CHAPTER 17

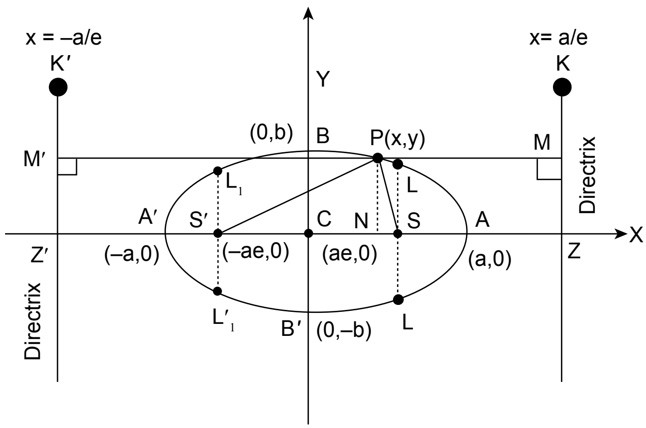
ELLIPSE

17.1 DEFINITION

Ellipse is the locus of a point which moves in a plane, such that the ratio of its distance from a fixed point (Focus) to its distance from the fixed line (Directrix) is always constant and equal to a quantity which is less than 1

17.2 STANDARD EQUATION OF ELLIPSE

Given focus S(ae, 0) and the x – (a/e) = 0 as directrix.



17.2.1 Focal Distance

Focal distance (SP) of a point P is given as:

$$\because \quad SP = e.PM = e\left(\frac{a}{e} - h\right) = a - eh \qquad \Rightarrow \quad \sqrt{(h-ae)^2 + k^2} = a - eh$$

$$\Rightarrow \quad a^2e^2 + h^2 - 2aeh + k^2 = a^2 + e^2h^2 - 2aeh$$

$\Rightarrow \quad a^2e^2 + h^2 + k^2 = a^2 + e^2h^2$

$\Rightarrow \quad h^2 - e^2h^2 + k^2 = a^2 - a^2e^2$

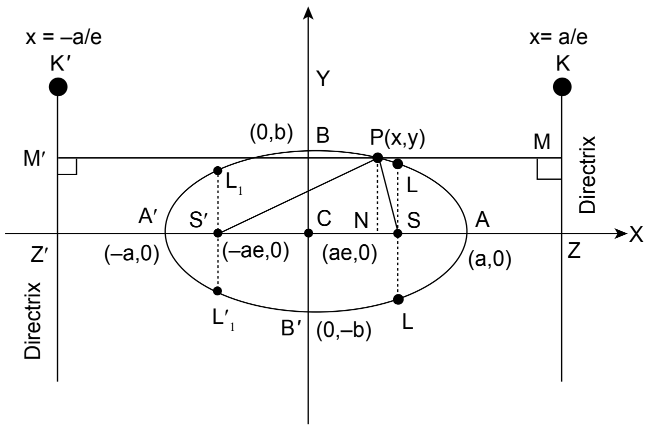
$\Rightarrow \quad h^2(1-e^2) + k^2 = a^2(1-e^2)$

$\Rightarrow \quad \dfrac{h^2}{a^2} + \dfrac{k^2}{a^2(1-e^2)} = 1.$

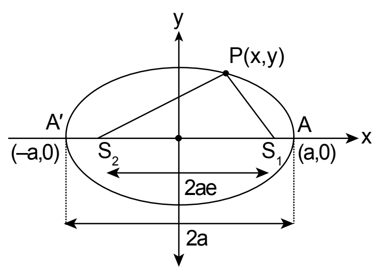
Let $a^2(1-e^2) = b^2$

$\Rightarrow \quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$

17.3 TRACING OF ELLIPSE



- **Equation of Ellipse:** $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$
- **Eccentricity:** $e = \sqrt{1 - \dfrac{b^2}{a^2}}$
- **Symmetry:** Since curve is even w.r.t. variable x and y the graph is symmetric about both the co-ordinate axes. There are two foci and two directrices.
- **Foci:** S_1 (ae, 0); S_2 (–ae, 0)
- **Directrices:** D_1: x = a/e; D_2: x = –a/e
- **Focal distances:** $S_1P = ePM = a - eh$, $S_2P = ePM' = e\left(\dfrac{a}{e} + h\right) = a + eh$
- AA′is called major axis. length = 2a, equation : y = 0.
- BB′is called minor axis length= 2b, equation x = 0.
- The point of intersection of major and minor is called centre. All the chords passing through the centre get bisected at the centre.
- **Normal chord:** Chord normal to the major axis is called normal chord, or double ordinate. If it passes through the focus, it is called latus rectum.
- **Length of** $LR = \dfrac{2b^2}{a}$ **equation of L.R. x = ae**



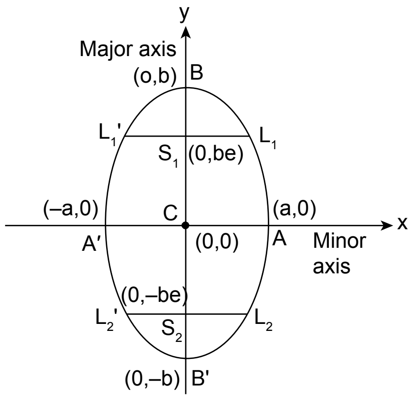
- Ellipse is a locus of the point that moves in such a manner so that the summation of its distances from two fixed points S_1 and S_2 (foci) remains constant (2a).
- $S_1P + S_2P = 2a$; where 2a is length of major axis.

  Case I: If $2a > S_1S_2 = 2ae$ locus ellipse.

  Case II: $S_1P + S_2P = S_1S_2$ locus segment SS′.

  Case III: $S_1P + S_2P < S_1S_1$ no locus.

  If equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; where b > a.



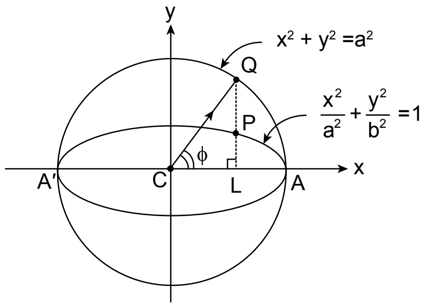
Eccentricity	$e = \sqrt{1-(a^2/b^2)}$	Major axis	$x = 0$
Length of Major axis	2b	Minor axis:	y = 0,
Length of Minor axis	2a	foci:	$(0, \pm be)$
L.R. :	$y = \pm be$, length of LR $= 2a^2/b$	Extremities	$(\pm a^2/b, be)$

Equation of ellipse where centre lies at (α, β) and major axis is parallel to the x-axis of length 2a and minor axis of 2b (a > b) $\frac{(x-\alpha)^2}{a^2}+\frac{(y-\beta)^2}{b^2}=1$.

Major axis:	$y = \beta$	Length of Major axis	2a
Minor axis :	$x = \alpha$	Length of Minor axis	2b
Foci	$S_1 = (a + ae, b)$ $S_2 = (a - ae, b)$	Directrix	$x = a + a/e$, $x = a - a/e$

- **Auxiliary Circle of an Ellipse:** A circle drawn on major axis of the ellipse as diameter is called Auxiliary circle of ellipse. Given the equation of ellipse $S:\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

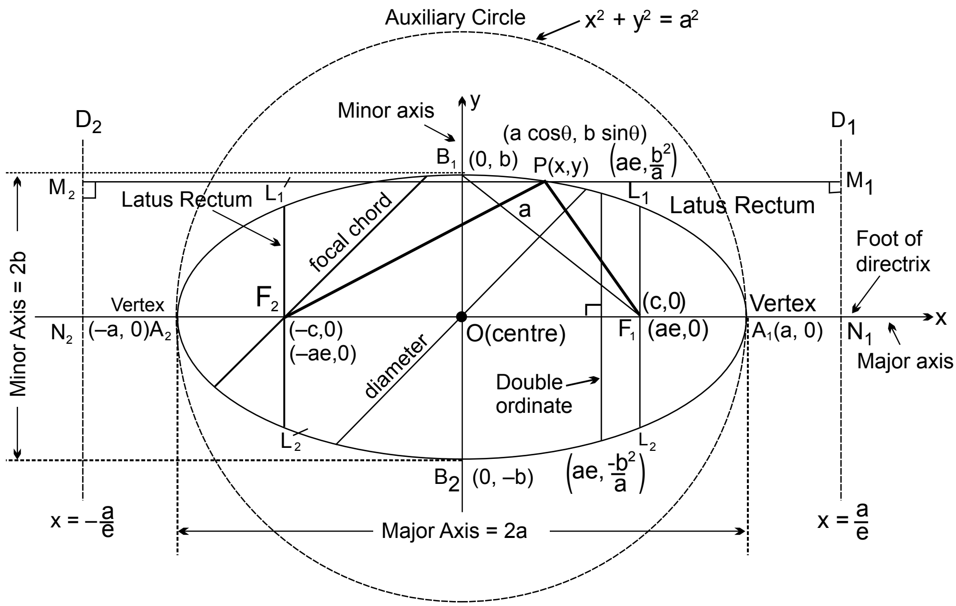
  The equation of auxiliary circle $x^2+y^2=a^2$.
- **Eccentric Angle:** Of any point P on the ellipse is angle (θ) made by CP′ with positive direction of major axis in anti-clockwise sense. (where C is centre and P′ is corresponding point of P on Auxiliary circle).



$\because \quad x_P' = x_P = a\cos\theta \qquad \Rightarrow \frac{a^2\cos^2\theta}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow y^2 = b^2\sin^2\theta$

❒ Parametric equation $x_p = a\cos\theta$ and $y_p = b\sin\theta \in [0, 2\pi)$ $(a\cos\theta, b\sin\theta)$, is called point θ an the ellipse.

17.4 PROPERTIES RELATED TO ELLIPSE AND AUXILIARY CIRCLE



❒ The ratio of ordinate of point P on ellipse and its corresponding point on AC is constant $\frac{PM}{P'M}=\frac{b\sin\theta}{a\cos\theta}=\frac{b}{a}$.

❒ The ratio of area of triangle inscribed in ellipse (ΔPQR) to the area of triangle (ΔP′Q′R′) formed by its corresponding point an A.C. is constant = b/a.

❒ The above property is valid even for an n-sided polygon inscribed in the ellipse. As $n \to \infty$ is the polygon that coincides with the ellipse and its corresponding polygon coincides with auxiliary circle.

17.4.1 Position of a Point with Respect to Ellipse $S: \frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$

A point $P(x_1,y_1)$ lies inside/on/outside of ellipse as $S_1 < 0/S_1 = 0/S_1 > 0$.

17.4.2 Position of a Line with Respect to Ellipse

The Straight line $y = mx + c$, cuts/touches/has no contact with ellipse.

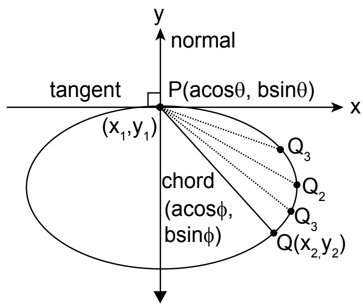
$S: \frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ as the equation $b^2x^2 + a^2(mx + c)^2 - a^2b^2 = 0$ has $D > 0/D = 0/D < 0$.

- Condition of tangency, $\pm\sqrt{a^2m^2+b^2}$. Thus all lines of the form $y=mx\pm\sqrt{a^2m^2+b^2}$ will always be tangent to the ellipse, where m is real.
- Equation of tangent in terms of slope also known as ever tangent $y=mx\pm\sqrt{a^2m^2+b^2}$ and point of contact is $\left(-\frac{a^2m}{c},\frac{b^2}{c}\right)$.
- Chord of ellipse joining point θ and ϕ:

  Slope of chord of joining point θ and ϕ $=-\frac{b}{a}\cot\left(\frac{\theta+\phi}{2}\right)$

  Equation of chord : $\frac{x}{a}\cos\frac{\theta+\phi}{2}+\frac{y}{b}\sin\frac{\theta+\phi}{2}=\cos\frac{\theta-\phi}{2}$

  Condition of focal chord: If Passes through (ae, 0) or (–ae, 0) $\Rightarrow \tan\frac{\theta}{2}\tan\frac{\phi}{2}=\frac{e-1}{e+1}$, or $\frac{e+1}{e-1}$



- Equation of tangent at θ (a cosθ, b sinθ): $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$
- **Equation of Normal at θ: Slope:** $m=\frac{a}{b}\tan\theta \Rightarrow$ Equation: $y-b\sin\theta=\frac{a\sin\theta}{b\cos\theta}(x-a\cos\theta)$

$\Rightarrow$ $ax\sec\theta - by\,\mathrm{cosec}\,\theta = a^2 - b^2 = a^2e^2$

- Equation of tangent $T:\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=0$ and equation of Normal $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2=a^2e^2$.

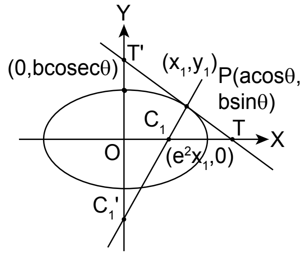
17.5 PROPERTIES OF TANGENTS AND NORMALS

- The slopes and equations of various tangents and normals are given by

Construction	Slope	Equation
Tangent at (x_1,y_1):	$-\frac{b^2x_1}{a^2y_1}$	$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=0$
Tangent at θ:	$-\frac{b}{a}\cot\theta$	$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$

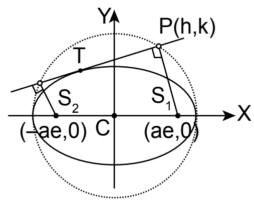
Construction	Slope	Equation
Normal at (x_1,y_1):	$\dfrac{a^2y_1}{b^2x_1}$	$\dfrac{a^2x}{x_1}-\dfrac{b^2y}{y_1}=\underbrace{a^2-b^2}_{a^2e^2}$
Normal at θ:	$\dfrac{a}{b}\tan\theta$	$ax\sec\theta-by\operatorname{cosec}\theta=\underbrace{a^2-b^2}_{a^2e^2}$

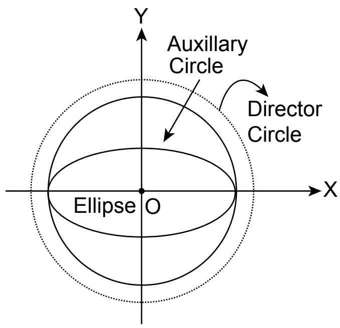
- Point of Intersection of Tangent: Point of intersection of tangent at 'ϕ' and 'θ' on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is $\left(\dfrac{a\cos\left(\dfrac{\theta+\phi}{2}\right)}{\cos\left(\dfrac{\theta-\phi}{2}\right)},\dfrac{b\sin\left(\dfrac{\theta+\phi}{2}\right)}{\cos\left(\dfrac{\theta-\phi}{2}\right)}\right)$



- Locus of foot of perpendicular from either foci upon any tangent, is auxiliary circle of ellipse.

- Locus of point of intersection of a perpendicular tangents is the director circle of ellipse, in fact the locus of point of intersection of perpendicular tangents (in case of conic sections other than parabola) is called, 'director circle of conic section'.

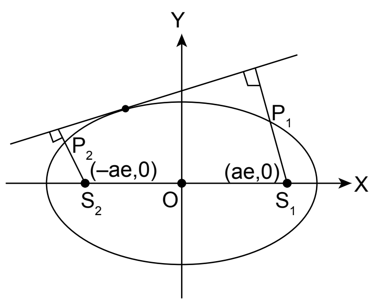


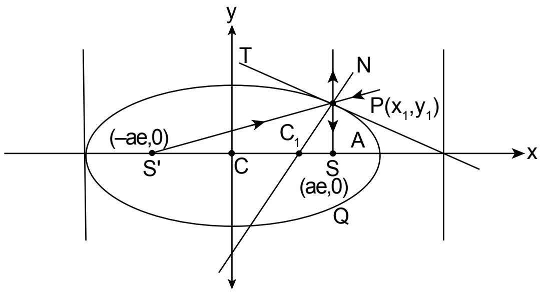


- Product of length of perpendiculars, from both foci upon any tangent, is constant (b^2) where b is length of semi-major axis of ellipse.

  product of the length's of the perpendiculars from either foci on a variable tangent to an Ellipse/Hyperbola = **(semi minor axis)²/(semi conjugate axis)² = b²**.

- Tangent at any point (P) bisects the external angle and normal at same point bisects the internal angle between focal distances of P. This refers to the well-known reflection property of the ellipse which states that rays from one are reflected through other focus and vice-versa.





- In general, four normals can be drawn to an ellipse from any point and if α, β, δ, γ, are the eccentric angles of these four co-normal points, then $\alpha + \beta + \delta + \gamma$ is an odd multiple of p.
- In general, there are four concyclic points on an ellipse and if α, β, δ, γ are the eccentric angles of these four points, then $\alpha + \beta + \delta + \gamma$ is an even multiple of p.
- The circle on any focal distance as diameter touches the auxiliary circle.
- The straight lines joining each focus to the foot of the perpendicular from the other focus upon the tangent at any point P meet on the normal PG and bisects it where G is the point where normal at P meets the major axis.
- Chord of contact: $T = \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0$
- Pair of tangents: $SS_1 = T^2 \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1\right)^2$
- Chord with a given middle point: $T = S_1 \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1\right) = \left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right) \Rightarrow \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$
- **Diameter**: The locus of the mid points of a system of parallel chords of an ellipse is called the diameter and the point where the diameter intersects the ellipse is called the vertex of the diameter.

  If $y = mx + c$ is the system of parallel chords to $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right)$, then the locus of the midpoint is given by $y = -\frac{b^2 x}{a^2 m}$.
- **Conjugate diameter:** Two diameters are said to be conjugate if each bisects all chords parallel to the other.
- If conjugate diameters are perpendicular to each other, then ellipse becomes a circle.
- The eccentric angles of the ends of a pair of conjugate diameters of an ellipse differ by a right angle.
- The sum of squares of any two conjugate semi-diameters of an ellipse is constant and is equal to sum of the squares of the semi-axes of the ellipse.
- The product of the focal distances of a point on an ellipse is equal to the square of the semi-diameter which is conjugate to the diameter through the point.
- The tangents at the extremities of a pair of conjugate diameters form a parallelogram whose area is constant and is equal to the area of rectangle formed by major and minor axis lengths.