Logarithmic differentiation

Q.1. If $e^{x+y} = xy$, show that $dy/dx = [y(1 - x)/x(y - 1)]$.

Solution : 1

We have $e^{\mathbf{x+y}} = xy$,

Taking log of both sides , we get

$(x + y) \log_e = \log_e x + \log_e y$

Or,$(x + y) = \log_e x + \text{loge } y$ [As, $\text{loge}_e = 1$]

Differentiating both sides with respect to x , we get

$1 + dy/dx = 1/x + (1/y)\, dy/dx$

Or,$(1 - 1/y)\, dy/dx = 1/x - 1$

Or,$\{(y - 1)/y\}dy/dx = \{(1 - x)/x\}$

Or,$dy/dx = \{(1 - x)/x\}\times\{y/(y - 1)$

$= \{y(1 - x)\}/\{x (y - 1)\}$. **[Proved.]**

Q.2. If $x^p y^p = (x + y)^{p + q}$. Prove that $dy/dx = y/x$.

Solution : 2

We have $x^p y^q = (x + y)^{\mathbf{p + q}}$,

Taking log of both sides , we get

$\log (x^p y^q) = \log (x + y)^{\mathbf{p + q}}$

Or, $\log x^p + \log y^q = (p + q) \log (x + y)$

Or, $p \log x + q \log y = (p+ q) \log (x + y)$

Differentiating both sides with respect to x , we get

$p/x + q/y\ (dy/dx) = (p + q)/(x + y)\ d/dx\ (x + y)$

Or, $p/x + q/y\ (dy/dx)$

$= (p + q)/(x + y)\ (1 + dy/dx) = (p + q)/(x + y) + \{(p + q)/(x + y)\}(dy/dx)$

Or, $[q/y - (p + q)/(x + y)]\ (dy/dx) = (p + q)/(x + y) - p/x$

Or, $(qx + qy - py - qy)/\{y(x + y)\}dy/dx = (px + qx - px - py)/\{x(x + y)\}$

Or, $\{(qx - py)/y\}dy/dx = (qx - py)/x$

Or, $dy/dx = y/x$.

Q.3. If $x^y\ y^x = 5$ show that :

$dy/dx = -\ [\{\log y + y/x\}/\{\log x + x/y\}]$.

Solution : 3

Taking log of both the sides , we get

logxy yx = log 5

Or, $y \log x + x \log y = \log 5$

Differentiating both sides with respect to x , we get

$y/x + \log x\ dy/dx + x/y\ dy/dx + \log y = 0$

Or, $(x/y + \log x)\ dy/dx = -\ (y/x + \log y)$

Or, $dy/dx = -\ [\{\log y + y/x\}/\{\log x + x/y\}]$. **[Proved.]**

Q.4. If $y = (\cos x)^{\cos x}$, find dy/dx .

Solution : 4

Taking log of both sides , we get

$\log y = \log (\cos x) \cos x$

$= \cos x \log (\cos x)$

Differentiating both sides with respect to x , we get

$1/y \; dy/dx = \cos x \,.\, (1/\cos x) \,.\, (-\sin x) + (-\sin x) \,.\, \log(\cos x)$

$= -\sin x - \sin x \log(\cos x)$

$= -\sin x \{1 + \log(\cos x)\}$

Therefore , $dy/dx = y(-\sin x)\{1 + \log(\cos x)\}$

Putting the value of y , we get

$dy/dx = (\cos x)^{\cos x} [-\sin x \{1 + \log(\cos x)\}]$

Q.5. If $y = x^{y}$, prove that : $x \; dy/dx = y^2/(1 - y \log x)$.

Solution : 5

We have , $y = x^{\mathbf{y}}$

Taking log of both sides , we get

$\log y = \log x^y = y \log x$

Differentiating both sides with respect to x , we get

$1/y \; dy/dx = y.\, 1/x + \log x \,.\, dy/dx$

Or, $1/y \; dy/dx - \log x \,.\, dy/dx = y/x$

Or, $(1/y - \log x) \; dy/dx = y/x$

Or, $\{(1 - y \log x)/y\} \; dy/dx = y/x$

Or, $dy/dx = (y/x)/\{(1 - y \log x)/y\}$

$= y^{\mathbf{2}}/\{(1 - y \log x)\, x\}$

Or, $x.dy/dx = y^{\mathbf{2}}/(1 - y \log x)$. **[Proved.]**

Q.6. Find dy/dx , when $x^y = e^{x - y}$.

Solution : 6

We have $x^{\mathbf{y}} = e^{\mathbf{x - y}}$,

Taking log of both sides, we get y

log x = (x − y) log e = x − y [log e = 1]

Or, y log x + y = x Or, y (1 + log x) = x

Or, y = x/(1 + log x)

Therefore , dy/dx = [(1 + log x) . 1 − x .(1/x)]/[(1 + log x)$^{\mathbf{2}}$]

= log x/(1+ log x)$^{\mathbf{2}}$.

Q.7. Differentiate x^ sin $^{-1}$ x w . r. t. x .

Solution : 7

Let y = x^ sin $^{\mathbf{-1}}$ x ,

Then log y = sin $^{\mathbf{-1}}$ x log x

Differentiating both sides w. r. t. x , we get

1/y . dy/dx = sin$^{\mathbf{-1}}$ x × (1/x) + {1/ √(1 − x$^{\mathbf{2}}$)}× log x

Or, dy/dx = y [sin $^{\mathbf{-1}}$ x /x + log x/√(1 − x$^{\mathbf{2}}$)]

Or, dy/dx = x^ sin $^{\mathbf{-1}}$ x [sin −1 x /x + log x/√(1 − x$^{\mathbf{2}}$)]