 + \frac{dy}{dx}\right)}$$

Substituting the above expression for h in the equation of the circle

$$\left(x - \frac{x + y\frac{dy}{dx}}{1 + \frac{dy}{dx}}\right)^2 + \left(y - \frac{x + y\frac{dy}{dx}}{1 + \frac{dy}{dx}}\right)^2 = \left(\frac{x + y\frac{dy}{dx}}{1 + \frac{dy}{dx}}\right)^2$$

or
$$(x-y)^2\left(\frac{dy}{dx}\right)^2 + (x-y)^2 = \left(x + y\frac{dy}{dx}\right)^2$$

or
$$(x-y)^2\left[1 + \left(\frac{dy}{dx}\right)^2\right] = \left[x + y\left(\frac{dy}{dx}\right)\right]^2.$$

Initial Value Problems A first order differential equation with a condition that $y = y_0$ when $x = x_0$ [written as $y(x_0) = y_0$] is known as an initial value problem. For example,

1. $\dfrac{dy}{dx} = \dfrac{x}{y}$; $y(0) = 1$

2. $\dfrac{dy}{dx} + 2xy = x^3$; $y(1) = 6$

3. $\dfrac{dy}{dx} + \dfrac{3y}{x} = e^x$; $y(0) = 4$

To solve such problems, we first obtain the general solution and find that particular value of the arbitrary constant in the general solution which satisfies the condition $y(x_0) = y_0$. This means that the solution of an initial value problem is a particular solution of the given differential equation.

First Order First Degree Equations The general form of the equation will be $\dfrac{dy}{dx} = f(x, y)$.

Separable Equations (or Variables Separable Type) Here, the given differential equation can be reduced to the form $f(y)dy = g(x)dx$. [Recall that $\dfrac{dy}{dx}$ may be thought as the ratio of the differential of y to the differential of x]. Direct integration of the relation with respect to the variable on each side gives general solution or, in other words, the general solution of the differential equation above may be written as $\int f(y)\,dy = \int g(x)\,dx + C$, where C is an arbitrary constant.

Example 3

Solve: $\dfrac{dy}{dx} = \sqrt{\dfrac{1+y^2}{1+x^2}}$.

Solution

$$\frac{dy}{dx} = \sqrt{\frac{1+y^2}{1+x^2}}$$

$$\frac{1}{\sqrt{1+y^2}}dy = \frac{1}{\sqrt{1+x^2}}dx$$

Integrating on both sides,

$$\int \frac{1}{\sqrt{1+y^2}}dy = \int \frac{1}{\sqrt{1+x^2}}dx.$$

$$\sinh^{-1}y = \sinh^{-1}x + c.$$

Example 4

Solve: $(x - xy^2)\dfrac{dy}{dx} + (y + x^2y) = 0$.

Solution

$$(x - xy^2)\frac{dy}{dx} + (y + x^2y) = 0$$

$$(x - xy^2)\,dy + (y + x^2y)\,dx = 0$$

$$x(1 - y^2)\,dy + y(1 + x^2)\,dx = 0$$

$$\frac{1-y^2}{y}dy + \frac{1+x^2}{x}dx = 0$$

Integrating on both sides,

$$\int\left(\frac{1}{y} - y\right)dy + \int\left(\frac{1}{x} + x\right)dx = 0$$

$$\log y - \frac{y^2}{2} + \log x + \frac{x^2}{2} = \log C$$

$$\log_e \frac{xy}{C} = \frac{y^2 - x^2}{2} \quad \Rightarrow \quad \frac{xy}{C} = e^{\left(\frac{y^2-x^2}{2}\right)}$$

$$\Rightarrow \quad xy = Ce^{\left(\frac{y^2-x^2}{2}\right)}$$

Example 5

Solve the initial value problem

$$y^2\frac{dy}{dx} = x^2 e^{y^3}, \quad y(1) = (0)$$

Solution

Given: $y^2 \dfrac{dy}{dx} = x^2 e^{y^3}$

$$y^2 e^{-y^3}\,dy = x^2 dx.$$

$$\int y^2 e^{-y^3}\,dy = \int x^2 dx$$

Let $e^{-y^3} = t \quad \Rightarrow \quad e^{-y^3} \cdot -3y^2 dy = dt$

$$-\frac{1}{3}\int dt = \int x^2 dx$$

$$\frac{-1}{3}t = \frac{x^3}{3} + c$$

$$-\frac{1}{3}e^{-y^3} = \frac{x^3}{3} + c.$$

Given: When $x = 1$, $y = 0$;

$$-\frac{1}{3}e^{\circ} = \frac{1}{3} + c$$

$$c = -\frac{2}{3}$$

$\therefore$ The solution is $-\dfrac{1}{3}e^{-y^3} = \dfrac{x^3}{3} - \dfrac{2}{3}$.

$$x^3 + e^{-y3} - 2 = 0.$$

Homogeneous Differential Equations

Homogeneous differential equation will be of the form $f(x, y)dy = g(x, y)dx$, where $f(x, y)$ and $g(x, y)$ are homogeneous functions in x and y of the same degree.

Definition

A function $F(x, y)$ in x and y is a homogeneous function in x and y of degree n(n, a rational number), if $F(x, y)$ can be expressed as $x^n\phi\left(\dfrac{y}{x}\right)$ or $y^n\psi\left(\dfrac{x}{y}\right)$.

1. $x^3+4x^2y-y^3=x^3\left(1+\frac{4y}{x}-\frac{y^3}{x^3}\right)$ is a homogeneous function in x and y of degree 3.

2. $x^3\tan\left(\frac{y}{x}\right)$ is a homogeneous function in x and y of degree 3.

3. $\frac{x+y}{2x-3y}$ is a homogeneous function in x and y of degree 0. We change the dependent variable y to v by the substitution $y=vx$. Then, $\frac{dy}{dx}=v+x\frac{dv}{dx}$.

On substitution y and $\frac{dy}{dx}$ in the given homogeneous equation, it reduces to the variables separable form.

Example 6

Solve: $x^2\frac{dy}{dx}=x^2+7xy+9y^2$.

Solution

$$x^2\frac{dy}{dx}=x^2+7xy+9y^2$$

$$\frac{dy}{dx}=1+\frac{7y}{x}+9\left(\frac{y}{x}\right)^2$$

Put $y=xv \Rightarrow \frac{dy}{dx}=v+\frac{dv}{dx}$

$$v+\frac{dv}{dx}=1+7v+9v^2$$

$$x\frac{dv}{dx}=9v^2+6v+1$$

$$\frac{1}{9v^2+6v+1}dv=\frac{1}{x}dx$$

Integrating on both sides,

$$\int\frac{1}{9v^2+6v+1}dv=\int\frac{1}{x}dx$$

$$\int\frac{1}{(3v+1)^2}dv=\int\frac{1}{x}dx\quad -\frac{1}{3(3v+1)}=\log x+\log c$$

$$=-\frac{1}{3\left(\frac{3y}{x}+1\right)}=\log_e cx=\frac{-x}{9y+3x}=\log_e cx$$

where C is an arbitrary constant.

Example 7

Solve $x\frac{dy}{dx}=y+x\sin\left(\frac{y}{x}\right)$

Solution

Given: $x\frac{dy}{dx}=y+x\sin\left(\frac{y}{x}\right)$

$$\frac{dy}{dx}=\frac{y}{x}+\sin\left(\frac{y}{x}\right) \tag{1}$$

Put $y=vx, \frac{dy}{dx}=v+x\frac{dv}{dx}$.

Substituting in (1) we get,

$$v+x\frac{dv}{dx}=v+\sin v$$

$$\Rightarrow \frac{xdv}{dx}=\sin v \quad\Rightarrow\quad \frac{1}{\sin v}dv=\frac{1}{x}dx$$

$$=\int\text{cosec } v\,dv=\int\frac{1}{x}dx$$

$\Rightarrow$ $\log(\text{cosec } v-\cot v)=\log x+\log c$

$\Rightarrow$ $\text{cosec } v-\cot v=cx$

$$\text{cosec}\left(\frac{y}{x}\right)-\cot\left(\frac{y}{x}\right)=cx.$$

Example 8

Solve $3y^2\,dx+(2xy+3x^2)\,dy=0$.

Solution

$3y^2\,dx+(2xy+3x^2)\,dy=0$.

$$\frac{dy}{dx}=\frac{-3y^2}{2xy+3x^2}$$

Put $y=vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

$$v+x\frac{dv}{dx}=\frac{-3v^2}{2v+3}$$

$$x\frac{dv}{dx}=\frac{-3v^2}{2v+3}-v$$

$$x\frac{dv}{dx}=\frac{-3v^2-2v^2-3v}{2v+3}$$

$$\frac{2v+3}{-5v^2-3v}dv=\frac{1}{x}dx$$

$$\Rightarrow \frac{2v+3}{v(5v+3)}dv+\frac{1}{x}dx=0$$

Integrating on both sides,

$$\Rightarrow \int\left[\frac{1}{v}-\frac{3}{5v+3}\right]dv+\int\frac{1}{x}dx=0$$

$$\Rightarrow \log v-\frac{3}{5}\log(5v+3)+\log x=\log c.$$

$$\Rightarrow 5\log v-3\log(5v+3)+5\log x=5\log c.$$

$$\Rightarrow \quad \log \frac{v^5}{(5v+3)^3} x^5 = \log c^5$$

$$\Rightarrow \quad \frac{y^5}{\left(5\frac{y}{x}+3\right)^3} = c_1, \text{ where } c_1 = c^5$$

$$\Rightarrow \quad \frac{y^5 x^3}{(5y+3x)^3} = c_1 \quad \Rightarrow \quad x^3 y^5 = c_1(5y+3x)^3$$

Exact Differential Equations

If M, as well as N, is a function in x and y, then the equation $Mdx + Ndy = 0$ is said to be an exact differential equation if there exists a function $f(x, y)$ such that

$$d(f(x, y)) = Mdx + Ndy.$$

That is, $$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = Mdx + Ndy$$

Example: $3x^2ydx + x^3dy = 0$ is an exact differential equation since there exists a function x^3y such that

$$d(x^3 y) = 3x^2ydx + x^3dy$$

The necessary and sufficient condition for an equation of the form $Mdx + Ndy = 0$ to be an exact equation is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

The solution of the exact differential equation

$$Mdx + Ndy = 0 \text{ is } U + \int \phi(y)\, dy = C$$

where $U = \int Mdx$ and $\phi(y) = N - \frac{\partial u}{\partial y}$

Or $\int^x Mdx + \int (\text{terms of } N \text{ not containing } x)dy = C$

Here $\int^x Mdx$ denotes integration of M with respect to x treating y as a constant.

Example 9

Find the solution of

$$(3x - 2y + 5)\, dx + (3y - 2x + 7)dy = 0.$$

Solution

$M = 3x - 2y + 5$, $N = 3y - 2x + 7$

$$\frac{\partial M}{\partial y} = -2\frac{\partial N}{\partial x} = -2. \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ The given equation is exact.

The solution is

$$\int^x Mdx + \int (\text{the terms of } N \text{ not containing } x)\, dy = C$$

$$\int^x (3x - 2y + 5)\, dx + \int (3y + 7)\, dy = C$$

$$\frac{3x^2}{2} - 2yx + 5x + \frac{3y^2}{2} + 7y = C$$

Example 10

Find the solution of $(e^y + 1) \cot x\, dx + e^y \log(\sin x)\, dy = 0$.

Solution

Given $(e^y + 1)\cot x\, dx + e^y \log(\sin x)dy = 0$

Let $M = (e^y + 1)\cot x$ and $N = e^y \log(\sin x)$

$$\frac{\partial M}{\partial y} = e^y \cot x \text{ and } \frac{\partial N}{\partial x} = e^y \cot x$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ The given equation is exact.

The solution is

$$\int^x Mdx + \int (\text{the terms of } N \text{ not containing } x)dy = C$$

$$\therefore \quad \int^x (e^y + 1)\cot x\, dx + \int 0\, dy = C$$

$$(e^y + 1)\log(\sin x) = C$$

Integrating factors: Let us say $M(x, y)dx + N(x, y)dy = 0$ be a non-exact differential equation. If it can be made exact by multiplying it by a suitable function $\mu(x, y)$, then $\mu(x, y)$ is called an integrating factor.

Methods to Find the Integrating Factors

Method I

If $Mdx + Ndy = 0$ is a homogeneous differential equation and $Mx + Ny \neq 0$, then $\frac{1}{Mx + Ny}$ is an integrating factor of $Mdx + Ndy = 0$

Example 11

Find the solution of $(x + 2y)dx + (y - 2x)dy = 0$.

Solution

Here $M = x + 2y$ and $N = y - 2x$

$$\frac{\partial M}{\partial y} = 2\frac{\partial N}{\partial x} = -2$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

The above equation is not an exact equation.

But M and N are homogeneous functions

$\therefore$ The integrating factor $= \frac{1}{Mx + Ny}$

$$(x + 2y)x + (y - 2x)y = x^2 + y^2 \tag{1}$$

Now by multiplying Eq. (1) by $\frac{1}{x^2 + y^2}$, it become an exact equation.

$$\left(\frac{x+2y}{x^2+y^2}\right)dx+\left(\frac{y-2x}{x^2+y^2}\right)dy=0$$

The solution is $U+\int\phi(y)\,dy=C$

$$U=\int^x M_1 dx,\ \text{where } M_1=\frac{x+2y}{x^2+y^2}$$

$$=\int^x \frac{x+2y}{x^2+y^2}\,dx$$

$$=\int\frac{x}{x^2+y^2}dx+2y\int\frac{1}{x^2+y^2}dx$$

$$=\frac{1}{2}\log(x^2+y^2)+2y\frac{1}{y}\tan^{-1}\left(\frac{x}{y}\right)$$

$$=\frac{1}{2}\log(x^2+y^2)+2\tan^{-1}\left(\frac{x}{y}\right)$$

Since in $N_1=\dfrac{y-2x}{x^2+y^2}$ there is no term independent of x, the solution is

$$\frac{1}{2}\log(x^2+y^2)+2\tan^{-1}\left(\frac{x}{y}\right)=C$$

Method 2

If the differential equation $Mdx+Ndy=0$ is of the form $y, f(xy)dx+x\,g(xy)dy=0$ and $Mx-Ny\neq 0$, then $\dfrac{1}{Mx-Ny}$ is an integrating factor of $Mdx+Ndy=0$.

Method 3

In the equation $Mdx+Ndy=0$,

if $\dfrac{1}{N}\left[\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}\right]=f(x)$, then $e^{\int f(x)\,dx}$ is an integrating factor of the given equation.

Similarly if $\dfrac{1}{M}\left[\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}\right]=g(y)$ then $e^{\int g(y)\,dy}$ is an integrating factor of the given equation.

Example 12

Find the solution $(x^2-y^2)dx+2xy\,dy=0$.

Solution

Given $(x^2-y^2)dx+2xy\,dy=0$ (1)

$$M=x^2-y^2 \text{ and } N=2xy$$

$$\frac{\partial M}{\partial y}=-2y \text{ and } \frac{\partial N}{\partial x}=2y$$

$$\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$$

$$\frac{1}{N}\left[\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right]=\frac{1}{2xy}[-2y-2y]$$

$$=\frac{-2}{x}=f(x)$$

Integrating factor (IF)

$$=e^{\int f(x)\,dx}$$

$$=e^{\int\frac{-2}{x}dx}=e^{\int-2\log x}=e^{\log\frac{1}{x^2}}=\frac{1}{x^2}$$

∴ Multiplying the given equation with $\dfrac{1}{x^2}$, we get

$$\left(\frac{x^2-y^2}{x^2}\right)dx+\frac{2xy}{x^2}dy=0$$

$$\left(\frac{x^2-y^2}{x^2}\right)dx+2\frac{y}{x}dy=0 \qquad (2)$$

$$M_1=\frac{x^2-y^2}{x^2} \text{ and } N_1=\frac{2y}{x}$$

$$\frac{\partial M_1}{\partial y}=\frac{-2y}{x^2},\ \text{and}\ \frac{\partial N_1}{\partial x}=\frac{-2y}{x^2}$$

$$\frac{\partial M_1}{\partial y}=\frac{\partial N_1}{\partial x}$$

∴ Eq. (2) is an exact equation and its solution is

$$\int^x M_1dx+\int(\text{the terms of } N_1 \text{ not containing } x)\,dy=C$$

$$\int^x\frac{x^2-y^2}{x^2}dx+\int 0dy=C$$

$$\Rightarrow\int^x 1-\frac{y^2}{x^2}dx=C \quad\Rightarrow\quad x+\frac{y^2}{x}=C.$$

Example 13

Find the solution of $xy^2dx+(y+y^2)dy=0$.

Solution

Given $xy^2dx+(y+y^2)dy=0$ (1)

$$Mdx+Ndy=0$$

$$M=xy^2;\quad N=y+y^2$$

$$\frac{\partial M}{\partial y}=2xy \text{ and } \frac{\partial N}{\partial x}=0$$

$$\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$$

$$\frac{1}{M}\left[\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right]=\frac{1}{xy^2}[-2xy]$$

$$=\frac{-2}{y}=g(y)$$

Integrating factor is $e^{\int g(y)dy}$

$$=e^{\int\frac{-2}{y}dy}=e^{-2\log y\,dy}=e^{\log\frac{1}{y^2}}=\frac{1}{y^2}$$

Multiplying Eq. (1) by $\dfrac{1}{y^2}$, we get $\dfrac{xy^2dx}{y^2}+\left(\dfrac{y+y^2}{y^2}\right)dy=0$

$$xdx + \left(\frac{1}{y} + 1\right)dy = 0$$

Integrating on both sides we get $\frac{x^2}{2} + \log y + y = C$

Linear Equations

Consider the linear differential equation $\frac{dy}{dx} + Py = Q$ (1)

where P and Q are functions of only x. We explain below, how such equations can be solved. Consider the equation

$$\frac{dy}{dx} + Py = 0 \quad (2)$$

The Eq. (2) is called the homogeneous linear equation corresponding to Eq. (1). We find the general solution of Eq. (2).

Eq. (2) is a variables separable type. We write it as

$$\frac{dy}{y} = -Pdx.$$

Integrating the above equation given.

$$\log y = -\int Pdx + \log C \text{ or } y = Ce^{-\int Pdx} \quad (3)$$

This represents the general solution of Eq. (2).

Eq. (3) may also be written as $ye^{\int Pdx} = c$.

Now, $\frac{d}{dx}(ye^{\int Pdx}) = 0$

That is, $e^{\int Pdx}\frac{dy}{dx} + ye^{\int Pdx} \times P = 0$ or $e^{\int Pdx}\left[\frac{dy}{dx} + Py\right] = 0.$

This means that if we multiply both sides of Eq. (2) by $e^{\int Pdx}$, the product

$e^{\int Pdx}\left[\frac{dy}{dx} + Py\right]$ is $\frac{d}{dx}\{ye^{\int Pdx}\}$. The factor $e^{\int Pdx}$ is called an integrating factor of Eq. (2).

Suppose we multiply both sides of Eq. (1) by $e^{\int Pdx}$, it is reduced to $\frac{d}{dx}(ye^{\int Pdx}) = \frac{d}{dx}\left(\int Qe^{\int Pdx}dx\right)$, $\left[\text{since } \frac{d}{dx}\left(\int Qe^{\int Pdx} dx\right) = Qe^{\int Pdx}\right]$. Hence, we get the general solution of Eq. (1) as

$$ye^{\int Pdx} = C + \int Qe^{\int Pdx} dx.$$

Example 14

Solve $\sin x\frac{dy}{dx} + y\cos x = 1$.

Solution

$$\sin x\frac{dy}{dx} + y\cos x = 1$$

$$\frac{dy}{dx} + (\cot x)y = \operatorname{cosec} x.$$

This is a linear equation in y

Here, $P = \cot x$, $Q = \operatorname{cosec} x$.

$$\int Pdx = \int \cot x\, dx = \log(\sin x)$$

$$\text{IF} = e^{\int Pdx} = e^{\log\sin x} = \sin x.$$

$\therefore$ The general solution is $y \cdot \text{IF} = \int \text{QIF}\, dx + c$

$$y\sin x = \int \operatorname{cosec} x \cdot \sin x\, dx + c$$

$$y\sin x = \int dx + c$$

$$y\sin x = x + c.$$

Example 15

Solve $(1+x^4)\frac{dy}{dx} + 4x^3y = \sin^3 x$.

Solution

Given: $(1+x^4)\frac{dy}{dx} + 4x^3y = \sin^3 x$

$$\frac{dy}{dx} + \frac{4x^3}{1+x^4}y = \frac{\sin^3 x}{1+x^4}$$

It is a linear differential equation in y.

Here, $P = \frac{4x^3}{1+x^4}$ and $Q = \frac{\sin^3 x}{1+x^4}$

$$\int Pdx = \int \frac{4x^3}{1+x^4}dx = \log(1+x^4)$$

$\text{IF} = e^{\int pdx} = e^{\log(1+x^4)} = 1 + x$

General solution

$y \cdot \text{IF} = \int Q \cdot \text{IF}\, dx + c.$

$$y(1+x^4) = \int \frac{\sin^3 x}{1+x^4}(1+x^4)\, dx + c$$

$$= \int \sin^3 x\, dx + C = \int \frac{3\sin x - \sin 3x}{4}dx + c$$

$$y(1+x^4) = \frac{\cos 3x}{12} - \frac{3}{4}\cos x + c$$

$$12y(1+x^4) = \cos 3x - 9\cos x + c$$

Example 16

Solve $x^2\left(\frac{dy}{dx} + y\right) = 4x^2 + 8 - 2y$.

Solution

Given: $x^2\left(\frac{dy}{dx} + y\right) = 4x^2 + 8 - 2y$

$$\frac{dy}{dx} + y = 4 + \frac{8}{x^2} - \frac{2y}{x^2}$$

$$\frac{dy}{dx} + y\left(1 + \frac{2}{x^2}\right) = 4 + \frac{8}{x^2}$$

Here, $P = 1 + \frac{2}{x^2}$ and $Q = 4 + \frac{8}{x^2}$

$$\int Pdx = \int 1 + \frac{2}{x^2}\, dx = x - \frac{2}{x}$$

$$\text{IF} = e^{\int P dx} = e^{\left(x - \frac{2}{x}\right)}$$

General solution is $y \cdot \text{IF} = \int Q \cdot \text{IF} dx + c$

$$ye^{x-\frac{2}{x}} = \int \left(4 + \frac{8}{x^2}\right) e^{x-\frac{2}{x}} dx + c$$

$$= 4\int \left(1 + \frac{2}{x^2}\right) e^{x-\frac{2}{x}} dx + c$$

(Put $e^{x-\frac{2}{x}} = t$

$$\Rightarrow \quad e^{x-\frac{2}{x}} \left(1 + \frac{2}{x^2}\right) dx = dt)$$

$$= 4\int dt + c = 4t + c$$

The general solution is

$$ye^{x-\frac{2}{x}} = 4e^{x-\frac{2}{x}} + c$$

Bernoulli's Linear Equations

An equation of the form $\frac{dy}{dx} + Py = Qy^n$ is called Bernoulli's linear equation, where P, Q are continuous functions in x.

Example 17

Solve $\frac{dy}{dx} + xy = -(3xy^2)$.

Solution

Given $\frac{dy}{dx} + xy = -(3xy^2)$

Throughout the equation dividing with y^2 we get

$$y^{-2}\frac{dy}{dx} + xy^{-1} = -3x \tag{1}$$

Let $y^{-1} = u \quad \Rightarrow \quad -y^{-2}\frac{dy}{dx} = \frac{du}{dx}$

The Eq. (1) becomes $\frac{-du}{dx} + xu = -3x$

$$\frac{du}{dx} - xu = 3x$$

The above equation is a linear differential equation in u.

$\therefore \quad \text{IF} = e^{\int P dx} = e^{-\int x dx} = e^{\frac{-x^2}{2}}$

$\therefore \quad$ Solution is $u \cdot \text{IF} = \int \text{QIF}\, dx$

$$u \cdot e^{\frac{-x^2}{2}} = \int 3xe^{\frac{-x^2}{2}} dx.$$

$$= -\int 3e^{-t} dt \text{ when } t = \frac{-x^2}{2}$$

$$= \frac{-3e^{-t}}{-1} = 3e^{-t}$$

$$u \cdot e^{\frac{-x^2}{2}} = 3e^{\frac{-x^2}{2}} + C$$

$$\frac{1}{y} = 3 + Ce^{\frac{x^2}{2}}$$

$$y = \frac{1}{3 + Ce^{\frac{x^2}{2}}}$$

Example 18

Solve $\frac{dy}{dx} + \frac{y}{x}\log y = \frac{y}{x^3}(\log y)^2$.

Solution

Given $\frac{dy}{dx} + \frac{y \log y}{x} = \frac{y(\log y)^2}{x^3}$

$$\Rightarrow \quad \frac{1}{y(\log y)^2}\frac{dy}{dx} + \frac{1}{x} \cdot \frac{1}{(\log y)} = \frac{1}{x^3} \tag{1}$$

Let $\frac{1}{\log y} = u$,

Differenting wrt x

$$\frac{-1}{(\log y)^2} \cdot \frac{1}{y}\frac{dy}{dx} = \frac{du}{dx}$$

$\therefore \quad$ Eq. (1) becomes

$$\Rightarrow \quad \frac{-du}{dx} + \frac{1}{x}u = \frac{1}{x^3} \quad \Rightarrow \quad \frac{du}{dx} - \frac{1}{x}u = \frac{-1}{x^3}$$

It is a linear equation in u.

Here $P = \frac{-1}{x}$ and $Q = \frac{-1}{x^3}$

$$\text{IF} = e^{\int P dx} = e^{-\int \frac{1}{x} dx} = e^{-\log x} = \frac{1}{x}$$

$\therefore \quad$ Solution is $u \cdot \text{IF} = \int Q\text{IF} dx + c$

$$\frac{1}{x}u = \int \frac{-1}{x^3} \cdot \frac{1}{x} dx + c$$

$$\frac{1}{x}u = -\int x^{-4} dx + c$$

$$\frac{1}{(\log y)x} = \frac{1}{3x^3} + c$$

Second Order Linear Differential Equations with Constant Co-efficients

The standard form of a second order linear differential equation with constant co-efficients is

$$a_0\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = F(x) \tag{1}$$

where a_0, a_1, a_2 are real constants and $F(x)$ is a function of only x. The second order equation,

$$a_0\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = 0 \tag{2}$$

represents the corresponding homogeneous equation.

Let $y = u(x)$ represent the general solution of Eq. (2) [$u(x)$ will contain two arbitrary constants]. This means that

$$a_0 \frac{d^2u}{dx^2} + a_1 \frac{du}{dx} + a_2 u = 0 \tag{3}$$

Let $y = v(x)$ represent a particular solution of the given equation of Eq. (1). We have, then,

$$a_0 \frac{d^2v}{dx^2} + a_1 \frac{dv}{dx} + a_2 v = F(x) \tag{4}$$

Substituting $y = u(x) + v(x)$ in Eq. (1),

$$a_0 \frac{d^2}{dx^2}(u+v) + a_1 \frac{d}{dx}(u+v) + a_2(u+v)$$

$$= \left(a_0 \frac{d^2u}{dx^2} + a_1 \frac{du}{dx} + a_2 u \right) + \left(a_0 \frac{d^2v}{dx^2} + a_1 \frac{dv}{dx} + a_2 v \right)$$

$= 0 + F(x)$ (by Eqs. (3) and (4))

$= F(x)$.

We infer that $y = u(x) + v(x)$ is the general solution of the Eq. (1). Thus, the general solution of Eq. (2) is the sum of the general solution of the corresponding homogeneous equation (2) and a particular solution of the given equation (1). $y = u(x)$ is called the *complementary function* of Eq. (2) and $y = v(x)$ is called a particular integral of Eq. (1). The general solution of Eq. (1) is given by $y = u(x) + v(x)$.

= [Complementary function] + [Particular integral]

= CF + PI (in short).

To find the complementary function of Eq. (1) or to obtain the general solution of the homogeneous equation (2): As $y = e^{mx}$ is a solution of $\frac{dy}{dx} - my = 0$, we assume $y = e^{mx}$ (for some value of m) to be a solution of Eq. (2).

Then, $a_0 \frac{d^2}{dx^2}(e^{mx}) + a_1 \frac{d}{dx}(e^{mx}) + a_2 e^{mx}$ must be equal to zero (or) $e^{mx}\{a_0 m^2 + a_1 m + a_2\} = 0$.

Since e^{mx} cannot be equal to zero, $a_0 m^2 + a_1 m + a_2 = 0$ (5)

Eq. (5) is called the auxiliary equation corresponding to (1) [or (2)]. Eq. (5) is quadratic in m and gives two values for m, which may be real or complex.

Case 1: Let the roots of Eq. (5) be real and distinct, say m_1 and m_2 ($m_1 \neq m_2$). Then, $y = e^{m_1 x}$ and $y = e^{m_2 x}$ are two distinct solutions of (2) or $y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$ (6)

(C_1 and C_2 are arbitrary constants) is the general solution of (2) or the complementary function of (1).

Case 2: Let the roots of (5) be real and equal and each equals to m_1.

$$\text{Let } \frac{d}{dx} \equiv D, \quad \frac{d^2}{dx^2} \equiv D^2.$$

Then Eq. (2) may be expressed as $(a_0 D^2 + a_1 D + a_2)y = 0$.

Since the roots of the auxiliary equation are equal and each equal to m_1, this reduces to

$$a_0(D - m_1)^2 y = 0 \text{ or } (D - m_1)^2 y = 0 \tag{7}$$

(since $a_0 \neq 0$)

Let $(D - m_1)y = Y_1$ (8)

Then, Eq. (7) becomes $(D - m_1)Y_1 = 0$. (9)

Now, Eq. (9) is reduced to $\frac{dY_1}{dx} - m_1 Y_1 = 0$, giving $Y_1 = C_1 e^{m_1 x}$ as the solution.

Substituting in Eq. (8), $\frac{dy}{dx} - m_1 y = c_1 e^{m_1 x}$ is a linear equation. The general solution is given by $ye^{-m_1 x} = c_2 + \int c_1 e^{m_1 x} \times e^{-m_1 x} dx = c_2 + c_1 x$

or $$y = c_2 e^{m_1 x} + c_1 x e^{m_1 x} = e^{m_1 x}(c_2 + c_1 x)$$

where c_1 and c_2 are arbitrary constants.

Case 3: Let the roots of (V) be complex. Let us assume the roots as the conjugate pairs $\alpha \pm i\beta$. (The co-efficients a_0, a_1, a_2 being real, roots occur in conjugate pairs).

The general solution is $y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}$

$$= c_1 e^{\alpha x}(\cos \beta x + i \sin \beta x) + c_2 e^{\alpha x}(\cos \beta x - i \sin \beta x)$$

$$= e^{\alpha x}\{(c_1 + c_2)\cos \beta x + i(c_1 - c_2)\sin \beta x\}$$

$$= e^{\alpha x}\{A_1 \cos \beta x + A_2 \sin \beta x).$$

where A and B are arbitrary constants. We may now summarize the nature of the complementary function of Eq. (1) as follows:

Roots of the Auxiliary Equation $a_0 m^2 + a_1 m + a_2 = 0$	Complementary Function of (1), or General Solution of (2)
Roots, real and distinct, say m_1, m_2	$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$
Roots, real and equal, say each equals m_1	$y = (c_1 + c_2 x)e^{m_1 x}$
Roots, complex, say $\alpha \pm i\beta$	$y = e^{\alpha x}\{c_1 \cos \beta x + c_2 \sin \beta x\}$
Roots, complex and repeated, say $m_1 = m_2 = \alpha + i\beta$ and $m_3 = m_4 = \alpha - i\beta$	$y = e^{\alpha x}[(c_1 + c_2 x)\cos \beta x + (c_3 + c_4 x)\sin \beta x$

Example 23

Obtain the complementary function of the equation $\frac{d^2y}{dx^2} - \frac{7dy}{dx} + 6y = x^4$.

Solution

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 6y = x^4$$

$\Rightarrow \quad (D^2 - 7D + 6)y = x^4$

Auxiliary equation is $m^2 - 7m + 6 = 0$

$$m = 1, 6.$$

$\therefore$ The complementary function of the given equation.

$$y = c_1 e^x + c_2 e^{6x}.$$

Example 24

Obtain the general solution of the equation $\frac{d^2y}{dx^2} - 10 \cdot \frac{dy}{dx} + 25y = 0.$

Solution

Given: $\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 25y = 0$

$\Rightarrow \quad (D^2 - 10D + 25)y = 0$

Auxiliary equation is $m^2 - 10m + 25 = 0$

The roots are $(m) = 5, 5$

$\therefore$ The general solution of the equation is $(c_1 + c_2x)e^{5x}$.

Example 25

Obtain the complementary function of the equation $\frac{d^2y}{dx^2} - 6 \cdot \frac{dy}{dx} + 10y = e^{3x}.$

Solution

Given: $\frac{d^2y}{dx^2} - 6 \cdot \frac{dy}{dx} + 10y = e^{3x}$

$\Rightarrow \quad (D^2 - 6D + 10)y = e^{3x}$

Auxiliary equation is $m^2 - 6m + 10 = 0$

$$m = \frac{6 \pm \sqrt{36-40}}{2} = \frac{6 \pm 2i}{2} = 3 \pm i$$

$\therefore$ The complementary function is given by $y_c = e^{3x}(c_1\cos x + c_2\sin x)$.

To find a particular integral of Eq. (1) or to find a particular solution of the Eq. (1):

$$a_0\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = F(x)$$

We may write the above as $(a_0D^2 + a_1D + a_2)y = F(x)$ or $f(D)\, y = F(x)$ where $f(D)$ stands for $(a_0D^2 + a_1D + a_2)$. Particular integral y is that function of x independent of arbitrary constants such that $f(D)$ on y or $f(D)\, y$ yields $F(x)$.

This is symbolically represented as $y = \frac{1}{f(D)}\{F(x)\}$.

Case 1: $F(x) = e^{kx}$ where k is a constant.

We have $D(e^{kx}) = ke^{kx}$, $D^2(e^{kx}) = k^2e^{kx}$ … or, in general, $g(D)\,(e^{kx}) = g(k)\, e^{kx}$ where $g(D)$ is a polynomial in D, in particular, $f(D)\,\{e^{kx}\} = f(k)\, e^{kx}$.

Since $\frac{1}{f(D)}e^{kx}$ is that function of x which when operated by $f(D)$ gives e^{kx}, it is clear that $\frac{1}{f(D)}e^{kx} = \frac{1}{f(k)}e^{kx}$ provided $f(k) \neq 0$. $f(k)$ reduces to zero when one or both the roots of the auxiliary equation $a_0m^2 + a_1m + a_2 = 0$, is k.

1. Suppose one of the roots is k. Then, $f(D) = a_0(D - k)(D - m_0)$, where $m_0 \neq k$. Particular integral

$$= \frac{1}{a_0(D-k)(D-m_0)}e^{kx}$$

$$= \frac{1}{D-k}\left\{\frac{1}{a_0(D-m_0)}e^{kx}\right\}$$

$$= \frac{1}{a_0(k-m_0)}\frac{1}{(D-k)}e^{kx}$$

Let $\frac{1}{(D-k)}e^{kx} = X_1$

Then $(D-k)X_1 = e^{kx}$ or $\frac{dX_1}{dx} - kX_1 = e^{kx}$

This is a linear equation and the particular solution of the above equation is xe^{kx}. Therefore, particular integral $= \frac{1}{a_0(k-m_0)}xe^{kx}$.

2. Suppose both the roots of the auxiliary equation are k. Then, particular integral

$$= \frac{1}{a_0(D-k)^2}[e^{kx}]$$

$$= \frac{1}{a_0(D-k)}\left[\frac{1}{(D-k)}e^{kx}\right]$$

$$= \frac{1}{a_0(D-k)}[xe^{kx}],$$

Use the result in (1) . Now, let $\frac{1}{D-k}(xe^{kx}) = X_2$

We have, therefore, $(D-k)X_2 = xe^{kx}$ or $\frac{dX_2}{dx} - kX_2 = xe^{kx}$ which is a linear equation.

Particular solution is $X_2 = \frac{x^2}{2}e^{kx}$ or, particular integral in this case is given by $y = \frac{x^2}{2}e^{kx}$.

Example 26

Solve the differential equation:

$$(D^2 + 5D + 6)y = e^{-4x}$$

Solution

$(D^2 + 5D + 6)y = e^{-4x}$

Auxiliary equation is $m^2 + 5m + 6 = 0$.

$$(m+3)(m+2) = 0.$$

$\therefore$ Roots are $m = -3, -2$.

Complementary function is $c_1e^{-3x} + c_2e^{-2x}$.

$$\text{Particular integral } = \frac{1}{D^2+5D+6}\cdot e^{-4x}$$

$$= \frac{1}{(-4)^2+5(-4)+6}e^{-4x} = \frac{e^{-4x}}{2}$$

$\therefore$ General solution is

$$y = c_1e^{-3x} + c_2e^{-2x} + \frac{e^{-4x}}{2}.$$

Example 27

Solve $(3D^2 - D - 10)y = 6e^{2x}$

Solution

Given $(3D^2 - D - 10)y = 6e^{2x}$

Auxiliary equation $3m^2 - m - 10 = 0$

$$m = 2, \frac{-5}{3}.$$

$\therefore$ Complementary function is

$$\text{CF} = c_1e^{2x} + c_2e^{\frac{-5}{3}x}$$

$$\text{PI} = \frac{1}{3D^2-D-10}6e^{2x}$$

$$= \frac{1}{(D-2)(3D+5)}6e^{2x}$$

$$= 6\frac{1}{D-2}\left[\frac{1}{3D+5}e^{2x}\right] = 6\frac{1}{(D-2)}\frac{1}{11}e^{2x}1$$

$$= \frac{6}{11}\frac{1}{(D-2)}e^{2x} = \frac{6}{11}xe^{2x}$$

$\therefore$ General solution is

$$y = c_1e^{2x} + c_2e^{-\frac{5}{3}x} + \frac{6}{11}xe^{2x}.$$

Example 28

Solve $(D^2 - 12D + 36)y = e^{6x}$

Solution

Given: $(D^2 - 12D + 36)y = e^{6x}$

Auxiliary equation is $m^2 - 12m + 36 = 0$.

$$m^2 - 12m + 36 = 0.$$

$$m = 6, 6$$

Complementary function (CF) $= (c_1 + c_2x)e^{6x}$

$$\text{PI} = \frac{1}{D^2-12D+36}e^{6x} = \frac{1}{(D-6)^2}e^{6x}$$

$$= \frac{x^2}{2!}e^{6x}$$

$\therefore$ General solution is $y = \text{CF} + \text{PI}$

$$= (c_1 + c_2x)e^{6x} + \frac{x^2}{2!}e^{6x0}$$

Case 2: $F(x) = \sin kx$ or $\cos kx$ where k is a constant.

We have $D\{\sin kx\} = k\cos kx$

$$D^2\{\sin kx\} = -k^2\sin kx$$

Similarly, $D^2\{\cos kx\} = -k^2\cos kx$

If $g(D^2)$ is a polynomial in D^2,

$$g(D^2)\{\sin kx \text{ or } \cos kx\} = g(-k^2)\sin kx \text{ or } g(-k^2)\cos kx.$$

$$\text{Hence, } \frac{1}{g(D^2)}\sin kx = \frac{1}{g(-k^2)}\sin kx \text{ and } \frac{1}{g(D^2)}\cos kx$$

$= \dfrac{1}{g(-k^2)}\cos kx$, provided $g(-k^2) \neq 0$.

We shall illustrate the above technique by considering two examples.

Example 29

Find the particular integral of the equation $(D^2 + 16)y = \cos 3x$.

Solution

$$\text{PI} = \frac{1}{D^2+16}\cos 3x = \frac{1}{-(3)^2+16}\cos 3x = \frac{\cos 3x}{7}$$

Example 30

Find the particular integral of the equation $(D^2 - 5D + 6)y = \sin 3x$.

Solution

$$\text{PI} = \frac{1}{D^2-5D+6}\sin 3x$$

$$= \frac{1}{-3^2-5D+6}\sin 3x$$

$$\frac{1}{-5D-3}\sin 3x$$

$$= -\frac{5D-3}{(5D+3)(5D-3)}\sin 3x$$

$$= \frac{(5D-3)}{-(25D^2-9)}\sin 3x = \frac{3-5D}{25\times(-9)-9}\sin 3x$$

$$= \frac{1}{-234}[(3-5D)\sin 3x]$$

$$= \frac{-1}{234}[3\sin 3x - 5D(\sin 3x)]$$

$$= \frac{-1}{234}[3\sin 3x - 15\cos 3x]$$

$$\text{PI} = \frac{15\cos 3x}{234} - \frac{3\sin 3x}{234}$$

NOTE

Suppose $g(-k^2) = 0$.

Let us discuss the technique of finding particular integral in this case.

Suppose we have to find $\dfrac{1}{D^2+k^2}[\sin kx]$.

By Euler's formula, $e^{ikx} = \cos kx + i \sin kx$ or $\sin kx$ = imaginary part of e^{ikx}.

$$\text{Particular integral} = \frac{1}{D^2+k^2}[\sin kx].$$

$$= \text{Imaginary part of } \frac{1}{D^2+k^2}(e^{ikx})$$

$$= \text{Imaginary part of } \frac{1}{(D-ik)\,(D+ik)} e^{ikx}$$

$$= \text{Imaginary part of } \frac{1}{D-ik}\left[\frac{e^{ikx}}{2ik}\right]$$

$$= \text{Imaginary part of } \frac{xe^{ikx}}{2ik}$$

$$= \text{Imaginary part of } \frac{x}{2ki}(\cos kx + i \sin kx)$$

$$= \text{Imaginary part of } \frac{x}{2k}(-i\cos kx + \sin kx)$$

$$= \frac{-x\cos kx}{2k}.$$

Similarly, if we have to find $\dfrac{1}{D^2+k^2}[\cos kx]$.

We write it as the real part of $\dfrac{1}{D^2+k^2}(e^{ikx})$

$$= \text{Real part of } \frac{1}{(D-ik)(D+ik)}(e^{ikx})$$

$$= \text{Real part of } \frac{x}{2k}(-i\cos kx + \sin kx)$$

$$= \frac{x\sin kx}{2k}.$$

$$\boxed{\begin{aligned} \frac{1}{D^2+k^2}\sin kx &= \frac{-x}{2k}\cos kx \\ \frac{1}{D^2+k^2}\cos kx &= \frac{x}{2k}\sin kx \end{aligned}}$$

Example 31

Solve the equation $(D^2 + 16)y = \sin 4x$.

Solution

Given: $(D^2 + 16)y = \sin 4x$

Auxiliary equation is $m^2 + 16 = 0$

$$m = \pm 4i$$

$\therefore$ CF $= c_1 \cos 4x + c_2 \sin 4x$

$$PI = \frac{1}{D^2+16}\sin 4x.$$

$$= -\frac{x}{2\cdot 4}\cos 4x$$

$$\left(\because \frac{1}{D^2+k^2}\sin kx = -\frac{x}{2kx}\cos kx\right)$$

$$= -\frac{x}{8}\cos 4x$$

General solution is y = CF + PI

$$= c_1\cos 4x + c_2 \sin 4x - \frac{x}{8}\cos 4x.$$

Cauchy's Homogeneous Linear Equations

An equation of the form

$$x^n \frac{d^n y}{dx^n} + p_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots p_n y = Q(x) \qquad (1)$$

where $p_1, p_2, \ldots, p_n$ are constants is called Cauchy's linear equation. To convert the above equation into linear differential equation with constant co-efficients, we substitute $x = e^z$ or $z = \log x$.

$\therefore \quad z = \log x,$

$$\Rightarrow \quad \frac{dz}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx}$$

$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{1}{x}$$

$$\frac{dy}{dz} = x\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\cdot\frac{dy}{dz}\right)$$

$$= \frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} = \frac{d}{dz}\left[\frac{dy}{dz} - y\right]$$

$$\text{Let } \frac{dy}{dz} = \theta y \quad \Rightarrow \quad x\frac{dy}{dx} = \theta y,\ x^2\frac{d^2y}{dx^2} = \theta(\theta-1)y$$

Similarly $x^3\dfrac{d^3y}{dx^3}=\theta(\theta-1)(\theta-2)y$, and so on.

Then Eq. (1) is changed into a linear differential equation.

We solve this by methods discussed earlier.

Example 32

Solve $x^2\dfrac{d^2y}{dx^2}+3x\dfrac{dy}{dx}-3y=0$

Solution

Let $x=e^z$ or $z=\log x$

Then $x\dfrac{dy}{dx}=\theta y;\ x^2\dfrac{d^2y}{dx^2}=\theta(\theta-1)y$

The above equation becomes

$$[\theta(\theta-1)+3\theta-3]y=0$$

$$\left\lfloor \theta^2+2\theta-3\right\rfloor y=0$$

Auxiliary equation is $m^2+2m-3=0$

$\Rightarrow\quad (m+3)(m-1)=0$

$\Rightarrow\quad m=-3, 1$

$\therefore\quad y=c_1e^{-3z}+c_2e^z$

$\quad\quad =c_1x^{-3}+c_2x.$

Example 33

Solve $x^3\dfrac{d^3y}{dx^3}+6x^2\dfrac{d^2y}{dx^2}+8\dfrac{dy}{dx}+2y=x^2\log x.$

Solution

Put $x=e^z$ or $z=\log x$. Then

$$x\frac{dy}{dx}=\theta y,\ x^2\frac{d^2y}{dx^2}=\theta(\theta-1)y,$$

$$x^3\frac{d^3y}{dx^3}=\theta(\theta-1)(\theta-2)y$$

The given equation becomes

$[\theta(\theta-1)(\theta-2)+6\theta(\theta-1)+8\theta+2]y=e^{2z}\cdot z$

$$(\theta^3+3\theta^2+4\theta+2)y=e^{2z}\cdot z$$

$$\text{AE}=m^3+3m^2+4m+2=0$$

$$(m+1)(m^2+2m+2)=0$$

$$m=-1 \text{ or } m=-1\pm i$$

$$\text{CF}=C_1e^{-z}+e^{-z}(C_2\cos z+C_3\sin z)$$

$$\text{PI}=\frac{1}{\theta^3+3\theta^2+4\theta+2}\cdot e^{2z}z$$

$$=e^{2z}\frac{1}{(\theta+2)^3+3(\theta+2)^2+4(\theta+2)+2}z$$

$$=e^{2z}\frac{1}{\theta^3+9\theta^2+28\theta+30}\cdot z$$

$$\frac{e^{2z}}{30}\left[1+\frac{\theta^3+9\theta^2+28\theta}{30}\right]^{-1}z$$

$$=\frac{e^{2z}}{30}\left[1-\frac{\theta^3+9\theta^2+28\theta}{30}\right]z$$

$$=\frac{e^{2z}}{30}z-\frac{28}{(30)^2}e^{2z}$$

$$y=\text{CF}+\text{PI}$$

$$=C_1e^{-z}+e^{-z}(C_2\cos z+C_3\sin z)+\frac{e^{2z}}{30}z-\frac{28}{(30)^2}e^{2z}$$

$$=\frac{C_1}{x}+\frac{1}{x}(C_2\cos(\log x)+C_3\sin(\log x))+\frac{x^2\log x}{30}-\frac{28}{900}x^2$$

Example 34

Solve $(2x-1)\dfrac{d^2y}{dx^2}+2(2x-1)\dfrac{dy}{dx}-100y=32(2x-1)^2$

Solution

Let $2x-1=u$

$$2=\frac{du}{dx}$$

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=2\frac{dy}{du}$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(2\frac{dy}{du}\right)$$

$$=2\frac{d}{du}\left(\frac{dy}{du}\right)\cdot\frac{du}{dx}=2^2\frac{d^2y}{du^2}$$

$\therefore$ The given equation becomes

$$2^2u^2\frac{d^2y}{du^2}+2\cdot 2u\frac{dy}{du}-100y=32u^2$$

$$u^2\frac{d^2y}{du^2}+u\frac{dy}{du}-25y=8u^2$$

Let $u=e^z$, $u\dfrac{dy}{dx}=0;\ x^2\dfrac{d^2y}{dx^2}=\theta(\theta-1)$

$$[\theta(\theta-1)+\theta-25]y=8e^{2z}$$

$$\left\lfloor\theta^2-25\right\rfloor y=8e^{2z}$$

$$\text{AE}=m^2-25=0\quad\Rightarrow\quad m=\pm 5$$

$$\text{CF}=C_1e^{-5z}+C_2e^{5z}$$

$$\text{PI}=\frac{1}{\theta^2-25}\cdot 8e^{2z}=8.e^{2z}\frac{1}{2^2-25}=\frac{-8}{21}e^{2z}$$

$$y=\text{CF}+\text{PI}=C_1e^{5z}+C_2e^{5z}-\frac{8}{21}e^{2z}$$

$$=C_1u^{-5}+C_2u^5-\frac{8}{21}u \text{ where } u=(2x-1).$$

Method of Variation of Parameters

An equation of the form $\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=R(x)$, where $P(x)$, $Q(x)$ and $R(x)$ are real valued functions of x, is called the linear equation of the second order with variable co-efficients.

The above equation is solved by the method of variation of parameters.

The method is explained below:

1. Find the solution of $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=0$ and let the solution be $y_c = C_1U(x) + C_2V(x)$
2. Write particular solution as follows:

   $y_p = AU(x) + BV(x)$

   where $A=\int\frac{-VR}{W}dx$

   and $B=\int\frac{UR}{W}dx$

   where $W=\begin{vmatrix} U & V \\ \frac{dU}{dx} & \frac{dV}{dx}\end{vmatrix}=U\frac{dV}{dx}-V\frac{dU}{dx}$ is called the Wronskian of U and V.
3. Then the solution is $y_c + y_p$

   i.e., $y = C_1U(x) + C_2V(x) + AU(x) + BV(x)$

Example 35

Solve the differential equation $(D^2 + 4)y = \sec 2x$ by variation of parameters.

Solution

Given $(D^2 + 4)y = \sec 2x$

$$\text{AE} = m^2 + 4 = 0 \quad \Rightarrow \quad m = \pm 2i$$

$$\text{CF} = y_c = C_1\cos 2x + C_2\sin 2x$$

$$\therefore\ U(x) = \cos 2x;\ V(x) = \sin 2x$$

$$y_p = AU(x) + BV(x)$$

$$W = U\frac{dV}{dx}-V\frac{dU}{dx}$$

$$=\cos 2x\frac{d}{dx}(\sin 2x)-\sin 2x\cdot\frac{d}{dx}(\cos 2x)$$

$$= 2\cos^2 2x + 2\sin^2 2x = 2$$

$$A=-\int\frac{VR}{W}dx=-\int\frac{\sin 2x\cdot\sec 2x}{2}dx$$

$$=-\int\frac{\tan 2x}{2}dx=\frac{1}{4}\log(\cos 2x)$$

$$B=\int\frac{UR}{W}dx=\int\frac{\cos 2x\cdot\sec 2x}{2}dx=\frac{1}{2}x$$

$$\therefore\quad y_p=\frac{1}{4}[\log(\cos 2x)]\cdot\cos 2x+\frac{1}{2}x\sin 2x$$

$$\therefore\quad y=y_c+y_p=C_1\cos 2x+C_2\sin 2x+\frac{1}{4}[\log(\cos 2x)]\cos 2x+\frac{1}{2}x\sin 2x.$$

Example 36

Solve the differential equation $y'' + 4y' + 4y = x^3e^{2x}$

Solution

Given equation

$$(D^2 + 4D + 4)y = x^3 e^{2x}$$

The auxiliary equation is

$$m^2 + 4m + 4 = 0$$

$$(m + 2)^2 = 0 \quad \Rightarrow \quad m = -2$$

$$y_C = C_1 e^{-2x} + C_2xe^{-2x}$$

Let $U(x) = e^{-2x}$ and $V(x) = xe^{-2x}$

$$y_p = AU(x) + BV(x)$$

$$A=-\int\frac{VR}{W}dx,\ B=\int\frac{UR}{W}dx$$

$$W=u\frac{dv}{dx}-v\frac{du}{dx}=e^{-2x}\frac{d}{dx}(xe^{-2x})-xe^{-2x}\frac{d}{dx}(e^{-2x})$$

$$=e^{-2x}\left\lfloor e^{-2x}-2xe^{-2x}\right\rfloor+2xe^{-2x}e^{-2x}=e^{-4x}$$

$$A=-\int\frac{UR}{\frac{udv}{dx}-\frac{vdu}{dx}}dx=-\int\frac{xe^{-2x}x^3\cdot e^{2x}}{e^{-4x}}dx$$

$$=-\int x^4e^{4x}dx$$

$$=-x^4\frac{e^{4x}}{4}+\frac{x^3e^{4x}}{4}-3\frac{x^2e^{4x}}{16}+6\frac{xe^{4x}}{16\times 4}-6\frac{e^{4x}}{16\times 16}$$

$$B=\int\frac{UR}{W}dx=\int\frac{e^{-2x}x^3e^{2x}}{e^{-4x}}dx=\int x^3e^{4x}dx$$

$$=x^3\frac{e^{4x}}{4}-\frac{3}{4}\left[x^2\frac{e^{4x}}{4}-2\frac{xe^{4x}}{16}+\frac{e^{4x}}{32}\right]$$

$$y = y_c + y_p = AU(X) + BV(x) + C_1e^{-2x} + C_2xe^{-2x}$$

$$=C_1e^{-2x}+C_2xe^{-2x}-x^4\frac{e^{2x}}{4}+\frac{x^3e^{2x}}{4}-\frac{3x^2e^{2x}}{16}$$

$$+\frac{6xe^{2x}}{64}-\frac{3e^{2x}}{3128}+\frac{x^4e^{2x}}{4}-\frac{3x^3e^{2x}}{16}+\frac{3}{32}x^2e^{2x}-\frac{3}{128}xe^{2x}$$

$$=C_1e^{-2x}+C_2xe^{-2x}-\frac{1}{16}x^3e^{2x}-\frac{3}{32}x^2e^{2x}\frac{9}{128}-\frac{xe^{2x}}{128}+\frac{9}{128}e^{2x}$$

Laplace Transforms

Let $f(t)$ be a given function defined for all $t \geq 0$. The Laplace transform of $F(t)$ is denoted by $L\{f(t)\}$ or $L\{f\}$ and is defined as $L\{f(t)\}=\int_0^\infty e^{-st}f(t)dt=F(s)$.

Here L is Laplace transform operator. $f(t)$ is the determining function depends on it. $F(s)$ is the function to be determined called generating function. e^{-st} is called kernel of the transform.

Some standard results of Laplace transforms are given below.

1. $L\{e^{at}\} = \frac{1}{s-a}, s > a$

2. $L\{e^{-at}\} = \frac{1}{s+a},$

3. (a) Let k be a constant $L\{k\} = \frac{k}{s}$

   (b) $L\{1\} = \frac{1}{s}, s > 0$

4. $L\{t^n\} = \frac{n!}{s^{n+1}}, s > 0$

5. $L\{\cos at\} = \frac{s}{s^2 + a^2}, s > 0$

6. $L\{\sin at\} = \frac{a}{s^2 + a^2}, s > 0$

7. $L\{\cosh at\} = \frac{s}{s^2 - a^2}, s > |a|$

8. $L\{\sinh at\} = \frac{a}{s^2 - a^2}, s > |a|$

9. $L\{t^n \cdot e^{at}\} = \frac{n!}{(s-a)^{n+1}}, n \in Z^+$

10. $L\left\{\frac{1}{t} f(t)\right\} = \int_s^\infty F(s)\,ds$

Example 37

Find the Laplace transform of the function

$$f(x) = 5e^{2x} + 7e^{-3x}$$

Solution

$$L\{f(x)\} = L(5e^{2x} + 7e^{-3x})$$
$$= 5L(e^{2x}) + 7L(e^{-3x})$$
$$L\{f(t)\} = 5 \cdot \frac{1}{s-2} + 7 \cdot \frac{1}{s+3}$$
$$= \frac{5}{s-2} + \frac{7}{s+3}.$$

Example 37

Find $L\{f(t)\}$ where

$$f(t) = 0,\ 0 < t < 1$$
$$= 1,\ 1 < t < 2$$
$$= t,\ t > 2.$$

Solution

As the given function is not defined at $t = 0$, 1 and 2

$$L\{f(t)\} = \int_0^\infty e^{-st} \cdot F(t)dt$$
$$= \int_0^1 e^{-st} \cdot 0dt + \int_1^2 e^{-st} \cdot 1dt + \int_2^\infty e^{-st} \cdot tdt$$
$$= \int_1^2 e^{-st} dt + \int_2^\infty e^{-st} \cdot tdt$$
$$= \frac{e^{-st}}{-s}]\int_1^2 + t \cdot \frac{e^{-st}}{-s}]\int_2^\infty - \int_2^\infty \frac{e^{-st}}{-s} \cdot dt$$
$$= -\frac{e^{-2s}}{s} + \frac{e^{-s}}{s} + \frac{2e^{-2s}}{s} + \frac{1}{s}\frac{e^{-st}}{-s}\Big]_2^\infty$$
$$= \frac{-e^{-2s}}{s} + \frac{e^{-s}}{s} + 2\frac{e^{-2s}}{s} + \frac{1}{s^2}e^{-2s}$$
$$= \frac{e^{-2s}}{s}\left(1 + \frac{1}{s}\right) + \frac{e^{-s}}{s}.$$

Example 39

Find the Laplace transform of the function

$$f(t) = \sin 2t,\ 0 < t < \pi\ = \frac{e^{-st}}{-s}\Bigg]_1^2 \int + t \cdot \frac{e^{-st}}{-s}\Bigg]_2^\infty \int - \int_2^\infty \frac{e^{-st}}{-s} \cdot dt$$
$$= 0,\ t > \pi$$

Solution

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$
$$= \int_0^\pi e^{-st} \cdot \sin 2tdt + \int_\pi^\infty e^{-st} \cdot 0dt$$
$$= \int_0^\pi e^{-st} \sin 2tdt$$
$$= \frac{e^{-st}}{s^2+4}[-s \sin 2t - 2\cos 2t]]_0^\pi$$
$$= \frac{2(1 - e^{-\pi s})}{s^2 + 4}.$$

Example 40

Find the Laplace transform of the function $f(t) = (\sin t + \cos t)^2$

Solution

$L\{(\sin t + \cos t)^2\} = L\{1 + \sin 2t\} = L\{1\} + L\{\sin 2t\} = \frac{1}{s} + \frac{2}{s^2+4}$

Some important (theorems) properties of Laplace transforms:

1. **Linear property:** Let f and g be any two functions of t and a_1, a_2 are constants, then $L\{a_1f(t) + a_2g(t)\} = a_1L\{f(t)\} + a_2L\{g(t)\}$
2. **First shifting property:** If $L\{f(t)\} = F(s)$ then $L\{e^{at} f(t) = F(s-a)$

   Example: $L\{e^{at}\cos ct\} = \dfrac{s-a}{(s-a)^2+c^2}$
3. **Change of scale property:** If $L\{f(t)\} = F(s)$ then $L\{f(at)\} = \left|\dfrac{1}{a}\right| F\left(\dfrac{s}{a}\right)$

   Example: We know

   $$L\{e^{at}\} = \frac{1}{s-a} = F(s)$$

   Then $L\{be^{at}\} = \dfrac{1}{|b|}F\left(\dfrac{s}{b}\right) = \dfrac{1}{|b|}\dfrac{1}{\dfrac{s}{b}-a} = \dfrac{1}{|b|}\cdot\dfrac{b}{s-ab}$
4. **Differentiation theorem:** If derivatives of $f(t)$ are continuous and $L\{f(t)\} = F(s)$ then $L\{f'(t)\} = sF(s) - f(0)$ and

   $L\{f^n(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-3}f''(0)\ldots f^{n-1}(0) = s^nF(s) - \sum_{r=0}^{n-1} s^{n-1-r}\cdot f^r(0)$ (f^r represents rth derivative of f)
5. **Multiplication theorem:** If $L\{f(t)\} = F(s)$ then $L\{t\cdot f(t)\} = -F'(s)$

   and $L\{t^n\cdot f(t)\} = (-1)^n \dfrac{d^n}{ds^n}[F(s)]$
6. **Division theorem:** If $L\{f(t)\} = F(s)$, then $L\left\{\dfrac{1}{t}f(t)\right\} = \displaystyle\int_s^\infty F(s)ds$
7. **Transforms of integrals (theorem)**

   If $L\{f(t)\} = F(s)$, then $L\{\displaystyle\int_0^t f(u)du\} = \dfrac{1}{s}F(s)$

Example 41

Find the Laplace transform of $te^{-2t}\sin^2 t$.

Solution

$$L\{\sin^2 t\} = \frac{1}{2}L\{1-\cos 2t\} = \frac{1}{2}\left\{\frac{1}{s} - \frac{s}{s^2+4}\right\}$$

$$\therefore \quad L\{t\cdot\sin^2 t\} = (-1)\frac{d}{ds}\left(\frac{1}{2}\left\{\frac{1}{s}-\frac{s}{s^2+4}\right\}\right)$$

(using multiplication theorem)

$$L\{t\sin^2 t\} = \frac{-1}{2}\left(\frac{-1}{s^2} - \frac{(s^2+4)-s(2s)}{(s^2+4)}\right)$$

$$= \frac{1}{2s^2} + \frac{4-s^2}{2(s^2+4)^2}$$

$$L\{e^{-2t}\cdot t\sin^2 t\} = \frac{1}{2(s+2)^2} + \frac{4-(s+2)^2}{2[(s+2)^2+4]^2}$$

(using shifting property)

$$= \frac{1}{2(s+2)^2} - \frac{4s+s^2}{2(s^2+4s+8)^2}.$$

Example 42

Find the Laplace transform of $\dfrac{\sin 2t - \cos 2t}{t}$.

Solution

$$L\{\sin 2t - \cos 2t\} = \frac{2}{s^2+4} - \frac{s}{s^2+4}$$

$$L\{\frac{\sin 2t - \cos 2t}{t}\} = \int_s^\infty \left(\frac{2}{s^2+4} - \frac{s}{s^2+4}\right)ds$$

(using division property)

$$= \frac{2}{2}\left(\tan^{-1}\frac{s}{2}\right)_s^\infty - \frac{1}{2}[\log(s^2+4)]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\frac{s}{2} + \frac{1}{2}\log(s^2+4)$$

$$= \cot^{-1}\frac{s}{2} + \frac{1}{2}\log(s^2+4).$$

Example 43

Find the Laplace transform of $\displaystyle\int_0^t \frac{\sin 2u}{u}du$.

Solution

$$L\{\sin 2u\} = \frac{2}{s^2+4}$$

and $\left\{\dfrac{\sin 2u}{u}\right\} = \displaystyle\int_s^\infty \frac{2}{s^2+4}ds$

(using division theorem)

$$\frac{2}{2}\tan^{-1}\frac{s}{2}\Big]_s^\infty = \frac{\pi}{2} - \tan^{-1}\frac{s}{2} = \cot^{-1}\frac{s}{2}$$

$$\therefore \quad L\left\{\int_0^t \frac{\sin 2u}{u}du\right\} = \frac{1}{s}\cot^{-1}\frac{s}{2}$$

(using transform of integral theorem).

Inverse Laplace Transforms

If $F(s)$ is the Laplace transform of the function $f(t)$ i.e., $L\{f(t)\} = F(s)$ then $f(t)$ is called the inverse Laplace transform of the function $F(s)$ and is written as $f(t) = L^{-1}\{F(s)\}$. Here L^{-1} is called inverse Laplace transformation operator.

Some important standard results for inverse Laplace transform.

1. $L^{-1}\left(\frac{1}{s}\right) = 1$

2. $L^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{t^n}{n!}$ where n is a positive integer

   or $L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!}$

3. $L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

4. $L^{-1}\left(\frac{1}{(s-a)^n}\right) = \frac{e^{at}t^{n-1}}{(n-1)!}$

5. $L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at$

6. $L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$

7. $L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cos hat$

8. $L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{1}{a}\sin hat$

9. $L^{-1}\left[\frac{1}{(s-a)^2+b^2}\right] = \frac{1}{b}e^{at}\sin bt$

10. $L^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = e^{at}\cos bt$

11. $L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = \frac{1}{2a^3}$ (sin $at - at$ cos at)

12. $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{1}{2a}\, t\sin at$

To find the inverse Laplace transform we use the following methods.

1. Using the following properties

   (a) If $L^{-1}\{F(s)\} = f(t)$, than $L^{-1}\{F(s-a)\} = e^{at}f(t)$

   (b) If $L^{-1}\{F(s)\} = f(t)$ and $f(0) = 0$; then

   (i) $L^{-1}\{sF(s)\} = \frac{d}{dt}(f(t))$

   (ii) $L^{-1}\{s^nF(s)\} = \frac{d^n}{dt^n}(f(t))$ if $f(0) = f^1(0) = f^{(n-1)}(0) = 0$

   (c) If $L^{-1}\{F(s)\} = f(t)$, then

   (i) $L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(t)dt$

   (ii) $L^{-1}\left\{\frac{F(S)}{s^2}\right\} = \int_0^t\left\{\int_0^t f(t)dt\right\}dt$

2. **Convolution theorem:** Let $f(t)$ and $g(t)$ be two functions and

   $L^{-1}\{F(s)\} = f(t)$ and $L^{-1}\{G(s)\} = f(t)$, then

   $$L^{-1}\{F(s)\cdot G(s)\} = \int_0^t f(x)g(t-x)dx$$

   It is denoted by $f(t) * g(t)$ here $*$ represents convolution.

3. **Unit step function:** This function is defined as $u(t-a) = H(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$ the Laplace transform of $H(t-a) = L\{H(t-a)\}$

   $$= \int_0^\infty e^{-st}u(t-a)dt = \frac{e^{-as}}{s}$$

NOTE

This is also called as Heavisides unit function

4. **Periodic function:** If $f(t)$ is a periodic function with period a i.e., $f(t+a) = f(t)$, then

   $$L\{f(t)\} = \frac{\int_0^a e^{-st}f(t)dt}{1-e^{-sa}}$$

5. **Using partial fractions:** If $F(s)$ is of the from $\frac{G(s)}{H(s)}$ where G and H are polynomials in S then break $F(s)$ into partial fractions and manipulate term by term.

6. **Heavisides expansion formula:** Let $F(s)$ and $G(s)$ be two polynomials in 's' where $F(s)$ has degree less than that of $G(s)$. If $G(s)$ has n distinct zeros a_r, $r = 1, 2, 3, \ldots, n$

   i.e., $G(s) = (s-\alpha_1)(s-\alpha_2)\ldots(s-\alpha_n)$, then

   $$L^{-1}\left[\frac{F(S)}{G(S)}\right] = \sum_{r=1}^{n}\frac{F(\alpha_r)}{G'(\alpha_r)}e^{\alpha_r t}$$

Transform of Special Functions

7. **Bessel function:**

   $$J_0(x) = 1 - \frac{x^2}{2} + \frac{x^4}{2^2\cdot 4^2} - \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \cdots$$

   then $L\{J_0(x)\} = \frac{1}{\sqrt{s^2+1}}$

8. Error function: Error function is denoted as $erf(t)$

$$erf(\sqrt{x}) = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{x}} e^{-t^2}dt,$$

then $L\{erf(\sqrt{x})\} = \frac{1}{s\sqrt{s+1}}$

9. Complex inversion (theorem) formula: If $f(t)$ has a continuous derivative and is of exponential order and $L\{f(t)\} = F(s)$ then $L^{-1}\{F(s)\}$ is given by

$$f(t) = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{st}F(s)ds,\ t>0 \text{ and } f(t)=0 \text{ for } t<0$$

NOTES

1. The above result is also known as Bromwich's integral formula
2. The integration is to be performed along a line $s=r$ in the complex plane where $s=x+iy$. The real number r is chosen so that $p=r$ lies to the right of all the singularities.

10. The Gamma function: If $n > 0$, then the gamma function is defined by $\Gamma(n) = \int_0^\infty u^{n-1}e^u du$

11. Exponential Integral: The exponential integral is denoted by

$$E_i(t) = = \int_t^\infty \frac{e^{-u}}{u}du$$

Example 44

Evaluate $L^{-1}\left\{\frac{e^{2-3s}}{(s+2)^{5/2}}\right\}$

Solution

We have

$$L^{-1}\left\{\frac{1}{(s+2)^{5/2}}\right\} = e^{-2t}L^{-1}\left\{\frac{1}{s^{5/2}}\right\}$$

$$= e^{-2t}\frac{t^{\frac{5}{2}-1}}{\Gamma\left(\frac{5}{2}\right)} = \frac{4t^{\frac{3}{2}}e^{-2t}}{3\sqrt{\pi}}$$

$$\therefore \quad L^{-1}\left\{\frac{e^{2-3s}}{(s+2)^{5/2}}\right\} = e^2L^{-1}\left\{\frac{e^{-3s}}{(s+2)^{5/2}}\right\}$$

$$= \frac{4}{3\sqrt{\pi}}(t-3)^{3/2}e^{-2(t-4)}\cdot H(t-3)$$

(when expressed in terms of Heaviside's unit step function)

Example 45

Evaluate $L^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\}$

Solution

$$L^{-1}\left\{\frac{3(s-1)+10}{(s-1)^2-4}\right\}$$

$$= L^{-1}\left\{\frac{3(s-1)}{(s-1)^2-4}+\frac{10}{(s-1)^2-4}\right\}$$

$$= 3L^{-1}\left(\frac{s-1}{(s-1)^2-4}\right)+10L^{-1}\left\{\frac{1}{(s-1)^2-4}\right\}$$

$$= 3e^tL^{-1}\left\{\frac{s}{s^2-2^2}\right\}+10e^tL^{-1}\left\{\frac{1}{s^2-2^2}\right\}.$$

$$= 3e^t\cosh 2t + 5e^t\sinh 2t = 4e^{3t} - e^{-t}$$

Example 46

Evaluate $L^{-1}\left\{\frac{1}{s(s^2+4)^2}\right\}$

Solution

$$L^{-1}\left\{\frac{1}{s^2}\cdot\frac{s}{(s^2+4)^2}\right\}$$

Let $F_1(s) = \frac{1}{s^2}$ and $F_2(s) = \frac{s}{(s^2+4)^2}$ so that

$$L^{-1}\{F_1(s)\} = L^{-1}\left\{\frac{1}{s^2}\right\} = t = f_1(t)$$

and $$L^{-1}\{F_2(s)\} = L^{-1}\left\{\frac{s}{\left(s^2+4\right)^2}\right\}$$

$$= = \frac{t\cdot\sin 2t}{4} = f_2(t) \text{ (say)}$$

∴ By convolution theorem, we have

$$L^{-1}\left\{\frac{1}{s^2}\cdot\frac{s}{(s^2+4)^2}\right\} = L^{-1}\{F_1(s)\cdot F_2(s)\}$$

$$= \int_0^t f_2(x)f_1(t-x)dx = \int_0^t\left(\frac{x}{4}\sin 2x\right)(t-x)dx$$

$$= \frac{t}{4}\int_0^t x\sin 2x dx - \frac{1}{4}\int_0^t x^2\sin 2x dx$$

$$= \frac{t}{4}\left(-\frac{x}{2}\cos 2x + \frac{1}{4}\sin 2x\right)_0^t$$

$$-\frac{1}{4}\left(-\frac{x^2}{2}\cos 2x + \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x\right)_0^t$$

$$= \frac{1}{16}(1 - t\sin 2t - \cos 2t)$$

Application of Laplace transforms to solutions of differential equations: Solution of ordinary differential equations with constant co-efficients:

Consider a linear differential equation with constant co-efficients

$(D^n + C_1D^{n-1} + C_2D^{n-2} +\cdots+ (C_{nt})y = F(t)$ (1)

where $F(t)$ is a function of the independednt variable t

Let $y(0) = A_1, y^1 (0) = A_2,\ldots, y^{n-1} (0) = A_{n-1}$ (2)

be the given initial or boundary conditions where $A_1, A_2 \ldots A_{n-1}$ are constants.

By taking the Lapalce transform on both sides of (1) and using the conditions (2), we obtain an algebraic equation known as subsidiary equation from which $y(s) = L\{y(t)\}$ is determined. The required solution is obtained by finding the inverse Laplace transform of $y(s)$.

Example 47

Solve $(D + 3)^2 y = 9e^{-3t}$, $y(0) = -1$ and $y'(0) = 9$.

Solution

The given equation can be written as

$$(D^2 + 6D + 9)y = 9e^{-3t}$$

applying Laplace transform we get

$\therefore L\{y''\} + 6L\{y'\} + 9L\{y\} = 9L\{e^{-3t}\}$

or $s^2L\{y\} - sy(0) - y'(0) + 6[sL\{y\} - y(0)] + 9L\{y\} = \dfrac{9}{s+3}$

or $s^2L\{y\} + s - 9 + 6s\,L\{y\} + 6 + 9L\{y\} = \dfrac{9}{s+3}$

$$\Rightarrow \quad (s^2 + 6s + 9)\,L\{y\} = \frac{9}{s+3} - s + 3$$

$$(s+3)^2 L\{y\} = \frac{18-s^2}{s+3}$$

$$L\{y\} = \frac{18-s^2}{(s+3)^3}$$

$$\therefore \quad y = L^{-1}\left\{\frac{9-(s+3)^2+6(s+3)}{(s+3)^3}\right\}$$

$$= e^{-3t}L^{-1}\left\{\frac{9-s^2+6s}{s^3}\right\}$$

$$= e^{-t}\left[L^{-1}\left\{\frac{9}{s^3}\right\} - L^{-1}\left\{\frac{1}{s}\right\} + 6L^{-1}\left\{\frac{1}{s^2}\right\}\right]$$

$$y = e^{-3t}\left(9\cdot\frac{t^2}{2!} - 1 + 6t\right)$$

$\therefore$ The required solution is

$$y = \frac{e^{-3t}}{2}(9t^2 + 12t - 2).$$

Exercises

1. The order and degree of the DE $\dfrac{d^2y}{dx^2} = n^2y$ respectively are
   (A) 1, 2 (B) 1, 1
   (C) 2, 2 (D) 2, 1

2. The differential equation whose solution is $y = mx + \dfrac{4}{m}$, where 'm' is parameter is
   (A) $x\left(\dfrac{dy}{dx}\right)^2 - y\dfrac{dy}{dx} + 4\dfrac{dy}{dx} = 0.$
   (B) $\left(\dfrac{dy}{dx}\right)^2 - \dfrac{dy}{dx} + 4 = 0.$
   (C) $x\dfrac{dy}{dx} - y + 4 = 0.$
   (D) $x\left(\dfrac{dy}{dx}\right)^2 + \dfrac{dy}{dx} + 4 = 0.$

3. If $y = c_1 \log x + c_2 \log c_3 + c_4 e^x + c_5$ is the general solution of a homogeneous linear differential equation, then the order of the equation is
   (A) 2 (B) 3
   (C) 4 (D) 5

4. Find the solution of $\tan y \sec^2 x\, dx + \tan x \sec^2 y dy = 0$ when $x = y = \dfrac{\pi}{4}$.
   (A) $\tan x \tan y = 1$
   (B) $\cot x \tan y = 1$
   (C) $\tan x \cot y = 1$
   (D) $\cot x \cot y = 1$

5. The general solution of the DE, $(e^x + 1)ydy = (y + 1) e^x dx$ is
   (A) $\log (e^x + 1) - \log (y + 1) + c = 0$
   (B) $\log (e^x + 1) = y - \log (y + 1) + c$
   (C) $\log (e^x - 1) + \log (y + 1) + c = 0$
   (D) $\log\left(\dfrac{e^x}{y+1}\right) = c$

6. Solve $\dfrac{dy}{dx} = |x|$
   (A) $y = \dfrac{x^2}{2} + c$ (B) $y = \dfrac{x^2}{2} + x + c$
   (C) $y = \dfrac{-x|x|}{2} + c$ (D) $y = \dfrac{x|x|}{2} + c$

7. Solve $(x+y)^2 \dfrac{dy}{dx} = k^2$.

(A) $y = \tan^{-1}(x+y)$

(B) $y = \sin^{-1}\left(\dfrac{x+y}{k}\right)+c$

(C) $y = k\tan^{-1}\left(\dfrac{x+y}{k}\right)+c$

(D) $y = \cot^{-1}\left(\dfrac{x+y}{k}\right)+c$

8. The general solution of the DE, $\dfrac{dy}{dx} = (3x+y+1)^2$ is

(A) $\sec^{-1}(3x+y+1) = x+c$

(B) $\dfrac{1}{\sqrt{3}}\tan^{-1}\left(\dfrac{3x+y+1}{\sqrt{3}}\right) = x+c$

(C) $\tan^{-1}(3x+y+1) = x+c$

(D) $\dfrac{2}{\sqrt{3}}\tan^{-1}\left(\dfrac{2x-y+1}{\sqrt{3}}\right) = x+c$

9. The general solution of $\dfrac{dy}{dx} = \dfrac{x-y}{x+y}$ is

(A) $x^2+xy+y^2 = k$
(B) $x^2-y^2 = k$
(C) $x^2-2xy-y^2 = k$
(D) $x^2y^2 = k$

10. The general solution of $\dfrac{dy}{dx} = \dfrac{x-2y+1}{2x-4y+3}$ is

(A) $x^2 - 4xy - 6y = c$
(B) $x^2 - 4xy + 4y^2 + 2x - 6y = c$
(C) $x^2 + 4xy + 4y^2 + 2x - 6y = c$
(D) $x^2 + 4xy - x + 6y = c$

11. The solution of the differential equation $2xy\,dy + (x^2 + y^2 + 1)dx = 0$ is

(A) $x^3 + xy^2 + 3x = c$
(B) $x^3 + 3xy^2 + x = c$
(C) $\dfrac{x^3}{3} + xy^2 + x = c$
(D) $3x^2 + y^2 + 2x = c$

12. The general solution of $ye^{xy}dx + (xe^{xy} + 2y)dy = 0$ is

(A) $e^x + y^2 = c$
(B) $e^{xy} + y^2 = c$
(C) $e^{y^2} + xy = c$
(D) $e^y + xy = c$

13. The solution of the differential equation $(3xy + 2y^2)dx + (x^2 + 2xy)dy = 0$ is

(A) $x^3y + x^2y = c$
(B) $x^3y + x^2y^2 = c$
(C) $x^2y + xy^2 = c$
(D) $2xy(x + y) = c$

14. The integrating factor of the equation $(x^2 + xy - y^2)dx + (xy - x^2)dy = 0$ is

(A) $\dfrac{1}{x^2}$
(B) $\dfrac{1}{x^3}$
(C) x^2
(D) x^3

15. The solution of $(1+x)\dfrac{dy}{dx} - xy = 1 - x$ satisfying the initial conditions at $x = 0$ and $y = 1$ is

(A) $1 + x = y + e^x$
(B) $y(1 + x) = x + e^x$
(C) $x + y = e^x$
(D) $x(1 + y) = ce^x$

Direction for questions 16 to 17:

Consider the differential equation $\dfrac{dy}{dx} + y\cot x = y^2 \sin x$

16. The integrating factor of the above equation is

(A) $\operatorname{cosec} x$
(B) $\sin x$
(C) $\cos x$
(D) $\sec x$

17. The solution of the above equation when $x = \dfrac{\pi}{2}$, $y = 1$ is

(A) $y \operatorname{cosec} x - x = \dfrac{\pi+2}{2}$

(B) $\dfrac{\operatorname{cosec} x}{y} + x = \dfrac{\pi+2}{2}$

(C) $y \operatorname{cosec} x + x = \dfrac{\pi-2}{2}$

(D) $\dfrac{\operatorname{cosec} x}{y} - x = \dfrac{\pi+2}{2}$

18. The general solution of $x\dfrac{dy}{dx} + y = y^2 \log x$ is

(A) $y = \log x + cx$
(B) $y = x + c\log x$
(C) $\dfrac{1}{y} = 1 + cx$
(D) $\dfrac{1}{y} = 1 + cx + \log x$

19. Consider the differential equation $\cos y\dfrac{dy}{dx} + 3x^2 \sin y = x^2$.

To convert the above equation into linear form the substituted variable is

(A) $z = \cos y$
(B) $z = \operatorname{cosec} y$
(C) $z = \sin y$
(D) $z = \sec y$

20. The solution of $(aD^2 + bD + c)\,y = 0$ whose auxiliary equation has its discriminant as zero and has 5 as one of its roots is

(A) $y = c_1e^{5x} + c_2e^{5x}$
(B) $y = c_1e^x + c_2e^x$
(C) $y = (c_1 + c_2x)e^{5x}$
(D) $y = c_1 + c_2x$

21. Find the general solution of $\dfrac{d^3y}{dx^3} + 3\dfrac{d^2y}{dx^2} - 4y = 0$.

(A) $y = (c_1 + c_2x)e^x + c_3e^{-2x}$
(B) $y = (c_1 + c_2x)e^{-2x} + c_3e^x$
(C) $y = (c_1 + c_2x)e^{2x} + c_3e^{-x}$
(D) $y = (c_1 + c_2x)e^{-x} + c_3e^{2x}$

22. The general solution of the differential equation $\frac{d^4x}{dt^4}+13\frac{d^2x}{dt^2}+36x=0$ is ______.

(A) $x = (c_1 + c_2t) \cos 2t + (c_3 + c_4t) \sin 3t$

(B) $x = c_1e^{2t} + c_2e^{-2t} + c_3e^{3t} + c_4e^{-3t}$

(C) $x = (c_1 + c_2t)\, e^{2t} + (c_3 + c_4t)\, e^{3t}$

(D) $x = c_1 \cos 2t + c_2 \sin 2t + c_3\cos 3t + c_4 \sin 3t$

23. The particular integral of $(D^2 - 4D + 3)y = e^{3x}$ is

(A) $\frac{xe^{3x}}{2}$ (B) e^{3x}

(C) $\frac{1}{2}e^{3x}$ (D) xe^{2x}

24. The particular integral of $(D^3 - 4D^2)y = 6$ is

(A) x^2 (B) $\frac{3}{4}x^2$

(C) $-\frac{3}{4}x^2$ (D) $\frac{-x^2}{4}$

25. The particular of integral of $(D^2 + 3D + 2)y = \cos 2x$ is

(A) $3 \sin 2x - \cos 2x$ (B) $\frac{3\sin 2x-\cos 2x}{20}$

(C) $\frac{\cos 2x-3\sin 2x}{10}$ (D) $\frac{\cos x-\sin 2x}{40}$

26. The particular integral of $(D^2 - D)\, y = x^2 - 2x + 4$ is

(A) $x^3 - 8x + 4$ (B) $-x^3 + 4x - 4$

(C) $\frac{x^3}{3} + 8x - 4$ (D) $\frac{-x^3}{3} - 4x - 4$

27. If $y_1 = e^{2x}$ and $y_2 = xe^{2x}$ are two solutions of a second order linear differential equation, then the Wronskian W of y_1 and y_2 is ______.

(A) e^{4x} (B) xe^{4x}

(C) $2e^{4x}$ (D) $2xe^{4x}$

28. The complementary function of the differential equation $\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y = 5e^{3x}$ is $y_c = c_1e^{-2x} + c_2e^{-3x}$ using the method of variation of parameters, its particular is found to be $y_p = A(x)\, e^{-2x} + B(x)\, e^{-3x}$. Then $A(x) =$

(A) $5e^{5x}$ (B) e^{5x}

(C) $\frac{1}{5}e^{-5x}$ (D) e^{-5x}

29. The solution of the DE $(D^2 + 1)y = 0$ given $x = 0$, $y = 2$ and $x=\frac{\pi}{2}$, $y=-2$ is

(A) $y = \sin x - \cos x$ (B) $y = 2(\cos x - \sin x)$

(C) $y = 2\cos x \sin x$ (D) $y = 2(e^x + e^{-x})$

30. Solve the equation

$$3x^2\frac{d^2}{dx^2}+x\frac{dy}{dx}-y=x^2.$$

(A) $y = C_1x^{-3} + C_2x^{-1} + x^3/7$

(B) $y = C_1x^3 + C_2x + x^2/7$

(C) $y = C_1x^{1/3} + C_2x^{-1} + x/7$

(D) $y = C_1x^{-1/3} + C_2x + x^2/7$

31. Laplace transform of $2\sin^2 2t =$ ______.

(A) $\frac{1}{s}+\frac{1}{s^2+16}$ (B) $\frac{s}{s^2+16}$

(C) $\frac{1}{s}-\frac{1}{s^2+16}$ (D) $\frac{1}{s}+\frac{1}{s^2+16}$

32. The Laplace transform of $(t + 1)^3$ is ______.

(A) $\frac{6-6s+3s^2-s^3}{s^3}$ (B) $\frac{6+6s+3s^2+s^3}{s}$

(C) $\frac{6(1+s+s^2+s^3)}{s^4}$ (D) $\frac{6+6s+3s^2+s^3}{s^4}$

33. The value of $L\,\{\sinh 3t \cos 3t\}$ ______.

(A) $\frac{s^2+18}{s^4+81}$ (B) $\frac{s^2+18}{s^4+324}$

(C) $\frac{3(s^2-18)}{s^4+324}$ (D) $\frac{3(s^2+18)}{s^4-324}$

34. The value of $L\{t^2\cos 3t\}$ is ______.

(A) $\frac{s^2-27}{(s^2+9)^4}$ (B) $\frac{2s(s^2-27)}{(s^2+9)^3}$

(C) $\frac{s^3-27}{(s^2+9)^4}$ (D) $\frac{s(s^3-27)}{(s^2+9)^3}$

35. Laplace transform of $\frac{\cos 4t}{t}$ ______.

(A) $\frac{64}{s^2+16}$ (B) $\frac{16}{(s^2+16)^2}$

(C) $\frac{8}{(s^2+16)^2}$ (D) Does not exist

36. The Laplace transform of the function defined by $f(t)=\begin{matrix} 2, & 0<t<1 \\ 1, & t>1 \end{matrix}$ is ______.

(A) $\frac{2-e^{-s}}{s}$ (B) $\frac{2-e^{-s}}{2}$

(C) $\frac{2+e^{-s}}{s}$ (D) $\frac{2+e^{-s}}{2}$

37. If $f(t) = t$; $0 < t < 3$ and $f(t+3) = f(t)$, then $L\{f(t)\}$ is

(A) $\dfrac{1}{s^2(1-e^{-3s})}[1+e^{3s}+e^{-3s}]$

(B) $\dfrac{1}{s(1-e^{-3s})}[1-e^{-3s}+se^{-3s}]$

(C) $\dfrac{1}{s^2(1-e^{-3s})}[1-e^{-3s}-3se^{-3s}]$

(D) $\dfrac{1}{s(1-e^{-3s})}[1-e^{-3s}-se^{-3s}]$

38. The value of $\int_0^\infty \dfrac{e^{-4t}-e^{-8t}}{t} dts$ is ______.

(A) log 2 (B) log 4
(C) log 8 (D) log 6

39. $\int_0^\infty t \cdot e^{-2t} \sin 3t dt$ = ______.

(A) $\dfrac{5}{169}$ (B) $\dfrac{10}{169}$

(C) $\dfrac{6}{169}$ (D) $\dfrac{12}{169}$

40. The inverse Laplace transform of $\left(\dfrac{1}{S^{9/2}}\right)$ is ______.

(A) $\dfrac{16}{105}\sqrt{\dfrac{t^7}{\pi}}$ (B) $\dfrac{8}{15}\sqrt{\dfrac{t^5}{\pi}}$

(C) $\dfrac{16}{35}\sqrt{\dfrac{t}{\pi}}$ (D) $\dfrac{8}{105}\sqrt{\dfrac{t^7}{\pi}}$

41. The value of $L^{-1}\left\{\dfrac{8}{3s-2}-\dfrac{4+2s}{16s^2-25}\right\}$ is ______.

(A) $\dfrac{8}{3}\sin h\dfrac{5t}{4}-\cos h\dfrac{5t}{4}$

(B) $\dfrac{8}{3}e^{2/3t}-\sin h\dfrac{5t}{4}-\cos h\dfrac{5t}{4}$

(C) $\dfrac{8}{3}e^{2/3t}-\dfrac{1}{5}\sin h\dfrac{5t}{4}-\dfrac{1}{8}\cos h\dfrac{5t}{4}$

(D) None of these

42. The inverse Laplace transform of $\dfrac{1}{s^2-8s+20}$ is ______.

(A) $\dfrac{e^{2t}}{2}\sin 2t$ (B) $\dfrac{e^{4t}}{2}\sin 2t$

(C) $e^{4t}\sin 2t$ (D) $e^{4t}\sin 4t$

43. The inverse Laplace transform of $\dfrac{1}{s^3(s^2+4)}$ is

(A) $\dfrac{1}{16}(2t^2+\cos 2t-1)$ (B) $2t^2-\cos 2t-1$

(C) $\dfrac{1}{16}(1-\cos 2t-4t^2)$ (D) $\dfrac{1}{8}(2+\cos 2t-4t^2)$

44. The inverse Laplace transform of $\dfrac{e^{-3s}}{(s-4)^5}$ when expressed in terms of Heaviside unit step function is ______.

(A) $\dfrac{1}{16}t^4e^{4(t-3)}H(t-3)$

(B) $\dfrac{1}{24}(t-3)^4e^{4t}H(t-3)$

(C) $\dfrac{1}{24}(t-3)^4e^{4(t-3)}H(t-3)$

(D) $\dfrac{1}{24}t^4e^{4t}H(t-3)$

45. The value of $L^{-1}\left\{\log\dfrac{s-4}{s+3}\right\}$ is

(A) $e^{4t}-e^{-3t}$ (B) $\dfrac{1}{t}(e^{4t}-e^{-3t})$

(C) $\dfrac{1}{t}(e^{-3t}-e^{4t})$ (D) $t(e^{-3t}-e^{4t})$

46. Using convolution theorem, the value of $\int_0^t \sin x \cos(t-x)dx$ is ______.

(A) $\dfrac{1}{2}\cos t$ (B) $\dfrac{t}{2}\sin t$

(C) $t\sin\dfrac{t}{2}$ (D) $t\cos\dfrac{t}{2}$

47. Solve $(D^4-16)y = 1$, $y = y' = y'' = y''' = 0$.

(A) $y = \dfrac{-1}{16}-[\cos h\, 2t+\sin h\, 2t]$

(B) $y = \dfrac{1}{32}(1-\cos h\, 2t+\cos 2t)$

(C) $y = \dfrac{-1}{16}+\dfrac{1}{32}(\cos h\, 2t-\sin t)$

(D) $y = \dfrac{-1}{16}+\dfrac{1}{32}(\cos h\, 2t+\cos 2t)$

48. Solve $(D^2 - 5D + 6)y = 1 - e^{-2t}$, $y = 1$, $y' = 0$ when $t = 0$.

(A) $y = \frac{1}{20}e^{-2t} + \frac{11}{4}e^{2t} - \frac{59}{30}e^{3t}$

(B) $y = \frac{1}{6} - \frac{1}{20}e^{-2t} + \frac{11}{4}e^{2t} - \frac{28}{15}e^{3t}$

(C) $y = \frac{1}{6} - \frac{1}{20}e^{2t} + \frac{11}{4}e^{-2t} + \frac{59}{30}e^{3t}$

(D) $y = \frac{1}{6} - \frac{1}{20}e^{2t} - \frac{11}{4}e^{-2t} + \frac{59}{30}e^{3t}$

PREVIOUS YEARS' QUESTIONS

1. The solution for the differential equation $\frac{dy}{dx} = x^2y$ with the condition that $y = 1$ at $x = 0$ is **[GATE, 2007]**

(A) $y = e^{\frac{1}{2x}}$ (B) $\ln(y) = \frac{x^3}{3} + 4$

(C) $\ln(y) = \frac{x^2}{2}$ (D) $y = e^{\frac{x^3}{3}}$

2. The general solution of $\frac{d^2y}{dx^2} + y = 0$ is **[GATE, 2008]**

(A) $y = P\cos x + Q\sin x$

(B) $y = P\cos x$

(C) $y = P\sin x$

(D) $y = P\sin^2 x$

3. Solution of $\frac{dy}{dx} = -\frac{x}{y}$ at $x = 1$ and $y = \sqrt{3}$ is **[GATE, 2008]**

(A) $x - y^2 = -2$ (B) $x + y^2 = 4$

(C) $x^2 - y^2 = -2$ (D) $x^2 + y^2 = 4$

4. Solution of the differential equation $3y\frac{dy}{dx} + 2x = 0$ represents a family of **[GATE, 2009]**

(A) ellipses (B) circles

(C) parabolas (D) hyperbolas

5. Laplace transform for the function $f(x) = \cosh(ax)$ is **[GATE, 2009]**

(A) $\frac{a}{s^2 - a^2}$ (B) $\frac{s}{s^2 - a^2}$

(C) $\frac{a}{s^2 + a^2}$ (D) $\frac{s}{s^2 + a^2}$

6. The order and degree of the differential equation $\frac{d^3y}{dx^3} + 4\sqrt{\left(\frac{dy}{dx}\right)^3} + y^2 = 0$ are respectively **[GATE, 2010]**

(A) 3 and 2 (B) 2 and 3

(C) 3 and 3 (D) 3 and 1

7. The solution to the ordinary differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$ is **[GATE, 2010]**

(A) $y = c_1e^{x} + c_2e^{-2x}$

(B) $y = c_1e^{3x} + c_2e^{2x}$

(C) $y = c_1e^{-3x} + c_2e^{2x}$

(D) $y = c_1e^{-3x} + c_2e^{-2x}$

8. The solution of the differential equation $\frac{dy}{dx} + \frac{y}{x} = x$, with the condition that $y = 1$ at $x = 1$, is **[GATE, 2011]**

(A) $y = \frac{2}{3x^2} + \frac{x}{3}$ (B) $y = \frac{x}{2} + \frac{1}{2x}$

(C) $y = \frac{2}{3} + \frac{x}{3}$ (D) $y = \frac{2}{3x} + \frac{x^2}{3}$

9. The solution of the ordinary differential equation $\frac{dy}{dx} + 2y = 0$ for the boundary condition, $y = 5$ at $x = 1$ is **[GATE, 2012]**

(A) $y = e^{-2x}$

(B) $y = 2e^{-2x}$

(C) $y = 10.95e^{-2x}$

(D) $y = 36.95e^{-2x}$

10. The integrating factor for the differential equation $\frac{dp}{dt} + k_2P = k_1L_0e^{kt}$ is **[GATE, 2014]**

(A) e^{-k_1t} (B) e^{-k_2t}

(C) e^{-k_1t} (D) e^{k_2t}

11. Consider the following differential equation:

$$x(ydx + xdy)\cos\frac{y}{x} = y(xdy - ydx)\sin\frac{y}{x}$$

Which of the following is the solution of the above equation (c is an arbitrary constant)? **[GATE, 2015]**

(A) $\frac{x}{y}\cos\frac{y}{x} = C$ (B) $\frac{x}{y}\sin\frac{y}{x} = C$

(C) $xy\cos\frac{y}{x} = C$ (D) $xy\sin\frac{y}{x} = C$

12. Consider the following second order linear differential equation $\frac{d^2y}{dx^2} = -12x^2 + 24x - 20$.

The boundary conditions are: at $x = 0$, $y = 5$ and at $x = 2$, $y = 21$

The value of y at $x = 1$ is ________. **[GATE, 2015]**

13. The respective expressions for complimentary function and particular integral part of the solution of the differential equation are **[GATE, 2016]**

(A) $\left[c_1 + c_2x + c_3\sin\sqrt{3x} + c_4\cos\sqrt{3x}\right]$ and $\left[3x^4 - 12x^2 + c\right]$

(B) $\left[c_2x + c_3\sin\sqrt{3x} + c_4\cos\sqrt{3x}\right]$ and $\left[5x^4 - 12x^2 + c\right]$

(C) $\left[c_1 + c_3\sin\sqrt{3x} + c_4\cos\sqrt{3x}\right]$ and $\left[3x^4 - 12x^2 + c\right]$

(D) $\left[c_1 + c_2x + c_3\sin\sqrt{3x} + c_4\cos\sqrt{3x}\right]$ and $\left[5x^4 - 12x^2 + c\right]$

Answer Keys

Exercises

1. D	**2.** A	**3.** B	**4.** A	**5.** B	**6.** D	**7.** C	**8.** B	**9.** C	**10.** B
11. C	**12.** B	**13.** B	**14.** B	**15.** B	**16.** A	**17.** B	**18.** D	**19.** C	**20.** C
21. B	**22.** D	**23.** A	**24.** C	**25.** B	**26.** D	**27.** A	**28.** B	**29.** B	**30.** D
31. C	**32.** D	**33.** C	**34.** B	**35.** D	**36.** A	**37.** C	**38.** A	**39.** D	**40.** A
41. C	**42.** B	**43.** A	**44.** C	**45.** C	**46.** B	**47.** D	**48.** B		

Previous Years' Questions

1. D	**2.** A	**3.** D	**4.** A	**5.** B	**6.** A	**7.** C	**8.** D	**9.** D	**10.** D
11. C	**12.** 18	**13.** A							