Session 4

General Equation of Second Degree, Important Theorems

General Equation of Second Degree

The equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

is the general equation of second degree and represents a conics (pair of straight lines, circle, parabola, ellipse, hyperbola). It contains six constants a, b, c, f, g, h.

i.e. $a =$ coefficient of x^2, $b =$ coefficient of y^2,

$c =$ constant term, $g =$ half the coefficient of x,

$f =$ half the coefficient of y,

$h =$ half the coefficient of xy.

Theorem The necessary and sufficient condition for

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

to represent a pair of straight lines is that

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \text{ or } \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

Proof Necessary condition : Let the equation be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{...(i)}$$

represent a pair of lines. Assuming that these lines are not parallel, we suppose further that their point of intersection is (x_1, y_1). Shifting the origin at (x_1, y_1) without rotating the coordinate axes, we have the Eq. (i) transforms to

$$a(X + x_1)^2 + 2h(X + x_1)(Y + y_1) + b(Y + y_1)^2 + 2g(X + x_1) + 2f(Y + y_1) + c = 0 \quad \text{...(ii)}$$

Now this Eq. (ii) represents a pair of lines through the new origin and consequently, it is homogeneous in X and Y.

Hence, the coefficients of X and Y and the constant term in Eq. (ii) must vanish separately.

i.e. coefficient of $X =$ coefficient of $Y =$ constant term $= 0$

$$\Rightarrow \quad ax_1 + hy_1 + g = 0 \quad \text{...(iii)}$$

$$hx_1 + by_1 + f = 0 \quad \text{...(iv)}$$

and $$ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0 \quad \text{...(v)}$$

Now, $ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0$

$\Rightarrow x_1(ax_1 + hy_1 + g) + y_1(hx_1 + by_1 + f) + (gx_1 + fy_1 + c) = 0$

$\Rightarrow x_1 \cdot 0 + y_1 \cdot 0 + gx_1 + fy_1 + c = 0$ [from Eqs. (iii) and (iv)]

$$\Rightarrow \quad gx_1 + fy_1 + c = 0 \quad \text{...(vi)}$$

On eliminating x_1, y_1 from Eqs. (iii), (iv) and (vi), we get the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

$\therefore abc + 2fgh - af^2 - bg^2 - ch^2 = 0,$

as the required condition.

Remarks

1. **Without using determinant** On solving Eqs. (iii) and (iv), we get

$$(x_1, y_1) = \left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2} \right)$$

and then substituting the values of x_1 and y_1 in Eqs. (vi), we obtain

$$g\left(\frac{hf - bg}{ab - h^2}\right) + f\left(\frac{gh - af}{ab - h^2}\right) + c = 0$$

$$\Rightarrow \quad abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

2. By making $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ homogeneous with the help of a new variable z, i.e.

$ax^2 + 2hxy + by^2 + 2gxz + 2fyz + cz^2 = 0$

Let $f(x, y, z) \equiv ax^2 + 2hxy + by^2 + 2gxz + 2fyz + cz^2 = 0$

$$\therefore \quad \frac{\partial f}{\partial x} = 2ax + 2hy + 2gz = 0$$

$$\frac{\partial f}{\partial y} = 2hx + 2by + 2fz = 0$$

$$\frac{\partial f}{\partial z} = 2gx + 2fy + 2cz = 0$$

and finally putting $z = 1$, we obtain equations

$$ax + hy + g = 0, hx + by + f = 0, gx + fy + c = 0$$

which are same as Eqs. (iii), (iv) and (vi), respectively.

3. If $ab - h^2 = 0$, the lines given by Eq. (i) are parallel. In this case, the method followed in the above proof fails and we follow the following method.

Aliter I : (Proof)

Let the lines represented by

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \qquad ...(i)$$

be $lx+my+n=0$ and $l'x+m'y+n'=0$

then $ax^2+2hxy+by^2+2gx+2fy+c$

$$\equiv (lx+my+n)(l'x+m'y+n') \qquad ...(ii)$$

Comparing the coefficients of similar terms in both sides of Eq. (ii), we get

$$\left\{\begin{matrix} ll'=a, mm'=b, nn'=c \\ lm'+l'm=2h, \ln'+l'n=2g, \\ mn'+m'n=2f \end{matrix}\right\} \qquad ...(iii)$$

We now eliminate l, m, n, l', m' and n' from these equations,

we have $$\begin{vmatrix} l & l' & 0 \\ m & m' & 0 \\ n & n' & 0 \end{vmatrix} \times \begin{vmatrix} l' & l & 0 \\ m' & m & 0 \\ n' & n & 0 \end{vmatrix}=0$$

[$\because$ each determinant $=0$]

$$\Rightarrow \begin{vmatrix} 2ll' & lm'+l'm & ln'+l'n \\ ml'+m'l & 2mm' & mn'+m'n \\ nl'+n'l & nm'+n'm & 2nn' \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{vmatrix}=0$$

$$\Rightarrow \Delta=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0$$

$\therefore \quad \Delta=abc+2fgh-af^2-bg^2-ch^2=0,$

which is the required necessary condition.

Remark

Without using determinant

Now, $(lm'+l'm)(\ln'+l'n)(mn'+m'n)=2h\cdot 2g\cdot 2f$

$\Rightarrow \quad 2ll'mm'nn'+ll'(m^2n'^2+m'^2n^2)$

$+mm'(n^2l'^2+n'^2l^2)+nn'(l^2m'^2+l'^2m^2) \quad =8fgh$

$\Rightarrow \quad 2ll'mm'nn'+ll'\{(mn'+m'n)^2-2mm'nn'\}$

$+mm'\{(nl'+n'l)^2-2nn'll'\}+nn'\{(lm'+l'm)^2-2ll'mm'\}$

$=8fgh \qquad$ [from Eq. (iii)]

$\Rightarrow 2abc+a(4f^2-4bc)+b(4g^2-4ca)+c(4h^2-4ab)=8fgh$

$\therefore \qquad abc+2fgh-af^2-bg^2-ch^2=0,$

which is the required necessary condition.

Aliter II : (Proof)

Given equation is

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \qquad ...(i)$$

Case I If $a\neq 0$, then writing Eq. (i) as a quadratic equation in x, we get

$$ax^2+2x(hy+g)+by^2+2fy+c=0$$

Solving, we have

$$x=\frac{-2(hy+g)\pm\sqrt{4(hy+g)^2-4a(by^2+2fy+c)}}{2a}$$

$$\therefore x=\frac{-(hy+g)\pm\sqrt{\{(h^2-ab)y^2+2(gh-af)y+(g^2-ac)\}}}{a}$$

Eq. (i), will represent two straight lines, if LHS of Eq. (i), can be resolved into two linear factors, therefore the expression under the square root should be a perfect square.

[$\because Ax^2+Bx+C=0$ is a perfect square $\Leftrightarrow B^2-4AC=0$]

Hence, $4(gh-af)^2-4(h^2-ab)(g^2-ac)=0$

or $\qquad abc+2fgh-af^2-bg^2-ch^2=0 \qquad ...(ii)$

Ths is called discriminant of the Eq. (i).

Case II If $a=0, b\neq 0$, then writing Eq. (i) as a quadratic equation in y

i.e. $\quad by^2+2y(hx+f)+2gx+c=0$

and proceeding above we get the condition

$$2fgh-bg^2-ch^2=0$$

which is condition obtained by putting $a=0$ in Eq. (ii).

Case III If $a=0, b=0$ but $h\neq 0$, then Eq. (i) becomes

$$2hxy+2gx+2fy+c=0$$

Multiplying by $\frac{h}{2}$ $\qquad$ [$\because h\neq 0$]

$$\Rightarrow \quad h^2xy+hgx+hfy+\frac{ch}{2}=0$$

$$\Rightarrow \quad (hx+f)(hy+g)=fg-\frac{ch}{2}$$

Above equation represents two straight lines, if

$$fg-\frac{ch}{2}=0 \quad \Rightarrow \quad 2fgh-ch^2=0$$

which is condition obtained by putting $a=0, b=0$ in Eq. (ii).

Hence in each case, the condition that

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

represents two straight lines is

$$abc+2fgh-af^2-bg^2-ch^2=0$$

which is the required necessary condition.

Sufficient condition (**Conversely**)

Here, we have to show that the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines.

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

i.e. lines $ax + hy + g = 0$, $hx + by + f = 0$, $gx + fy + c = 0$ are concurrent.

Let the point of concurrency be (x_1, y_1).

Then, $$ax_1 + hy_1 + g = 0 \quad \text{...(iii)}$$
$$hx_1 + by_1 + f = 0 \quad \text{...(iv)}$$
and $$gx_1 + fy_1 + c = 0 \quad \text{...(v)}$$

Now, shifting the origin at (x_1, y_1) without rotating the coordinate axes the equation

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ reduces to

$$a(X + x_1)^2 + 2h(X + x_1)(Y + y_1) + b (Y + y_1)^2 + 2g(X + x_1) + 2f(Y + y_1) + c = 0$$

$$\Rightarrow \; aX^2 + 2hXY + bY^2 + 2X(ax_1 + hy_1 + g) + 2Y(hx_1 + by_1 + f) + x_1(ax_1 + hy_1 + g) + y_1(hx_1 + by_1 + f) + (gx_1 + fy_1 + c) = 0$$

$$\Rightarrow \; aX^2 + 2hXY + bY^2 + 0 + 0 + 0 + 0 + 0 = 0$$

[from Eqs. (iii), (iv) and (v)]

i.e. $$aX^2 + 2hXY + bY^2 = 0$$

It is homogeneous equation of second degree. So, it represents a pair of straight lines through the new origin.

Hence, the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines, if

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$$

Some useful identities If $y = m_1x + c_1$, $y = m_2x + c_2$ be lines represented by Eq. (i). Then,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c$$
$$= b(y - m_1x - c_1)(y - m_2x - c_2)$$
$$= b(y^2 - (m_1 + m_2)xy + m_1m_2x^2 + (m_1c_2 + m_2c_1)x - (c_1 + c_2)y + c_1c_2)$$

On equating coefficients, we get

$$m_1 + m_2 = -\frac{2h}{b}, m_1m_2 = \frac{a}{b}, m_1c_2 + m_2c_1 = \frac{2g}{b},$$

$$c_1 + c_2 = -\frac{2f}{b}, c_1c_1 = \frac{c}{b}$$

These five relations are very useful to solve many problems.

Important Theorems

Theorem 1 The angle between the lines represented by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

is given by

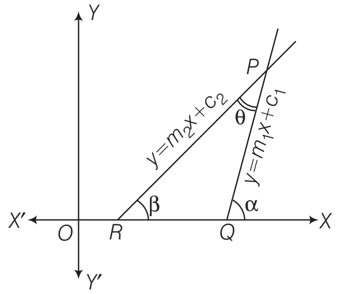
$$\theta = \tan^{-1}\left\{\frac{2\sqrt{(h^2 - ab)}}{|a + b|}\right\}.$$

Proof Let $y = m_1x + c_1$

and $y = m_2x + c_2$

be the lines represented by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$



where, $m_1 = \tan\alpha, m_2 = \tan\beta$

Then, $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

$$\equiv (y - m_1x - c_1)(y - m_2x - c_2)$$

Comparing coefficients of like powers, we obtain

$$m_1 + m_2 = -\frac{2h}{b}, m_1m_2 = \frac{a}{b}$$

Now, if θ be the acute angle between the lines $y = m_1x + c_1$ and $y = m_2x + c_2$, then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| = \frac{\sqrt{(m_1 + m_2)^2 - 4m_1m_2}}{|1 + m_1m_2|}$$

$$= \frac{\sqrt{\left(-\frac{2h}{b}\right)^2 - 4\left(\frac{a}{b}\right)}}{\left|1 + \frac{a}{b}\right|} = \frac{2\sqrt{(h^2 - ab)}}{|a + b|}$$

$$\therefore \quad \theta = \tan^{-1}\left\{\frac{2\sqrt{(h^2 - ab)}}{|a + b|}\right\}$$

Corollary 1. The angle between the lines represented by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

is the same as the angle between the lines represented by

$$ax^2 + 2hxy + by^2 = 0$$

***Corollary* 2.** The lines represented by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

are perpendicular iff $a + b = 0$ and parallel iff $h^2 = ab$.

Theorem 2 The lines represented by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

will be coincident, if $h^2 - ab = 0, g^2 - ac = 0$

and $f^2 - bc = 0.$

Proof Let the lines represented by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

be $lx + my + n = 0$ and $l'x + m'y + n' = 0$

then $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

$$\equiv (lx + my + n)(l'x + m'y + n')$$

Comparing the coefficients of similar terms in both sides, we get

$$\left.\begin{array}{l} ll' = a, mm' = b, nn' = c \\ lm' + l'm = 2h, \ln' + l'n = 2g \\ mn' + m'n = 2f \end{array}\right\}$$

$\because$ Lines $lx + my + n = 0$ and $l'x + m'y + n' = 0$ are coincident, then $\frac{l}{l'} = \frac{m}{m'} = \frac{n}{n'}$.

Taking the ratios in pairs, then

$$lm' - l'm = 0, mn' - m'n = 0, \ln' - l'n = 0$$

$$\Rightarrow \sqrt{(lm' + l'm)^2 - 4ll'mm'} = 0,$$

$$\sqrt{(mn' + m'n)^2 - 4mm'nn'} = 0$$

and $\sqrt{(ln' + l'n)^2 - 4ll'nn'} = 0$

i.e. $\sqrt{(4h^2 - 4ab)} = 0, \sqrt{(4f^2 - 4bc)} = 0$

and $\sqrt{(4g^2 - 4ac)} = 0$

i.e. $h^2 - ab = 0, f^2 - bc = 0, g^2 - ac = 0$

Theorem 3 The point of intersection of the lines represented by

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is

$$\left(\sqrt{\left(\frac{f^2 - bc}{h^2 - ab}\right)}, \sqrt{\left(\frac{g^2 - ca}{h^2 - ab}\right)}\right) \text{or} \left(\frac{bg - hf}{h^2 - ab}, \frac{af - gh}{h^2 - ab}\right).$$

Proof Let the lines represented by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

be $lx + my + n = 0$ and $l'x + m'y + n' = 0$

then $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

$$\equiv (lx + my + n)(l'x + m'y + n')$$

Comparing the coefficients of similar terms in both sides

then $ll' = a, mm' = b, nn' = c$

$$lm' + l'm = 2h, \ln' + l'n = 2g, mn' + m'n = 2f$$

$$\Rightarrow (lm' - l'm) = \sqrt{(lm' + l'm)^2 - 4ll'mm'} = 2\sqrt{(h^2 - ab)}$$

$$\Rightarrow (nl' - n'l) = \sqrt{(ln' + l'n)^2 - 4ll'nn'} = 2\sqrt{(g^2 - ac)}$$

and $(mn' - m'n) = \sqrt{(mn' + m'n)^2 - 4mm'nn'}$

$$= 2\sqrt{(f^2 - bc)}$$

Now, solving $lx + my + n = 0$ and $l'x + m'y + n' = 0$

then $\frac{x}{(mn' - m'n)} = \frac{y}{(nl' - n'l)} = \frac{1}{(lm' - l'm)}$

$$\Rightarrow \frac{x}{2\sqrt{(f^2 - bc)}} = \frac{y}{2\sqrt{(g^2 - ac)}} = \frac{1}{2\sqrt{(h^2 - ab)}}$$

$$\therefore (x, y) = \left(\sqrt{\left(\frac{f^2 - bc}{h^2 - ab}\right)}, \sqrt{\left(\frac{g^2 - ac}{h^2 - ab}\right)}\right)$$

Also, $x = \sqrt{\left(\frac{f^2 - bc}{h^2 - ab}\right)} = \frac{\sqrt{(f^2 - bc)(h^2 - ab)}}{(h^2 - ab)}$

$$= \frac{\sqrt{f^2h^2 - abf^2 - bch^2 + b(abc)}}{(h^2 - ab)}$$

$$= \frac{\sqrt{f^2h^2 - abf^2 - bch^2 + b(af^2 + bg^2 + ch^2 - 2fgh)}}{(h^2 - ab)}$$

$$[\because abc + 2fgh - af^2 - bg^2 - ch^2 = 0]$$

$$= \frac{\sqrt{(f^2h^2 + b^2g^2 - 2bfgh)}}{(h^2 - ab)} = \frac{\sqrt{(bg - hf)^2}}{(h^2 - ab)}$$

$$= \left(\frac{bg - hf}{h^2 - ab}\right)$$

Similarly, $y = \left(\frac{af - gh}{h^2 - ab}\right)$

Hence, $(x, y) = \left(\frac{bg - hf}{h^2 - ab}, \frac{af - gh}{h^2 - ab}\right).$

Remembering Method (For second point)

Since, $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

taking first two rows (repeat first column)

$$\begin{matrix} a & & h & & g & & a \\ & \times & & \times & & \times & \\ h & & b & & f & & h \end{matrix}$$

$\Rightarrow \quad ab - h^2, hf - bg, gh - af$

$\Rightarrow \quad h^2 - ab, bg - hf, af - gh$

$\Rightarrow \quad 1, \dfrac{bg - hf}{h^2 - ab}, \dfrac{af - gh}{h^2 - ab}$

Hence, point of intersection is $\left(\dfrac{bg - hf}{h^2 - ab}, \dfrac{af - gh}{h^2 - ab} \right)$.

OR

Cofactors of third column are C_{13}, C_{23}, C_{33}

$$\because \quad C_{13} = \begin{vmatrix} h & b \\ g & f \end{vmatrix} = hf - bg$$

$$C_{23} = -\begin{vmatrix} a & h \\ g & f \end{vmatrix} = hg - af$$

and $$C_{33} = \begin{vmatrix} a & h \\ h & b \end{vmatrix} = ab - h^2$$

Point of intersection is $\left(\dfrac{C_{13}}{C_{33}}, \dfrac{C_{23}}{C_{33}} \right)$ i.e.

$$\left(\frac{hf - bg}{ab - h^2}, \frac{hg - af}{ab - h^2} \right) \quad \text{or} \quad \left(\frac{bg - hf}{h^2 - ab}, \frac{af - hg}{h^2 - ab} \right).$$

Remembering Mehod (For first point)

Cofactors of leading diagonal are

C_{11}, C_{22}, C_{33}

$$\because \quad C_{11} = \begin{vmatrix} b & f \\ f & c \end{vmatrix} = bc - f^2,$$

$$C_{22} = \begin{vmatrix} a & g \\ g & c \end{vmatrix} = ac - g^2$$

and $$C_{33} = \begin{vmatrix} a & h \\ h & b \end{vmatrix} = ab - h^2$$

$\therefore$ Point of intersection are $\left(\sqrt{\dfrac{C_{11}}{C_{33}}}, \sqrt{\dfrac{C_{22}}{C_{33}}} \right)$

i.e. $$\left(\sqrt{\left(\frac{bc - f^2}{ab - h^2} \right)}, \sqrt{\left(\frac{ac - g^2}{ab - h^2} \right)} \right)$$

or $$\left(\sqrt{\left(\frac{f^2 - bc}{h^2 - ab} \right)}, \sqrt{\left(\frac{g^2 - ac}{h^2 - ab} \right)} \right)$$

Theorem 4 The pair of bisectors of the lines represented by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

is $$\frac{(x - \alpha)^2 - (x - \beta)^2}{(a - b)} = \frac{(x - \alpha)(y - \beta)}{h}$$

where (α, β) be the point of intersection of the pair of straight lines represented by Eq. (i).

Proof Since (α, β) be the point of intersection of the lines represented by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \text{...(i)}$$

Shifting the origin at (α, β) without rotating the coordinate axes, the Eq. (i) reduces to

$$a(X + \alpha)^2 + 2h(X + \alpha)(Y + \beta) + b(Y + \beta)^2 + 2g(X + \alpha) + 2f(Y + \beta) + c = 0$$

$[\because x = X + \alpha \text{ and } y = Y + \beta]$

$$\Rightarrow (aX^2 + 2hXY + bY^2) + 2X(a\alpha + h\beta + g) + 2Y(h\alpha + b\beta + f) + a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c = 0 \qquad \text{...(ii)}$$

This equation represents a pair of straight lines passing through the new origin. So, it must be homogeneous equation of second degree in X and Y.

$$\therefore \quad a\alpha + h\beta + g = 0 \qquad \text{...(iii)}$$

$$h\alpha + b\beta + f = 0 \qquad \text{....(iv)}$$

and $a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c = 0$ ...(v)

Now, from Eq. (ii), $aX^2 + 2hXY + bY^2 = 0$ ...(vi)

The equation of the bisectors of the angles between the lines given by Eq. (vi) is

$$\frac{X^2 - Y^2}{a - b} = \frac{XY}{h} \qquad \text{...(vii)}$$

[with reference to new origin]

Replacing X by $x - \alpha$ and Y by $y - \beta$ in Eq. (vii), then

$$\frac{(x - \alpha)^2 - (y - \beta)^2}{(a - b)} = \frac{(x - \alpha)(y - \beta)}{h}$$

[with reference to old origin]

which is the required equation of the bisectors of the angles between the lines given by Eq. (i).

Remark

If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

represents two straight lines, then the equation of lines through the origin and parallel to them is $ax^2 + 2hxy + by^2 = 0$.

Example 20 For what value of λ does the equation $12x^2 - 10xy + 2y^2 + 11x - 5y + \lambda = 0$ represent a pair of straight lines? Find their equations and the angle between them.

Sol. Comparing the given equation with the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

we get $a = 12, h = -5, b = 2, g = \frac{11}{2}, f = -\frac{5}{2}$ and $c = \lambda$

If the given equation represents a pair of straight lines, then

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$\Rightarrow \quad 12 \times 2 \times \lambda + 2 \times \left(-\frac{5}{2}\right) \times \frac{11}{2} \times (-5) - 12 \times \frac{25}{4} - 2 \times \frac{121}{4} - \lambda \times 25 = 0$$

$\therefore \quad \lambda = 2$, also $h^2 - ab = 25 - 24 = 1 > 0$

$\therefore$ The given equation will represent a pair of straight lines, if $\lambda = 2$.

To find the two lines

First method

Substituting $\lambda = 2$ in the given equation, we get

$$12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0 \quad \text{...(i)}$$

Since, $\quad 12x^2 - 10xy + 2y^2 = 2(3x - y)(2x - y)$

factors of Eq. (i) can be taken as

$$2(3x - y + l)(2x - y + m)$$
$$= 12x^2 - 10xy + 2y^2 + 2(2l + 3m)x + 2(-l - m)y + 2lm$$

On comparing, $2l + 3m = \frac{11}{2}, l + m = \frac{5}{2}, lm = 1$

Solving, we get $l = 2, m = \frac{1}{2}$.

Thus, the factors of Eq. (i) are

$$2(3x - y + 2)\left(2x - y + \frac{1}{2}\right) = 0$$

or $\quad (3x - y + 2)(4x - 2y + 1) = 0$.

$\therefore$ The two straight lines represented by the given equation are

$$3x - y + 2 = 0 \text{ and } 4x - 2y + 1 = 0.$$

Second Method

Writing Eq. (i) as quadratic equation in x, we get

$$12x^2 + (-10y + 11)x + 2y^2 - 5y + 2 = 0$$

$\therefore$

$$x = \frac{-(-10y + 11) \pm \sqrt{(-10y + 11)^2 - 48(2y^2 - 5y + 2)}}{24}$$

i.e. $\quad 24x = (10y - 11) \pm \sqrt{(4y^2 + 20y + 25)}$

$$= (10y - 11) \pm (2y + 5)$$

$\therefore \quad 24x = 12y - 6$, i.e. $4x - 2y + 1 = 0$

and $\quad 24x = 8y - 16$, i.e. $3x - y + 2 = 0$

are the required lines.

To find the angle between the lines

If θ be the angle between the lines, then

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{|a + b|}$$
$$= \frac{2\sqrt{25 - 24}}{|12 + 2|} = \frac{1}{7}$$

$$\therefore \quad \theta = \tan^{-1}\left(\frac{1}{7}\right).$$

Example 21 ***Prove that the equation $8x^2 + 8xy + 2y^2 + 26x + 13y + 15 = 0$ represents a pair of parallel straight lines. Also, find the perpendicular distance between them.***

Sol. Given equation is

$$8x^2 + 8xy + 2y^2 + 26x + 13y + 15 = 0 \quad \text{...(i)}$$

Writing Eq. (i) as quadratic equation in x, we get

$$8x^2 + 2x(4y + 13) + 2y^2 + 13y + 15 = 0$$

$$\therefore \quad x = \frac{-2(4y + 13) \pm \sqrt{4(4y + 13)^2 - 32(2y^2 + 13y + 15)}}{16}$$

$$\Rightarrow \quad x = \frac{-(4y + 13) \pm \sqrt{(4y + 13)^2 - 8(2y^2 + 13y + 15)}}{8}$$

$$\Rightarrow \quad x = \frac{-(4y + 13) \pm 7}{8}$$

$\Rightarrow \quad 8x = -4y - 13 + 7$, i.e. $4x + 2y + 3 = 0$

and $\quad 8x = -4y - 13 - 7$, i.e. $2x + y + 5 = 0$

i.e. the given Eq. (i) represents two straight lines

$$2x + y + 5 = 0$$

and $\quad 4x + 2y + 3 = 0$

i.e. $\quad 2x + y + \frac{3}{2} = 0$

both lines are parallel.

$$\therefore \text{ Distance between them} = \frac{\left|5 - \frac{3}{2}\right|}{\sqrt{2^2 + 1^2}} = \frac{7}{2\sqrt{5}}$$

Aliter : Here, $\Delta = 8 \times 2 \times 15 + 2 \times \frac{13}{2} \times 13 \times 4 - 8 \times \left(\frac{13}{2}\right)^2 - 2 \times (13)^2 - 15 \times (4)^2 = 0$

and $\quad h^2 = (4)^2 = 16 = 8 \times 2 = ab$

$\therefore$ Given equation

$$8x^2 + 8xy + 2y^2 + 26x + 13y + 15 = 0 \quad \text{...(i)}$$

represents two parallel straight lines.

Since, $\quad 8x^2 + 8xy + 2y^2 = 2(2x + y)^2$

factors of Eq. (i) can be taken as

$$2(2x + y + l)(2x + y + m)$$
$$= 8x^2 + 8xy + 2y^2 + 2(2m + 2l)x + 2(m + l)y + 2lm$$

On comparing, we get $l + m = \frac{13}{2}$ and $lm = \frac{15}{2}$

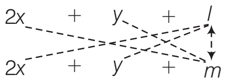
$\therefore$ Distance between them

$$= \frac{|l - m|}{\sqrt{(2^2 + 1^2)}} = \frac{\sqrt{(l + m)^2 - 4lm}}{\sqrt{5}}$$

$$= \frac{\sqrt{\frac{169}{4} - \frac{60}{2}}}{\sqrt{5}} = \frac{7}{2\sqrt{5}}$$

Remark

For comparing coefficients write equation in form



coefficient of x is $2m + 2l$, cofficient of y is $l + m$ and coefficients of constant term is lm.

i.e. $l + m = \frac{13}{2}$, $lm = \frac{15}{2}$

Example 22 ***Find the combined equation of the straight lines passing through the point (1, 1) and parallel to the lines represented by the equation*** $x^2 - 5xy + 4y^2 + x + 2y - 2 = 0$.

Sol. Given equation of lines is

$$x^2 - 5xy + 4y^2 + x + 2y - 2 = 0. \quad \text{...(i)}$$

Since, $x^2 - 5xy + 4y^2 = (x - 4y)(x - y)$

Factors of Eq. (i) taken as $(x - 4y + l)(x - y + m)$.

Now, equation of line through (1, 1) and parallel to

$x - 4y + l = 0$ is $x - 4y + \lambda = 0$

i.e. $1 - 4 + \lambda = 0$

$\therefore$ $\lambda = 3$

then line is $x - 4y + 3 = 0$ ...(ii)

and equation of line through (1, 1) and parallel to

$x - y + m = 0$ is $x - y + \mu = 0$

i.e. $1 - 1 + \mu = 0$

$\therefore$ $\mu = 0$

then line is $x - y = 0$ ...(iii)

Hence, equation of lines Eqs. (ii) and (iii) is

$$(x - 4y + 3)(x - y) = 0$$

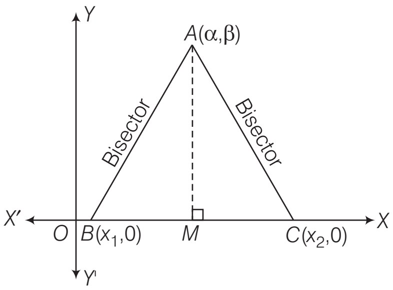
i.e. $x^2 - 5xy + 4y^2 + 3x - 3y = 0$

Example 23 ***If*** $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ***represents a pair of lines, prove that the area of the triangle formed by their bisectors and axis of x is*** $\frac{\sqrt{(a-b)^2 + 4h^2}}{|2h|} \cdot \left|\frac{ca - g^2}{ab - h^2}\right|$.

Sol. Given $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ...(i)

The point of intersection of the lines given by Eq. (i) are

$$\alpha = \sqrt{\left(\frac{f^2 - bc}{h^2 - ab}\right)}, \beta = \sqrt{\left(\frac{g^2 - ca}{h^2 - ab}\right)}$$



Hence, equation of the bisectors of the lines given by Eq. (i) is

$$\frac{(x - \alpha)^2 - (y - \beta)^2}{a - b} = \frac{(x - \alpha)(y - \beta)}{h}$$

For X-axis, $y = 0$.

$$\therefore \quad \frac{(x - \alpha)^2 - \beta^2}{a - b} = \frac{-\beta(x - \alpha)}{h}$$

or $h(x - \alpha)^2 + \beta(x - \alpha)(a - b) - h\beta^2 = 0$ ...(ii)

Eq. (ii) is a quadratic in $(x - \alpha)$ and let two values of x be x_1 and x_2, so that its roots are

$x_1 - \alpha$ and $x_2 - \alpha$

$\therefore$ $(x_1 - \alpha) + (x_2 - \alpha) = \text{Sum of roots} = \frac{-\beta(a - b)}{h}$

$(x_1 - \alpha)(x_2 - \alpha) = \text{Products of roots} = -\beta^2$

$\therefore$ $|x_2 - x_1| = |(x_2 - \alpha) - (x_1 - \alpha)|$

$$= \sqrt{[(x_2 - \alpha) + (x_1 - \alpha)]^2 - 4(x_2 - \alpha)(x_1 - \alpha)}$$

$\therefore$ $|x_2 - x_1| = \sqrt{\left\{\frac{\beta^2 (a - b)^2}{h^2} + 4\beta^2\right\}}$

$$= \left|\frac{\beta}{h}\right| \sqrt{(a - b)^2 + 4h^2}$$

$\therefore$ Area of $\Delta ABC = \frac{1}{2}|BC||AM|$

$$= \frac{1}{2}|x_2 - x_1||\beta|$$

$$= \frac{1}{2}\left|\frac{\beta}{h}\right|\sqrt{(a - b)^2 + 4h^2} \times |\beta|$$

$$= \frac{\sqrt{(a - b)^2 + 4h^2}}{|2h|} \cdot \beta^2$$

$$= \frac{\sqrt{(a - b)^2 + 4h^2}}{|2h|} \cdot \frac{ca - g^2}{ab - h^2}$$

Exercise for Session 4

1. If $\lambda x^2 + 10xy + 3y^2 - 15x - 21y + 18 = 0$ represents a pair of straight lines. Then, the value of λ is

(a) -3 (b) 3 (c) 4 (d) -4

2. The point of intersection of the straight lines given by the equation $3y^2 - 8xy - 3x^2 - 29x + 3y - 18 = 0$ is

(a) $\left(1, \frac{1}{2}\right)$ (b) $\left(1, -\frac{1}{2}\right)$ (c) $\left(-\frac{3}{2}, \frac{5}{2}\right)$ (d) $\left(-\frac{3}{2}, -\frac{5}{2}\right)$

3. If the equation $12x^2 + 7xy - py^2 - 18x + qy + 6 = 0$ represents two perpendicular lines, then the value of p and q are

(a) 12, 1 (b) 12, -1 (c) 12, $\frac{23}{2}$ (d) 12, $-\frac{23}{2}$

4. If the angle between the two lines represented by $2x^2 + 5xy + 3y^2 + 7y + 4 = 0$ is $\tan^{-1}(m)$, then m is equal to

(a) $-\frac{1}{5}$ (b) $\frac{1}{5}$ (c) $-\frac{3}{5}$ (d) $\frac{3}{5}$

5. The equation of second degree $x^2 + 2\sqrt{2}xy + 2y^2 + 4x + 4\sqrt{2}\,y + 1 = 0$ represents a pair of straight lines, the distance between them is

(a) 2 (b) $2\sqrt{3}$ (c) 4 (d) $4\sqrt{3}$

6. Find the area of the parallelogram formed by the lines
$2x^2 + 5xy + 3y^2 = 0$ and $2x^2 + 5xy + 3y^2 + 3x + 4y + 1 = 0$.

7. Find the locus of the incentre of the triangle formed by
$xy - 4x - 4y + 16 = 0$ and $x + y = a$ ($a > 4, a \neq 4\sqrt{2}$ and a is the parameter).

8. If the equation $2hxy + 2gx + 2fy + c = 0$ represents two straight lines, then show that they form a rectangle of area $\frac{|fg|}{h^2}$ with the coordinate axes.

9. Find the area of the triangle formed by the lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ and axis of x.

10. Find the equations of the straight lines passing through the point (1, 1) and parallel to the lines represented by the equation $x^2 - 5xy + 4y^2 + x + 2y - 2 = 0$.

Answers

Exercise for Session 4

1. (b) **2.** (d) **3.** (a, d) **4.** (b) **5.** (a)

6. 1 sq unit **7.** $x - y = 0$ **9.** $\dfrac{|g^2 - ac|}{|a|\sqrt{h^2 - ab}}$

10. $x^2 - 5xy + 4y^2 + 3x - 3y = 0$