Remainder Theorem and Factor Theorem 7

STUDY NOTES

- The value of a polynomial $p(x)$ at $x = \alpha$ is obtained by substituting $x = \alpha$ in the given polynomial and is denoted by $p(\alpha)$.
- *Remainder Theorem*
  If a polynomial $p(x)$ is divided by $(x - \alpha)$, then the remainder is $p(\alpha)$.
- When $p(x)$ is divided by $(x + \alpha)$, then remainder = $p(-\alpha)$.
- When $p(x)$ is divided by $(ax + b)$, then remainder = $p\left[-\frac{b}{a}\right]$
- A polynomial $g(x)$ is called a factor of polynomial $p(x)$ if $g(x)$ divides $p(x)$ exactly, i.e., on dividing $p(x)$ by $g(x)$, we get 0 as remainder.
- *Factor Theorem*
  Let $p(x)$ be a polynomial and α be a real number. Then, $(x - \alpha)$ is a factor of $p(x)$ if $p(\alpha) = 0$
- $(x + \alpha)$ is a factor of $p(x)$ if $p(-\alpha) = 0$
- $(ax + b)$ is a factor of $p(x)$ if $p\left(\frac{-b}{a}\right) = 0$

QUESTION BANK

A. Multiple Choice Questions [1 Mark]

Choose the correct option:

1. $(x + 1)$ is a factor of :
(a) $x^3 + 4x^2 + 5x + 2$ (b) $x^3 - 4x^2 - 5x + 2$ (c) $x^3 - 3x^2 - 7x - 1$ (d) $2x^3 - 4x^2 - 5x + 1$

2. On dividing $mx^3 + 9x^2 + 4x - 10$ by $(x + 3)$, the remainder is 5. The value of m is :
(a) -1 (b) 2 (c) -2 (d) 3

3. When $x^4 + 1$ is divided by $(x + 1)$, the remainder is :
(a) 0 (b) 2 (c) 1 (d) -1

4. If a polynomial $p(x)$ is divided by $(x + \alpha)$, then the remainder is :
(a) $p(\alpha)$ (b) $p(-\alpha)$ (c) $\alpha p(x)$ (d) $p(x) + \alpha$

5. The remainder when $x^4 - x^3 + x^2 - x + 1$ is divided by $(x - 1)$ is :
(a) 0 (b) 1 (c) -1 (d) 2

6. If $(x - 2)$ is a factor $x^2 + mx - 2$, then the value of m is :
(a) 1 (b) -1 (c) 0 (d) -2

7. When $f(x)$ is divided by $(ax - b)$, then the remainder is :
(a) $f(a)$ (b) $f(ab)$ (c) $f(-b/a)$ (d) $f(b/a)$

8. When a polynomial $f(x)$ is divided by $(3x + 4)$, the remainder is :
(a) $f\left(\frac{3}{4}\right)$ (b) $f\left(\frac{4}{3}\right)$ (c) $f\left(-\frac{3}{4}\right)$ (d) $f\left(-\frac{4}{3}\right)$

9. If $(x - 2)$ is a factor of $x^2 - 4x + m$, then the value of m is :
(a) -4 (b) 4 (c) 0 (d) -1

10. If $(x - 1)$ is a factor of $x^3 + 2x^2 - x + k$, then the value of k is :
(a) 1 (b) 2 (c) -2 (d) -1

11. If $(2x - 1)$ is a factor of $f(x) = 2x^2 + px - 5$, then the value of p is :
(a) 10 (b) 9 (c) 8 (d) 5

12. If $(x - 2)$ is a factor of $f(x) = x^3 + kx^2 - 5x - 6$, then the value of k is :
(a) 2 (b) 1 (c) -1 (d) -2

13. Given $f(x) = 3x^2 - 5x + p$. If $(x - 2)$ is a factor of $f(x)$, then the value of p is :
(a) -2 (b) 2 (c) -1 (d) 1

14. If $(x - 1)$ and $(x + 2)$ are factors of $f(x) = x^3 + 10x^2 + ax + b$, then the value of b is :
(a) 10 (b) 12 (c) 18 (d) -18

15. When the polynomials $f(x) = (px^3 + 3x^2 - 3)$ and $g(x) = (2x^3 - 5x + p)$ are divided by $(x - 4)$, they leave the same remainder. The value of p is :
(a) -1 (b) 0 (c) 1 (d) 2

16. Given $f(x) = x^3 + (kx + 8)\,x + k$. If $f(x)$ is divided by $(x + 1)$, the remainder is :
(a) $2k - 9$ (b) $2k + 9$ (c) $2k$ (d) 0

17. If $(x - 3)$ is a factor of $f(x) = x^2 - kx + 12$, then the other factor of $f(x)$ is :
(a) $x - 4$ (b) $x - 1$ (c) $x - 2$ (d) $x + 4$

18. $(x + 2)$ and $(x + 3)$ are factors of $f(x) = x^3 + ax + b$, then the value of $f(-3)$ is :
(a) 1 (b) 2 (c) -1 (d) 0

19. Given $f(x) = (3k + 2)\,x^3 + (k - 1)$. If $f(x)$ is divided by $(2x + 1)$, the remainder is :
(a) $f\left(\frac{1}{2}\right)$ (b) $f\left(-\frac{1}{2}\right)$ (c) $f(2)$ (d) $f(-2)$

20. On dividing $x^2 - 4x + m$ by $(x - 2)$, the remainder is -1. The value of m is :
(a) 1 (b) 2 (c) -2 (d) 3

Answers

1. (a)	**2.** (b)	**3.** (b)	**4.** (b)	**5.** (b)	**6.** (b)	**7.** (d)	**8.** (d)	**9.** (b)	**10.** (c)
11. (b)	**12.** (a)	**13.** (a)	**14.** (d)	**15.** (c)	**16.** (a)	**17.** (a)	**18.** (d)	**19.** (b)	**20.** (d)

B. Short Answer Type Questions [3 Marks]

1. Using remainder theorem, find the value of k if on dividing $2x^3 + 3x^2 - kx + 5$ by $(x - 2)$, leaves a remainder 7.

Sol. Since, $x - 2$ is a factor of $p(x) = 2x^3 + 3x^2 - kx + 5$, so
$x - 2 = 0 \Rightarrow x = 2$
$p(2) = 7 \;\Rightarrow\; 2\,(2)^3 + 3(2)^2 - 2k + 5 = 7$
$\Rightarrow 16 + 12 + 5 - 2k = 7 \;\Rightarrow k = 13.$

2. What must be subtracted from $16x^3 - 8x^2 + 4x + 7$ so that the resulting expression has $(2x + 1)$ as a factor?

Sol. Let us find the remainder when $p(x) = 16x^3 - 8x^2 + 4x + 7$ is divided by $(2x + 1)$, we find $p\left(-\frac{1}{2}\right)$

$$p\left(-\frac{1}{2}\right) = 16 \times \left(\frac{-1}{2}\right)^3 - 8\times\left(-\frac{1}{2}\right)^2 + 4 \times \left(-\frac{1}{2}\right) + 7 = 1$$

Since the remainder is 1, so 1 must be subtracted from $p(x)$, so that the resulting expression has $(2x + 1)$, as a factor.

3. When divided by $(x - 3)$, the polynomials $f(x) = x^3 - px^2 + x + 6$ and $g(x) = 2x^3 - x^2 - (p + 3)\,x - 6$ leave the same remainder. Find the value of p. Also, find the remainder.

Sol. $f(x) = x^3 - px^2 + x + 6$
$f(3) = (3)^3 - p(3)^2 + 3 + 6 = 36 - 9p$
and $g(x) = 2x^3 - x^2 - (p + 3)x - 6$
Then $g(3) = 2(3)^3 - (3)^2 - (p + 3)3 - 6 = 30 - 3p$
$f(3) = g(3)$
$\Rightarrow 36 - 9p = 30 - 3p \;\Rightarrow 6p = 6 \Rightarrow p = 1$
Remainder $= 36 - 9 \times 1 = 27.$

4. Find the value of k, if $(x - 2)$ is a factor of $x^3 + 2x^2 - kx + 10$. Hence, determine whether $(x + 5)$ is also a factor.

Sol. $\therefore$ $(x - 2)$ is a factor of $p(x) = x^3 + 2x^2 - kx + 10$, so, $p(2) = 0$.
$\Rightarrow (2)^3 + 2(2)^2 - k \times 2 + 10 = 0$

$\Rightarrow 8 + 8 + 10 - 2k = 0 \ \Rightarrow -2k = -26 \Rightarrow k = 13.$

So, $p(x) = x^3 + 2x^2 - 13x + 10$

Now, we find $p(-5)$

Since $p(-5) = 0$, so $(x + 5)$ is a factor of $p(x)$.

$p(-5) = (-5)^3 + 2(-5)^2 - 13 \times (-5) + 10 = -125 + 50 + 65 + 10 = 0.$

5. Find a, if the two polynomials $ax^3 + 3x^2 - 9$ and $2x^3 + 4x + a$ leave the same remainder when divided by $(x + 3)$. Also, find the remainder.

Sol. Let $p(x) = ax^3 + 3x^2 - 9$

$\therefore\ p(-3) = a(-3)^3 + 3(-3)^2 - 9 = -27a + 18$

and let $q(x) = 2x^3 + 4x + a$

$\therefore\ q(-3) = 2(-3)^3 + 4(-3) + a = a - 66$

Given, $p(-3) = q(-3)$

$\Rightarrow -27a + 18 = a - 66$

$\Rightarrow -28a = -84 \Rightarrow a = 3.$

$\therefore$ Remainder $= -27 \times 3 + 18 = -63.$

6. If $p(x) = 2x^3 + 3x^2 - ax - b$ is divided by $(x - 1)$, the remainder is 6, then find the value of $a + b$.

Sol. Let $p(x) = 2x^3 + 3x^2 - ax - b$

$\Rightarrow p(1) = 2(1)^3 + 3(1)^2 - a \times 1 - b = 6$

$\Rightarrow 2 + 3 - (a + b) = 6 \quad \Rightarrow a + b = -1$

7. Find the value of p if $x^3 + px + 2p - 2$ is exactly divisible by $(x + 1)$.

Sol. Let $f(x) = x^3 + px + 2p - 2$

$\therefore\ f(-1) = (-1)^3 + p(-1) + 2p - 2 = 0$

$\Rightarrow -1 - p + 2p - 2 = 0 \ \Rightarrow p = 3.$

C. Long Answer Type Questions [4 Marks]

1. If $(x + 2)$ and $(x + 3)$ are factors of $x^3 + ax + b$, find the values of a and b.

Sol. Let $f(x) = x^3 + ax + b$

$f(-2) = 0 \ \Rightarrow\ (-2)^3 + a(-2) + b = 0$

$\Rightarrow -2a + b = 8$ ...(i)

$\therefore\ f(-3) = 0 \Rightarrow (-3)^3 + a(-3) + b = 0$

$\Rightarrow -3a + b = 27$ ...(ii)

Solving eq. (i) and (ii), we get, $a = -19$ and $b = -30$.

2. Use factor theorem to factorise $6x^3 + 17x^2 + 4x - 12$ completely.

Sol. Let $f(x) = 6x^3 + 17x^2 + 4x - 12$

$f(-2) = 6(-2)^3 + 17(-2)^2 + 4(-2) - 12 = -48 + 68 - 8 - 12 = 0$

$\therefore\ (x + 2)$ is a factor of $f(x)$.

Now $6x^3 + 17x^2 + 4x - 12$

$= 6x^3 + 12x^2 + 5x^2 + 10x - 6x - 12$

$= 6x^2(x + 2) + 5x(x + 2) - 6(x + 2)$

$= (x + 2)(6x^2 + 5x - 6)$

$= (x + 2)(6x^2 + 9x - 4x - 6)$

$= (x + 2)[3x(2x + 3) - 2(2x + 3)]$

$= (x + 2)(3x - 2)(2x + 3).$

3. If $(x - 2)$ is a factor of $2x^3 - x^2 - px - 2$,

(a) find the value of p. (b) with the value of p, factorise the above expression completely.

Sol. $\because\ x - 2$ is a factor of given polynomial

(a) Let $f(x) = 2x^3 - x^2 - px - 2$, then

$f(2) = 2(2)^3 - (2)^2 - p(2) - 2 = 0$

$\Rightarrow 16 - 4 - 2p - 2 = 0 \ \Rightarrow p = 5.$

(b) $f(x) = 2x^3 - x^2 - 5x - 2$
$= 2x^3 - 4x^2 + 3x^2 - 6x + x - 2$
$= 2x^2(x - 2) + 3x(x - 2) + 1(x - 2)$
$= (x - 2)(2x^2 + 3x + 1)$
$= (x - 2)(2x^2 + 2x + x + 1)$
$= (x - 2)[2x(x + 1) + 1(x + 1)]$
$= (x - 2)(x + 1)(2x + 1)$

4. Given that $(x + 2)$ and $(x + 3)$ are factors of $2x^3 + ax^2 + 7x - b$. Determine the values of a and b.

Sol. Let $f(x) = 2x^3 + ax^2 + 7x - b$
$\therefore f(-2) = 2(-2)^3 + a(-2)^2 + 7(-2) - b = 0$
$\Rightarrow -16 + 4a - 14 - b = 0$
$\Rightarrow 4a - b = 30$ (i)
$\therefore f(-3) = 2(-3)^3 + a(-3)^2 + 7(-3) - b = 0$
$\Rightarrow 9a - b = 75$ (ii)
Solving (i) and (ii), we get
$a = 9$ and $b = 6$.

5. Using the remainder theorem, factorise completely the following polynomial : $3x^3 + 2x^2 - 19x + 6$.

Sol. Let $f(x) = 3x^3 + 2x^2 - 19x + 6$
$f(2) = 3(2)^3 + 2(2)^2 - 19 \times 2 + 6 = 24 + 8 - 38 + 6 = 0$.
So, $(x - 2)$ is a factor of $f(x)$.
Now, $3x^3 + 2x^2 - 19x + 6$
$= 3x^3 - 6x^2 + 8x^2 - 16x - 3x + 6$
$= 3x^2(x - 2) + 8x(x - 2) - 3(x - 2)$
$= (x - 2)(3x^2 + 8x - 3)$
$= (x - 2)(3x^2 + 9x - x - 3)$
$= (x - 2)(3x(x + 3) - 1(x - 3)]$
$= (x - 2)(x + 3)(3x - 1)$.

6. If $(x - 2)$ is a factor of the expression, $2x^3 + ax^2 + bx - 14$ and when the expression is divided by $(x - 3)$, it leaves a remainder 52, find the values of a and b.

Sol. Let $f(x) = 2x^3 + ax^2 + bx - 14$
$\therefore f(2) = 2(2)^3 + a(2)^2 + b(2) - 14 = 0$
$\Rightarrow 16 + 4a + 2b - 14 = 0$
$\Rightarrow 2a + b = -1$ (i)
$f(3) = 2(3)^3 + a(3)^2 + b(3) - 14 = 52$
$\Rightarrow 54 + 9a + 3b = 66$
$\Rightarrow 3a + b = 4$ (ii)
Solving (i) and (ii) , we get
$a = 5$ and $b = -11$.

7. Using the remainder theorem and factor theorem, factorise the following polynomial : $x^3 + 10x^2 - 37x + 26$.

Sol. Let $f(x) = x^3 + 10x^2 - 37x + 26$
$f(1) = (1)^3 + 10(1)^2 - 37 \times 1 + 26 = 1 + 10 - 37 + 26 = 0$
So, $(x - 1)$ is a factor of $f(x)$.
$x^3 + 10x^2 - 37x + 26$
$= x^3 - x^2 + 11x^2 - 11x - 26x + 26$
$= x^2(x - 1) + 11x(x - 1) - 26(x - 1)$
$= (x - 1)(x^2 + 11x - 26)$
$= (x - 1)(x^2 + 13x - 2x - 26)$
$= (x - 1)[x(x + 13) - 2(x + 13)]$
$= (x - 1)(x + 13)(x - 2)$.

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