Chapter 4
Determinants

Exercise 4.4

Q. 1 A

Write Minors and Cofactors of the elements of following determinants:

$$\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$$

Answer:

Minor: Minor of an element aij of a determinant is the determinant obtained by removing i^{th} row and j^{th} column in which element a_{ij} lies. It is denoted by M_{ij}.

Cofactor: Cofactor of an element aij, $A_{ij} = (-1)^{i+j} M_{ij}$.

Minor of element $a_{ij} = M_{ij}$

$a_{11} = 2$, Minor of element $a_{11} = M_{11} = 3$

Here removing 1st row and 1st column from the determinant we are left out with 3 so $M_{11} = 3$.

Similarly, finding other Minors of the determinant

$a_{12} = -4$, Minor of element $a_{12} = M_{12} = 0$

$a_{21} = 0$, Minor of element $a_{21} = M_{21} = -4$

$a_{22} = 3$, Minor of element $a_{22} = M_{22} = 2$

Cofactor of an element aij, $A_{ij} = (-1)^{i+j} \times M_{ij}$

$A_{11} = (-1)^{1+1} \times M_{11} = 1 \times 3 = 3$

$A_{12} = (-1)^{1+2} \times M_{12} = (-1) \times 0 = 0$

$A_{21} = (-1)^{2+1} \times M_{11} = (-1) \times (-4) = 4$

$A_{22} = (-1)^{2+2} \times M_{22} = 1 \times 2 = 2$

Q. 1 B

Write Minors and Cofactors of the elements of following determinants:

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

Answer:

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

Minor of an element $a_{ij} = M_{ij}$

$a_{11} = a$, Minor of element $a_{11} = M_{11} = d$

Here removing 1st row and 1st column from the determinant we are left out with d so $M_{11} = d$.

Similarly, finding other Minors of the determinant

$a_{12} = c$, Minor of element $a_{12} = M_{12} = b$

$a_{21} = b$, Minor of element $a_{21} = M_{21} = c$

$a_{22} = d$, Minor of element $a_{22} = M_{22} = a$

Cofactor of an element a_{ij}, $A_{ij} = (-1)^{i+j} \times M_{ij}$

$A_{11} = (-1)^{1+1} \times M_{11} = 1 \times d = d$

$A_{12} = (-1)^{1+2} \times M_{12} = (-1) \times b = -b$

$A_{21} = (-1)^{2+1} \times M_{11} = (-1) \times c = -c$

$A_{22} = (-1)^{2+2} \times M_{22} = 1 \times a = a$

Q. 2 A

Write Minors and Cofactors of the elements of following determinants:

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Answer:

Minor of an element $a_{ij} = M_{ij}$

$a_{11} = 1$, Minor of element $a_{11} = M_{11} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = (1 \times 1) - (0 \times 0) = 1$

Here removing 1st row and 1st column from the determinant we are left out with the determinant $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$. Solving this we get $M_{11} = 1$

Similarly, finding other Minors of the determinant

$a_{12} = 0$, Minor of element $a_{12} = M_{12} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = (0 \times 1) - (0 \times 0) = 0$

$a_{13} = 0$, Minor of element $a_{13} = M_{13} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = (0 \times 0) - (1 \times 0) = 0$

$a_{21} = 0$, Minor of element $a_{21} = M_{21} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = (0 \times 1) - (0 \times 0) = 0$

$a_{22} = 1$, Minor of element $a_{22} = M_{22} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = (1 \times 1) - (0 \times 0) = 1$

$a_{23} = 0$, Minor of element $a_{23} = M_{23} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = (1 \times 0) - (0 \times 0) = 0$

$a_{31} = 0$, Minor of element $a_{31} = M_{31} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = (0 \times 0) - (0 \times 1) = 0$

$a_{32} = 0$, Minor of element $a_{32} = M_{32} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = (1 \times 0) - (0 \times 0) = 0$

$a_{33} = 1$, Minor of element $a_{33} = M_{33} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = (1 \times 1) - (0 \times 0) = 1$

Cofactor of an element a_{ij}, $A_{ij} = (-1)^{i+j} \times M_{ij}$

$A_{11} = (-1)^{1+1} \times M_{11} = 1 \times 1 = 1$

$A_{12} = (-1)^{1+2} \times M_{12} = (-1) \times 0 = 0$

$A_{13} = (-1)^{1+3} \times M_{13} = 1 \times 0 = 0$

$A_{21} = (-1)^{2+1} \times M_{21} = (-1) \times 0 = 0$

$A_{22} = (-1)^{2+2} \times M_{22} = 1 \times 1 = 1$

$A_{23} = (-1)^{2+3} \times M_{23} = (-1) \times 0 = 0$

$A_{31} = (-1)^{3+1} \times M_{31} = 1 \times 0 = 0$

$A_{32} = (-1)^{3+2} \times M_{32} = (-1) \times 0 = 0$

$A_{33} = (-1)^{3+3} \times M_{33} = 1 \times 1 = 1$

Q. 2 B

Write Minors and Cofactors of the elements of following determinants:

$$\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$$

Answer:

$$\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$$

Minor of an element $a_{ij} = M_{ij}$

$a_{11} = 1$, Minor of element $a_{11} = M_{11} = \begin{vmatrix} 5 & -1 \\ 1 & 2 \end{vmatrix} = (5 \times 2) - ((-1) \times 1)$ $= 10 + 1 = 11$

Here removing 1st row and 1st column from the determinant we are left out with the determinant $\begin{vmatrix} 5 & -1 \\ 1 & 2 \end{vmatrix}$. Solving this we get $M_{11} = 11$

Similarly, finding other Minors of the determinant

a_{12} = 0, Minor of element a_{12} = M_{12} = $\begin{vmatrix} 3 & -1 \\ 0 & 2 \end{vmatrix}$ = (3 × 2) – ((-1) × 0) = (6 - 0) = 6

a_{13} = 4, Minor of element a_{13} = M_{13} = $\begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix}$ = (3 × 1) – (5 × 0) = 3 - 0 = 3

a_{21} = 3, Minor of element a_{21} = M_{21} = $\begin{vmatrix} 0 & 4 \\ 1 & 2 \end{vmatrix}$ = (0 × 2) – (4 × 1) = 0 – 4 = -4

a_{22} = 5, Minor of element a_{22} = M_{22} = $\begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix}$ = (1 × 2) – (4 × 0) = 2 – 0 = 2

a_{23} = -1, Minor of element a_{23} = M_{23} = $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$ = (1 × 1) – (0 × 0) = 1

a_{31} = 0, Minor of element a_{31} = M_{31} = $\begin{vmatrix} 0 & 4 \\ 5 & -1 \end{vmatrix}$ = (0 × (-1)) – (4 × 5) = 0 – 20 = -20

a_{32} = 1, Minor of element a_{32} = M_{32} = $\begin{vmatrix} 1 & 4 \\ 3 & -1 \end{vmatrix}$ = (1 × (-1)) – (4 × 3) = -1 – 12 = -13

a_{33} = 2, Minor of element a_{33} = M_{33} = $\begin{vmatrix} 1 & 0 \\ 3 & 5 \end{vmatrix}$ = (1 × 5) – (0 × 3) = (5 – 0) = 5

Cofactor of an element aij, Aij = $(-1)^{i+j}$ × Mij

$A_{11} = (-1)^{1+1} \times M_{11} = 1 \times 11 = 11$

$A_{12} = (-1)^{1+2} \times M_{12} = (-1) \times 6 = -6$

$A_{13} = (-1)^{1+3} \times M_{13} = 1 \times 3 = 3$

$A_{21} = (-1)^{2+1} \times M_{21} = (-1) \times (-4) = 4$

$A_{22} = (-1)^{2+2} \times M_{22} = 1 \times 2 = 2$

$A_{23} = (-1)^{2+3} \times M_{23} = (-1) \times 1 = -1$

$A_{31} = (-1)^{3+1} \times M_{31} = 1 \times (-20) = -20$

$A_{32} = (-1)^{3+2} \times M_{32} = (-1) \times (-13) = 13$

$A_{33} = (-1)^{3+3} \times M_{33} = 1 \times 5 = 5$

Q. 3

Using Cofactors of elements of second row, evaluate $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

Answer:

To evaluate a determinant using cofactors, Let

$B = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Expanding along Row 1

$B = (-1)^{1+1} \times a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} \times a_{12} \times \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} \times a_{13} \times \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

$B = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}$

[Where Aij represents cofactors of aij of determinant B.]

B = Sum of product of elements of R_1 with their corresponding cofactors

Similarly, the determinant can be solved by expanding along column

So, B = sum of product of elements of any row or column with their corresponding cofactors

$$\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

Cofactors of second row

$A_{21} = (-1)^{2+1} \times M_{21} = (-1) \times \begin{vmatrix} 3 & 8 \\ 2 & 3 \end{vmatrix} = (-1) \times (3 \times 3 - 8 \times 2) = (-1) \times (-7) = 7$

$A_{22} = (-1)^{2+2} \times M_{22} = 1 \times \begin{vmatrix} 5 & 8 \\ 1 & 3 \end{vmatrix} = (5 \times 3 - 8 \times 1) = 7$

$A_{23} = (-1)^{2+3} \times M_{23} = (-1) \times \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = (-1) \times (5 \times 2 - 3 \times 1) = (-1) \times 7 = -7$

[Where $Aij = (-1)^{i+j} \times M_{ij}$, M_{ij} = Minor of ith row & jth column]

Therefore,

$\Delta = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$

$\Delta = 2 \times 7 + 1 \times (-7) = 14 - 7 = 7$

Ans: $\Delta = 7$

Q. 4

Using Cofactors of elements of third column, evaluate.

$$\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$$

Answer:

To evaluate a determinant using cofactors, Let

$$B = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expanding along Row 1

$B = (-1)^{1+1} \times a_{11} \times \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} \times a_{12} \times \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} \times a_{13} \times \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

$B = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}$

[Where Aij represents cofactors of aij of determinant B.]

B = Sum of product of elements of R_1 with their corresponding cofactors

Similarly, the determinant can be solved by expanding along column

So, B = sum of product of elements of any row or column with their corresponding cofactors

$$\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$$

Cofactors of third column

$A_{13} = (-1)^{1+3} \times M_{13} = 1 \times \begin{vmatrix} 1 & y \\ 1 & z \end{vmatrix} = 1 \times (1 \times z - 1 \times y) = (z - y)$

$A_{23} = (-1)^{2+3} \times M_{23} = (-1) \times \begin{vmatrix} 1 & x \\ 1 & z \end{vmatrix} = (-1) \times (1 \times z - 1 \times x) = -(z - x) = (x - z)$

$A_{33} = (-1)^{3+3} \times M_{33} = 1 \times \begin{vmatrix} 1 & x \\ 1 & y \end{vmatrix} = 1 \times (1 \times y - 1 \times x) = (y - x)$

[Where $A_{ij} = (-1)^{i+j} \times M_{ij}$, M_{ij} = Minor of ith row & jth column]

Therefore,

$\Delta = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}$

$\Delta = yz (z - y) + zx (x - z) + xy (y - x) = z [y (z - y) + x (x - z)] + xy (y - x)$

$\Delta = z (yz - y^2 + x^2 - xz) + xy (y - x) = z [(yz - xz) + (x^2 - y^2)] + xy (y - x)$

$\Delta = z [z \times (y - x) + (x + y) \times (x - y)] + xy (y - x)$

$\Delta = z \times (y - x) \times (z – x - y) + xy (y - x)$

$\Delta = (y - x) \times (z^2 – xz – yz + xy)$

$\Delta = (y - x) \times [z (z - x) – y (z - x)] = (y - x) \times (z - y) \times (z - x)$

$\Delta = (x - y) (y - z) (z - x)$

Ans: $\Delta = (x - y) (y - z) (z - x)$

Q. 5

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & aa_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and A_{ij} is Cofactors of aij, then value of Δ is given by

A. $a_{11} A_{31} + a_{12} A_{32} + a_{13} A_{33}$

B. $a_{11} A_{11} + a_{12} A_{21} + a_{13} A_{31}$

C. $a_{21} A_{11} + a_{22} A_{12} + a_{23} A_{13}$

D. $a_{11} A_{11} + a_{21} A_{21} + a_{31} A_{31}$

Answer:

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expanding along Column 1

$\Delta = B = (-1)^{1+1} \times a_{11} \times \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{2+1} \times a_{21} \times \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + +$
$(-1)^{3+1} \times a_{31} \times \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$

$\Delta = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$