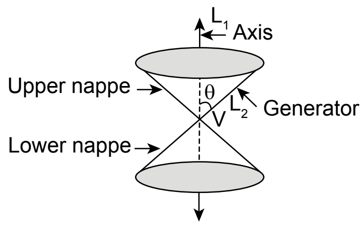
CHAPTER 16

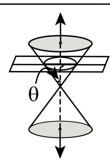
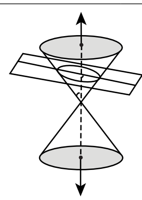
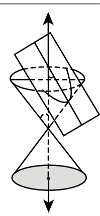
PARABOLA

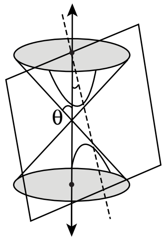
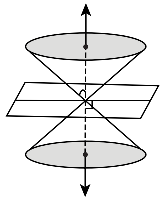
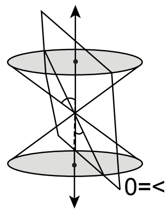
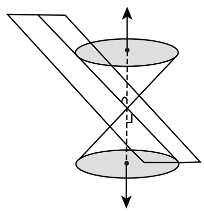
16.1 INTRODUCTION TO CONIC SECTIONS

A conic section or conic is the locus of a point which moves in a plane so that its distance from a fixed point is in a constant ratio to its perpendicular distance from a fixed straight line. Conic sections are section obtained when a pair of two vertical cones with same vertex are intersected by a plane in various orientation.



- The point V is called vertex and the line L_1 is Axis. The rotating line L_2 is called as generator of the cone the vertex separates the cone into two parts known as nappes.
- Nature of conic sections depends on the position of the intersecting plane with respect to the cone and the angle ϕ made by it with the vertical axis of the cone.

Circle	When $\phi = 90°$, the section is a circle.	
Ellipse	When, $\theta < \phi < 90°$ the section is an ellipse.	
Parabola	If plane is parallel to a generator of the cone (i.e., when $\phi = \theta$), then section is a parabola.	

Hyperbola	When $0 \le \phi < \theta$, the plane cuts through both the nappes and the curves of intersection is hyperbola.		
Degenerated Conics	When the plane cuts at the vertex of the cone, we have the different cases:	When $\theta < \phi \le 90°$, then the section is a point.	
		When $0 \le \phi < \theta$, then the section is a pair of intersecting straight lines. It is the degenerated case of a hyperbola.	
		When $\phi = \theta$, then the plane contains a generator of the cone and the section is a coincident straight line.	

16.1.1 Definition of Various Terms Related to Conics

Focus: The fixed point is called the focus of the conic section.

Eccentricity: The constant ratio (e) is called the eccentricity of the conic section.

Directrix: The fixed straight line is called the directrix.

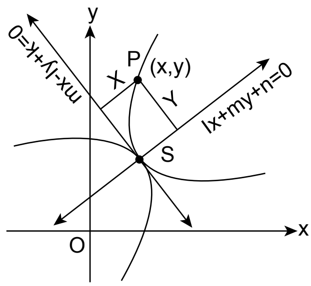
Axis: The straight line passing through the focus and perpendicular to the directrix is called the axis of the conic section.

Vertex: The point of intersection of conic and the axis are called vertices of conic section.

Centre: The point which bisects every chord of the conic passing through it is called centre of the conic.

Double Ordinate: A chord, perpendicular to the axis, is called double ordinate (normal chord) of the conic section. The double ordinate passing through the focus is called the **latus rectum**.

16.1.2 General Equation of a Conic



If the focus is (α, β) and the directrix is ax + by + c = 0, then the equation of the conic section whose eccentricity is e, is given by

According to the definition of conic $\frac{SP}{PM} = \text{costant} = e$ or

SP = e PM; $\sqrt{(x-\alpha)^2+(y-\beta)^2} = e\frac{|ax+by+c|}{\sqrt{(a^2+b^2)}}$; where P(x, y) is a point lying on the conic, or $(x-\alpha)^2+(y-\beta)^2 = e^2\frac{(ax+by+c)^2}{(a^2+b^2)}$.

The equation of conics is represented by the general equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

We know that the discriminant of the above equation is represented by Δ; where

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 \text{ or } \Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

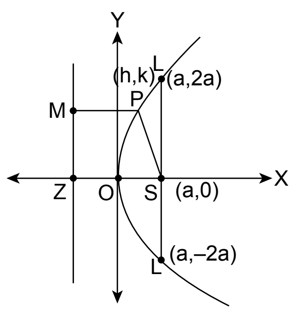
Case I: When Δ = 0, then the equation represents degenerate conic

Condition	Conic
Δ = 0 and $h^2 - ab = 0$	A pair of coincident lines or parallel lines
Δ = 0 and $h^2 - ab > 0$	A pair of intersecting straight lines
Δ = 0 and $h^2 - ab < 0$	Imaginary pair of straight lines with real point of intersection also known as point locus.

Case II: When Δ ≠ 0 the equation represents non-degenerate conic

Condition	Conic
Δ ≠ 0 and h = 0, a = b	A circle
Δ ≠ 0 and $h^2 - ab = 0$	A parabola
Δ ≠ 0 and $h^2 - ab < 0$	An ellipse or empty set
Δ ≠ 0 and $h^2 - ab > 0$	A hyperbola
Δ ≠ 0 and $h^2 - ab > 0$ and a + b = 0	A rectangular hyperbola

16.2 PARABOLA



A parabola is the locus of a point which moves in a plane so that its distance from a fixed point (called **focus**) is equal to its distance from a fixed straight line (called **directrix**). It is the conic with e = 1.

16.2.1 Standard Equation

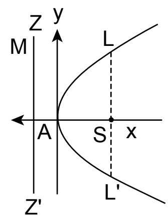
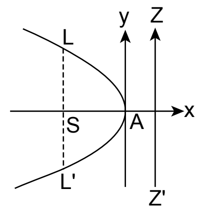
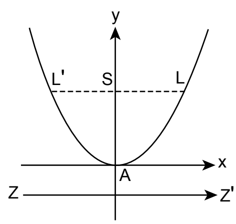
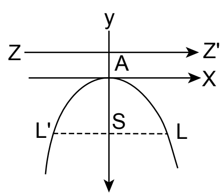
Given S(a, 0) as focus and the line x + a = 0 as directrix.

- **Standard Equation:** Given S(a, 0) as focus and the line x + a = 0 as directrix.

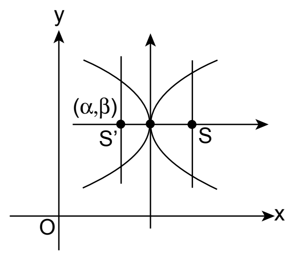
- **Focal distance:** $\because$ SP = PM = a + h $\Rightarrow \sqrt{(h-a)^2 + k^2} = a + h$

  $\Rightarrow a^2 + h^2 - 2ah + k^2 = a^2 + h^2 + 2ah \Rightarrow k^2 = 4ah \Rightarrow y^2 = 4ax$

  Equation of parabola: $y^2 = 4ax$, $a > 0$
- Opening rightwards, passing through origin.
- **Parametric equation:** $x = at^2 y = 2$ at where $t \in \mathbb{R}$
- **Focus:** S(a, o)
- vertex : (0, 0)
- Axis: y = 0
- Directrix: x + a = 0
- T.V.: x = 0
- Focal distance=a + h
- **Latus rectum:** Equation x – a = 0 and length 4a, extremities (a, ±2a).

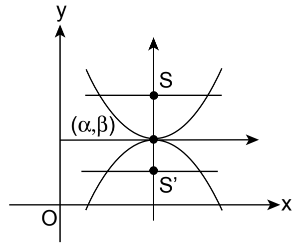
Equation (a > o)	Axis	Focus	Directrix	Latus rectum	Graph
$y^2 = 4ax$ $x = at^2$ $y = 2at$	$y = 0$	$(a, 0)$	$x + a = 0$	$x = a$, 4a $(a, \pm 2a)$	
$y^2 = -4ax$ $x = -at^2$ $y = 2at$	$y = 0$	$(-a, 0)$	$x - a = 0$	$x = -a$, 4a $(-a, \pm 2a)$	
$x^2 = 4ay$ $y = at^2$ $x = 2at$	$x = 0$	$(0, a)$	$y + a = 0$	$y = a$, 4a $(\pm 2a, a)$	
$x^2 = -4ay$ $y = -at^2$ $x = 2at$	$x = 0$	$(0, -a)$	$y - a = 0$	$y = -a$, 4a $(\pm 2a, -a)$	

Equation of parabola with length of L.R. (latus rectum) = 4a, vertex at (a, b) and axis is given as $(y - \beta)^2 = \pm 4a(x - \alpha)$.



- Focus: $(\alpha \pm a, \beta)$
- Axis: $y - \beta = 0$
- T.V. (transverse axis): $x - \alpha = 0$
- Parametric equation: $(\alpha + at^2, \beta + 2at); (\alpha - at^2, \beta + 2at)$
- Directrix: $x = \alpha \mp a$
- Extremetric: $(\alpha - a, \beta \pm 2a)$ $(\alpha - a, \beta \pm 2a)$
- Focus lies at 1/4th of the latus rectum away from vertex along axis towards parabola.

Equation of parabola with length of L.R. = 4a, vertex at (a, b) and axis parallel to y-axis is given as $(x - \alpha)^2 = \pm 4a(y - \beta)$:

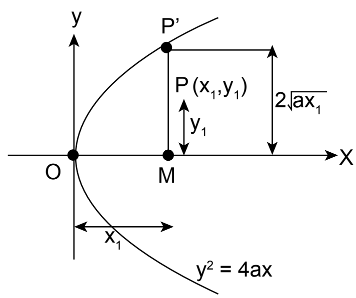


- **Focus:** $(\alpha, \beta \pm a)$
- **Axis:** $x - \alpha = 0$
- **T.V.:** $y - \beta = 0$
- **Directrix:** $y = \beta \mp a$
- **Ends of LR:** $(\alpha \pm 2a, \beta + a); (\alpha \pm 2a, \beta - a)$
- **Parametric equation:** $(\alpha + 2at, y = \beta + at^2); (\alpha + 2at, y = \beta - at^2)$

Note:

Equation of general parabola with axis $lx + my + n = 0$ and T.V. is $mx - ly + k = 0$ and L.R. is of length 4a is given as $(lx + my + n)^2 = \pm L.R.\sqrt{l^2 + m^2}(mx - ly + k)$.

16.2.2 Position of Point w.r.t. Parabola



The region towards focus is defined as inside region of parabola and towards directrix is outside region of parabola.

Given a parabola $y^2 = 4ax$ and a point $P(x_1, y_1)$

- Point P lies inside $\Leftrightarrow S_1 < 0$
- Point P lies on parabola $\Leftrightarrow S_1 = 0$
- Point P lies outside parabola $\Leftrightarrow S_1 > 0$

16.2.3 Position of Line w.r.t. Parabola

Whether the straight line y = mx + c cuts/touches/has no contact with the parabola $y^2 = 4ax$ can be determined by solving the parabola and straight line together.

$y^2 - 4a\left(\frac{y-c}{m}\right) = 0$	$(mx + c)^2 - 4ax = 0$ which is $m^2x^2 + (2cm - 4a)x + c^2 = 0$
$\Rightarrow y^2 - \frac{4a}{m}y + \frac{4ac}{m} = 0$	(i) $D > 0 \Rightarrow$ line cuts at two distinct point $\Rightarrow y_1 + y_2 = \frac{4a}{m}$ and $y_1.y_2 = \frac{4ac}{m}$
	(ii) $D = 0 \Rightarrow$ line touches the parabola
Condition of tangency: D =0	
$\frac{16a}{m^2}(a - cm) = 0 \Rightarrow c = \frac{a}{m}$	(iii) $D < 0 \Rightarrow$ line has no contact

$\Rightarrow$ $y = mx + \dfrac{a}{m}$ $\forall\ m \in \mathbb{R} \sim \{0\}$ known as family of tangent with slope m is tangent to the parabola $y^2 = 4ax$.

$\Rightarrow$ Point of contact $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$ $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right) \Leftrightarrow (at^2, 2at) \Rightarrow m = \dfrac{1}{t}$.

$\Rightarrow$ Parametric equation of tangent at point 't' is given as $y = \dfrac{x}{t} + at \Rightarrow yt = x + at^2$.

16.3 CHORDS OF PARABOLA AND ITS PROPERTIES

Given a parabola $y^2 = 4ax$, let AB be the chord joining $A(x_1, y_1)$ and $B(x_2, y_2)$

$\because$ $y_1^2 = 4ax_1$ and $y_2^2 = 4ax_2 \Rightarrow y_2{}^2 - y_1{}^2 = 4a(x_2 - x_1) \Rightarrow \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{4a}{y_1 + y_2}$

$\Rightarrow$ Slope of chord $AB = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{4a}{y_1 + y_2} = \dfrac{2a}{\dfrac{y_1 + y_2}{2}}$

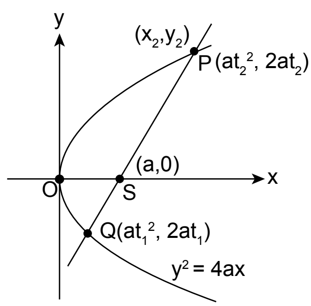
- ❒ Equation of chord: $y - y_1 = \dfrac{4a}{y_1 + y_2}(x - x_1)$
- ❒ Condition to be a focal chord $\Rightarrow y_1y_2 = -4a^2$ and $x_1x_2 = a^2$, i.e., $t_1.t_2 = -1$.

16.3.1 Chord of Parabola in Parametric Form

Slope of chord $= \dfrac{2}{t_1 + t_2}$; Equation of chord: $y - 2at_1 = \dfrac{2}{t_1 + t_2}\left(x - at_1^2\right)$

For focal chord, Put $y = 0$, $x = a$ $\Rightarrow 0 = 2a(1 + t_1t_2) \Rightarrow t_1.t_2 = -1$

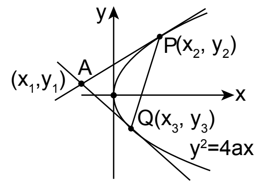
16.3.2 Properties of Focal Chord



A focal chord is basically a chord passing through the focus of the parabola

- ❒ Extremeties of focal chord: $P(at^2, 2at)$ and $Q\left(\dfrac{a}{t^2}, -\dfrac{2a}{t}\right)$.
- ❒ Segments of focal chord: $SP = l_1 = a + at^2$, $SQ = l_2 = a + \dfrac{a}{t^2}$.
- ❒ H.M of segments of focal chord is semi latus-rectum: 2a.
- ❒ Length of focal chord $\Rightarrow L = a\left(t + \dfrac{1}{t}\right)^2$.
- ❒ Slope of focal chord : $\dfrac{2t}{t \quad 1}$.
- ❒ Equation of focal chord: $y = \dfrac{2t}{t^2 - 1}(x - a)$.

Notes:



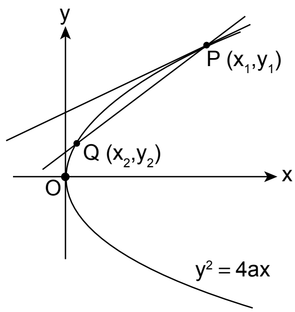
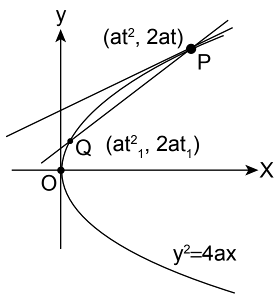
(i) Equation of chord with mid-point M (x_1, y_1):

$\Rightarrow$ $yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$, *i.e.,* $T = S_1$.

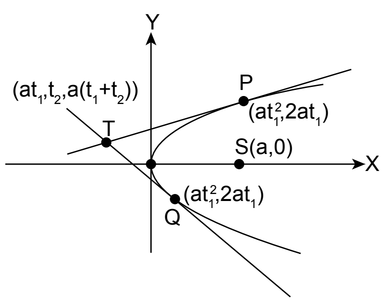
(ii) Equation of a chord of contact formed by joining the points of contacts of the tangents drawn form point A to the parabola

Chord of contact is $yy_1 - 2a(x + x_1) = 0$, *i.e.,* $T = 0$.

16.4 TANGENT OF PARABOLA AND ITS PROPERTIES

Tangent to a parabola **at $P(x_1, y_1)$**		$\Rightarrow$ $T : yy_1 - 2a(x + x_1) = 0$
Tangent to the parabola at **$P(at^2, 2at)$**		$\Rightarrow$ $yt = x + at^2$

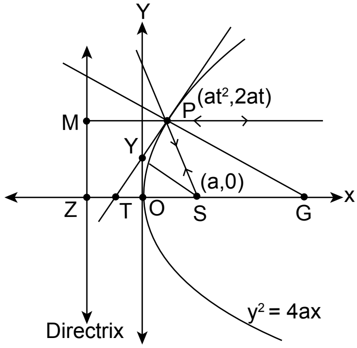
16.4.1 Properties of Tangents of a Parabola



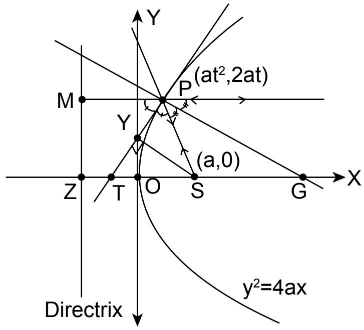
- If the point of intersection of tangents at t_1 and t_2 on the parabola be T, then T $(at_1 t_2, a (t_1 + t_2))$.
- If T be the point of intersection of tangent at P and Q, then SP, ST, SQ are in GP.
  i.e., $ST = \sqrt{SPSQ}$

Consider the parabola shown in the diagram below:

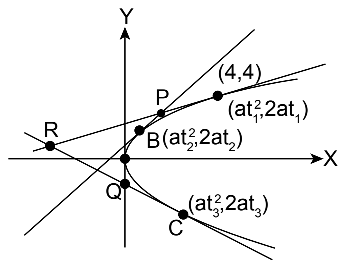
- Coordinate of T: $(-at^2, 0)$, coordinate of Y : $(0, at)$
- $SP = ST = PM = SG = a + at^2$
- $\angle MPT = \angle STP = \angle SPT = \theta$



- **Reflection Property of Parabola:** Light rays emerging from focus after reflection become parallel to the axis of parabolic mirror and all light rays coming parallel to axis of parabola converge at focus.



- Foot of perpendicular from focus upon any tangent lies at Y(0, at) on the tangent at vertex (T.V.)
- SY is median and DSPT is isosceles. SY is altitude, i.e., SY is perpendicular to PT. $\angle TSY = \angle YSP = \pi/2 - \theta$ and SY = MY $\Rightarrow$ SPMT is rhombus.
- Points A, B and C lie on the parabola $y^2 = 4ax$. The tangents to the parabola at A, B and C taken in pairs intersect at points P, Q and R respectively. then the ratio of the areas of the ΔABC and ΔPQR is 2 : 1.



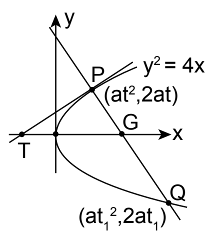
- Tangent at any point on parabola bisects the internal angle between focal distances SP and PM.

$\therefore$ Normal at P bisects the external angle between SP and PM.

- The portion of the tangent intercepted between axis and point of contact is bisected by tangent at vertex.
- Y is the mid-point of PT, SY is median and DSPT is isosceles. SY is altitude. $\angle TSY = \angle YSP = \pi/2 - \theta$ and SY = MY $\Rightarrow$ SPMT is rhombus.
- Equation of a pair of tangents to the parabola form $P(x_1,y_1)$; $SS_1 = T^2$

$(y^2 - 4ax)(y_1^2 - 4ax_1) = [yy_1 - 2a(x + x_1)]^2$.

16.5 NORMALS AND THEIR PROPERTIES



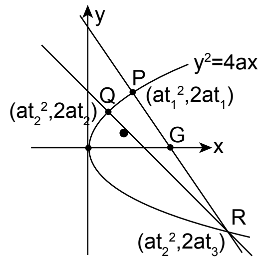
Given a parabola $y^2 = 4ax$, at point 't'

Slope of normal : $m = -t$

Equation of normal : $y - 2at = -t(x - at^2)$

$\Rightarrow \quad y + xt = 2at + at^3$

16.5.1 Properties



- Coordinate of G = $(2a + at^2, 0)$.
- If the normal at P(t) meets the parabola at $Q(t_1)$, then $t = -t - -$.
- If the normal to the parabola $y^2 = 4ax$, at point $P(t_1)$ and $Q(t_2)$, cuts the parabola at some point $R(t_3)$, then
  (i) $t_1t_2 = 2$ (ii) $t_3 = -(t_1 + t_2)$

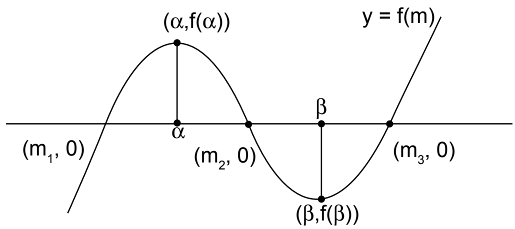
16.5.2 Normals in Terms of Slope

Since Equation of normal: $y + xt = 2at + at^3$ at $(at^2, 2at)$
Put $t = -m \Rightarrow y = mx - 2am - am^3$; where foot of normal is $(am^2, -2am)$.

- From any point P(h, k) in the plane of the parabola three normals can be drawn to the parabola. The foot of these normals are called co-normal points of the parabola.

$\Rightarrow$ Sum of ordinate of foot of conformal points $y_P + y_Q + y_R = -2a(m_1 + m_2 + m_3) = 0$; where m_1, m_2, m_3 are the slopes of the three normals.

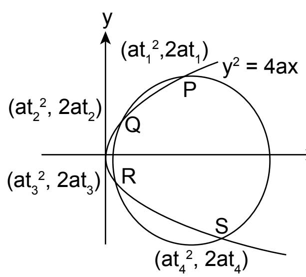
- Sum of the slopes of the concurrent normals to a parabola is zero. Centroid of the triangle joining the co-normal point P, Q, R lies on the axis of the parabola.
- Necessary condition for existence of three real normal through the point (h, k), is $h > 2a$ if $a > 0$ and $h < 2a$ if $a < 0$.
  But the converse of statement is not true, i.e., if $h > 2a$ if $a > 0$ and $h < 2a$, if $a < 0$ does not necessarily implies that the three normals are real.
- Sufficient condition for 3 real normals from (h, k): $f(m) = am^3 + (2a - h)m + k$, it has 3 real and distinct roots.
  If $f'(m) = 3am^2 + 2a - h = 0$ has 2 real and distinct roots, i.e.,



$\therefore$ $m = \pm\sqrt{\dfrac{h-2a}{3a}}$ say α, β; sufficient condition for 3 real slopes is $f(\alpha).\ f(\beta) < 0$.

$\Rightarrow$ $f(\alpha)\ .\ f(\beta) < 0$ $\Rightarrow$ $f(\alpha).(-\alpha) < 0$ $\Rightarrow$ $27ak^2 < 4(h - 2a)^3$

- Atmost there are four concylic point on the parabola and sum of ordinates of these points vanishes.



$\Rightarrow$ Sum of ordinates of four concyclic points on parabola. Since $2a(t_1 + t_2 + t_3 + t_4) = 0$.

- Pair of chord obtained by joining any four concyclic points are equally inclined to the axis of the parabola.
- Circle passing through any three co-normal points on the parabola also passes through the vertex of the parabola

Table representing the equations of tangents in different forms and related terms.

Equation	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Tangent in point form	$yy_1 = 2a(x + x_1)$	$yy_1 = -2a(x + x_1)$	$xx_1 = 2a(y + y_1)$	$xx_1 = -2a(y + y_1)$
Parametric co-ordinate	$(at^2, 2at)$	$(-at^2, 2at)$	$(2at, at^2)$	$(2at, -at^2)$
Tangent in parametric form	$ty = x + at^2$	$ty = -x + at^2$	$tx = y + at^2$	$tx = -x + at^2$
Point of contact in terms of slope (m)	$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$	$\left(-\frac{a}{m^2}, \frac{-2a}{m}\right)$	$(2am, am^2)$	$(-2am, -am^2)$
Condition of tangency	$c = \frac{a}{m}$	$c = -\frac{a}{m}$	$c = -am^2$	$c = am^2$
Tangent in slope form	$y = mx + \frac{a}{m}$	$y = mx - \frac{a}{m}$	$y = mx - am^2$	$y = mx + am^2$

Table representing the equations of tangents in different forms and related terms to parabolas having vertex at (h, k) and axes parallel to co-ordinate axes.

Equation	$(y-k)^2 = 4a(x-h)$	$(y-k)^2 = -4a(x-h)$	$(x-h)^2 = 4a(y-k)$	$(x-h)^2 = -4a(y-k)$
Tangent in point form	$(y - y_1)(y - k) = 2a(x - x_1)$	$(y - y_1)(y - k) = -2a\,(x - x_1)$	$(x - x_1)(x_1 - h) = 2a\,(y - y_1)$	$(x - x_1)(x_1 - h) = -2a(y - y_1)$
Parametric co-ordinate	$(h + at^2, k + 2at)$	$(h - at^2, k + 2at)$	$(h + 2at, k + at^2)$	$(h + 2at, k - at^2)$
Tangent in parametric form	$t(y - k) = (x - h) + at^2$	$t(y - k) = -(x - h) + at^2$	$t(x - h) = (y - k) + at^2$	$t(x - h) = -(y - k) + at^2$
Point of contact in terms of slope (m)	$\left(h + \frac{a}{m^2}, k + \frac{2a}{m}\right)$	$\left(h - \frac{a}{m^2}, k - \frac{2a}{m}\right)$	$(h + 2am, k + am^2)$	$(h - 2am, k - am^2)$
Condition of tangency	$c + mh = k + \frac{a}{m}$	$c + mh = k - \frac{a}{m}$	$c + mh = k - am^2$	$c + mh = k + am^2$
Tangent in slope form	$y = mx - mh + k + \frac{a}{m}$	$y = mx - mh + k - \frac{a}{m}$	$y = mx - mh + k - am^2$;	$y = mx - mh + k + am^2$

Table representing the equations of normal and related terms to standard parabolas indifferent forms.

Equation of Parabola	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Equation of normal in point form	$y - y_1 = \frac{-y_1}{2a}(x - x_1)$	$y - y_1 = \frac{y_1}{2a}(x - x_1)$	$y - y_1 = \frac{-2a}{x_1}(x - x_1)$	$y - y_1 = \frac{2a}{x_1}(x - x_1)$
Parametric co-ordinate	$(at^2, 2at)$	$(-at^2, 2at)$	$(2at, at^2)$	$(2at, -at^2)$
Normal in parametric form	$y + tx = 2at + at^3$	$y - tx = 2at + at^3$	$x + ty = 2at + at^3$	$x - ty = 2at + at^3$
Point of contact in terms of slope (m)	$(am^2, -2am)$	$(-am^2, 2am)$	$\left(-\frac{2a}{m}, \frac{a}{m^2}\right)$	$\left(\frac{2a}{m}, \frac{a}{m^2}\right)$
Condition of normality	$c = -2am - am^3$	$c = 2am + am^3$	$c = 2a + \frac{a}{m^2}$	$c = -2a - \frac{a}{m^2}$
Normal in slope form	$y = mx - 2am - am^3$	$y = mx + 2am + am^3$	$y = mx + 2a + \frac{a}{m^2}$	$y = mx - 2a - \frac{a}{m^2}$

Equation of Parabola	$(y-k)^2 = 4a(x-h)$	$(y-k)^2 = -4a(x-h)$	$(x-h)^2 = 4a(y-k)$	$(x-h)^2 = -4a(y-k)$
Equation of normal in point form	$y - y_1 = \frac{-(y_1 - k)}{2a}(x - x_1)$	$y - y_1 = \frac{(y_1 - k)}{2a}(x - x_1)$	$y - y_1 = \frac{-2a}{x_1 - h}(x - x_1)$	$y - y_1 = \frac{2a}{x_1 - h}(x - x_1)$
Normal in parametric form	$(y - k) + t(x - h) = 2at + at^3$	$(y - k) - t(x - h) = 2at + at^3$	$(x - h) + t(y - k) = 2at + at^3$	$(x - h) - t(y - k) = 2at + at^3$
Point of contact in terms of slope (m)	$(h + am^2, k - 2am)$	$(h - am^2, k + 2am)$	$\left(h - \frac{2a}{m}, k + \frac{a}{m^2})\right)$	$\left(h + \frac{2a}{m}, k - \frac{a}{m^2})\right)$
Condition of normality	$c = k - mh - 2am - am^3$	$c = k - mh + 2am + am^3$	$c = k - mh + 2a + \frac{a}{m^2}$	$c = k - mh - 2a - \frac{a}{m^2}$
Normal in slope form	$(y - k) = m(x - h) - 2am - am^3$	$(y - k) = m(x - h) + 2am + am^3$	$(y - k) = m(x - h) + 2a + \frac{a}{m^2}$	$(y - k) = m(x - h) - 2a - \frac{a}{m^2}$