JEE Type Solved Examples : Single Option Correct Type Questions

- This section contains **10 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Ex. 1 *Number of words of 4 letters that can be formed with the letters of the word IIT JEE, is*

(a) 42 (b) 82 (c) 102 (d) 142

Sol. (*c*) There are 6 letters I, I, E, E, T, J

The following cases arise:

***Case* I** All letters are different

$$^4P_4 = 4! = 24$$

***Case* II** Two alike and two different

$$^2C_1 \times {}^3C_2 \times \frac{4!}{2!} = 72$$

***Case* III** Two alike of one kind and two alike of another kind

$$^2C_2 \times \frac{4!}{2!\,2!} = 6$$

Hence, number of words $= 24 + 72 + 6 = 102$

Aliter

Number of words = Coefficient of x^4 in

$$4!\left(1 + \frac{x}{1!} + \frac{x^2}{2!}\right)^2 (1 + x)^2$$

$= $ Coefficient of x^4 in $6[(1 + x)^2 + 1]^2(1 + x)^2$

$= $ Coefficient of x^4 in $6[(1 + x)^6 + 2(1 + x)^4 + (1 + x)^2]$

$= 6[{}^6C_4 + 2 \cdot {}^4C_4 + 0] = 6(15 + 2) = 102$

Ex. 2 *Let y be element of the set $A = \{1, 2, 3, 5, 6, 10, 15, 30\}$ and x_1, x_2, x_3 be integers such that $x_1 x_2 x_3 = y$, the number of positive integral solutions of $x_1 x_2 x_3 = y$, is*

(a) 27 (b) 64 (c) 81 (d) 256

Sol. (*b*) Number of solutions of the given equations is the same as the number of solutions of the equation

$$x_1\, x_2\, x_3\, x_4 = 30 = 2 \times 3 \times 5$$

Here, x_4 is infact a dummy variable.

If $x_1\, x_2\, x_3 = 15$, then $x_4 = 2$ and if $x_1 x_2\, x_3 = 5$, then $x_4 = 6$, etc.

Thus, $\quad x_1\, x_2\, x_3\, x_4 = 2 \times 3 \times 5$

Each of 2, 3 and 5 will be factor of exactly one of x_1, x_2, x_3, x_4 in 4 ways.

$\therefore$ Required number $= 4^3 = 64$

Ex. 3 *The number of positive integer solutions of $a + b + c = 60$, where a is a factor of b and c, is*

(a) 184 (b) 200 (c) 144 (d) 270

Sol. (*c*) $\because$ a is a factor of b and $c \Rightarrow a$ divides 60

$\therefore \quad a = 1, 2, 3, 4, 5, 6, 10, 12, 15, 30 \quad [\because a \neq 60]$

and $b = ma$, $c = na$, when $m, n \geq 1$

$\because \quad a + b + c = 60$

$\Rightarrow \quad a + ma + na = 60 \quad \Rightarrow \quad m + n = \left(\frac{60}{a} - 1\right)$

$\therefore$ Number of solutions $= {}^{\frac{60}{a} - 1 - 1}C_{2-1} = \left(\frac{60}{a} - 2\right)$

Hence, total number of solutions for all values of a

$= 58 + 28 + 18 + 13 + 10 + 8 + 4 + 3 + 2 + 0 = 144$

Ex. 4 *The number of times the digit 3 will be written when listing the integers from 1 to 1000, is*

(a) 269 (b) 271 (c) 300 (d) 302

Sol. (*c*) Since, 3 does not occur in 1000. So, we have to count the number of times 3 occurs, when we list the integers from 1 to 999.

Any number between 1 and 999 is of the form xyz, where $0 \leq x, y, z \leq 9$.

Let us first count the number in which 3 occurs exactly once. Since, 3 can occur at one place in ${}^3C_1$ ways, there are ${}^3C_1 \times 9 \times 9 = 243$ such numbers. Next 3 can occur in exactly two places in ${}^3C_2 \times 9 = 27$ such numbers. Lastly, 3 can occur in all three digits in one number only. Hence, the number of times, 3 occurs is $1 \times 243 + 2 \times 27 + 3 \times 1 = 300$

Ex. 5 *Number of points having position vector $a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$, where $a, b, c \in \{1, 2, 3, 4, 5\}$ such that $2^a + 3^b + 5^c$ is divisible by 4, is*

(a) 70 (b) 140
(c) 210 (d) 280

Sol. (*a*) $\because 2^a + 3^b + 5^c = 2^a + (4 - 1)^b + (4 + 1)^c$

$$= 2^a + 4k + (-1)^b + (1)^c$$

$$= 2^a + 4k + (-1)^b + 1$$

I. $a = 1$, $b = $ even, $c = $ any number

II. $a \neq 1$, $b = $ odd, $c = $ any number

$\therefore$ Required number of ways $= 1 \times 2 \times 5 + 4 \times 3 \times 5 = 70$

[$\because$ even numbers = 2, 4; odd numbers = 1, 3, 5 and any numbers = 1, 2, 3, 4, 5]

Ex. 6 *Number of positive unequal integral solutions of the equation $x + y + z = 12$ is*

(a) 21 (b) 42 (c) 63 (d) 84

Sol. (*b*) We have, $x + y + z = 12$ ...(i)

Assume $x < y < z$. Here, $x, y, z \geq 1$

$\therefore$ Solutions of Eq. (i) are
(1, 2, 9), (1, 3, 8), (1, 4, 7), (1, 5, 6), (2, 3, 7), (2, 4, 6) and (3, 4, 5).

Number of positive integral solutions of Eq. (i) = 7 but x, y, z can be arranged in 3! = 6

Hence, required number of solutions = $7 \times 6 = 42$

Aliter

Let $x = \alpha, y - x = \beta, z - y = \gamma$

$\therefore\ x = \alpha, y = \alpha + \beta, z = \alpha + \beta + \gamma$

From Eq. (i), $3\alpha + 2\beta + \gamma = 12; \alpha, \beta, \gamma \geq 1$

$\therefore$ Number of positive integral solutions of Eq. (i)

= Coefficient of λ^{12} in

$$(\lambda^3 + \lambda^6 + \lambda^9 + \lambda^{12} + \ldots)(\lambda^2 + \lambda^4 + \lambda^6 + \lambda^8 + \lambda^{10} + \lambda^{12} + \ldots)(\lambda + \lambda^2 + \lambda^3 + \ldots + \lambda^{12})$$

= Coefficient of λ^6 in $(1 + \lambda^3 + \lambda^6)(1 + \lambda^2 + \lambda^4 + \lambda^6)(1 + \lambda + \lambda^2 + \lambda^3 + \lambda^4 + \lambda^5 + \lambda^6)$

= Coefficient of λ^6 in $(1 + \lambda^2 + \lambda^4 + \lambda^6 + \lambda^3 + \lambda^5 + \lambda^6) \times (1 + \lambda + \lambda^2 + \lambda^3 + \lambda^4 + \lambda^5 + \lambda^6)$

$= 1 + 1 + 1 + 1 + 1 + 1 + 1 = 7$

but x, y, z can be arranged in 3! = 6

Hence, required number of solutions = $7 \times 6 = 42$

Ex. 7 *12 boys and 2 girls are to be seated in a row such that there are atleast 3 boys between the 2 girls. The number of ways this can be done is $\lambda \times 12!$, the value of λ is*

(a) 55 (b) 110 (c) 20 (d) 45

Sol. (*b*) Let P = Number of ways, 12 boys and 2 girls are seated in a row

$= 14! = 14 \times 13 \times 12! = 182 \times 12!$

P_1 = Number of ways, the girls can sit together

$= (14 - 2 + 1) \times 2! \times 12! = 26 \times 12!$

P_2 = Number of ways, one boy sits between the girls

$= (14 - 3 + 1) \times 2! \times 12! = 24 \times 12!$

P_3 = Number of ways, two boys sit between the girls

$= (14 - 4 + 1) \times 2! \times 12! = 22 \times 12!$

$\therefore$ Required number of ways $= (182 - 26 - 24 - 22) \times 12!$

$= 110 \times 12! = \lambda \times 12!$ [given]

$\therefore\ \lambda = 110$

Ex. 8 *A is a set containing n elements. A subset P of A is chosen. The set A is reconstructed by replacing the elements of P. A subset Q of A is again chosen, the number of ways of choosing so that $(P \cup Q)$ is a proper subset of A, is*

(a) 3^n (b) 4^n (c) $4^n - 2^n$ (d) $4^n - 3^n$

Sol. (*d*) Let $A = \{a_1, a_2, a_3, \ldots, a_n\}$

a general element of A must satisfy one of the following possibilities.

[here, general element be $a_i (1 \leq i \leq n)$]

(i) $a_i \in P, a_i \in Q$ (ii) $a_i \in P, a_i \notin Q$

(iii) $a_i \notin P, a_i \in Q$ (iv) $a_i \notin P, a_i \notin Q$

Therefore, for one element a_i of A, we have four choices (i), (ii), (iii) and (iv).

$\therefore$ Total number of cases for all elements = 4^n

and for one element a_i of A, such that $a_i \in P \cup Q$, we have three choices (i), (ii) and (iii).

$\therefore$ Number of cases for all elements belong to $P \cup Q = 3^n$

Hence, number of ways in which atleast one element of A does not belong to

$$P \cup Q = 4^n - 3^n.$$

Ex. 9 *Let N be a natural number. If its first digit (from the left) is deleted, it gets reduced to $\frac{N}{29}$. The sum of all the digits of N is*

(a) 14 (b) 17
(c) 23 (d) 29

Sol. (*a*) Let $N = a_n\ a_{n-1}\ a_{n-2} \ldots a_3\ a_2\ a_1\ a_0$

$= a_0 + 10a_1 + 10^2 a_2 + \ldots + 10^{n-1} a_{n-1} + 10^n a_n$ ...(i)

Then, $\frac{N}{29} = a_{n-1}\ a_{n-2}\ a_{n-3} \ldots a_3\ a_2\ a_1\ a_0$

$= a_0 + 10a_1 + 10^2 a_2 + \ldots + 10^{n-2} a_{n-2} + 10^{n-1} a_{n-1}$

or $N = 29(a_0 + 10a_1 + 10^2 a_2 + \ldots + 10^{n-2} a_{n-2} + 10^{n-1} a_{n-1})$ ...(ii)

From Eqs. (i) and (ii), we get

$10^n \cdot a_n = 28(a_0 + 10a_1 + 10^2 a_2 + \ldots + 10^{n-1} a_{n-1})$

$\Rightarrow$ 28 divides $10^n \cdot a_n \Rightarrow a_n = 7, n \geq 2 \Rightarrow 5^2 = a_0 + 10a_1$

The required N is 725 or 7250 or 72500, etc.

$\therefore$ The sum of the digits is 14.

Ex. 10 *If the number of ways of selecting n cards out of unlimited number of cards bearing the number 0, 9, 3, so that they cannot be used to write the number 903 is 93, then n is equal to*

(a) 3 (b) 4
(c) 5 (d) 6

Sol. (*c*) We cannot write 903.

If in the selection of n cards, we get either

(9 or 3), (9 or 0), (0 or 3), (only 0), (only 3) or (only 9).

For (9 or 3) can be selected $= 2 \times 2 \times 2 \times \ldots \times n$ factors $= 2^n$

Similarly, (9 or 0) or (0 or 3) can be selected $= 2^n$

In the above selection (only 0) or (only 3) or (only 9) is repeated twice.

$\therefore$ Total ways $= 2^n + 2^n + 2^n - 3 = 93$

$\Rightarrow\ 3 \cdot 2^n = 96 \Rightarrow 2^n = 32 = 2^5$

$\therefore\ n = 5$

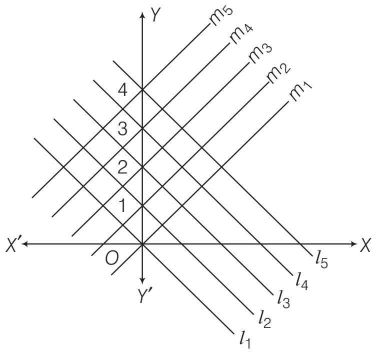
JEE Type Solved Examples :
More than One Correct Option Type Questions

- This section contains **5 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of **which more than one** may be correct.

Ex. 11 *In a plane, there are two families of lines $y = x + r$, $y = -x + r$, where $r \in \{0, 1, 2, 3, 4\}$. The number of squares of diagonals of the length 2 formed by the lines is*

(a) 9 (b) 16 (c) $\frac{3}{2} \cdot {}^4C_2$ (d) $5_{C_2} + {}^3P_2$

Sol. (*a*, *c*) There are two sets of five parallel lines at equal distances. Clearly, lines like l_1, l_3, m_1 and m_3 form a square whose diagonal's length is 2.



$\therefore$ The number of required squares $= 3 \times 3 = 9 = \frac{3}{2} \cdot {}^4C_2$

[$\because$ choices are $(l_1, l_2), (l_2, l_4)$ and (l_3, l_5) for one set, etc.]

Ex. 12 *Number of ways in which three numbers in AP can be selected from $1, 2, 3, \ldots, n$, is*

(a) $\left(\frac{n-1}{2}\right)^2$, if n is even (b) $\frac{n(n-2)}{4}$, if n is even

(c) $\frac{(n-1)^2}{4}$, if n is odd (d) $\frac{n(n+1)}{2}$, if n is odd

Sol. (*b*, *c*) If a, b, c are in AP, then $a + c = 2b$

a and c both are odd or both are even.

***Case* I** If ***n* is even**

Let $n = 2m$ in which m are even and m are odd numbers.

$\therefore$ Number of ways $= {}^mC_2 + {}^mC_2 = 2 \cdot {}^mC_2 = 2 \cdot \frac{m(m-1)}{2}$

$$= \frac{n}{2}\left(\frac{n}{2} - 1\right) = \frac{n(n-2)}{4} \qquad [\because n = 2m]$$

***Case* II** If ***n* is odd**

Let $n = 2m + 1$ in which m are even and $m + 1$ are odd numbers.

$\therefore$ Number of ways $= {}^mC_2 + {}^{m+1}C_2$

$$= \frac{m(m-1)}{2} + \frac{(m+1)m}{2} = m^2 = \frac{(n-1)^2}{4} \qquad [\because n = 2m + 1]$$

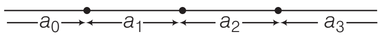
Ex. 13 *If n objects are arranged in a row, then number of ways of selecting three of these objects so that no two of them are next to each other, is*

(a) ${}^{n-2}C_3$ (b) ${}^{n-3}C_3 + {}^{n-3}C_2$

(c) $\frac{(n-2)(n-3)(n-4)}{6}$ (d) ${}^nC_2$

Sol. (*a*, *b*, *c*) Let a_0 be the number of objects to the left of the first object chosen, a_1 be the number of objects between the first and the second, a_2 be the number of objects between the second and the third and a_3 be the number of objects to the right of the third object. Then,

$$a_0, a_3 \geq 0 \text{ and } a_1, a_2 \geq 1$$



also $a_0 + a_1 + a_2 + a_3 = n - 3$

Let $a = a_0 + 1$, $b = a_3 + 1$, then $a \geq 1$, $b \geq 1$ such that $a + a_1 + a_2 + b = n - 1$

The total number of positive integral solutions of this equation is ${}^{n-1-1}C_{4-1} = {}^{n-2}C_3 = {}^{n-3}C_3 + {}^{n-3}C_2$

$$= \frac{(n-2)(n-3)(n-4)}{1 \cdot 2 \cdot 3}$$

Ex. 14 *Given that the divisors of $n = 3^p \cdot 5^q \cdot 7^r$ are of the form $4\lambda + 1$, $\lambda \geq 0$. Then,*

(a) $p + r$ is always even (b) $p + q + r$ is even or odd

(c) q can be any integer (d) if p is even, then r is odd

Sol. (*a*, *b*, *c*)

$\because$ $3^p = (4-1)^p = 4\lambda_1 + (-1)^p$,

$5^q = (4+1)^q = 4\lambda_2 + 1$

and $7^r = (8-1)^r = 8\lambda_3 + (-1)^r$

Hence, both p and r must be odd or both must be even. Thus, $p + r$ is always even. Also, $p + q + r$ can be odd or even.

Ex. 15 *Number of ways in which 15 identical coins can be put into 6 different bags*

(a) is coefficient of x^{15} in $x^6(1 + x + x^2 + \ldots \infty)^6$, if no bag remains empty

(b) is coefficient of x^{15} in $(1 - x)^{-6}$

(c) is same as number of the integral solutions of $a + b + c + d + e + f = 15$

(d) is same as number of non-negative integral solutions of $\sum_{i=1}^{6} x_i = 15$

Sol. (*a*, *b*, *d*) Let bags be x_1, x_2, x_3, x_4, x_5 and x_6, then $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 15$.

$\therefore$ For no bags remains empty, number of ways

$= $ Coefficient of x^{15} in $(x^1 + x^2 + x^3 + \ldots \infty)^6$

$= $ Coefficient of x^{15} in $x^6(1 + x + x^2 + \ldots \infty)^6$

$= $ Coefficient of x^9 in $(1 - x)^{-6}$

In option (c), it is not mentioned that solution is positive integral.

JEE Type Solved Examples :
Passage Based Questions

- This section contains **3 solved passages** based upon each of the passage **3 multiple choice** examples have to be answered. Each of these examples has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Passage I
(Ex. Nos. 16 to 18)

All the letters of the word 'AGAIN' be arranged and the words thus formed are known as 'Simple Words'. Further two new types of words are defined as follows:

(i) ***Smart word:*** *All the letters of the word 'AGAIN' are being used, but vowels can be repeated as many times as we need.*

(ii) ***Dull word:*** *All the letters of the word 'AGAIN' are being used, but consonants can be repeated as many times as we need.*

16. If a vowel appears in between two similar letters, the number of simple words is

(a) 12 (b) 6 (c) 36 (d) 14

17. Number of 7 letter smart words is

(a) 1500 (b) 1050 (c) 1005 (d) 150

18. Number of 7 letter dull words in which no two vowels are together, is

(a) 402 (b) 420 (c) 840 (d) 42

Sol.

16. (*b*)

N	A-I-A	G

$\therefore$ Required number of simple words $= 3! = 6$

17. (*b*)

A	G	A	I	N	A	A
					I	I
					A	I

$\therefore$ Number of 7 letter smart words

$$= \frac{7!}{4!} + \frac{7!}{2!\,3!} + \frac{7!}{3!\,2!} = 210 + 420 + 420 = 1050$$

18. (*b*) Now, 3 vowels A, I, A are to be placed in the five available places.

$$\begin{cases} \times \text{N} \times \text{G} \times \text{N} \times \text{N} \times \\ \quad \text{OR} \\ \times \text{N} \times \text{G} \times \text{G} \times \text{G} \times \\ \quad \text{OR} \\ \times \text{N} \times \text{G} \times \text{G} \times \text{N} \times \end{cases}$$

Hence, required number of ways

$$= {}^5C_3 \times \frac{3!}{2!} \times \left\{\frac{4!}{3!} + \frac{4!}{3!} + \frac{4!}{2!\,2!}\right\}$$

$$= 30(4 + 4 + 6) = 420$$

Passage II (Ex. Nos. 19 to 21)

Consider a polygon of sides 'n' which satisfies the equation $3 \cdot {}^nP_4 = {}^{n-1}P_5$.

19. Rajdhani express travelling from Delhi to Mumbai has n stations enroute. Number of ways in which a train can be stopped at 3 stations if no two of the stopping stations are consecutive, is

(a) 20 (b) 35 (c) 56 (d) 84

20. Number of quadrilaterals that can be formed using the vertices of a polygon of sides 'n' if exactly 1 side of the quadrilateral is common with side of the n-gon, is

(a) 96 (b) 100 (c) 150 (d) 156

21. Number of quadrilaterals that can be made using the vertices of the polygon of sides 'n' if exactly two adjacent sides of the quadrilateral are common to the sides of the n-gon, is

(a) 50 (b) 60 (c) 70 (d) 80

Sol. $\because \quad 3 \cdot {}^nP_4 = {}^{n-1}P_5$

It is clear that $n \geq 6$.

$\therefore \quad 3 \cdot n(n-1)(n-2)(n-3) = (n-1)(n-2)(n-3)(n-4)(n-5)$

$\Rightarrow \quad (n-1)(n-2)(n-3)(n^2 - 12n + 20) = 0$

$\Rightarrow \quad (n-1)(n-2)(n-3)(n-10)(n-2) = 0$

$\therefore \quad n = 10, n \neq 1, 2, 3 \qquad [\because n \geq 6]$

$\Rightarrow \quad n = 10$

19. (*d*) Let a_0 be the number of stations to the left of the station I chosen, a_1 be the number of stations between the station I and station II, a_2 be the number of stations between the station II and station III and a_3 be the number of stations to the right of the third station. Then,

$$a_0, a_3 \geq 0 \text{ and } a_1, a_2 \geq 1$$

Also, $a_0 + a_1 + a_2 + a_3 = n + 1 - 3$

Let $a = a_0 + 1, b = a_3 + 1$, then $a, b \geq 1$ such that $a + a_1 + a_2 + b = n$

$\therefore$ Required number of ways $= {}^{n-1}C_{4-1} = {}^9C_3 \quad$ [here, $n = 10$]

$= 84$

20. (*c*) Number of quadrilaterals of which exactly one side is the side of the n-gon

$$= n \times {}^{n-4}C_2 = 10 \times {}^6C_2 = 150 \qquad [\because n = 10]$$

21. (*a*) Number of quadrilaterals of which exactly two adjacent sides of the quadrilateral are common to the sides of the n-gon

$$= n \times {}^{n-5}C_1 = n(n-5) = 10 \times 5 \qquad [\because n = 10]$$

$$= 50$$

Passage III
(Ex. Nos. 22 to 23)

Consider the number $N = 2016$.

22. Number of cyphers at the end of ${}^NC_{N/2}$ is

(a) 0 (b) 1
(c) 2 (d) 3

23. Sum of all even divisors of the number N is

(a) 6552 (b) 6448
(c) 6048 (d) 5733

Sol.

22. (*c*) $\because \quad {}^NC_{N/2} = {}^{2016}C_{1008} = \dfrac{(2016)!}{[(1008)!]^2}$

$$E_5(2016!) = \left[\frac{2016}{5}\right] + \left[\frac{2016}{5^2}\right] + \left[\frac{2016}{5^3}\right] + \left[\frac{2016}{5^4}\right]$$

$$= 403 + 80 + 16 + 3 = 502$$

and $\quad E_5(1008!) = \left[\dfrac{1008}{5}\right] + \left[\dfrac{1008}{5^2}\right] + \left[\dfrac{1008}{5^3}\right] + \left[\dfrac{1008}{5^4}\right]$

$$= 201 + 40 + 8 + 1 = 250$$

Hence, the number of cyphers at the end of ${}^{2016}C_{1008}$

$$= 502 - 250 - 250 = 2$$

23. (*b*) $\because \quad N = 2016 = 2^5 \cdot 3^2 \cdot 7^1$

$\therefore$ Sum of all even divisors of the number N

$$= (2 + 2^2 + 2^3 + 2^4 + 2^5)(1 + 3 + 3^2)(1 + 7^1) = 6448$$

JEE Type Solved Examples :
Single Integer Answer Type Questions

- This section contains **2 examples.** The answer to each example is a **single digit integer** ranging from **0** to **9** (both inclusive).

● ***Ex. 24*** *If* $\dbinom{18}{r-2} + 2\dbinom{18}{r-1} + \dbinom{18}{r} \geq \dbinom{20}{13}$, *then the number of values of r are*

Sol. (7) We have, $\dbinom{18}{r-2} + 2\dbinom{18}{r-1} + \dbinom{18}{r} \geq \dbinom{20}{13}$

It means that ${}^{18}C_{r-2} + 2 \cdot {}^{18}C_{r-1} + {}^{18}C_r \geq {}^{20}C_{13}$

$\Rightarrow \quad ({}^{18}C_{r-2} + {}^{18}C_{r-1}) + ({}^{18}C_{r-1} + {}^{18}C_r) \geq {}^{20}C_7$

$\Rightarrow \quad {}^{19}C_{r-1} + {}^{19}C_r \geq {}^{20}C_7$

$\Rightarrow \quad {}^{20}C_r \geq {}^{20}C_7$

or $\quad {}^{20}C_r \geq {}^{20}C_{13}$

$\Rightarrow \quad 7 \leq r \leq 13$

$\therefore \quad r = 7, 8, 9, 10, 11, 12, 13$

Hence, the number of values of r are 7.

● ***Ex. 25*** *If* λ *be the number of* 3-*digit numbers are of the form xyz with* $x < y$, $z < y$ *and* $x \neq 0$, *the value of* $\dfrac{\lambda}{30}$ *is*

Sol. (8) Since, $x \geq 1$, then $y \geq 2 \qquad [\because x < y]$

If $y = n$, then x takes values form 1 to $n - 1$ and z can take the values from 0 to $n - 1$ (i.e., n values).

Thus, for each values of $y (2 \leq y \leq 9)$, x and z take $n(n-1)$ values.

Hence, the 3-digit numbers are of the form xyz

$$= \sum_{n=2}^{9} n(n-1) = \sum_{n=1}^{9} n(n-1) \qquad [\because \text{at } n = 1, n(n-1) = 0]$$

$$= \sum_{n=1}^{9} n^2 - \sum_{n=1}^{9} n$$

$$= \frac{9(9+1)(18+1)}{6} - \frac{9(9+1)}{2}$$

$$= 285 - 45$$

$$= 240 = \lambda \qquad \text{[given]}$$

$$\therefore \quad \frac{\lambda}{30} = 8$$

JEE Type Solved Examples :
Matching Type Questions

- This section contains **2 examples.** Examples 26 and 27 have four statements (A, B and C) given in **Column I** and four statements (p, q, r and s) in **Column II.** Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II.**

Ex. 26

	Column I		Column II
(A)	The sum of the factors of 8! which are odd and are the form $3\lambda + 2, \lambda \in N$, is	(p)	384
(B)	The number of divisors of $n = 2^7 \cdot 3^5 \cdot 5^3$ which are the form $4\lambda + 1, \lambda \in N$, is	(q)	240
(C)	Total number of divisors of $n = 2^5 \cdot 3^4 \cdot 5^{10} \cdot 7^6$ which are the form $4\lambda + 2, \lambda \geq 1$, is	(r)	11
(D)	Total number of divisors of $n = 3^5 \cdot 5^7 \cdot 7^9$ which are the form $4\lambda + 1, \lambda \geq 0$, is	(s)	40

Sol. (A) $\to$ s; (B) $\to$ r; (C) $\to$ p; (D) $\to$ q

(A) Here, $8! = 2^7 \cdot 3^2 \cdot 5^1 \cdot 7^1$

So, the factors may be 1, 5, 7, 35 of which 5 and 35 are of the form $3\lambda + 2$.

$\therefore$ Sum is 40.

(B) Number of odd numbers $= (5+1)(3+1) = 24$

Required number = 12, but 1 is included.

$\therefore$ Required number of numbers $= 12 - 1 = 11$ of the form $4\lambda + 1$.

(C) Here, $4\lambda + 2 = 2(2\lambda + 1)$

$\therefore$ Total divisors $= 1 \cdot 5 \cdot 11 \cdot 7 - 1 = 384$

[$\because$ one is subtracted because there will be case when selected powers of 3, 5 and 7 are zero]

(D) Here, any positive integer power of 5 will be in the form of $4\lambda + 1$ when even powers of 3 and 7 will be in the form of $4\lambda + 1$ and odd powers of 3 and 7 will be in the form of $4\lambda - 1$.

$\therefore$ Required divisors $= 8(3 \cdot 5 + 3 \cdot 5) = 240$

Ex. 27

	Column I		Column II
(A)	Four dice (six faced) are rolled. The number of possible outcomes in which atleast one die shows 2, is	(p)	210
(B)	Let A be the set of 4-digit numbers $a_1a_2a_3a_4$, where $a_1 > a_2 > a_3 > a_4$. Then, $n(A)$ is equal to	(q)	480
(C)	The total number 3-digit numbers, the sum of whose digits is even, is equal to	(r)	671
(D)	The number of 4-digit numbers that can be formed from the digits 0, 1, 2, 3, 4, 5, 6, 7, so that each number contains digit 1, is	(s)	450

Sol. (A) $\to$ r; (B) $\to$ p; (C) $\to$ s; (D) $\to$ q

(A) The number of possible outcomes with 2 on atleast one die
= The total number of outcomes with 2 on atleast one die
= (The total number of outcomes) – (The number of outcomes in which 2 does not appear on any dice)
$= 6^4 - 5^4 = 1296 - 625 = 671$

(B) Any selection of four digits from the 10 digits 0, 1, 2, 3,..., 9 gives one number. So, the required number of numbers is ${}^{10}C_4$ i.e., 210.

(C) Let the number be $n = pqr$. Since, $p + q + r$ is even, p can be filled in 9 ways and q can be filled in 10 ways.
r can be filled in number of ways depending upon what is the sum of p and q.
If $(p+q)$ is odd, then r can be filled with any one of five odd digits.
If $(p+q)$ is even, then r can be filled with any one of five even digits.
In any case, r can be filled in five ways.
Hence, total number of numbers is $9 \times 10 \times 5 = 450$

(D) After fixing 1 at one position out of 4 places, 3 places can be filled by ${}^7P_3$ ways. But for some numbers whose fourth digit is zero, such type of ways is ${}^6P_2$. Therefore, total number of ways is ${}^7P_3 - {}^6P_2 = 480$

JEE Type Solved Examples :
Statement I and II Type Questions

- **Directions** Example numbers 28 and 29 are Assertion-Reason type examples. Each of these examples contains two statements:
  Statement-1 (Assertion) and **Statement-2** (Reason)
  Each of these examples also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below.
  (a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
  (b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
  (c) Statement-1 is true, Statement-2 is false
  (d) Statement-1 is false, Statement-2 is true

- ***Ex. 28*** **Statement-1** *Number of rectangles on a chess-board is* ${}^8C_2 \times {}^8C_2$.

Statement-2 *To form a rectangle, we have to select any two of the horizontal lines and any two of the vertical lines.*

Sol. (*d*) In a chessboard, there are 9 horizontal lines and 9 vertical lines.

$\therefore$ Number of rectangles of any size are ${}^9C_2 \times {}^9C_2$.

Hence, Statement-1 is false and Statement-2 is true.

- ***Ex. 29*** **Statement-1** *If* $f:\{a_1, a_2, a_3, a_4, a_5\} \to \{a_1, a_2, a_3, a_4, a_5\}$, *$f$ is onto and $f(x) \neq x$ for each $x \in \{a_1, a_2, a_3, a_4, a_5\}$, is equal to* 44.

- **Statement-2** *The number of derangement for n objects is* $n!\sum_{r=0}^{n} \frac{(-1)^r}{r!}$.

Sol. (*a*) $\because D_n = n!\sum_{r=0}^{n} \frac{(-1)^r}{r!} = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{(-1)^n}{n!}\right)$

$$\therefore \quad D_5 = 5!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right)$$
$$= 120\left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}\right)$$
$$= 6 - 20 + 5 - 1$$
$$= 65 - 21$$
$$= 44$$

Hence, Statement-1 is true, Statement-2 is true and Statement-2 is a correct explanation for Statement-1.

Subjective Type Examples

- In this section, there are **17 subjective** solved examples.

- ***Ex. 30*** *Solve the inequality*

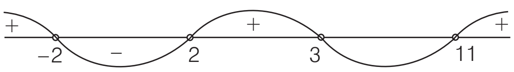
$${}^{x-1}C_4 - {}^{x-1}C_3 - \frac{5}{4}\,{}^{x-2}A_2 < 0, x \in N.$$

Sol. We have, ${}^{x-1}C_4 - {}^{x-1}C_3 - \frac{5}{4}\,{}^{x-2}A_2 < 0$

$$\Leftrightarrow \frac{(x-1)(x-2)(x-3)(x-4)}{1\cdot2\cdot3\cdot4} - \frac{(x-1)(x-2)(x-3)}{1\cdot2\cdot3} - \frac{5}{4}\cdot(x-2)(x-3) < 0$$

$$\Leftrightarrow (x-1)(x-2)(x-3)(x-4) - 4(x-1)(x-2)(x-3) - 30(x-2)(x-3) < 0$$

$$\Leftrightarrow (x-2)(x-3)\{(x-1)(x-4) - 4(x-1) - 30\} < 0$$



$$\Leftrightarrow \quad (x-2)(x-3)\{x^2 - 9x - 22\} < 0$$

$$\Leftrightarrow \quad (x-2)(x-3)(x+2)(x-11) < 0$$

From wavy curve method

$$x \in (-2,2) \cup (3,11)$$

but $\quad x \in N$

$\therefore \quad x = 1, 4, 5, 6, 7, 8, 9, 10 \quad$ ...(i)

From inequality,

$$x - 1 \geq 4, x - 1 \geq 3, x - 2 \geq 2$$

or $\quad x \geq 5, x \geq 4, x \geq 4$

Hence, $\quad x \geq 5 \quad$ ...(ii)

From Eqs. (i) and (ii), solutions of the inequality are

$$x = 5, 6, 7, 8, 9, 10.$$

- ***Ex. 31*** *Find the sum of the series*

$(1^2 + 1)1! + (2^2 + 1)\,2! + (3^2 + 1)\,3! + \ldots + (n^2 + 1)n!$.

Sol. Let $S_n = (1^2 + 1)1! + (2^2 + 1)2! + (3^2 + 1)3! + \ldots + (n^2 + 1)n!$

$\therefore$ nth term $T_n = (n^2 + 1)n!$

$$= \{(n+1)(n+2) - 3(n+1) + 2\}n!$$

$$T_n = (n+2)! - 3(n+1)! + 2n!$$

Putting $n = 1, 2, 3, 4, \ldots, n$

Then, $T_1 = 3! - 3 \cdot 2! + 2 \cdot 1!$

$T_2 = 4! - 3 \cdot 3! + 2 \cdot 2!$

$T_3 = 5! - 3 \cdot 4! + 2 \cdot 3!$

$T_4 = 6! - 3 \cdot 5! + 2 \cdot 4!$

$\ldots\ldots\ldots\ldots\ldots\ldots$

$\ldots\ldots\ldots\ldots\ldots\ldots$

$T_{n-1} = (n+1)! - 3n! + 2(n-1)!$

$T_n = (n+2)! - 3(n+1)! + 2n!$

$\therefore \quad S_n = T_1 + T_2 + T_3 + \ldots + T_n$

$= (n+2)! - 2(n+1)!$ [the rest cancel out]

$= (n+2)(n+1)! - 2(n+1)!$

$= (n+1)!(n+2-2)$

$= n(n+1)!$

● **Ex. 32** *Find the negative terms of the sequence*

$$x_n = \frac{{}^{n+4}P_4}{P_{n+2}} - \frac{143}{4P_n}.$$

Sol. We have,

$$x_n = \frac{{}^{n+4}P_4}{P_{n+2}} - \frac{143}{4P_n}$$

$$\therefore \quad x_n = \frac{(n+4)(n+3)(n+2)(n+1)}{(n+2)!} - \frac{143}{4n!}$$

$$= \frac{(n+4)(n+3)(n+2)(n+1)}{(n+2)(n+1)n!} - \frac{143}{4n!}$$

$$= \frac{(n+4)(n+3)}{n!} - \frac{143}{4n!} = \frac{(4n^2 + 28n - 95)}{4n!}$$

$\because \ x_n$ is negative

$$\therefore \quad \frac{(4n^2 + 28n - 95)}{4n!} < 0$$

which is true for $n = 1, 2$.

Hence, $x_1 = -\dfrac{63}{4}$ and $x_2 = -\dfrac{23}{8}$ are two negative terms.

● **Ex. 33** *How many integers between* 1 *and* 1000000 *have the sum of the digits equal to* 18*?*

Sol. Integers between 1 and 1000000 will be 1, 2, 3, 4, 5 or 6 digits and given sum of digits = 18

Thus, we need to obtain the number of solutions of the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 18 \quad \ldots\text{(i)}$$

where, $0 \le x_i \le 9$, $i = 1, 2, 3, 4, 5, 6$

Therefore, the number of solutions of Eq. (i), will be

$= \text{Coefficient of } x^{18} \text{ in } (x^0 + x^1 + x^2 + x^3 + \ldots + x^9)^6$

$= \text{Coefficient of } x^{18} \text{ in } \left(\dfrac{1 - x^{10}}{1 - x}\right)^6$

$= \text{Coefficient of } x^{18} \text{ in } (1 - x^{10})^6 (1 - x)^{-6}$

$= \text{Coefficient of } x^{18} \text{ in } (1 - 6x^{10})(1 + {}^6C_1 x + {}^7C_2 x^2 + \ldots + {}^{13}C_8 x^8 + \ldots + {}^{23}C_{18} x^{18} + \ldots)$

$= {}^{23}C_{18} - 6 \cdot {}^{13}C_8 = {}^{23}C_5 - 6 \cdot {}^{13}C_5$

$= \dfrac{23 \cdot 22 \cdot 21 \cdot 20 \cdot 19}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - 6 \cdot \dfrac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$

$= 33649 - 7722 = 25927$

● **Ex. 34** *How many different car licence plates can be constructed, if the licences contain three letters of the English alphabet followed by a three digit number,*

(i) if repetition are allowed?

(ii) if repetition are not allowed?

Sol. (i) Total letters = 26 (i.e., $A, B, C, \ldots, X, Y, Z$)

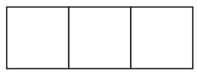
and total digit number = 10 (i.e., 0, 1, 2, ..., 9)

If three letters on plate is represented by, then first place can be filled = 26

Second place can again be filled = 26

[$\because$ repetition are allowed]

and third place can again be filled = 26



Hence, three letters can be filled $= 26 \times 26 \times 26$

$= (26)^3$ ways

and three digit numbers on plate by 999 ways

(i.e., 001, 002, ..., 999)

Hence, by the principle of multiplication, the required number of ways $= (26)^3 (999)$ ways

(ii) Here, three letters out of 26 can be filled $= {}^{26}P_3$

[$\because$ repetition are not allowed]

and three digit can be filled out of $10 = {}^{10}P_3$

[$\because$ repetition are not allowed]

Hence, required number of ways $= ({}^{26}P_3)({}^{10}P_3)$ ways.

● **Ex. 35** *A man has* 7 *relatives,* 4 *of them are ladies and* 3 *gentlemen, has wife, has also* 7 *relatives,* 3 *of them are ladies and* 4 *are gentlemen. In how many ways can they invite a dinner party of* 3 *ladies and* 3 *gentlemen so that there are* 3 *of them man's relatives and* 3 *of the wife's relatives?*

Sol. The four possible ways of inviting 3 ladies and 3 gentlemen for the party with the help of the following table :

Man's relatives		Wife's relatives		Number of ways
4 Ladies	3 Gentlemen	3 Ladies	4 Gentlemen	
0	3	3	0	${}^4C_0 \times {}^3C_3 \times {}^3C_3 \times {}^4C_0 = 1$
1	2	2	1	${}^4C_1 \times {}^3C_2 \times {}^3C_2 \times {}^4C_1 = 144$
2	1	1	2	${}^4C_2 \times {}^3C_1 \times {}^3C_1 \times {}^4C_2 = 324$
3	0	0	3	${}^4C_3 \times {}^3C_0 \times {}^3C_0 \times {}^4C_3 = 16$

$\therefore$ Required number of ways to invite $= 1 + 144 + 324 + 16$
$= 485$

Ex. 36 *A team of ten is to be formed from 6 male doctors and 10 nurses of whom 5 are male and 5 are female. In how many ways can this be done, if the team must have atleast 4 doctors and atleast 4 nurses with atleast 2 male nurses and atleast 2 female nurses?*

Sol.

6 Doctors	5 Male nurses	5 Female nurses	Number of ways of selection
4	4	2	${}^6C_4 \times {}^5C_4 \times {}^5C_2 = 750$
4	3	3	${}^6C_4 \times {}^5C_3 \times {}^5C_3 = 1500$
4	2	4	${}^6C_4 \times {}^5C_2 \times {}^5C_4 = 750$
5	3	2	${}^6C_5 \times {}^5C_3 \times {}^5C_2 = 600$
5	2	3	${}^6C_5 \times {}^5C_2 \times {}^5C_3 = 600$
6	2	2	${}^6C_6 \times {}^5C_2 \times {}^5C_2 = 100$
			$\therefore$ Total ways = 4300

Ex. 37 *A number of four different digits is formed with the help of the digits 1,2,3,4,5,6,7 in all possible ways.*

(i) How many such numbers can be formed?
(ii) How many of these are even?
(iii) How many of these are exactly divisible by 4?
(iv) How many of these are exactly divisible by 25?

Sol. Here total digit = 7 and no two of which are alike

(i) Required number of ways = Taking 4 out of 7

$$= {}^7P_4 = 7 \times 6 \times 5 \times 4 = 840$$

(ii) For even number last digit must be 2 or 4 or 6. Now the remaining three first places on the left of 4-digit numbers are to be fillled from the remaining 6-digits and this can be done in

$${}^6P_3 = 6 \cdot 5 \cdot 4 = 120 \text{ ways}$$

and last digit can be filled in 3 ways.

$\therefore$ By the principle of multiplication, the required number of ways

$$= 120 \times 3 = 360$$

(iii) For the number exactly divisible by 4, then last two digit must be divisible by 4, the last two digits are *viz.*, 12, 16, 24, 32, 36, 52, 56, 64, 72, 76. Total 10 ways.

Now, the remaining two first places on the left of 4-digit numbers are to be filled from the remaining 5-digits and this can be done in ${}^5P_2 = 20$ ways.

Hence, by the principle of multiplication, the required number of ways

$$= 20 \times 10 = 200$$

(iv) For the number exactly divisible by 25, then last two digit must be divisible by 25, the last two digits are *viz.*, 25, 75. Total 2 ways.

Now, the remaining two first places on the left of 4-digit number are to be filled from the remaining 5-digits and this can be done in ${}^5P_2 = 20$ ways.

Hence, by the principle of multiplication, the required number of ways

$$= 20 \times 2 = 40$$

Ex. 38 *India and South Africa play One Day International Series until one team wins 4 matches. No match ends in a draw. Find in how many ways the series can be won?*

Sol. Taking I for India and S for South Africa. We can arrange I and S to show the wins for India and South Africa, respectively.

For example., *ISSSS* means first match is won by India which is followed by 4 wins by South Africa. This is one way in which series can be won.

Suppose, South Africa wins the series, then last match is always won by South Africa.

	Wins of I	Wins of S	Number of ways
(i)	0	4	1
(ii)	1	4	$\frac{4!}{3!} = 4$
(iii)	2	4	$\frac{5!}{2!3!} = 10$
(iv)	3	4	$\frac{6!}{3!3!} = 20$

$\therefore$ Total number of ways = 35

In the same number of ways, India can win the series.

$\therefore$ Total number of ways in which the series can be won

$$= 35 \times 2 = 70$$

● **Ex. 39** *Let n and k be positive integers such that $n \geq \frac{k(k+1)}{2}$. Find the number of solutions $(x_1, x_2, \ldots, x_k)$, $x_1 \geq 1, x_2 \geq 2, \ldots, x_k \geq k$ all integers satisfying $x_1 + x_2 + \ldots + x_k = n$.*

Sol. We have, $x_1 + x_2 + \ldots + x_k = n$ ...(i)

Now, let $y_1 = x_1 - 1,\ y_2 = x_2 - 2, \ldots, y_k = x_k - k$

$\therefore \quad y_1 \geq 0, y_2 \geq 0, \ldots, y_k \geq 0$

On substituting the values $x_1, x_2, \ldots, x_k$ in terms of $y_1, y_2, \ldots, y_k$ in Eq. (i), we get

$$y_1 + 1 + y_2 + 2 + \ldots + y_k + k = n$$

$$\Rightarrow \quad y_1 + y_2 + \ldots + y_k = n - (1 + 2 + 3 + \ldots + k)$$

$$\therefore \quad y_1 + y_2 + \ldots + y_k = n - \frac{k(k+1)}{2} = A \text{ (say)} \qquad \ldots\text{(ii)}$$

The number of non-negative integral solutions of the Eq. (ii) is

$$= {}^{k+A-1}C_A = \frac{(k+A-1)!}{A!(k-1)!}$$

where, $\quad A = n - \frac{k(k+1)}{2}$

● **Ex. 40** *Find the number of all whole numbers formed on the screen of a calculator which can be recognised as numbers with (unique) correct digits when they are read inverted. The greatest number formed on its screen is* 999999.

Sol. The number can use digits 0, 1, 2, 5, 6, 8 and 9 because they can be recognised as digits when they are see inverted.

A number can't begin with ,therefore all numbers having at unit's digit should no be counted. (when those numbers will be read inverted they will begin with).

No. of digits	Total numbers
1	7
2	$6^2 = 36$
3	$6 \times 7 \times 6 = 252$
4	$6 \times 7^2 \times 6 = 1764$
5	$6 \times 7^3 \times 6 = 12348$
6	$6 \times 7^4 \times 6 = 86436$
$\therefore$	Total = 100843

● **Ex. 41** *How many different numbers which are smaller than 2×10^8 and are divisible by 3, can be written by means of the digits* 0, 1 *and* 2*?*

Sol. 12, 21 ... 122222222 are form the required numbers we can assume all of them to be nine digit in the form $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9$ and can use 0 for $a_1; a_2$ and a_0 and $a_{0,}\ a_1, a_2$ and a_3... and so on to get 8-digit, 7-digit, 6-digit numbers etc. a_1 can assume one of the 2 values of 0 or 1. $a_2, a_3, a_4, a_5, a_6, a_7, a_8$ can assume any of 3 values 0, 1, 2.

The number for which $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = a_8 = a_9 = 0$ must be eleminated. The sum of first 8-digits i.e., $a_1 + a_2 + \ldots + a_8$ can be in the form of $3n - 2$ or $3n - 1$ or $3n$.

In each case a_9 can be chosen from 0,1,2 in only 1 way so that the sum of all 9-digits in equal to $3n$.

$\therefore$ Total numbers $= 2 \times 3^7 \times 1 - 1 = 4374 - 1 = 4373.$

● **Ex. 42** *There are n straight lines in a plane such that n_1 of them are parallel in one direction, n_2 are parallel in different direction and so on, n_k are parallel in another direction such that $n_1 + n_2 + \ldots + n_k = n$. Also, no three of the given lines meet at a point. Prove that the total number of points of intersection is*

$$\frac{1}{2}\left\{n^2 - \sum_{r=1}^{k} n_r^2\right\}.$$

Sol. Total number of points of intersection when no two of n given lines are parallel and no three of them are concurrent, is ${}^nC_2$. But it is given that there are k sets of $n_1, n_2, n_3, \ldots, n_k$ parallel lines such that no line in one set is parallel to line in another set.

Hence, total number of points of intersection

$$= {}^nC_2 - ({}^{n_1}C_2 + {}^{n_2}C_2 + \ldots + {}^{n_k}C_2)$$

$$= \frac{n(n-1)}{2} - \left\{\frac{n_1(n_1-1)}{2} + \frac{n_2(n_2-1)}{2} + \ldots + \frac{n_k(n_k-1)}{2}\right\}$$

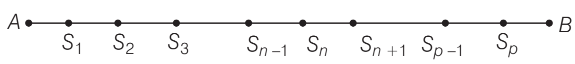
$$= \frac{n(n-1)}{2} - \frac{1}{2}\{(n_1^2 + n_2^2 + \ldots + n_k^2) - (n_1 + n_2 + \ldots + n_k)\}$$

$$= \frac{n(n-1)}{2} - \frac{1}{2}\left\{\sum_{r=1}^{k} n_r^2 - n\right\}$$

$$= \frac{n^2}{2} - \frac{1}{2}\sum_{r=1}^{k} n_r^2 = \frac{1}{2}\left\{n^2 - \sum_{r=1}^{k} n_r^2\right\}$$

● **Ex. 43** *There are p intermediate stations on a railway line from one terminus to another. In how many ways a train can stop at* 3 *of these interediate stations, if no two of these stopping stations are to be consecutive?*

Sol.



Let there be p intermediate stations between two terminus stations A and B as shown above.

Number of ways the train can stop in three intermediate stations = ${}^pC_3$

These are comprised of two exclusive cases *viz.*

(i) atleast two stations are consecutive.

(ii) now two of which is consecutive.

Now, there are $(p-1)$ pairs of consecutive intermediate stations.

In order to get a station trio in which atleast two stations are consecutive, each pair can be associated with a third station in $(p-2)$ ways. Hence, total number of ways in which 3 stations consisting of atleast two consecutive stations, can be chosen in $(p-1)(p-2)$ ways. Among these, each triplet of consecutive stations occur twice.

For example, the pair (S_n, S_{n-1}) when combined with S_{n+1} and the pair (S_n, S_{n+1}) when combined with S_{n-1} gives the same triplet and is counted twice. So, the number of three consecutive stations trio should be subtracted.

Now, number of these three consecutive stations trio is $(p-2)$.

Hence, the number of ways the triplet of stations consisting of atleast two consecutive stations can be chosen in

$$= \{(p-1)(p-2)-(p-2)\} \text{ ways}$$
$$= (p-2)^2 \text{ ways}$$

Therefore, number of ways the train can stop in three consecutive stations

$$= {}^pC_3 - (p-2)^2 = \frac{p(p-1)(p-2)}{1\cdot 2\cdot 3} - (p-2)^2$$

$$= (p-2)\left[\frac{p^2-p-6p+12}{6}\right] = \frac{(p-2)(p^2-7p+12)}{6}$$

$$= \frac{(p-2)(p-3)(p-4)}{1\cdot 2\cdot 3} = {}^{(p-2)}C_3$$

Ex. 44 *How many different 7-digit numbers are there and sum of whose digits is even?*

Sol. Let us consider 10 successive 7-digit numbers

$$a_1a_2a_3a_4a_5a_6\ 0,$$
$$a_1a_2a_3a_4a_5a_6\ 1,$$
$$a_1a_2a_3a_4a_5a_6\ 2,$$
$$\ldots\ldots\ldots\ldots$$
$$a_1a_2a_3a_4a_5a_6\ 9$$

where, a_1, a_2, a_3, a_4, a_5 and a_6 are some digits. We see that half of these 10 numbers, i.e. 5 numbers have an even sum of digits.

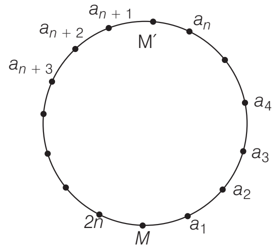
The first digit a_1 can assume 9 different values and each of the digits a_2, a_3, a_4, a_5 and a_6 can assume 10 different values.

The last digit a_7 can assume only 5 different values of which the sum of all digits is even.

$\therefore$ There are $9 \times 10^5 \times 5 = 45 \times 10^5$, 7-digit numbers the sum of whose digits is even.

Ex. 45 *There are 2n guests at a dinner party. Supposing that the master and mistress of the house have fixed seats opposite one another and that there are two specified guests who must not be placed next to one another. Find the number of ways in which the company can be placed.*

Sol. Let the M and M' represent seats of the master and mistress respectively and let $a_1, a_2, a_3, \ldots, a_{2n}$ represent the $2n$ seats.



Let the guests who must not be placed next to one another be called P and Q.

Now, put P at a_1 and Q at any position, other than a_2, say at a_3, then remaining $(2n-2)$ guests can be arranged in the remaining $(2n-2)$ positions in $(2n-2)!$ ways. Hence, there will be altogether $(2n-2)(2n-2)!$ arrangements of the guests, when P is at a_1.

The same number of arrangements when P is at a_n or a_{n+1} or a_{2n}. Thus, for these positions $(a_1, a_n, a_{n+1}, a_{2n})$ of P, there are altogether $4(2n-2)(2n-2)!$ ways. ...(i)

If P is at a_2, then there are altogether $(2n-3)$ positions for Q. Hence, there will be altogether $(2n-3)(2n-2)!$ arrangements of the guests, when P is at a_2.

The same number of arrangements can be made when P is at any other position excepting the four positions

$$a_1, a_n, a_{n+1}, a_{2n}.$$

Hence, for these $(2n-4)$ positions of P, there will be altogether

$(2n-4)(2n-3)(2n-2)!$ arrangements of the guests ...(ii)

Hence, from Eqs. (i) and (ii), the total number of ways of arranging the guests

$$= 4(2n-2)(2n-2)! + (2n-4)(2n-3)(2n-2)!$$
$$= (4n^2-6n+4)(2n-2)!$$

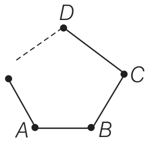
Ex. 46 *Find the number of triangles whose angular points are at the angular points of a given polygon of n sides, but none of whose sides are the sides of the polygon.*

Sol. A polygon of n sides has n angular points. Number of triangles formed from these n angular points $= {}^nC_3$.

These are comprised of two exclusive cases *viz.*

(i) atleast one side of the triangle is a side of the polygon.

(ii) no side of the triangle is a side of the polygon.



Let AB be one side of the polygon. If each angular point of the remaining $(n-2)$ points are joined with A and B, we get a triangle with one side AB.

$\therefore$ Number of triangles of which AB is one side $=(n-2)$

Likewise, number of triangles of which BC is one side $=(n-2)$ and of which atleast one side is the side of the polygon $=n(n-2)$.

Out of these triangle, some are counted twice. For example , the triangle when C is joined with AB is ΔABC, is taken when AB is taken as one side. Again triangle formed when A is joined with BC is counted when BC is taken as one side.

Number of such triangles $= n$

So, the number of triangles of which one side is the side of the triangle

$$= n(n-2) - n = n(n-3)$$

Hence, the total number of required triangles

$$= {}^nC_3 - n(n-3) = \frac{1}{6} n(n-4)(n-5)$$

Ex. 47 *Prove that $(n!)!$ is divisible by $(n!)^{(n-1)!}$.*

Sol. First we show that the product of p consecutive positive integers is divisible by $p!$. Let the p consecutive integers be $m, m+1, m+2, \ldots, m+p-1$. Then,

$$m(m+1)(m+2)\ldots(m+p-1) = \frac{(m+p-1)!}{(m-1)!}$$

$$= p!\,\frac{(m+p-1)!}{(m-1)!\,p!}$$

$$= p!\cdot{}^{m+p-1}C_p$$

Since, ${}^{m+p-1}C_p$ is an integer.

$$\therefore\quad {}^{m+p-1}C_p = \frac{m(m+1)(m+2)\ldots(m+p-1)}{p!}$$

Now, $(n!)!$ is the product of the positive integers from 1 to $n!$. We write the integers from 1 to $n!$ is $(n-1)!$ rows as follows:

1	2	3	...	n
$n+1$	$n+2$	$n+3$	...	$2n$
$2n+1$	$2n+2$	$2n+3$	...	$3n$
:	:	:	:	:
$n!-n+1$	$n!-n+2$	$n!-n+3$	....	$n!$

Each of these $(n-1)!$ rows contain n consecutive positive integers. The product of the consecutive integers in each row is divisible by $n!$. Thus, the product of all the integers from 1 to $n!$ is divisible by $(n!)^{(n-1)!}$.