QUADRATIC EQUATION

1. INTRODUCTION :

A function f defined by $f(x) = a_n x^n + a_{n-1}x^{n-1} + ...+ a_1 x + a_0$

where $a_0, a_1, a_2, ..., a_n \in R$ is called a polynomial of degree n with real coefficients ($a_n \neq 0$, $n \in W$). If $a_0, a_1, a_2, ..., a_n \in C$, it is called a polynomial with complex coefficients.

The algebraic expression of the form $ax^2 + bx + c$, $a \neq 0$ is called a quadratic expression, because the highest order term in it is of second degree. Quadratic equation means, $ax^2 + bx + c = 0$. In general whenever one says zeroes of the expression $ax^2 + bx + c$, it implies roots of the equation $ax^2 + bx + c = 0$, unless specified otherwise.

A quadratic equation has exactly two roots which may be real (equal or unequal) or imaginary.

★	a quadratic equation if	$a \neq 0$	Two Roots
★	a linear equation if	$a = 0, b \neq 0$	One Root
★	a contradiction if	$a = b = 0, c \neq 0$	No Root
★	an identity if	$a = b = c = 0$	Infinite Roots

Note : A quadratic equation cannot have three or more roots & if it has , it becomes an identity.If $ax^2 + bx + c = 0$ is an identity $\Leftrightarrow a = b = c = 0$

2. ROOTS OF QUADRATIC EQUATION & RELATION BETWEEN ROOTS & CO-EFFICIENTS :

(a) The general form of quadratic equation is $ax^2 + bx + c = 0$, $a \neq 0$.

The roots can be found in following manner :

$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0 \quad \Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} \quad \Rightarrow \quad \boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

This expression can be directly used to find the two roots of a quadratic equation.

(b) The expression $b^2 - 4ac \equiv D$ is called the discriminant of the quadratic equation.

(c) If α & β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then :

(i) $\alpha + \beta = -b/a$ (ii) $\alpha\beta = c/a$ (iii) $|\alpha - \beta| = \sqrt{D}/|a|$

(d) A quadratic equation whose roots are α & β is $(x - \alpha)(x - \beta) = 0$ i.e.

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$ i.e. x^2 – (sum of roots) x + product of roots = 0.

SOLVED EXAMPLE

Example 1: For what value of 'a', the equation $(a^2 - a - 2)x^2 + (a^2 - 4)x + (a^2 - 3a + 2) = 0$, will have more than two solutions ?

Solution: $a^2 - a - 2 = 0$, $a^2 - 4 = 0$, $a^2 - 3a + 2 = 0$

$\Rightarrow$ $a = 2, -1$ and $a = \pm 2$ and $a = 1, 2 \Rightarrow a = 2$

Now $(x^2 + x + 1)a^2 - (x^2 + 3)a - (2x^2 + 4x - 2) = 0$ will be an identity if $x^2 + x + 1 = 0$ & $x^2 + 3 = 0$ & $2x^2 + 4x - 2 = 0$ which is not possible.

Example 2 : If α, β are the roots of a quadratic equation $x^2 - 3x + 5 = 0$, then the equation whose roots are $(\alpha^2 - 3\alpha + 7)$ and $(\beta^2 - 3\beta + 7)$ is -

(A) $x^2 + 4x + 1 = 0$ (B) $x^2 - 4x + 4 = 0$ (C) $x^2 - 4x - 1 = 0$ (D) $x^2 + 2x + 3 = 0$

Solution : Since α, β are the roots of equation $x^2 - 3x + 5 = 0$

So $\alpha^2 - 3\alpha + 5 = 0$

$\beta^2 - 3\beta + 5 = 0$

$\therefore$ $\alpha^2 - 3\alpha = -5$

$\beta^2 - 3\beta = -5$

Putting in $(\alpha^2 - 3\alpha + 7)$ & $(\beta^2 - 3\beta + 7)$ ....(i)

$-5 + 7, -5 + 7$

$\therefore$ 2 and 2 are the roots.

$\therefore$ The required equation is

$x^2 - 4x + 4 = 0$. **Ans. (B)**

Example 3 : If α and β are the roots of $ax^2 + bx + c = 0$, find the value of $(a\alpha + b)^{-2} + (a\beta + b)^{-2}$.

Solution : We know that $\alpha + \beta = -\frac{b}{a}$ & $\alpha\beta = \frac{c}{a}$

$$(a\alpha + b)^{-2} + (a\beta + b)^{-2} = \frac{1}{(a\alpha+b)^2} + \frac{1}{(a\beta+b)^2}$$

$$= \frac{a^2\beta^2 + b^2 + 2ab\beta + a^2\alpha^2 + b^2 + 2ab\alpha}{(a^2\alpha\beta + ba\beta + ba\alpha + b^2)^2} = \frac{a^2(\alpha^2+\beta^2) + 2ab(\alpha+\beta) + 2b^2}{(a^2\alpha\beta + ab(\alpha+\beta) + b^2)^2}$$

($\alpha^2 + \beta^2$ can always be written as $(\alpha + \beta)^2 - 2\alpha\beta$)

$$= \frac{a^2\left[(\alpha+\beta)^2 - 2\alpha\beta\right] + 2ab(\alpha+\beta) + 2b^2}{(a^2\alpha\beta + ab(\alpha+\beta) + b^2)^2} = \frac{a^2\left[\frac{b^2-2ac}{a^2}\right] + 2ab\left(-\frac{b}{a}\right) + 2b^2}{\left(a^2\frac{c}{a} + ab\left(-\frac{b}{a}\right) + b^2\right)^2} = \frac{b^2 - 2ac}{a^2c^2}$$

Alternatively :

If α & β are roots of $ax^2 + bx + c = 0$

then, $a\alpha^2 + b\alpha + c = 0$

$$\Rightarrow \quad a\alpha + b = -\frac{c}{\alpha}$$

same as $a\beta + b = -\frac{c}{\beta}$

$$\therefore \quad (a\alpha + b)^{-2} = (\alpha\beta + b)^{-2} = \frac{\alpha^2}{c^2} + \frac{\beta^2}{c^2}$$

$$= \frac{(\alpha+\beta)^2 - 2\alpha\beta}{c^2}$$

$$= \frac{(-b/a)^2 - 2(c/a)}{c^2} = \frac{b^2 - 2ac}{a^2c^2}$$

SOLVED EXAMPLE

Example 4 : The least prime integral value of '2a' such that the roots α, β of the equation $2x^2 + 6x + a = 0$ satisfy the inequality $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} < 2$ is

'2a' का न्यूनतम अभाज्य पूर्णांक मान होगा जबकि समीकरण $2x^2 + 6x + a = 0$ के मूल α, β असमिका $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} < 2$ को सन्तुष्ट करते है–

Solution: $2x^2 + 6x + a = 0$

Its roots are α, β $\because$ इसके मूल α, β है, $\Rightarrow \alpha + \beta = -3$ & $\alpha\beta = \frac{a}{2}$

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} < 2$$

$$\Rightarrow \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} < 2 \quad \Rightarrow \frac{9-a}{a} < 1 \quad \Rightarrow \frac{2a-9}{a} > 0$$

$$\Rightarrow a \in (-\infty, 0) \cup \left(\frac{9}{2}, \infty\right) \quad \Rightarrow \quad 2a = 11 \text{ is least prime.}$$

Example 5 : If both roots of $x^2 - 32x + c = 0$ are prime numbers then possible values of c are

(A) 60 (B) 87 (C) 247 (D) 231

Solution: Split 32 into sum of two primes 32 = 2 + 30 = 3 + 29 = 5 + 27 = 7 + 25 = 11 + 21 = 13 + 19.

32 = 2 + 30 = 3 + 29 = 5 + 27 = 7 + 25 = 11 + 21 = 13 + 19. **Ans. (BC)**

Self practice problems : 1

(i) Find the roots of following equations :

(a) $x^2 + 3x + 2 = 0$

(b) $x^2 - 8x + 16 = 0$

(c) $x^2 - 2x - 1 = 0$

(ii) Find the roots of the equation $a(x^2 + 1) - (a^2 + 1)x = 0$, where $a \neq 0$.

(iii) Solve : $\frac{6-x}{x^2-4} = 2 + \frac{x}{x+2}$

(iv) If the roots of $4x^2 + 5k = (5k + 1)x$ differ by unity, then find the values of k.

Ans. (i) (a) $-1, -2$; (b) 4 ; (c) $1 \pm \sqrt{2}$; (ii) $a, \frac{1}{a}$;

(iii) $\frac{7}{3}$ (iv) $3, -\frac{1}{5}$

3. THEORY OF EQUATIONS :

Let $\alpha_1, \alpha_2, \alpha_3, ... \alpha_n$ are roots of the equation, $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... a_{n-1}x + a_n = 0$, where $a_0, a_1, ... a_n$ are constants and $a_0 \neq 0$.

$$f(x) = a_0(x - \alpha_1)(x - \alpha_2)(x - \alpha_3) .... (x - \alpha_n)$$

$$\therefore \quad a_0x^n + a_1x^{n-1} + ... a_{n-1}x + a_n = a_0(x - \alpha_1)(x - \alpha_2) ..... (x - \alpha_n)$$

Comparing the coefficients of like powers of x, we get

$$\sum \alpha_i = -\frac{a_1}{a_0} = S_1 \qquad \text{(say)}$$

or $$S_1 = -\frac{\text{coefficient of } x^{n-1}}{\text{coefficient of } x^n}$$

$$S_2 = \sum_{i \neq j} \alpha_i\alpha_j = (-1)^2 \frac{a_2}{a_0}$$

$$S_3 = \sum_{i \neq j \neq k} \alpha_i\alpha_j\alpha_k = (-1)^3 \frac{a_3}{a_0}$$

$\vdots$

$$S_n = \alpha_1\alpha_2 ......\alpha_n = (-1)^n \frac{a_n}{a_0} = (-1)^n \frac{\text{constant term}}{\text{coefficient of } x^n}$$

where S_k denotes the sum of the product of root taken k at a time.

For Cubic equation : If α, β, γ are roots of a cubic equation $ax^3 + bx^2 + cx + d = 0$, then

$$\alpha + \beta + \gamma = -\frac{b}{a}, \ \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} \ \text{ and } \ \alpha\beta\gamma = -\frac{d}{a}$$

4. TRANSFORMATION OF THE EQUATION :

Let $ax^2 + bx + c = 0$ be a quadratic equation with two roots α and β. If we have to find an equation whose roots are $f(\alpha)$ and $f(\beta)$, i.e. some expression in α & β, then this equation can be found by finding α in terms of y. Now as α satisfies given equation, put this α in terms of y directly in the equation.

$$y = f(\alpha)$$

By transformation, $\alpha = g(y)$

$$a(g(y))^2 + b(g(y)) + c = 0$$

This is the required equation in y.

SOLVED EXAMPLE

Example 6 : If the roots of $ax^2 + bx + c = 0$ are α and β, then find the equation whose roots are :

(a) $\frac{-2}{\alpha}, \frac{-2}{\beta}$ (b) $\frac{\alpha}{\alpha+1}, \frac{\beta}{\beta+1}$ (c) α^2, β^2

Solution : (a) $\frac{-2}{\alpha}, \frac{-2}{\beta}$

put, $y = \frac{-2}{\alpha} \Rightarrow \alpha = \frac{-2}{y}$

$a\left(-\frac{2}{y}\right)^2 + b\left(\frac{-2}{y}\right) + c = 0 \Rightarrow cy^2 - 2by + 4a = 0$

Required equation is $cx^2 - 2bx + 4a = 0$

(b) $\frac{\alpha}{\alpha+1}, \frac{\beta}{\beta+1}$

put, $y = \frac{\alpha}{\alpha+1} \Rightarrow \alpha = \frac{y}{1-y}$

$\Rightarrow a\left(\frac{y}{1-y}\right)^2 + b\left(\frac{y}{1-y}\right) + c = 0 \Rightarrow (a + c - b)y^2 + (-2c + b)y + c = 0$

Required equation is $(a + c - b)x^2 + (b - 2c)x + c = 0$

(c) α^2, β^2

put $y = \alpha^2 \Rightarrow \alpha = \sqrt{y}$

$ay + b\sqrt{y} + c = 0$

$b^2y = a^2y^2 + c^2 + 2acy \Rightarrow a^2y^2 + (2ac - b^2)y + c^2 = 0$

Required equation is $a^2x^2 + (2ac - b^2)x + c^2 = 0$

Example 7 : If two roots are equal, find the roots of $4x^3 + 20x^2 - 23x + 6 = 0$.

Solution : Let roots be α, α and β

$\therefore \alpha + \alpha + \beta = -\frac{20}{4} \Rightarrow 2\alpha + \beta = -5$ .... (i)

$\therefore \alpha . \alpha + \alpha\beta + \alpha\beta = -\frac{23}{4} \Rightarrow \alpha^2 + 2\alpha\beta = -\frac{23}{4}$ & $\alpha^2\beta = -\frac{6}{4}$

from equation (i)

$\alpha^2 + 2\alpha(-5 - 2\alpha) = -\frac{23}{4} \Rightarrow \alpha^2 - 10\alpha - 4\alpha^2 = -\frac{23}{4} \Rightarrow 12\alpha^2 + 40\alpha - 23 = 0$

$\therefore \alpha = 1/2, -\frac{23}{6}$ when $\alpha = \frac{1}{2}$

$\alpha^2\beta = \frac{1}{4}(-5 - 1) = -\frac{3}{2}$

when $\alpha = -\frac{23}{6} \Rightarrow \alpha^2\beta = \frac{23\times 23}{36}\left(-5 - 2x\left(-\frac{23}{6}\right)\right) \neq -\frac{3}{2} \Rightarrow \alpha = \frac{1}{2}$ $\beta = -6$

Hence roots of equation = $\frac{1}{2}, \frac{1}{2}, -6$

Example 8 : If α, β, γ are the roots of $x^3 - px^2 + qx - r = 0$, find :

(i) $\sum \alpha^3$ (ii) $\alpha^2(\beta+\gamma)+\beta^2(\gamma+\alpha)+\gamma^2(\alpha+\beta)$

Solution : We know that $\alpha+\beta+\gamma = p$

$\alpha\beta+\beta\gamma+\gamma\alpha = q$

$\alpha\beta\gamma = r$

(i) $\alpha^3+\beta^3+\gamma^3 = 3\alpha\beta\gamma+(\alpha+\beta+\gamma)\{(\alpha+\beta+\gamma)^2-3(\alpha\beta+\beta\gamma+\gamma\alpha)\}$

$= 3r + p\{p^2 - 3q\} = 3r + p^3 - 3pq$

(ii) $\alpha^2(\beta+\gamma)+\beta^2(\alpha+\gamma)+\gamma^2(\alpha+\beta) = \alpha^2(p-\alpha)+\beta^2(p-\beta)+\gamma^2(p-\gamma)$

$= p(\alpha^2+\beta^2+\gamma^2)-3r-p^3+3pq = p(p^2 - 2q) - 3r - p^3 + 3pq = pq - 3r$

Example 9 : If $b^2 < 2ac$ and a, b, c, d $\in$R, then prove that $ax^3 + bx^2 + cx + d = 0$ has exactly one real root.

Solution : Let α, β, γ be the roots of $ax^3 + bx^2 + cx + d = 0$

Then $\alpha+\beta+\gamma = -\dfrac{b}{a}$

$\alpha\beta + \beta\gamma + \gamma\alpha = \dfrac{c}{a}$

$\alpha\beta\gamma = \dfrac{-d}{a}$

$\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha) = \dfrac{b^2}{a^2} - \dfrac{2c}{a} = \dfrac{b^2-2ac}{a^2}$

$\Rightarrow$ $\alpha^2 + \beta^2 + \gamma^2 < 0$, which is not possible if all α, β, γ are real. So atleast one root is non-real, but complex roots occurs in pair. Hence given cubic equation has two non-real and one real roots.

Example 10: If the roots of $ax^3 + bx^2 + cx + d = 0$ are α, β, γ then find equation whose roots are $\dfrac{1}{\alpha\beta}, \dfrac{1}{\beta\gamma}, \dfrac{1}{\gamma\alpha}$.

Solution : Put $y = \dfrac{1}{\alpha\beta} = \dfrac{\gamma}{\alpha\beta\gamma} = -\dfrac{a\gamma}{d}$ $\quad (\because \alpha\beta\gamma = -\dfrac{d}{a})$

Put $x = -\dfrac{dy}{a}$

$\Rightarrow$ $a\left(-\dfrac{dy}{a}\right)^3 + b\left(-\dfrac{dy}{a}\right)^2 + c\left(-\dfrac{dy}{a}\right) + d = 0$

Required equation is $d^2x^3 - bdx^2 + acx - a^2 = 0$

<u>Self practice problems</u> : 2

(i) Let α, β be two of the roots of the equation $x^3 - px^2 + qx - r = 0$. If $\alpha+\beta = 0$, then show that pq = r

(ii) If two roots of $x^3 + 3x^2 - 9x + c = 0$ are equal, then find the value of c.

(iii) If α, β, γ be the roots of $ax^3 + bx^2 + cx + d = 0$, then find the value of

(a) $\sum \alpha^2$ (b) $\sum \dfrac{1}{\alpha}$ (c) $\sum \alpha^2(\beta+\gamma)$

(iv) If α, β are the roots of $ax^2 + bx + c = 0$, then find the equation whose roots are

(a) $\frac{1}{\alpha^2}, \frac{1}{\beta^2}$ (b) $\frac{1}{a\alpha+b}, \frac{1}{a\beta+b}$ (c) $\alpha+\frac{1}{\beta},\ \beta+\frac{1}{\alpha}$

(v) If α, β are roots of $x^2 - px + q = 0$, then find the quadratic equation whose root are $(\alpha^2-\beta^2)(\alpha^3-\beta^3)$ and $\alpha^2\beta^3+\alpha^3\beta^2$.

Ans. (ii) –27, 5

(iii) (a) $\frac{1}{a^2}(b^2-2ac)$, (b) $-\frac{c}{d}$, (c) $\frac{1}{a^2}(3ad-bc)$

(vi) (a) $c^2y^2 + y(2ac - b^2) + a^2 = 0$; (b) $acx^2 - bx + 1 = 0$; (c) $acx^2 + (a + c)bx + (a + c)^2 = 0$

(v) $x^2 - p(p^4 - 5p^2q + 5q^2)x + p^2q^2(p^2 - 4q)(p^2 - q) = 0$

5. NATURE OF ROOTS :

(a) Consider the quadratic equation $ax^2 + bx + c = 0$ where $a, b, c \in R$ & $a \neq 0$ then ;

$$x = \frac{-b \pm \sqrt{D}}{2a}$$

(i) $D > 0 \Leftrightarrow$ roots are real & distinct (unequal).

(ii) $D = 0 \Leftrightarrow$ roots are real & coincident (equal)

(iii) $D < 0 \Leftrightarrow$ roots are imaginary.

(b) Consider the quadratic equation $ax^2 + bx + c = 0$ where $a, b, c \in Q$ & $a \neq 0$ then ;

(i) If D is a perfect square, then roots are rational.

(ii) $a = 1$, $b, c \in I$ & D is square of an integer $\Rightarrow$ Roots are integral.

Note :

(i) Every equation of nth degree ($n \geq 1$) has exactly n root & if the equation has more than n roots, it is an identity.

(ii) If β is a root of the equation $f(x) = 0$, then $(x-\beta)$ is a factor of $f(x)$.

(iii) If the coefficients in the equation are all rational & $p + \sqrt{q}$ is one of its roots, then $p-\sqrt{q}$ is also a root where $p, q \in Q$ and q is not a perfect square.

(iv) If the coefficients of the equation $f(x) = 0$ are all real and $\alpha + i\beta$ is its root, then $\alpha - i\beta$ is also a root. i.e. imaginary roots occur in conjugate pairs.

(v) Every equation $f(x) = 0$ of degree odd has atleast one real root of a sign opposite to that of its last term.

(vi) If there be any two real numbers ‘a’ & ‘b’ such that f(a) & f(b) are of opposite signs, then $f(x) = 0$ must have atleast one real root between ‘a’ and ‘b’.

SOLVED EXAMPLE

Example 11: The quadratic equation whose one root is $\frac{1}{2+\sqrt{5}}$ will be

(A) $x^2 + 4x - 1 = 0$ (B) $x^2 - 4x - 1 = 0$ (C) $x^2 + 4x + 1 = 0$ (D) None of these

Solution : Given root $= \frac{1}{2+\sqrt{5}} = \sqrt{5}-2$

So the other root = $-\sqrt{5} - 2$ Then sum of the roots = – 4, product of the roots = –1

Hence the equation is $x^2 + 4x - 1 = 0$

Example 12: For what values of m the equation $(1 + m)x^2 - 2(1 + 3m)x + (1 + 8m) = 0$ has equal roots.

Solution : Given equation is $(1 + m)x^2 - 2(1 + 3m)x + (1 + 8m) = 0$ ....(i)

Let D be the discriminant of equation (i).

Roots of equation (i) will be equal if D = 0.

or $4(1 + 3m)^2 - 4(1 + m)(1 + 8m) = 0$

or $4(1 + 9m^2 + 6m - 1 - 9m - 8m^2) = 0$

or $m^2 - 3m = 0$ or, $m(m - 3) = 0$

$\therefore$ m = 0, 3.

Example 13: If the equation $(k - 2)x^2 - (k - 4)x - 2 = 0$ has difference of roots as 3 then the value of k is-

(1) 1, 3 (2) 3, 3/2 (3) 2, 3/2 (4) 3/2, 1

Sol. $\alpha + \beta = \frac{(k-4)}{(k-2)}, \alpha\beta = \frac{-2}{k-2}$

$|\alpha - \beta| = \sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$

$\therefore$ $|\alpha - \beta| = \sqrt{\left(\frac{k-4}{k-2}\right)^2 + \frac{8}{(k-2)}}$

$= \frac{\sqrt{k^2 + 16 - 8k + 8(k-2)}}{(k-2)}$

$3 = \frac{\sqrt{k^2 + 16 - 8k + 8k - 16}}{(k-2)}$

$\Rightarrow$ $3k - 6 = \pm k \Rightarrow k = 3, 3/2$

Example 14 : If the coefficient of the quadratic equation are rational & the coefficient of x^2 is 1, then find the equation one of whose roots is $\tan\frac{\pi}{8}$.

Solution : We know that $\tan\frac{\pi}{8} = \sqrt{2} - 1$

Irrational roots always occur in conjugational pairs.

Hence if one root is $(-1 + \sqrt{2})$, the other root will be $(-1 - \sqrt{2})$. Equation is

$(x - (-1 + \sqrt{2}))(x - (-1 - \sqrt{2})) = 0$ $\Rightarrow$ $x^2 + 2x - 1 = 0$

Example 15: If α, β are roots of $Ax^2 + Bx + C = 0$ and α^2, β^2 are roots of $x^2 + px + q = 0$, then p is equal to-

(1) $\frac{(B^2 - 2AC)}{A^2}$ (2) $\frac{(2AC - B^2)}{A^2}$ (3) $\frac{(B^2 - 4AC)}{A^2}$ (4) $(4AC - B^2)A^2$

Solution : $\alpha + \beta = -\frac{B}{A}, \alpha\beta = \frac{C}{A}$

$\alpha^2 + \beta^2 = -p, \alpha^2\beta^2 = q$ $\therefore (\alpha+\beta)^2 = \frac{B^2}{A^2}$

$\Rightarrow$ $(\alpha^2 + \beta^2) + 2\alpha\beta = \frac{B^2}{A^2} \Rightarrow -p + \frac{2C}{A} = \frac{B^2}{A^2} \Rightarrow p = \frac{2CA - B^2}{A^2}$

Example 16: Find all the integral values of a for which the quadratic equation $(x-a)(x-10)+1=0$ has integral roots.

Solution : Here the equation is $x^2-(a+10)x+10a+1=0$. Since integral roots will always be rational it means D should be a perfect square.

From (i) $D = a^2 - 20a + 96$.

$\Rightarrow$ $D = (a-10)^2 - 4$ $\Rightarrow$ $4 = (a-10)^2 - D$

If D is a perfect square it means we want difference of two perfect square as 4 which is possible only when $(a-10)^2 = 4$ and $D = 0$.

$\Rightarrow$ $(a-10) = \pm 2$ $\Rightarrow$ $a = 12, 8$

Self practice problems : 3

(i) If $2+\sqrt{3}$ is a root of the equation $x^2+bx+c=0$, where $b, c \in Q$, find b, c.

(ii) For the following equations, find the nature of the roots (real & distinct, real & coincident or imaginary).

(a) $x^2-6x+10=0$ (b) $x^2-(7+\sqrt{3})x+6(1+\sqrt{3})=0$ (c) $4x^2+28x+49=0$

(iii) If ℓ, m are real and $\ell \neq m$, then show that the roots of $(\ell - m)x^2 - 5(\ell + m)x - 2(\ell - m) = 0$ are real and unequal.

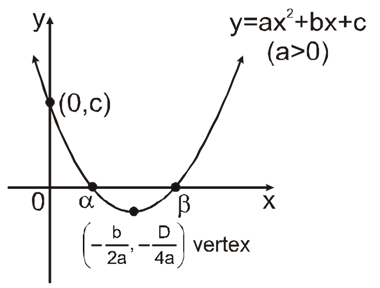
Ans. (i) $b=-4, c=1$; (ii) (a) imaginary; (b) real & distinct ; (c) real & coincident

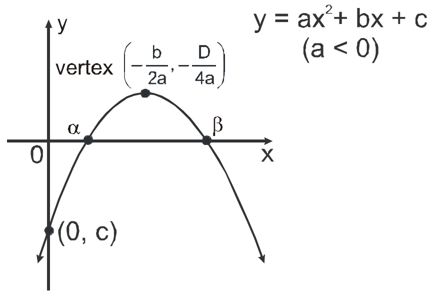
GRAPH OF QUADRATIC POLYNOMIAL:

★ the graph between x, y is always a parabola.

★ the co–ordinate of vertex are $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$

★ If $a > 0$ then the shape of the parabola is concave upwards & if $a < 0$ then the shape of the parabola is concave downwards.





★ the parabola intersect the y–axis at point (0, c).

★ the x–co–ordinate of point of intersection of parabola with x–axis are the real roots of the quadratic equation f (x) = 0. Hence the parabola may or may not intersect the x–axis.

6. RANGE OF QUADRATIC POLYNOMIAL $f(x) = ax^2 + bx + c$.

(i) Range :

If $a > 0$ $\Rightarrow$ $f(x) \in \left[-\frac{D}{4a}, \infty\right)$

If $a < 0$ $\Rightarrow$ $f(x) \in \left(-\infty, -\frac{D}{4a}\right]$

Hence maximum and minimum values of the expression f (x) is $-\frac{D}{4a}$ in respective cases and it occurs at $x = -\frac{b}{2a}$ (at vertex).

(ii) Range in restricted domain:

Given $x \in [x_1, x_2]$

(a) If $-\dfrac{b}{2a} \notin [x_1, x_2]$ then,

$$f(x) \in [\min\{f(x_1), f(x_2)\},\ \max\{f(x_1), f(x_2)\}]$$

(b) If $-\dfrac{b}{2a} \in [x_1, x_2]$ then,

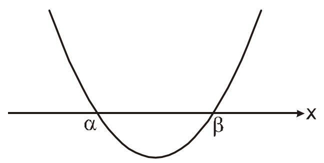
$$f(x) \in \left[\min\left\{f(x_1), f(x_2), -\frac{D}{4a}\right\},\ \max\left\{f(x_1), f(x_2), -\frac{D}{4a}\right\}\right]$$

7. SIGN OF QUADRATIC POLYNOMIAL :

The value of expression $f(x) = ax^2 + bx + c$ at $x = x_0$ is equal to y–co–ordinate of the point on parabola $y = ax^2 + bx + c$ whose x–co–ordinate is x_0. Hence if the point lies above the x–axis for some $x = x_0$, then $f(x_0) > 0$ and vice–versa.

We get six different positions of the graph with respect to x–axis as shown.

(i)



Conclusions :

(a) $a > 0$

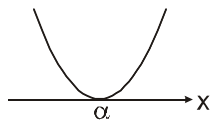
(b) $D > 0$

(c) Roots are real & distinct.

(d) $f(x) > 0$ in $x \in (-\infty, \alpha) \cup (\beta, \infty)$

(e) $f(x) < 0$ in $x \in (\alpha, \beta)$

(ii)



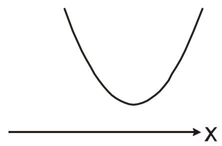
(a) $a > 0$

(b) $D = 0$

(c) Roots are real & equal.

(d) $f(x) > 0$ in $x \in R - \{\alpha\}$

(iii)



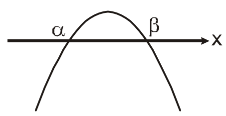
(a) $a > 0$

(b) $D < 0$

(c) Roots are imaginary.

(d) $f(x) > 0\ \forall\ x \in R$.

(iv)



(a) $a < 0$

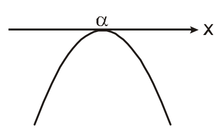
(b) $D > 0$

(c) Roots are real & distinct.

(d) $f(x) < 0$ in $x \in (-\infty, \alpha) \cup (\beta, \infty)$

(e) $f(x) > 0$ in $x \in (\alpha, \beta)$

(v)



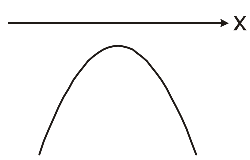
(a) $a < 0$

(b) $D = 0$

(c) Roots are real & equal.

(d) $f(x) < 0$ in $x \in R - \{\alpha\}$

(vi)



(a) $a < 0$

(b) $D < 0$

(c) Roots are imaginary.

(d) $f(x) < 0\ \forall\ x \in R$.

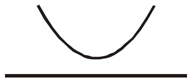
SOLVED EXAMPLE

Example 17 : The value of the expression $x^2 + 2bx + c$ will be positive for all real x if -

(A) $b^2 - 4c > 0$ (B) $b^2 - 4c < 0$ (C) $c^2 < b$ (D) $b^2 < c$

Solution : As a > 0, so this expression will be positive if D < 0

so $4b^2 - 4c < 0$



$b^2 < c$ **Ans. (D)**

Example 18 : The minimum value of the expression $4x^2 + 2x + 1$ is -

(A) 1/4 (B) 1/2 (C) 3/4 (D) 1

Solution : Since a = 4 > 0

therefore its minimum value = $-\dfrac{D}{4a} = \dfrac{4(4)(1)-(2)^2}{4(4)} = \dfrac{16-4}{16} = \dfrac{12}{16} = \dfrac{3}{4}$ **Ans. (C)**

Example 19 : Find the range of following quadratic expressions :

(i) $f(x) = -x^2 + 2x + 3$ $\forall\ x \in R$ **Ans.** $(-\infty, 4]$

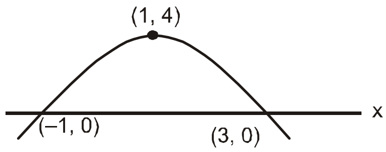
(ii) $f(x) = x^2 - 2x + 3$ $\forall\ x \in [0, 3]$ **Ans.** $[2, 6]$

(iii) $f(x) = x^2 - 4x + 6$ $\forall\ x \in (0, 1]$ **Ans.** $[3, 6)$

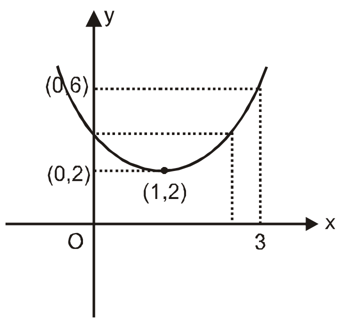
Solution : (i) $y = -x^2 + 2x + 3 = -(x^2 - 2x - 3) = -(x - 3)(x + 1)$

Here a < 0 and D > 0

$\Rightarrow$ Range is $(-\infty, 4]$



(ii) $f(x) = x^2 - 2x + 3$ $\forall$ $x \in [0, 3]$



$y \in [2, 6]$ $\forall$ $x \in [0, 3]$ **Ans.**

Aliter :

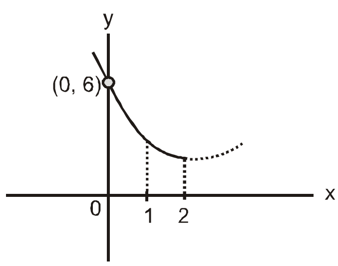
$f(x) = x^2 - 2x + 3 = (x - 1)^2 + 2$

Since $0 \le x \le 3 \Rightarrow -1 \le x - 1 \le 2 \Rightarrow 0 \le (x - 1)^2 \le 4$

$\Rightarrow$ $2 \le (x - 1)^2 + 2 \le 6 \Rightarrow 2 \le f(x) \le 6$

$\therefore$ Range of f(x) is [2, 6].

(iii) $y = x^2 - 4x + 6$; $\forall$ $x \in (0, 1]$



Here a > 0 and D < 0

f(0) = 6 $\Rightarrow$ f(1) = 3 $\Rightarrow$Clearly for $x \in (0, 1]$

$\Rightarrow$ $y \in [3, 6)$

Example 20 : Let $f(x) = ax^2 + bx + c > 0, \forall x \in R$ or $f(x) < 0, \forall x \in R$. Which of the following is/are **CORRECT** ?

(A) If $a + b + c > 0$ then $f(x) > 0, \forall x \in R$ (B) If $a + c < b$ then $f(x) < 0, \forall x \in R$

(C) If $a + 4c > 2b$ then $f(x) < 0, \forall x \in R$ (D) $ac > 0$.

Solution $f(x) > 0 \ \forall x \in R$ or $f(x) < 0 \ \forall x \in R$ hence $D < 0$

its graph can be

(i) or (ii)

(A) $f(1) > 0$ graph (i) will be possible

so $f(x) > 0 \ \forall x \in R$

(B) $f(-1) < 0$ graph (ii) will be possible so $f(x) < 0 \ \forall x \in R$

(C) $f\left(-\frac{1}{2}\right) > 0$ so $f(x) < 0 \ \forall x \in R$

so not possible

(D) $a > 0 c > 0$ (graph (i))

$a < 0 c < 0$ (graph (ii))

in both cases $ac > 0$ **(Ans. ABCD)**

Example 21 : Find the range of values of k, such that $f(x) = \dfrac{kx^2 + 2(k+1)x + (9k+4)}{x^2 - 8x + 17}$ is always negative.

Solution: We can see for $x^2 - 8x + 17$ $D = 64 - 4(17) < 0$

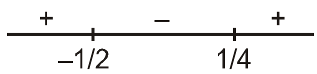
$\therefore$ $x^2 - 8x + 17$ is always +ve $\Rightarrow$ If $f(x) < 0$

$\therefore$ $kx^2 + 2(k + 1)x + (9k + 4) < 0 \Rightarrow$ $k < 0$ ......(1)

& $4(k + 1)^2 - 4k(9k + 4) < 0 \Rightarrow k^2 + 1 + 2k - 9k^2 - 4k < 0 \Rightarrow -8k^2 - 2k + 1 < 0$

$8k^2 + 2k - 1 > 0 \Rightarrow 8k^2 + 4k - 2k - 1 > 0 \Rightarrow 4k(2k + 1) - 1(2k + 1) > 0$

$(2k + 1)(4k - 1) > 0$



.......(2)

combining (1) & (2) we get $k \in \left(-\infty, -\frac{1}{2}\right)$

Example 22 : If x be real, then find the range of the following rational expressions :

(i) $y = \dfrac{x^2 + x + 1}{x^2 + 1}$ **Ans.** $\left[\frac{1}{2}, \frac{3}{2}\right]$

(ii) $y = \dfrac{x^2 - 2x + 9}{x^2 - 2x - 9}$ **Ans.** $\left(-\infty, \frac{-4}{5}\right] \cup (1, \infty)$

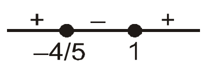
Solution : **(i)** $(y - 1)x^2 - x + y - 1 = 0$ $\because$ $x \in R$ $\therefore$ $D \geq 0$

$\Rightarrow$ $1 - 4(y - 1)^2 \geq 0 \Rightarrow (1 + 2y - 2)(1 - 2y + 2) \geq 0 \Rightarrow (2y - 1)(2y - 3) \leq 0 \Rightarrow \frac{1}{2} \leq y \leq \frac{3}{2}$

(ii) $y(x^2 - 2x - 9) = x^2 - 2x + 9 \Rightarrow (y - 1)x^2 - 2(y - 1)x - (y + 1)9 = 0$

If $y = 1 \Rightarrow -(2)9 = 0$ contradiction.

$\therefore$ $y \neq 1 D \geq 0$ $\Rightarrow$ $(5y + 4)(y - 1) \geq 0$



$y \in \left(-\infty, \frac{-4}{5}\right] \cup (1, \infty)$

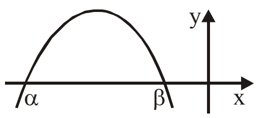
Self practice problems : 3

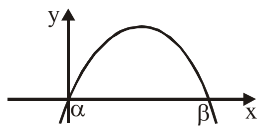
(i) Find the minimum value of :

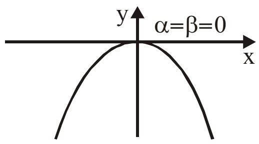
(a) $y = x^2 + 2x + 2$ (b) $y = 4x^2 - 16x + 15$

(ii) For following graphs of $y = ax^2 + bx + c$ with $a,b,c \in R$, comment on the sign of :

(i) a (ii) b (iii) c (iv) D (v) $\alpha + \beta$ (vi) $\alpha\beta$

(1) 

(2) 

(3) 

(iii) Given the roots of equation $ax^2 + bx + c = 0$ are real & distinct, where $a,b,c \in R^+$, then the vertex of the graph will lie in which quadrant.

(iv) Find the range of 'a' for which :

(a) $ax^2 + 3x + 4 > 0 \quad \forall\, x \in R$ (b) $ax^2 + 4x - 2 < 0 \quad \forall\, x \in R$

(v) Prove that the expression $\dfrac{8x-4}{x^2+2x-1}$ cannot have values between 2 and 4, in its domain.

Ans. **(i)** **(a)** 1 **(b)** –1

(ii)

(1)	(i) $a < 0$	(ii) $b < 0$	(iii) $c < 0$	(iv) $D > 0$	(v) $\alpha+\beta<0$	(vi) $\alpha\beta > 0$
(2)	(i) $a < 0$	(ii) $b > 0$	(iii) $c = 0$	(iv) $D > 0$	(v) $\alpha+\beta>0$	(vi) $\alpha\beta = 0$
(3)	(i) $a < 0$	(ii) $b = 0$	(iii) $c = 0$	(iv) $D = 0$	(v) $\alpha+\beta=0$	(vi) $\alpha\beta = 0$

(iii) Third quadrant

(iv) (a) $a > 9/16$ (b) $a < -2$

8. LOCATION OF ROOTS :

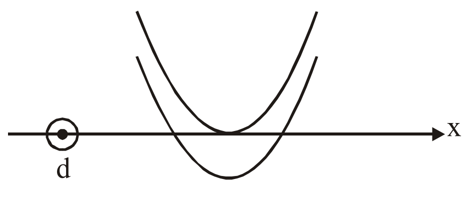
This article deals with an elegant approach of solving problems on quadratic equations when the roots are located/ specified on the number line with variety of constraints :

Consider the quadratic equation **$ax^2 + bx + c = 0$ with $a > 0$ and let $f(x) = ax^2 + bx + c$**

Case-1 :

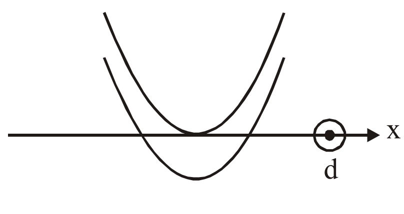
(A) Both roots of the quadratic equation are greater than a specific number (say d). The necessary and sufficient condition for this are :

(i) $D \geq 0$; (ii) $f(d) > 0$; (iii) $-\dfrac{b}{2a} > d$

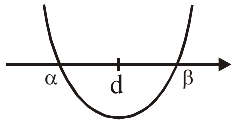


(B) When both roots of the quadratic equation are less than a specific number d than the necessary and sufficient condition will be :

(i) $D \geq 0$; (ii) $f(d) > 0$; (iii) $-\dfrac{b}{2a} < d$



Case-2 :

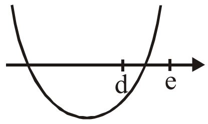


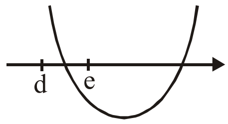
Both roots lie on either side of a fixed number say (d).

The necessary and sufficient condition for this are :

$f(d) < 0$

Case-3 :



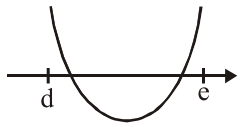


Exactly one root lies in the interval (d, e).

The necessary and sufficient condition for this are :

$f(d) \,.\, f(e) < 0$

Case-4 :



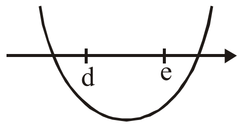
When both roots are lies between the number d and e (d < e).

The necessary and sufficient condition for this are :

(i) $D \geq 0$; (ii) $f(d) > 0$; (iii) $f(e) > 0$

(iv) $d < -\dfrac{b}{2a} < e$

Case-5 :



One root is greater than e and the other roots is less than d (d < e).

The necessary and sufficient condition for this are :

$f(d) < 0$ and $f(e) < 0$

Note : If $a < 0$ in the quadratic equation $ax^2 + bx + c = 0$ then we divide the whole equation by 'a'. Now assume $x^2 + \dfrac{b}{a}x + \dfrac{c}{a}$ as f(x). This makes the coefficient of x^2 positive and hence above cases are applicable.

SOLVED EXAMPLE

Example 23: Find the values of the parameter 'a' for which the roots of the quadratic equation $x^2 + 2(a - 1)x + a + 5 = 0$ are

(i)	real and distinct	(ii)	equal
(iii)	opposite in sign	(iv)	equal in magnitude but opposite in sign
(v)	positive	(vi)	negative
(vii)	greater than 3	(viii)	smaller than 3

(ix) such that both the roots lie in the interval (1, 3)

Solution : Let $f(x) = x^2 + 2(a - 1)x + a + 5 = Ax^2 + Bx + C$ (say)

$\Rightarrow$ $A = 1$, $B = 2(a - 1)$, $C = a + 5$.

Also $D = B^2 - 4AC = 4(a - 1)^2 - 4(a + 5) = 4(a + 1)(a - 4)$

(i) $D > 0$

$\Rightarrow$ $(a + 1)(a - 4) > 0$ $\Rightarrow$ $a \in (-\infty, -1) \cup (4, \infty)$.

(ii) $D = 0$

$\Rightarrow$ $(a + 1)(a - 4) = 0$ $\Rightarrow$ $a = -1, 4$.

(iii) This means that 0 lies between the roots of the given equation.

$\Rightarrow$ $f(0) < 0$ and $D > 0$ i.e. $a \in (-\infty, -1) \cup (4, \infty)$

$\Rightarrow$ $a + 5 < 0 \Rightarrow a < -5 \Rightarrow a \in (-\infty, -5)$.

(iv) This means that the sum of the roots is zero

$\Rightarrow$ $-2(a - 1) = 0$ and $D > 0$ i.e. $a \in -(-\infty, -1) \cup (4, \infty)$ $\Rightarrow$ $a = 1$

which does not belong to $(-\infty, -1) \cup (4, \infty)$

$\Rightarrow$ $a \in \phi$.

(v) This implies that both the roots are greater than zero

$\Rightarrow\quad -\frac{B}{A} > 0,\ \frac{C}{A} > 0, D \geq 0 \quad\Rightarrow\quad -(a-1) > 0,\ a+5 > 0,\ a \in (-\infty, -1]\cup[4, \infty)$

$\Rightarrow\quad a < 1, -5 < a,\ a \in (-\infty, -1]\cup[4, \infty) \quad\Rightarrow\quad a \in (-5, -1].$

(vi) This implies that both the roots are less than zero

$\Rightarrow\quad -\frac{B}{A} < 0,\ \frac{C}{A} > 0, D \geq 0 \quad\Rightarrow\quad -(a-1) < 0,\ a+5 > 0,\ a \in (-\infty, -1]\cup[4, \infty)$

$\Rightarrow\quad a > 1, a > -5,\ a \in (-\infty, -1]\cup[4, \infty) \quad\Rightarrow\quad a \in [4, \infty).$

(vii) In this case

$-\frac{B}{2a} > 3$, A.f(3) > 0 and $D \geq 0$.

$\Rightarrow\quad -(a-1) > 3,\ 7a + 8 > 0$ and $a \in (-\infty, -1]\cup[4, \infty)$

$\Rightarrow\quad a < -2,\ a > -8/7$ and $a \in (-\infty, -1]\cup[4, \infty)$

Since no value of 'a' can satisfy these conditions simultaneously, there can be no value of a for which both the roots will be greater than 3.

(viii) In this case

$-\frac{B}{2a} < 3$, A.f(3) > 0 and $D \geq 0$.

$\Rightarrow\quad a > -2,\ a > -8/7$ and $a \in (-\infty, -1]\cup[4, \infty) \Rightarrow\quad a \in (-8/7, -1]\cup[4, \infty)$

(ix) In this case

$1 < -\frac{B}{2A} < 3$, A.f(1) > 0, A.f(3) > 0, $D \geq 0$.

$\Rightarrow\quad 1 < -1(a-1) < 3,\ 3a + 4 > 0,\ 7a + 8 > 0,\ a \in (-\infty, -1]\cup[4, \infty)$

$\Rightarrow\quad -2 < a < 0,\ a > -4/3,\ a > -8/7,\ a \in (-\infty, -1]\cup[4, \infty) \Rightarrow a \in \left(-\frac{8}{7}, -1\right]$

Example 24 : Find all the values of 'p', so that exactly one root of the equation $x^2 - 2px + p^2 - 1 = 0$, lies between the numbers 2 and 4, and no root of the equation is either equal to 2 or equal to 4.

Solution: (i) $D > 0$

$4a^2 - 4(a^2 - 1) > 0$

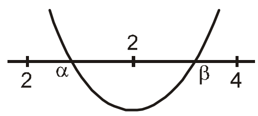
$4 > 0 \qquad \forall\ x \in R$

(ii) f(2) f(4) < 0

$(4 - 4a + a^2 - 1)\ (16 - 8a + a^2 - 1) < 0$

$(a - 3)^2\ (a - 1)\ (a - 5) < 0$

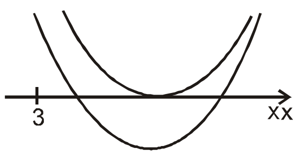
$a \in (1, 5) - \{3\}$



Example 25: If both roots of the equation $x^2 - 6ax + 2 - 2a + 9a^2 = 0$ exceed 3, then show that $a > 11/9$.

Solution. For both roots to exceed 3

(i) $D \geq 0 \Rightarrow 36a^2 - 8 + 8a - 36a^2 \geq 0 \Rightarrow a \geq 1$



(ii) $f(3) > 0 \Rightarrow 9 - 18a + 2 - 2a + 9a^2 > 0 \Rightarrow 9a^2 - 20a + 11 > 0 \Rightarrow a \in (-\infty, 1) \cup \left(\frac{11}{9}, \infty\right)$

(iii) $\frac{-b}{2a} > 3 \Rightarrow 3a > 3 \Rightarrow a > 1 \qquad \therefore \qquad$ (i) $\cap$ (ii) $\cap$ (iii) $\Rightarrow a > \frac{11}{9}$.

Example 26 : If α & β are the two distinct roots of $x^2 + 2(K - 3)x + 9 = 0$, then find the values of K such that $\alpha, \beta \in (-6, 1)$.

Solution $x^2 + 2(k - 3)x + 9 = 0$ ......(i)

Roots α, β of equation (i) are distinct & lies between –6 and 1

$D > 0 \quad \Rightarrow \quad 4(K - 3)^2 - 36 > 0 \quad \Rightarrow \quad k(k - 6) > 0$

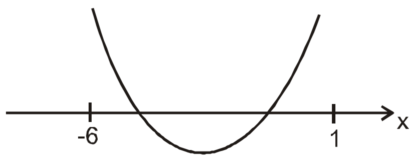
$\Rightarrow \quad k \in (-\infty, 0) \cup (6, \infty)$ ......(ii)

$f(1) > 0 \Rightarrow 1 + 2(k - 3) + 9 > 0$

$\Rightarrow \quad 2k + 4 > 0 \Rightarrow k \in (-2, \infty)$ ......(iii)

$f(-6) > 0 \Rightarrow 36 - 12(k - 3) + 9 > 0$

$\Rightarrow \quad 4k - 27 < 0 \Rightarrow \quad k \in \left(-\infty, \frac{27}{4}\right)$ ......(iv)



$-6 < -\frac{b}{2a} < 1 \Rightarrow -6 < \frac{-2(K-3)}{2} < 1 \Rightarrow -1 < k - 3 < 6 \quad \Rightarrow 2 < k < 9$ ......(v)

$(ii) \cap (iii) \cap (iv) \cap (v) \Rightarrow \quad k \in \left(6, \frac{27}{4}\right)$.

<u>Self practice problems</u> : 4

(i) If α, β are roots of $7x^2 + 9x - 2 = 0$, find their position with respect to following $(\alpha < \beta)$:

(a) –3 (b) 0 (c) 1

(ii) If $a > 1$, roots of the equation $(1 - a)x^2 + 3ax - 1 = 0$ are -

(A) one positive one negative (B) both negative

(C) both positive (D) both non-real

(iii) Find the set of value of a for which the roots of the equation $x^2 - 2ax + a^2 + a - 3 = 0$ are less than 3.

(iv) If α, β are the roots of $x^2 - 3x + a = 0$, $a \in R$ and $\alpha < 1 < \beta$, then find the values of a.

(v) If α, β are roots of $4x^2 - 16x + \lambda = 0$, $\lambda \in R$ such that $1 < \alpha < 2$ and $2 < \beta < 3$, then find the range of λ.

Ans. (i) $-3 < \alpha < 0 < \beta < 1$; (ii) C ; (iii) $a < 2$; (iv) $a < 2$; (v) $12 < \lambda < 16$

9. COMMON ROOTS OF TWO QUADRATIC EQUATIONS :

(a) Only one common root.

Let α be the common root of $ax^2 + bx + c = 0$ & $a'x^2 + b'x + c' = 0$ then

$a\alpha^2 + b\alpha + c = 0$ & $a'\alpha^2 + b'\alpha + c' = 0$. By Cramer's Rule $\frac{\alpha^2}{bc' - b'c} = \frac{\alpha}{a'c - ac'} = \frac{1}{ab' - a'b}$

Therefore, $\alpha = \frac{ca' - c'a}{ab' - a'b} = \frac{bc' - b'c}{a'c - ac'}$

So the condition for a common root is $(ca' - c'a)^2 = (ab' - a'b)(bc' - b'c)$.

(b) If both roots are same then $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$.

SOLVED EXAMPLE

Example 27 : Find p and q such that $px^2 + 5x + 2 = 0$ and $3x^2 + 10x + q = 0$ have both roots in common.

Solution : $a_1 = p$, $b_1 = 5$, $c_1 = 2$

$a_2 = 3$, $b_2 = 10$, $c_2 = q$

We know that :

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \Rightarrow \frac{p}{3} = \frac{5}{10} = \frac{2}{q} \Rightarrow p = \frac{3}{2};\ q = 4$$

Example 28 : Find the value of 'a' so that $x^2 - 11x + a = 0$ and $x^2 - 14x + 2a = 0$ have a common root.

Solution Given equation are $x^2 - 11x + a = 0$ ......(i)

$x^2 - 14x + 2a = 0$ ......(ii)

Multiplying equation (i) by 2 and then subtracting, we get $x^2 - 8x = 0 \Rightarrow x = 0, 8$

If $x = 0$, $a = 0$ If $x = 8, a = 24$

Example 29 : If the quadratic equations $ax^2 + bx + c = 0$ ($a, b, c \in R$, $a \neq 0$) and $x^2 + 4x + 5 = 0$ have a common root, then a, b, c must satisfy the relations:

(A) $a > b > c$ (B) $a < b < c$

(C) $a = k$; $b = 4k$; $c = 5k$ ($k \in R$, $k \neq 0$) (D) $b^2 - 4ac$ is negative.

Solution: $\because$ D of $x^2 + 4x + 5 = 0$ is less than zero

$\Rightarrow$ both the roots are imaginary $\Rightarrow$ both the roots of quadratic are same

$\Rightarrow$ $b^2 - 4ac < 0$ & $\frac{a}{1} = \frac{b}{4} = \frac{c}{5} = k$ $\Rightarrow$ $a = k, b = 4k, c = 5k$.

Example 30 : If $x^2 + px + q = 0$ and $x^2 + qx + p = 0$, $(p \neq q)$ have a common root, show that $1 + p + q = 0$; show that their other roots are the roots of the equation $x^2 + x + pq = 0$.

Solution: Let α is the common root hence $\alpha^2 + p\alpha + q = 0 \Rightarrow \alpha^2 + q\alpha + p = 0$

$$\frac{\alpha^2}{p^2 - q^2} = \frac{\alpha}{q - p} = \frac{1}{q - p} \Rightarrow \alpha^2 = -(p + q),\quad \alpha = 1 \Rightarrow -(p + q) = 1 \Rightarrow p + q + 1 = 0$$

Let other roots be β and δ then $\alpha + \beta = -p$, $\alpha\beta = q \Rightarrow \alpha + \delta = -q$, $\alpha\delta = p$

$$\beta - \delta = q - p,\ \frac{\beta}{\delta} = \frac{q}{p} \Rightarrow \frac{\beta - \delta}{\delta} = \frac{q - p}{p} \Rightarrow \frac{q - p}{\delta} = \frac{q - p}{p} \Rightarrow \delta = p \Rightarrow \beta = q$$

Equation having β, δ as roots

$x^2 - (\beta + \delta)x + \beta\delta = 0 \Rightarrow x^2 - (p + q)x + pq = 0 \Rightarrow x^2 + x + pq = 0$ [$\because p + q = -1$]

<u>Self practice problems</u> : 5

(i) If $x^2 + bx + c = 0$ & $2x^2 + 9x + 10 = 0$ have both roots in common, find b & c.

(ii) If $x^2 - 7x + 10 = 0$ & $x^2 - 5x + c = 0$ have a common root, find c.

(iii) Show that $x^2 + (a^2 - 2)x - 2a^2 = 0$ and $x^2 - 3x + 2 = 0$ have exactly one common root for all $a \in R$.

Ans. (i) $b = \frac{9}{2}, c = 5$; (ii) $c = 0, 6$

Exercise # 1

PART-I : SUBJECTIVE QUESTIONS

Section (A) : Quadratic Equation and Relation between the its roots and coefficients ;

A-1 If the equation $(\lambda^2 - 5\lambda + 6)x^2 + (\lambda^2 - 3\lambda + 2)x + (\lambda^2 - 4) = 0$ has more than two roots, then find the value of λ ? Does there exist a real value of 'x' for which the above equation will be an identity in 'λ' ?

A-2 If α and β are the roots of the equation $2x^2 + 3x + 4 = 0$, then find the values of

(i) $\alpha^2 + \beta^2 - 3$ (ii) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha} + \alpha \times \beta$

A-3 If α, β are roots of $x^2 - px + q = 0$ and $\alpha - 2, \beta + 2$ are roots of $x^2 - px + r = 0$, then find the value of $\dfrac{(r+4-q)^2}{p^2 - 4q}$

A-4 If α, β are the roots of quadratic equation $x^2 + px + q = 0$ and γ, δ are the roots of $x^2 + px - r = 0$, then find $(\alpha - \gamma) . (\alpha - \delta)$

A-5. If $m \neq n$ but $m^2 = 5m - 3, n^2 = 5n - 3$, then find the equation whose roots are $\dfrac{m}{n}$ and $\dfrac{n}{m}$.

A-6. For the equation $3x^2 + px + 3 = 0$, $p > 0$ if one of the roots is square of the other, then find the the value of p

A-7. Find the value of the expression $x^3 - x^2 + 3x + 8$ when $x = 1 + 2\sqrt{-1}$.

A-8 Let α and β be the roots of the equation $x^2 - 5x - 1 = 0$, then find the value of $\dfrac{\alpha^{15} + \alpha^{11} + \beta^{15} + \beta^{11}}{\alpha^{13} + \beta^{13}}$.

Section (B) : Relation between roots and coefficients; Higher Degree Equations

B-1. If α, β and γ are the roots of the equation $x^3 + 3x^2 - 4x - 2 = 0$, then find the values of the following expressions:

(i) $\alpha^2 + \beta^2 + \gamma^2$ (ii) $\alpha^3 + \beta^3 + \gamma^3$ (iii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$

B-2. If two roots of the equation $x^3 - px^2 + qx - r = 0$, $(r \neq 0)$ are equal in magnitude but opposite in sign, then prove that $pq = r$

B-3. Solve the equation $18x^3 + 81x^2 + \lambda x + 60 = 0$, one root being half the sum of the other two. Hence find the value of λ.

B-4. If α, β, γ are roots of equation $x^3 - 6x^2 + 10x - 3 = 0$, then find equation with roots $2\alpha + 1$, $2\beta + 1$, $2\gamma + 1$.

B-5. If α, β, γ are the roots of the equation $x^3 + 4x + 1 = 0$, then find the value of $(\alpha + \beta)^{-1} + (\beta + \gamma)^{-1} + (\gamma + \alpha)^{-1}$.

B-6. If α, β, γ are roots of $x^3 - 3x + 1 = 0$ then find the value of

(i) $\alpha^3 + \beta^3 + \gamma^3$ (ii) $\alpha^4 + \beta^4 + \gamma^4$ (iii) $\alpha^5 + \beta^5 + \gamma^5$

Section (C) : Nature of Roots

C-1. If $4 + i\sqrt{3}$ is a root of the equation $x^2 + px + q = 0$, where $p, q \in R$, then find the ordered pair (p, q).

C-2. If $p(q - r) x^2 + q(r - p) x + r(p - q) = 0$ has equal roots, the prove that $\frac{2}{q} = \frac{1}{p} + \frac{1}{r}$.

C-3. If $ax^2 + bx + 10 = 0$ does not have real & distinct roots, find the minimum value of $5a–b$.

C-4. If the roots of the equation $x^2 - 2cx + ab = 0$ are real and unequal, then prove that the roots of $x^2 - 2(a + b) x + a^2 + b^2 + 2c^2 = 0$ will be imaginary.

C-5. For what values of k the expression $kx^2 + (k + 1)x + 3$ will be a perfect square of a linear polynomial.

C-6. If the roots of the equation $x^2 + 2cx + b = 0$ are real and distinct and they differ by at most 2m, then find interval of b

C-7. Show that if roots of equation $(a^2 - bc) x^2 + 2(b^2 - ac) x + c^2 - ab = 0$ are equal then either $b = 0$ or $a^3 + b^3 + c^3 = 3abc$

C-8. Find the value of a for which one root of the quadratic equation $(a^2 - 5a + 3)x^2 + (3a - 1)x + 2 = 0$ is twice as large as other

C-9. Let $p, q, r, s \in R$, $x^2 + px + q = 0$, $x^2 + rx + s = 0$ such that $2 (q + s) = pr$ then prove that at least one of the equation have real roots.

C-10. If $p, q, r \in R$, then prove that the roots of the equation $\frac{1}{x-p} + \frac{1}{x-q} + \frac{1}{x-r} = 0$ are always real and cannot have roots if $p = q = r$.

C-11. If the roots of the equation $\frac{1}{(x+a)} + \frac{1}{(x+b)} = \frac{1}{c}$ are equal in magnitude but opposite in sign, show that $a + b = 2c$ & that the product of the roots is equal to $(-1/2)(a^2 + b^2)$.

C-12. The value of m for which one of the roots of $x^2 - 3x + 2m = 0$ is double of one of the roots $x^2 - x + m = 0$ is

C-13. **(i)** If $-2 + i\beta$, $\beta \in R - \{0\}$ is a root of $x^3 + 63x + \lambda = 0$, $\lambda \in R$ then find roots of equation.

(ii) If $\frac{-1}{2} + i\beta$, $\beta \in R - \{0\}$ is a root of $2x^3 + bx^2 + 3x + 1 = 0$, $b \in R$, then find the value(s) of b.

C-14. Solve the equation $x^4 + 4x^3 + 4x^2 - 4 = 0$, one root being $-1 + \sqrt{-1}$.

C-15. Sketch the graph of the following

(i) $f(x) = 2x^3 - 9x^2 + 12x - \frac{9}{2}$ (ii) $f(x) = 2x^3 - 9x^2 + 12x - 3$

C-16. Find the values of p for which the equation $x^4 - 14x^2 + 24x - 3 - p = 0$ have

(i) Two distinct negative real root (ii) Two real roots of opposite sign

(iii) Four distinct real roots (iv) No real roots

Section (D) : Range of quadratic expression and sign of quadratic expression

D-1. Draw the graph of the following expressions :-

(i) $y = x^2 - x + 2$ (ii) $y = -x^2 + 3x + 1$ (iii) $y = x^2 + 6x + 9$

D-2. If $y = x^2 - 2x - 3$, then find the range of y when :

(i) $x \in R$ (ii) $x \in [0,3]$ (iii) $x \in [-2,0]$

D-3. For $x \in R$, find the set of values attainable by

(i) $\dfrac{x^2 - 3x + 4}{x^2 + 3x + 4}$. (ii) $\dfrac{x^2 + 2x + 1}{x^2 + 2x + 7}$ (iii) $\dfrac{x^2 + x - 3}{x^2 + x}$

D-4. Find set of all real values of a such that $f(x) = \dfrac{(2a-1)x^2 + 2(a+1)x + (2a-1)}{x^2 - 2x + 40}$ is always negative.

D-5. Let $x^2 + y^2 + xy + 1 \geq a(x + y)$ $\forall$ x, y $\in$ R, then find the number of possible integer(s) in the range of a.

Section (E) : Location of Roots

E-1. Find all possible values of a for which exactly one root of $x^2 - (a+1)x + 2a = 0$ lies in interval (0,3).

E-2. Find value of k for which one root of equation $x^2 - (k+1)x + k^2 + k - 8 = 0$ exceeds 2 & other is less than 2.

E-3. If the roots of the quadratic equation $(4p - p^2 - 5)x^2 - (2p - 1)x + 3p = 0$ lie on either side of unity, then find the number of integral values of p.

E-4. Boot roots of $(a^2 - 1)x^2 + 2ax + 1 = 0$ belong to the interval (0,1), then find exhaustive set of values of 'a'.

E-5. Find the values of a > 0 for which both the roots equation $ax^2 - (a + 1)x + a - 2 = 0$ are greater than 3.

E-6. Find the values of a for which one root of equation $x^2 - (a + 1)x + a^2 + a - 8 = 0$ is greater than 2 and the other root smaller than 2.

E-7. If the equation $2x^3 + 9x^2 - 24x + 15 - \lambda = 0$ have

(i) 1 solution in $(1, \infty)$ then find λ

(ii) 2 solutions in $(0, \infty)$ then find λ

Section (F) : Common Roots

F-1. Find the possible value(s) of a for which the equations $x^2 + ax + 1 = 0$ and $x^2 + x + a = 0$ have atleast one common root.

F-2. If a, b, p, q are non–zero real numbers, then two equations $2a^2x^2 - 2abx + b^2 = 0$ and $p^2x^2 + 2pqx + q^2 = 0$ then prove that the equations have no common root.

F-3. If the equations $k(6x^2 + 3) + rx + 2x^2 - 1 = 0$ and $6k(2x^2 + 1) + px + 4x^2 - 2 = 0$ have both roots common, then find the value of $(2r - p)$.

F-4. If one of the roots of the equation $ax^2 + bx + c = 0$ be reciprocal of one of the roots of $a_1x^2 + b_1x + c_1 = 0$, then prove that $(aa_1 - cc_1)^2 = (bc_1 - ab_1)(b_1c - a_1b)$.

PART-II : OBJECTIVE QUESTIONS

Section (A) : Relation between the roots and coefficients quadratic equation

A-1 The roots of the equation $(\alpha-\beta)x^2 + (\beta-\gamma)x + (\gamma-\alpha) = 0$ are

(A) $\frac{\beta-\gamma}{\alpha-\beta}$, 1 (B) $\frac{\gamma-\alpha}{\alpha-\beta}$, 1 (C) $\frac{\alpha-\beta}{\gamma-\alpha}$, 1 (D) $\frac{\beta-\gamma}{\gamma-\alpha}$, 1

A-2 Let a, b, c be real numbers with $a \neq 0$. If α, β are the roots of the equation $ax^2 + bx + c = 0$, then the roots of $a^3x^2 + abcx + c^3 = 0$ in terms of α, β are

(A) α^2,β^2 (B) $\alpha^2\beta,\alpha\beta^2$ (C) α^3,β^3 (D) $\alpha^3\beta$, $\alpha\beta^3$

A-3 If the sum of the roots of quadratic equation $(a + 1)x^2 + (2a + 3)x + (3a + 4) = 0$ is -1, then the product of the roots is

(A) 1 (B) 2 (C) 3 (D) 4

A-4. Consider the following statements :

S_1 : If the roots of $x^2 - \alpha x + \beta = 0$ are two consecutive integers, then value of $\alpha^2 - 4\beta$ is equal to 1.

S_2 : If α, β are roots of $x^2 - x + 3 = 0$ then value of $\alpha^4 + \beta^4$ is equal 7.

S_3 : If α, β, γ are the roots of $x^3 - 7x^2 + 16x - 12 = 0$ then value of $\alpha^2 + \beta^2 + \gamma^2$ is equal to 17.

State, in order, whether S_1, S_2, S_3 are true or false

(A) TTT (B) FTF (C) TFT (D) FTT

A-5. If the roots of the equation $x^2 + mx + n = 0$ are cubes of the roots of the equation $x^2 + px + q = 0$, then

(A) $m = p^3 + 3pq$ (B) $m = p^3 - 3pq$ (C) $m + n = p^3$ (D) $\frac{m}{n} = \left(\frac{p}{q}\right)^3$

A-6. Let $f(x) = x^2 - a(x + 1) - b = 0$, $a, b \in R - \{0\}$, $a + b \neq 0$. If α and β are roots of equation $f(x) = 0$, then the value of $\frac{1}{\alpha^2 - a\alpha} + \frac{1}{\beta^2 - a\beta} - \frac{2}{a+b}$ is equal to

(A) 0 (B) 1 (C) 2 (D) 3

A-7. The set of possible values of λ for which $x^2 - (\lambda^2 - 5\lambda + 5)x + (2\lambda^2 - 3\lambda - 4) = 0$ has roots, whose sum and product are both less than 1, is

(A) $\left(-1, \frac{5}{2}\right)$ (B) (1, 4) (C) $\left[1, \frac{5}{2}\right]$ (D) $\left(1, \frac{5}{2}\right)$

Section (B) : Relation between roots and coefficients ; Higher Degree Equations

B-1. If a and b be two real roots of the equation $x^3 + px^2 + qx + r = 0$ satisfying the relation $ab + 1 = 0$, then the value of $r^2 + pr + q$ equals

(A) 1 (B) 2 (C) -1 (D) -2

B-2. If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$, then

$\left(\alpha - \frac{1}{\beta\gamma}\right)\left(\beta - \frac{1}{\gamma\alpha}\right)\left(\gamma - \frac{1}{\alpha\beta}\right) \times \alpha\beta\gamma$ equals

(A) $\frac{(r+1)^3}{r^2}$ (B) $\frac{(r+1)^3}{r}$ (C) $\frac{r}{(r+1)^3}$ (D) $\frac{r^2}{(r+1)^3}$

B-3. The equation $24x^3 - 14x^2 - 63x + \lambda = 0$ have one root being double of another then the value(s) of λ are
(A) 45 or –25 (B) 25 or 45 (C) –45 or 25 (D) –45 or –25

B-4. Let $\alpha, \beta, \gamma, \delta$ be the roots of $(x - a)(x - b)(x - c)(x - d) = e$, $e \neq 0$, then the roots of the equation $(x - \alpha)(x - \beta)(x - \gamma)(x - \delta) + e = 0$ are :
(A) $a + 1, b + 1, c + 1, d + 1$ (B) a, b, c, d
(C) $a - 1, b - 1, c - 1, d - 1$ (D) $\frac{a}{b}, \frac{b}{c}, \frac{c}{d}, \frac{d}{a}$

B-5. If coefficients of biquadratic equation are all distinct and belong to the set $\{-10, -6, 3, 5, 8\}$, then equation has
(A) at least two real roots
(B) four real roots, two are conjugate surds and other two are also conjugate surds
(C) four imaginary roots
(D) None of these

B-6. If α, β & γ are the roots of the equation $x^3 + x + 1 = 0$, then the value of $\frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1-\gamma}$ is
(A) $\frac{1}{2}$ (B) $-\frac{1}{2}$ (C) $-\frac{1}{3}$ (D) $\frac{1}{3}$

Section (C) : Nature of Roots

C-1. If a, b, c are integers and $b^2 = 4(ac + 6d^2)$, $d \in N$, then roots of the quadratic equation $ax^2 + bx + c = 0$ are
(A) Irrational (B) Rational & different (C) Complex conjugate (D) Rational & equal

C-2. Let a, b and c be real numbers such that $9a + 3b + c = 0$ and $ab > 0$. Then the equation $ax^2 + bx + c = 0$ has
(A) real roots (B) imaginary roots (C) exactly one root (D) none of these

C-3. Consider the equation $x^2 + 2x - n = 0$, where $n \in N$ and $n \in [1, 100]$. Total number of different values of 'n' so that the given equation has integral roots, is
(A) 4 (B) 6 (C) 8 (D) 9

C-4. If α & β $(\alpha < \beta)$ are the roots of the equation $x^2 + bx + c = 0$, where $c < 0 < b$, then
(A) $0 < \alpha < \beta$ (B) $\alpha < 0 < \beta^2 < \alpha^2$ (C) $\alpha < \beta < 0$ (D) $\alpha < 0 < \alpha^2 < \beta^2$

C-5. The number of positive zeroes of $y = 12x^3 - 4x^2 - 3x + 1$ is/are
(A) 1 (B) 2 (C) 3 (D) 0

C-6. The equation $x^4 - 2x^2 = a$ has four distinct real roots, then
(A) $a \in (2, 3)$ (B) $a \in (-1, 0)$ (C) $a \in (2, 4)$ (D) $a \in (-2, -1)$

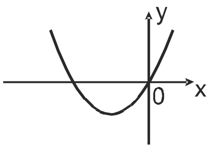
Section (D) : Range of quadratic expression and sign of quadratic expression

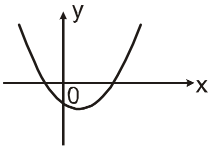
D-1. If $y = -2x^2 - 6x + 9$, then
(A) maximum value of y is 13.5 and it occurs at $x = -1.5$
(B) minimum value of y is 13.5 and it occurs at $x = -1.5$
(C) maximum value of y is –11 and it occurs at $x = 2$
(D) minimum value of y is –11 and it occurs at $x = 2$

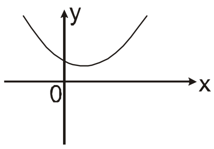
D-2. If a and b are the non-zero distinct roots of $x^2 + ax + b = 0$, then the least value of $x^2 + ax + b + 1$ is
(A) $\frac{3}{2}$ (B) $\frac{9}{4}$ (C) $-\frac{9}{4}$ (D) $-\frac{5}{4}$

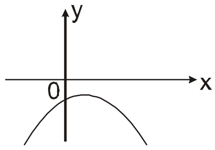
D-3. The largest natural number a for which the maximum value of $f(x) = a - 1 + 2x - x^2$ is smaller than the minimum value of $g(x) = x^2 - 2ax + 10 - 2a$ is
(A) 1 (B) – 2 (C) –1 (D) 2

D-4. Which of the following graph represents the expression $f(x) = a x^2 + b x + c$ $(a \neq 0)$ when $a > 0$, $b < 0$ & $c < 0$?

(A) 

(B) 

(C) 

(D) 

D-5. The entire graph of the expression $y = -x^2 + kx - x - 9$ is strictly below the x-axis if and only if
(A) $k < 7$ (B) $-5 < k < 7$ (C) $k > -5$ (D) $-2 < k < 4$

D-6. If $a, b \in R$, $a \neq 0$ and the quadratic equation $ax^2 - bx + 1 = 0$ has imaginary roots then $a + b + 1$ is:
(A) positive (B) negative C) zero (D) depends on the sign of b

D-7. The equation, $e^x = -2x^2 + 6x - 9$ has:
(A) no solution (B) one solution (C) two solutions (D) infinite solutions

D-8. If $(\lambda^2 + \lambda - 2)x^2 + (\lambda + 2)x < 1$ for all $x \in R$, then λ belongs to the interval
(A) $(-2, 1)$ (B) $\left[-2, \frac{2}{5}\right)$ (C) $\left(\frac{2}{5}, 1\right)$ (D) none of these

D-9. Let $x^2 + (a - b)x + (1 - a - b) = 0$, $a, b \in R$. The value of 'a' for which Roots are imaginary $\forall\, b \in R$ is/are
(A) $a > 1$ (B) $a > 2$ (C) $a < 1$ (D) $a \in \phi$

Section (E) : Location of Roots

E-1. If $b > a$, then the equation $(x - a)(x - b) - 1 = 0$, has:
(A) both roots in $[a, b]$ (B) both roots in $(-\infty, a)$
(C) both roots in $[b, \infty)$ (D) one root in $(-\infty, a)$ & other in (b, ∞)

E-2. All the values of m for which both the roots of the equation $x^2 - 2mx + m^2 - 1 = 0$ are greater than -2 but less than 4 lie in the interval
(A) $-2 < m < 0$ (B) $m > 3$ (C) $-1 < m < 3$ (D) $1 < m < 4$

E-3. If x_1, x_2 be the roots of $4x^2 - 16x + \lambda = 0$, where $\lambda \in R$, such that $1 < x_1 < 2$ and $2 < x_2 < 3$, then the number of integral solutions of λ is
(A) 5 (B) 6 (C) 2 (D) 3

E-4. The real values of 'a', so that the roots of the equation
$(a^2 - a + 2)x^2 + 2(a - 3)x + 9(a^4 - 16) = 0$ are of opposite sign is/are.
(A) $a \in (-2, 2)$ (B) $a \in (-4, -2)$ (C) $a \in (2, 7)$ (D) $a \in (7, 9)$

E-5. The values of a for which one root of equation $(a - 5)x^2 - 2ax + a - 4 = 0$ is smaller than 1 and other greater than 2 is/are.
(A) $5 > a$ (B) $a < 24$ (C) $5 < a < 24$ (D) $a < 27$

Section (F) : Common Roots

F-1. If $3x^2 - 17x + 10 = 0$ and $x^2 - 5x + \beta = 0$ has a common root, then sum of all possible real values of β is
(A) 0 (B) $-\frac{29}{9}$ (C) $\frac{26}{9}$ (D) $\frac{29}{3}$

F-2. If the equations $x^2 + ax + 12 = 0$, $x^2 + bx + 15 = 0$ & $x^2 + (a + b)x + 36 = 0$ have a common positive root, then
(A) $ab = 26$ (B) $a + b = -15$ (C) $ab = 30$ (D) $a + b = 15$.

F-3. If $ax^2 + bx + c = 0$ and $cx^2 + bx + a = 0$ have a common root and a, b, c are non-zero real numbers, then the value of $\frac{a^3 + b^3 + c^3}{abc}$ is.
(A) 2 (B) 3 (C) 4 (D) 5

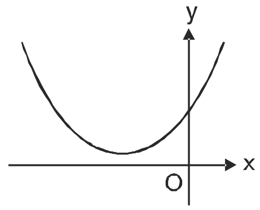
PART-III : MATCH THE COLUMN

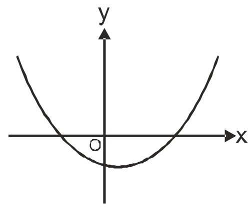
1. If $ax^2 + bx + c = 0$ where $a \neq 0$ is satisfied by α, β, α^2 and β^2, where $\alpha\beta \neq 0$. Let set S be the set of all possible unordered pairs (α, β).
Then match the following lists:

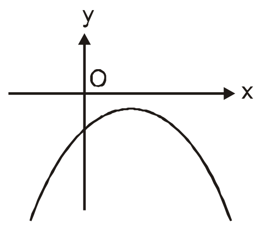
	Column-I		Column-II
(A)	The number of elements in set S is	(P)	2
(B)	The sum of all possible values of $(\alpha + \beta)$ of the pair (α , β) in set S is	(Q)	3
(C)	The sum of all possible values of $\alpha\beta$ of the pair (α , β) in set S is	(R)	4
(D)	The sum of all possible values of $\alpha^2 + \beta^2$ of of the pair (α, β) in set S is, where $\alpha,\beta \in R$ is	(S)	1

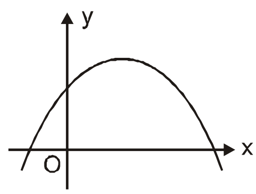
2. If graph of the expression $f(x) = ax^2 + bx + c$ $(a \neq 0)$ are given in column-II, then Match the items in column-I with in column-II (where $D = b^2 - 4ac$)

Column-I **Column-II**

(A) $\frac{abc}{D} < 0$ (P)


(B) $\frac{abc}{D} > 0$ (Q)


(C) $abc < 0$ (R)


(D) $abc > 0$ (S)


3. Match the following.

	Column-I		Column-II
(A)	If $x^2 + x - a = 0$ has integral roots and $a \in N$, then a can be equal to	(P)	2
(B)	If the equation $ax^2 + 2bx + 4c = 16$ has no real roots and $a + c > b + 4$ then integral value of c may be equal to	(Q)	12
(C)	If $x^2 + 2bx + 9b - 14 = 0$, has only negative roots, then integral values of b may be	(R)	72
(D)	Find the value of the expression $2x^3 + 2x^2 - 7x + 72$ when $x = \frac{-1+\sqrt{15}}{2}$	(S)	20

Exercise # 2

PART-I : OBJECTIVE QUESTIONS

1. If α, β are the roots of $ax^2 + bx + c = 0$ $(a \neq 0)$ and $\alpha + k$, $\beta + k$ are the roots of, $Ax^2 + Bx + C = 0$ $(A \neq 0)$ for some constant k, then

(A) $k = \frac{1}{2}\left(\frac{B}{A} - \frac{b}{a}\right)$ (B) $k = \frac{1}{2}\left(\frac{b}{a} + \frac{B}{A}\right)$ (C) $\frac{b^2 - 4ac}{a^2} = \frac{B^2 - 4AC}{A^2}$ (D) $\frac{b^2 + 4ac}{a^2} = \frac{B^2 + 4AC}{A^2}$

2. Let α, β be the roots of the equation $x^2 + ax + b = 0$ and γ, δ be the roots of $x^2 - ax + b - 2 = 0$. If $\alpha\beta\gamma\delta = 24$ and $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} = \frac{5}{12}$, then the value of a is.

(A) 5 (B) 8 (C) 9 (D) 10

3. If one root of the equation $4x^2 + 2x - 1 = 0$ is 'α', then

(A) α can be equal to $\frac{-2 + \sqrt{5}}{4}$ (B) α can be equal to $\frac{1 + \sqrt{5}}{4}$

(C) other root is $4\alpha^3 - 3\alpha$. (D) other root is $4\alpha^3 + 3\alpha$

4. If α, β are roots of $x^2 + 3x + 1 = 0$, then

(A) $(7 - \alpha)(7 - \beta) = 0$ (B) $(2 - \alpha)(2 - \beta) = 10$

(C) $\frac{\alpha^2}{3\alpha + 1} + \frac{\beta^2}{3\beta + 1} = -2$ (D) $\left(\frac{\alpha}{1 + \beta}\right)^2 + \left(\frac{\beta}{\alpha + 1}\right)^2 = 16$

5. If a, b, c, d $\in$ R, then equaton $(x^2 + ax + 4b)(x^2 - cx - 2b)(x^2 - 2dx + 3b) = 0$ has

(A) 6 real roots (B) at least two real roots

(C) 2 real and 4 imaginary roots (D) 4 real and 2 imaginary roots

6. If α, β are the real and distinct roots of $x^2 + \ell x + m = 0$ and α^4, β^4 are the roots of $x^2 - nx + t = 0$, then the equation $x^2 - 4mx + 2m^2 - n = 0$ has always

(A) two non real roots (B) two negative roots

(C) two positive roots (D) one positive root and one negative root

7. If one root of the equation $ax^2 + bx + c = 0$ is equal to q^{th} power of the other root, then

(A) $(ac^q)^{1/(q-1)} + (a^qc)^{1/(q-1)} + b = 0$. (B) $(ac^q)^{1/(q+1)} - (a^qc)^{1/(q+1)} + b = 0$.

(C) $(ac^q)^{1/(q+1)} + (a^qc)^{1/(q+1)} + b = 0$. (D) $(ac^q)^{1/(q+1)} + (a^qc)^{1/(q+1)} - b = 0$.

8. Let a, b, c $\in$ R with a > 0 such that the equation $ax^2 + bcx + b^3 + c^3 - 4abc = 0$ has non-real roots. If $P(x) = ax^2 + bx + c$ and $Q(x) = ax^2 + cx + b$, then

(A) P(x) > 0 for all x $\in$ R and Q (x) < 0 for all x $\in$ R

(B) P(x) < 0 for all x $\in$ R and Q (x) > 0 for all x $\in$ R

(C) neither P(x) > 0 for all x $\in$ R nor Q(x) > 0 for all x $\in$ R

(D) exactly one of P(x) or Q(x) is positive for all real x

9. If the roots of the equation $x^3 + Px^2 + Qx - 19 = 0$ are each one more than the roots of the equaton $x^3 - Ax^2 + Bx - C = 0$, where A, B, C, P & Q are constants, then find the value of $\frac{A + B + C}{10}$:-

(A) 1.8 (B) 2.8 (C) 1.9 (D) 2.9

10. If one root of the equation $t^2 - (12x)t - (f(x) + 64x) = 0$ is twice of other, then the maximum value of the function f(x), where x $\in$ R is.

(A) 28 (B) 30 (C) 32 (D) 34

11. Consider $y = \frac{2x}{1+x^2}$, where x is real, then find the range of expression $y^2 + 3y - 1$:-

(A) $\left[-\frac{9}{2}, 0\right]$ (B) $\left[\frac{9}{4}, 0\right]$ (C) $[-3, 3]$ (D) $\left[-\frac{9}{8}, 0\right]$

12. The values of a for which the expression $\frac{ax^2+3x-4}{3x-4x^2+a}$ assumes all real values for real values of x.

(A) [1,7] (B) (1,7] (C) (1,7) (D) [1,7)

13. The range of a for which the equation $x^2 + ax - 4 = 0$ has its smaller root in the interval (–1, 2) is

(A) $(-\infty, -3)$ (B) (0, 3) (C) $(0, \infty)$ (D) $(-\infty, -3) \cup (0, \infty)$.

14. The set of values of a for which $ax^2 + (a - 2)x - 2$ is negative for exactly two integral x, is

(A) (0, 2) (B) [1, 2) (C) (1, 2] (D) (0, 2]

15. If there exists at least one common x which satisfies the equations $x^2 + 3x + 5 = 0$ and $ax^2 + bc + c = 0$; a, b, c $\in$ N, then the minimum value of a + b + c is

(A) 6 (B) 7 (C) 8 (D) 9

16. If $x^3 + 3x^2 - 9x + c$ is of the form $(x - \alpha)^2 (x - \beta)$, then c can be equal to

(A) 27 (B) – 27 (C) 8 (D) –5

17. The complete set of values of the parameter a for which the equation
$(x^2 + 4x + 6)^2 - (a - 3)(x^2 + 4x + 6)(x^2 + 4x + 5) + (a - 4)(x^2 + 4x + 5)^2 = 0$ has at least one real solution belong to

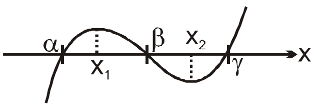
(A) (9, 12] (B) (5, 6] (C) (7, 8] (D) (3, 5]

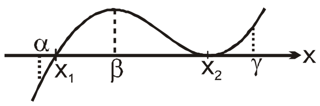
18. If the equation $ax^2 + bx + c = x$ has no real roots, then the equation $a(ax^2 + bx + c)^2 + b(ax^2 + bx + c) + c = x$ will have

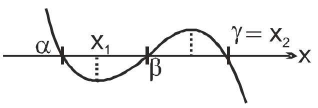
(A) four real roots (B) no real root

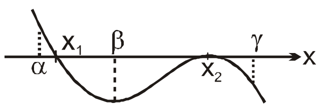
(C) at least two real roots (D) None of these

19. If f(x) is cubic polynomial with real coefficients, $\alpha < \beta < \gamma$ and $x_1 < x_2$ be such that $f(\alpha) = f(\beta) = f(\gamma) = f'(x_1) = f'(x_2) = 0$ then possible graph of y = f(x) is (assuming y-axis vertical)

(A) 

(B) 

(C) 

(D) 

20. Consider the following statements.

S_1 : The equation $2x^2 + 3x + 1 = 0$ has irrational roots.

S_2 : Let $f(x) = \frac{3}{x-2} + \frac{4}{x-3} + \frac{5}{x-4}$, then f(x) = 0 has exactly one real root in (3, 4)

S_3 : If $x^2 + 3x + 5 = 0$ and $ax^2 + bx + c = 0$ have a common root and a, b, c $\in$ N, then the minimum value of (a + b + c) is 10.

S_4 : The value of the biquadratic expression $x^4 - 8x^3 + 18x^2 - 8x + 2$, when $x = 2 + \sqrt{3}$, is 2

Which of the following is/are **CORRECT** ?

(A) S_1 (B) S_2 (C) S_3 (D) S_4

21. If the equation $x^2 + 9y^2 - 4x + 3 = 0$ is satisfied for real values of x and y, then

(A) $x \in (1, 3)$ & $y \in \left(-\frac{1}{3}, \frac{1}{3}\right)$ (B) $x \in (-1, 3)$ & $y \in \left(-\frac{1}{3}, \frac{1}{3}\right)$

(C) $x \in (1, 4)$ & $y \in \left(-\frac{1}{3}, \frac{1}{3}\right)$ (D) $x \in (-1, 4)$ & $y \in \left(-\frac{1}{3}, \frac{1}{3}\right)$

PART-II : NUMERICAL QUESTIONS

1. If the equation $x^2 + 2\alpha x + \alpha^2 - 1 = 0$ and $x^2 + 2\beta x + \beta^2 - 1 = 0$ have a common root ($\alpha > \beta$), then the value of the expression $\dfrac{2\alpha^2 - 4\alpha\beta - (\alpha - \beta) + (\alpha - \beta)\beta^2}{4}$ is :-

2. If α and β be the roots of the equation $x^2 - px + q = 0$, and $\dfrac{\alpha^2}{\beta^2} + \dfrac{\beta^2}{\alpha^2} = \dfrac{p^4}{q^2} - \dfrac{4p^2}{q} + \dfrac{5\lambda}{3}$, then find the value of λ

3. Given that $x^2 - 3x + 1 = 0$, then find the value of the expression y = $\dfrac{\left(x^9 + x^7 + x^{-9} + x^{-7}\right)}{100}$

4. If a, b are the roots of $x^2 + px + 1 = 0$ and c, d are the roots of $x^2 + qx + 1 = 0$. Then find the value of $\dfrac{7}{3}[(q^2 - p^2)/(a - c)(b - c)(a + d)(b + d)]$

5. If $\alpha > \beta > 0$ and $\alpha^3 + \beta^3 + 27\alpha\beta = 729$ then the quadratic equation $\alpha x^2 + \beta x - 9 = 0$ has roots a, b (a < b), then find the value of $\dfrac{6b - \alpha a}{10}$.

6. If the product of all real values of x satisfying $\left(a + \sqrt{b}\right)^{x^2 - 15} + \left(a - \sqrt{b}\right)^{x^2 - 15} = 2a$, where $a^2 - b = 1$ is 100k then find k.

7. If α, β are the roots of the equation $ax^2 + bx\ \ c = 0$ and $S_n = \alpha^n + \beta^n$ ($n \geq 2$), then find $aS_{n+1} + bS_n + cS_{n-1}$

8. If roots of the equation $x^2 - 10ax - 11b = 0$ are c and d and those of $x^2 - 10cx - 11d = 0$ are a and b, then find the value of $\dfrac{a + b + c + d}{100}$. (where a, b, c, d are all distinct numbers)

9. If one root of equation $(\ell - m)x^2 + \ell x + 1 = 0$ be double of the other and if ℓ be real, then the find maximum value of m

10. Let $P(x) = \dfrac{5}{3} - 6x - 9x^2$ and $Q(y) = -4y^2 + 4y + \dfrac{13}{2}$. If there exists unique pair of real numbers (x, y) such that P(x) Q(y) = 20, then the value of 3(x + y) is

11. If a, b, c are non-zero real numbers, then find minimum value of the expression $\left(\dfrac{(a^4 + 3a^2 + 1)(b^4 + 5b^2 + 1)(c^4 + 7c^2 + 1)}{10a^2b^2c^2}\right)$

12. Let $f(x) = x^3 + x + 1$. Suppose $P(x)$ be a cubic polynomial such that $P(0) = -1$ and the zeros of $P(x)$ are the squares of the roots of $f(x) = 0$. Then find value of $\frac{P(4)}{10}$ is

13. If $f(x)$ is a polynomial of degree four with leading coefficient one satisfying $f(1) = 1$, $f(2) = 2$, $f(3) = 3$, then find $\frac{f(-1)+f(5)}{f(0)+f(4)}$

14. The equations $x^3 + 5x^2 + qx + p = 0$ and $x^3 + 7x^2 + qx + s = 0$ have two roots in common. If the third root of each equation is represented by x_1 and x_2 respectively, then find $\frac{(x_1+x_2)^2}{10}$

15. All the values of k for which the quadratic polynomial $f(x) = -2x^2 + kx + k^2 + 5$ has two distinct zeros and only one of them satisfying $0 < x < 2$, lie in the interval (a, b). Find the value of $(a + 10b)$.

16. If $(y^2 - 5y + 3)(x^2 + x + 1) < 2x$ for all $x \in R$, then the interval in which y lies is (α, β). Find $\frac{\alpha+\beta}{2}$

17. If the range of function $f(x) = \frac{x+1}{k+x^2}$ contains the interval $[0, 1]$, then find the maximum value of k.

18. The values of k, for which the equation $x^2 + 2(k-1)x + k + 5 = 0$ possess atleast one positive root, are $(-\infty, -b]$ then find the value of $\frac{6b}{5}$.

19. The equations $x^2 - ax + b = 0$, $x^3 - px^2 + qx = 0$, where $a, b, p, q \in R - \{0\}$ have one common root & the second equation has two equal roots. Find value of $\frac{ap}{8(q+b)}$.

20. The condition on a, b, c, d such that equations $2ax^3 + bx^2 + cx + d = 0$ and $2ax^2 + 3bx + 4c = 0$ have a common root is $(ad + 4bc)^2 = \lambda (bd + 4c^2)(b^2 - ac)$ then find λ :-

21. If $P(x) = x^2 + 2xy + 2x + my - 3$ have two linear factor for two values of m which are m_1 and m_2 then find $\frac{m_1^2+m_2^2}{4(m_1+m_2)}$

PART - III : ONE OR MORE THAN ONE CORRECT OPTIONS

1. If both the roots of the equation $ax^2 + bx + c = 0$ have negative real parts then

(A) $a > 0$, $b > 0$ & $c > 0$ (B) $a < 0$, $b < 0$ & $c < 0$
(C) $a > 0$, $b > 0$ & $c < 0$ (D) $a < 0$, $b > 0$ & $c > 0$

2. Consider the quadratic equation $(a + c - b)x^2 + 2cx + (b + c - a) = 0$ where a, b, c are distinct real number and $a + c - b \neq 0$. If both the roots of the equation are rational then the numbers which must be rational are

(A) a, b, c (B) $\frac{c}{a-b}$ (C) $\frac{b+c-a}{a+c-b}$ (D) $\frac{b^2}{c-a}$

3. If α, β are the roots of the equation $\lambda(x^2 - x) + x + 5 = 0$ and if λ_1 and λ_2 are the two values of λ for which the roots α, β are connected by the relation $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{4}{5}$, then

(A) $\frac{\lambda_1}{\lambda_2}+\frac{\lambda_2}{\lambda_1} = 252$ (B) $\frac{\lambda_1}{\lambda_2}+\frac{\lambda_2}{\lambda_1} = 254$ (C) $\frac{\lambda_1^2}{\lambda_2}+\frac{\lambda_2^2}{\lambda_1} = 4042$ (D) $\frac{\lambda_1^2}{\lambda_2}+\frac{\lambda_2^2}{\lambda_1} = 4048$

4. If α, β are the roots of equation $x^2 - p(x + 1) - c = 0$, then

(A) $(\alpha + 1)(\beta + 1) = 1 - c$ (B) $(\alpha + 1)(\beta + 1) = 1 + c$.

(C) $\frac{\alpha^2+2\alpha+1}{\alpha^2+2\alpha+c}+\frac{\beta^2+2\beta+1}{\beta^2+2\beta+c} = 1$ (D) $\frac{\alpha^2+2\alpha+1}{\alpha^2+2\alpha+c}+\frac{\beta^2+2\beta+1}{\beta^2+2\beta+c} = -1$

5. The equation $(ay - bx)^2 + 4xy = 0$ has rational solutions x, y for

(A) a = 1/2, b = 2 (B) a = 4, b = 1/8 (C) a = 1, b = 3/4 (D) a = 2, b = 1

6. $f(x) = ax^2 + bx + c = 0$ has real roots and its coefficients are odd positive integers then

(A) f(x) = 0 always has irrational roots

(B) discriminant is a perfect square

(C) if ac = 1, then equation must have exactly one root 'α' such that $\alpha \in (-1, 0)$

(D) Equation f(x) = 0 has rational roots

7. Let α and β be roots of $x^2 - 6((\sin t)^2 - 2\sin t + 2)x - 2 = 0$ with $\alpha > \beta$. If $a_n = \alpha^n - \beta^n$ for $n \geq 1$, then

(A) the minimum value of $\frac{a_{100}-2a_{98}}{a_{99}}$ (where $t \in R$) 6

(B) the minimum value of $\frac{a_{100}-2a_{98}}{a_{99}}$ (where $t \in R$) 7

(C) the maximum value of $\frac{a_{100}-2a_{98}}{a_{99}}$ (where $t \in R$) 36

(D) the maximum value of $\frac{a_{100}-2a_{98}}{a_{99}}$ (where $t \in R$) 30

8. Let 'm' be a real number, and suppose that two of the three solutions of the cubic equation $x^3 + 3x^2 - 34x = m$ differ by 1. Then possible value of 'm' is/are

(A) 120 (B) 80 (C) –48 (D) –32

9. The real numbers x_1, x_2, x_3 satisfying the equation $x^3 - x^2 + \beta x + \gamma = 0$ are in A.P. if

(A) $\beta \leq \frac{1}{3}$ (B) $\beta > \frac{1}{3}$ (C) $\gamma \geq -\frac{1}{27}$ (D) $\gamma < -\frac{1}{27}$

10. If $-5 + i\beta$, $-5 + i\gamma$, $\beta^2 \neq \gamma^2$; $\beta, \gamma \in R$ are roots of $x^3 + 15x^2 + cx + 860 = 0$, $c \in R$, then

(A) c = 222

(B) all the three roots are imaginary

(C) two roots are imaginary but not complex conjugate of each other.

(D) $-5 + 7i\sqrt{3}$, $-5 - 7i\sqrt{3}$ are imaginary roots.

11. If α, β are the roots of the equation $x^2 + px + q = 0$ and also of the equation $x^{2n} + p^n x^n + q^n = 0$ and $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ are the roots of the equation $x^n + 1 + (x + 1)^n = 0$ then n can be

(A) 4 (B) 3 (C) 12 (D) $\sqrt{2}$

12. If $\alpha, \beta, \gamma, \delta$ are the roots of the equation $x^4 - Px^3 + Px^2 + Qx + R = 0$, where P, Q & R are real numbers, then

(A) the minimum value of $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ is -1

(B) the minimum value of $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ is -2

(C) the minimum value of $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ occure at P = 1

(D) the minimum value of $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ occure at P = 2

13. The inequation, $1 + \log_5(x^2 + 1) \geq \log_5(ax^2 + 4x + a)$ is valid for all real x then

(A) maximum value of a is 3 (B) maximum value of a is 5

(C) minimum value of a is 2 (D) minimum value of a is 3

14. If the equation $ax^2 + bx + c = 0$, a, b, c $\in$ R have non real roots, then

(A) $c(a - b + c) > 0$ (B) $c(a + b + c) > 0$

(C) $c(4a - 2a + c) > 0$ (D) none of these

15. For real x, the function $\frac{(x-a)(x-b)}{x-c}$ will assume all real values provided

(A) a > b > c (B) a < b < c (C) a > c > b (D) a < c < b

16. Let $P(x) = x^{64} - x^{45} + x^{36} - x^{21} + x^{20} - x^{17} + 1$. Which of the following are **CORRECT** ?

(A) Number of negative roots of P(x) = 0 are zero.

(B) Number of imaginary roots of P(x) = 0 are 64.

(C) Number of real roots of P(x) = 0 are zero.

(D) Number of imaginary roots of P(x) + P(–x) = 0 are 64.

17. If a < b < c < d, then the equation $(x - a)(x - c) + 2(x - b)(x - d) = 0$ have

(A) real and distinct roots (B) exactly one root between (a, b)

(C) exactly one root betwen (c, d) (D) both the roots between (a, d)

18. Let $x_1 < \alpha < \beta < \gamma < x_4$, $x_1 < x_2 < x_3$. If f(x) is a cubic polynomial with real coefficients such that $(f(\alpha))^2 + (f(\beta))^2 + (f(\gamma))^2 = 0$, $f(x_1) f(x_2) < 0$, $f(x_2) f(x_3) < 0$ and $f(x_1) f(x_3) > 0$ then which of the following are **CORRECT** ?

(A) $\alpha \in (x_1, x_2)$, $\beta \in (x_2, x_3)$ and $\gamma \in (x_3, x_4)$ (B) $\alpha \in (x_1, x_3)$, $\beta, \gamma \in (x_3, x_4)$

(C) $\alpha, \beta \in (x_1, x_2)$ and $\gamma \in (x_4, \infty)$ (D) $\alpha \in (x_1, x_3)$, $\beta \in (x_2, x_3)$ and $\gamma \in (x_2, x_4)$

19. Let p, q $\in$ I, such that $p^2 - 3p^2q^2 = 30q^2 + 517$ then -

(A) $3p^2q^2 = 488$ (B) $3p^2q^2 = 588$ (C) $3p^2 + q^2 = 51$ (D) $3q^2 + p^2 = 61$

20. If $x^2 - (a - 3)x + a = 0$ has at least one positive root then a may belong to

(A) $(-\infty, 0)$ (B) $(-\infty, 0) \cup [7, \infty)$ (C) $[9, \infty)$ (D) $(-\infty, 0) \cup [10, \infty)$

21. Suppose that the three quadratic equation $ax^2 - 2bx + c = 0$, $bx^2 - 2cx + a = 0$ and $cx^2 - 2ax + b = 0$ all have only postive roots. Then

(A) $b^2 = ca$ (B) $c^2 = ab$ (C) $a^2 = bc$ (D) a = b = c

22. If $x^2 + \lambda x + 1 = 0$, $\lambda \in (-2, 2)$ and $4x^3 + 3x + 2c = 0$ have a common root then $c + \lambda$ can be

(A) $\frac{1}{2}$ (B) $-\frac{1}{2}$ (C) 0 (D) $\frac{3}{2}$

23. If every pair from among the equations $x^2 + ax + bc = 0$, $x^2 + bx + ca = 0$, and $x^2 + cx + ab = 0$ has a common root, where a, b, c are non zero numbers, then
(A) the sum of the three common roots is $-(1/2)(a + b + c)$
(B) the sum of the three common roots is $2(a + b + c)$
(C) one of the values of the product of the three common roots is abc
(D) the product of the three common roots is $a^2b^2c^2$

24. If α, β, γ are real and $\alpha^2 + \beta^2 + \gamma^2 = 1$, then $\alpha\beta + \beta\gamma + \gamma\alpha$ may lies in the interval:
(A) [3, 4] (B) [4, 5] (C) $\left[-\frac{1}{2}, 1\right]$ (D) [–1, 2]

25. Consider the equation $3x^4 - 8x^3 - 6x^2 + 24x - \lambda = 0$
(A) If the equation have one solution in $(0, \infty)$ then $\lambda \in (0, 8) \cup (13. \infty)$
(B) If the equation has two solution in $(1, \infty)$ then $\lambda \in (8, 13)$
(C) If the equation has three solution in $(-1, \infty)$ then $\lambda \in (8, 13)$
(D) If the equation has four solution in $(-1, \infty)$ then $\lambda \in (8, 13)$

PART - IV : COMPREHENSION

COMPREHENSION - 1

$af(\mu) < 0$ is the necessary and sufficient condition for a particular real number μ to lie between the roots of quadratic equation $f(x) = 0$, where $f(x) = ax^2 + bx + c$. Again if $f(\mu_1) f(\mu_2) < 0$, then exactly one of the roots will lie between μ_1 and μ_2.

1. If $(b)^2 > (a + c)^2$, then
(A) one root of f(x) = 0 is positive, the other is negative
(B) exactly one of the roots of f(x) = 0 lies in (–1,1)
(C) 1 lies between the roots of f(x) = 0
(D) both the roots of f(x) = 0 are less than 1

2. If $a(a + b + c) < 0 < (a + b + c)c$, then
(A) one root is less than 0, the other is greater than 1
(B) exactly one of the roots lies in (0, 1)
(C) both the root lie in (0, 1)
(D) at least one of the roots lies in (0,1)

3. If $(a + b + c)c < 0 < a(a + b + c)$, then
(A) one root is less than 0, the other is greater than 1
(B) one root lies in $(-\infty, 0)$ and other in (0, 1)
(C) both the roots lie in (0, 1)
(D) one root lies in (0, 1) and other in $(1, \infty)$

COMPREHENSION - 2

Consider the equation $x^4 + 2\lambda x^2 + 8 = 0$. This can be solved by substituting $x^2 = t$ such equations are called as pseudo quadratic equations.

4. If the equation has only two real roots, then set of values of λ is
(A) $(-\infty, -2\sqrt{2})$ (B) $(-2\sqrt{2}, 2\sqrt{2})$ (C) $\{2\sqrt{2}\}$ (D) ϕ

5. If the equation has four real and distinct roots, then λ lies in the interval
(A) $(-\infty, -6) \cup (2\sqrt{2}, \infty)$ (B) $(0, \infty)$
(C) $(-\infty, -2\sqrt{2})$ (D) $(-\infty, -6)$

6. If the equation has no real root, then λ lies in the interval
(A) $(-\infty, 0)$ (B) $(-\infty, -\sqrt{2})$ (C) $(6, \infty)$ (D) $(-2\sqrt{2}, \infty)$

COMPREHENSION - 3

To solve equation of type,

$ax^{2m} + bx^{2m-1} + cx^{2m-2} + \ldots\ldots\ldots + kx^m + \ldots\ldots\ldots + cx^2 + bx + a = 0, \quad (a \neq 0) \rightarrow \quad (I)$

divide by x^m and rearrange terms to obtain

$$a\left(x^m + \frac{1}{x^m}\right) + b\left(x^{m-1} + \frac{1}{x^{m-1}}\right) + c\left(x^{m-2} + \frac{1}{x^{m-2}}\right) + \ldots\ldots\ldots + k = 0$$

Substitutions like

$t = x + \frac{1}{x}$ or $t = x - \frac{1}{x}$ helps transforming equation into a reduced degree equation.

7. Roots of equation $x^4 - 8x^3 + 17x^2 - 8x + 1 = 0$ are

(A) $\frac{-3 \pm \sqrt{5}}{2}, \frac{5 \pm \sqrt{21}}{2}$

(B) $\frac{3 \pm \sqrt{5}}{2}, \frac{5 \pm \sqrt{21}}{2}$

(C) $\frac{-3 \pm \sqrt{5}}{2}, \frac{-5 \pm \sqrt{21}}{2}$

(D) $\frac{3 \pm \sqrt{5}}{2}, \frac{-5 \pm \sqrt{21}}{2}$

8. Roots of equation $x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1 = 0$ are

(A) $1, \frac{3 \pm \sqrt{5}}{2}, \frac{1 \pm i\sqrt{3}}{2}$

(B) $1, \frac{5 \pm \sqrt{3}}{2}, \frac{3 \pm i}{2}$

(C) $1, \frac{3 \pm \sqrt{5}}{2}, \frac{3 \pm i}{2}$

(D) $1, \frac{5 \pm \sqrt{3}}{2}, \frac{1 \pm i\sqrt{3}}{2}$

9. If $(x + 1)(x + 2)(x + 3)(x + 6) = 3x^2$, then the equation has

(A) all imaginary roots

(B) two imaginary and two rational roots

(C) All rational roots

(D) Two imaginary and two irrational roots

Exercise # 3

PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

*** Marked Questions may have more than one correct option.**

1. Let p and q be real numbers such that $p \neq 0$, $p^3 \neq q$ and $p^3 \neq -q$. If α and β are nonzero complex numbers satisfying $\alpha + \beta = -p$ and $\alpha^3 + \beta^3 = q$, then a quadratic equation having $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ as its roots is

[IIT-JEE 2010, Paper-1, (3, –1)/ 84]

(A) $(p^3 + q) x^2 - (p^3 + 2q)x + (p^3 + q) = 0$
(B) $(p^3 + q) x^2 - (p^3 - 2q)x + (p^3 + q) = 0$
(C) $(p^3 - q) x^2 - (5p^3 - 2q)x + (p^3 - q) = 0$
(D) $(p^3 - q) x^2 - (5p^3 + 2q)x + (p^3 - q) = 0$

2. Let α and β be the roots of $x^2 - 6x - 2 = 0$, with $\alpha > \beta$. If $a_n = \alpha^n - \beta^n$ for $n \geq 1$, then the value of $\frac{a_{10} - 2a_8}{2a_9}$ is **[IIT-JEE 2011, Paper-1, (3, –1), 80]**

(A) 1 (B) 2 (C) 3 (D) 4

3. A value of b for which the equations **[IIT-JEE 2011, Paper-2, (3, –1), 80]**
$x^2 + bx - 1 = 0$
$x^2 + x + b = 0$
have one root in common is

(A) $-\sqrt{2}$ (B) $-i\sqrt{3}$ (C) $i\sqrt{5}$ (D) $\sqrt{2}$

4. The quadratic equation p(x) = 0 with real coefficients has purely imaginary roots. Then the equation p(p(x)) = 0 has **[JEE (Advanced) 2014, Paper-2, (3, –1)/60]**
(A) only purely imaginary roots
(B) all real roots
(C) two real and two purely imaginary roots
(D) neither real nor purely imaginary roots

5. Let S be the set of all non-zero real numbers α such that the quadratic equation $\alpha x^2 - x + \alpha = 0$ has two distinct real roots x_1 and x_2 satisfying the inequality $|x_1 - x_2| < 1$. Which of the following intervals is(are) a subset(s) of S ? **[JEE (Advanced) 2015, P-2 (4, –2)/ 80]**

(A) $\left(-\frac{1}{2}, -\frac{1}{\sqrt{5}}\right)$ (B) $\left(-\frac{1}{\sqrt{5}}, 0\right)$ (C) $\left(0, \frac{1}{\sqrt{5}}\right)$ (D) $\left(\frac{1}{\sqrt{5}}, \frac{1}{2}\right)$

6. Let $-\frac{\pi}{6} < \theta < -\frac{\pi}{12}$. Suppose α_1 and β_1 are the roots of the equation $x^2 - 2x\sec\theta + 1 = 0$ and α_2 and β_2 are the roots of the equation $x^2 + 2x\tan\theta - 1 = 0$. If $\alpha_1 > \beta_1$ and $\alpha_2 > \beta_2$, then $\alpha_1 + \beta_2$ equals

[JEE (Advanced) 2016-P-1]

(A) $2(\sec\theta - \tan\theta)$ (B) $2\sec\theta$ (C) $-2\tan\theta$ (D) 0

Comprehension (Q.7 to 8)

Let p,q be integers and let α,β be the roots of the equation, $x^2 - x - 1 = 0$, where $\alpha \neq \beta$. For $n = 0,1,2,....$, let $a_n = p\alpha^n + q\beta^n$. **[JEE (Advanced) 2017-P-2]**

FACT : If a and b are rational numbers and $a + b\sqrt{5} = 0$, then a = 0 = b.

7. If $a_4 = 28$, then p + 2q =

(A) 14 (B) 7 (C) 12 (D) 21

8. a_{12} =

(A) $2a_{11} + a_{10}$ (B) $a_{11} - a_{10}$ (C) $a_{11} + a_{10}$ (D) $a_{11} + 2a_{10}$

9. Let α and β be the roots of $x^2 - x - 1 = 0$, with $\alpha > \beta$. For all positive integers n, define

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},\quad n \geq 1$$

$b_1 = 1$ and $b_n = a_{n-1} + a_{n+1}$, $n \geq 2$.

Then which of the following options is/are correct ? **[JEE (Advanced) 2019-P-1]**

(A) $a_1 + a_2 + a_3 + ... + a_n = a_{n+2} - 1$ for all $n \geq 1$

(B) $\sum_{n=1}^{\infty} \frac{a_n}{10^n} = \frac{10}{89}$

(C) $\sum_{n=1}^{\infty} \frac{b_n}{10^n} = \frac{8}{89}$

(D) $b_n = \alpha^n + \beta^n$ for all $n \geq 1$

PART - II : JEE (MAIN) / AIEEE PROBLEMS (PREVIOUS YEARS)

1. Sachin and Rahul attempted to solve a quadratic equaiton. Sachin made a mistake in writing down the constant term and ended up in roots (4, 3). Rahul made a mistake in writing down coefficient of x to get roots (3, 2). The correct roots of equation are : **[AIEEE- 2011, II, (4, –1), 120]**

(1) 6, 1 (2) 4, 3 (3) –6 , –1 (4) –4 , –3

2. Let for $a \neq a_1 \neq 0$, $f(x) = ax^2 + bx + c$, $g(x) = a_1x^2 + b_1x + c_1$ and $p(x) = f(x) - g(x)$. If $p(x) = 0$ only for $x = -1$ and $p(-2) = 2$, then the value of p(2) is : **[AIEEE- 2011, II, (4, –1), 120]**

(1) 3 (2) 9 (3) 6 (4) 18

3. The equation $e^{\sin x} - e^{-\sin x} - 4 = 0$ has : **[AIEEE- 2012 (4, –1), 120]**

(1) infinite number of real roots (2) no real roots

(3) exactly one real root (4) exactly four real roots

4 . If the equations $x^2 + 2x + 3 = 0$ and $ax^2 + bx + c = 0$, $a,b,c \in R$, have a common root, then a : b : c is **[AIEEE - 2013, (4, –¼), 120]**

(1) 1 : 2 : 3 (2) 3 : 2 : 1 (3) 1 : 3 : 2 (4) 3 : 1 : 2

5. If $a \in R$ and the equation $-3(x - [x])^2 + 2(x - [x]) + a^2 = 0$ (where [x] denotes the greatest integer $\leq x$) has no intgeral solution, then all possible values of a lie in the interval : **[JEE(Main) 2014, (4, – ¼), 120]**

(1) (–2, –1) (2) $(-\infty, -2) \cup (2, \infty)$ (3) $(-1, 0) \cup (0, 1)$ (4) (1, 2)

6. Let α and β be the roots of equation $px^2 + qx + r = 0$, $p \neq 0$. If p, q ,r are in the A.P. and $\frac{1}{\alpha} + \frac{1}{\beta} = 4$, then the value of $|\alpha - \beta|$ is : **[JEE(Main) 2014, (4, – ¼), 120]**

(1) $\frac{\sqrt{34}}{9}$ (2) $\frac{2\sqrt{13}}{9}$ (3) $\frac{\sqrt{61}}{9}$ (4) $\frac{2\sqrt{17}}{9}$

7. Let α and β be the roots of equation $x^2 - 6x - 2 = 0$. If $a_n = \alpha^n - \beta^n$, for $n \geq 1$, then the value of $\frac{a_{10} - 2a_8}{2a_9}$ is equal to :

[JEE(Main) 2015, (4, – ¼), 120]

(1) 6 (2) – 6 (3) 3 (4) –3

8. If both the roots of the quadratic equation $x^2 - mx + 4 = 0$ are real and distinct and they lie in the interval [1,5], then m lies in the interval: **[JEE(Main) 2019, Jan]**

(1) (4,5) (2) (3,4) (3) (5,6) (4) (–5,–4)

9. The number of all possible positive integral values of α for which the roots of the quadratic equation, $6x^2 - 11x + \alpha = 0$ are rational numbers is : **[JEE(Main) 2019, Jan]**

(1) 2 (2) 5 (3) 3 (4) 4

10. The least positive value of 'a' for which the equation $2x^2 + (a - 10)x + \frac{33}{2} = 2a$ has real roots is

[JEE-Mains - 2020(April)]

Answers

Exercise # 1

PART - I

SECTION-(A)

A-1 $\lambda = 2$, No real value of x.

A-2 (i) $\frac{-19}{4}$ (ii) $\frac{9}{8}$ **A-3** 4

A-4 $-(q + r)$

A-5. $3x^2 - 19x + 3 = 0$. **A-6.** 3

A-7. **3** **A-8** 27

SECTION-(B)

B-1. (i) 17 (ii) –57 (iii) –2

B-3. roots are $\frac{-4}{3}, -\frac{3}{2}, \frac{-5}{3}$, $\lambda = 121$

B-4. $x^3 - 15x^2 + 67x - 77 = 0$.

B-5. 4

B-6. (i) –3 (ii) 15 (iii) –15

SECTION-(C)

C-1. $(-8, 19)$ **C-3** $5a - b \geq -2$

C-5. $5 \pm 2\sqrt{6}$ **C-6.** $[c^2 - m^2, c^2)$

C-8. 2/3 **C-12.** $\{0, -2\}$

C-13. (i) $4, \ -2 \pm i5\sqrt{3}$ (ii) 3 or 4

C-14. $-1 \pm \sqrt{3}, -1 \pm \sqrt{-1}$

C-16. (i) $(-120, -3)$ (ii) $(-3, 5) \cup (8, \infty)$ (iii) $(5, 8)$ (iv) $(-\infty, -120)$

SECTION-(D)

D-2. (i) $y \in [-4,\infty)$ (ii) $y \in [-4, 0]$
(iii) $y \in [-3, 5]$

D-3. (i) $\left[\frac{1}{7}, 7\right]$ (ii) $[0, 1]$
(iii) $(-\infty, 1) \cup [13, \infty)$

D-4. $a < 0$

D-5. 3

SECTION-(E)

E-1. $a \in (-\infty,0] \cup (6,\infty)$

E-3. 2

E-4. $(-\infty, -2) \cup (1, \infty)$

E-5. no value of a is

E-6. $-2 < a < 3$

E-7. (i) $(2, \infty)$ (ii) $(2, 15)$

SECTION-(F)

F-1 $a = 1, -2$

F-3. 0

PART - II

SECTION-(A)

A-1	(B)	**A-2**	(B)
A-3	(B)	**A-4.**	(A)
A-5.	(B)	**A-6.**	(A)
A-7.	(D)		

SECTION-(B)

B-1.	(C)	**B-2.**	(B)
B-3.	(A)	**B-4.**	(B)
B-5.	(A)	**B-6.**	(C)

SECTION-(C)

C-1.	(A)	**C-2.**	(A)
C-3.	(D)	**C-4.**	(B)
C-5.	(B)	**C-6.**	(B)

SECTION-(D)

D-1.	(A)	**D-2.**	(D)
D-3.	(A)	**D-4.**	(B)
D-5.	(B)	**D-6.**	(A)
D-7.	(A)	**D-8.**	(B)
D-9.	(D)		

SECTION-(E)

E-1.	(D)	**E-2.**	(C)
E-3.	(D)	**E-4.**	(A)
E-5.	(C)		

SECTION-(F)

F-1.	(C)	**F-2.**	(B)
F-3.	(B)		

PART - III

1.	$A \rightarrow q$; $B \rightarrow s$; $C \rightarrow s$; $D \rightarrow r$
2.	$A \rightarrow s, p,q$; $B \rightarrow q$; $C \rightarrow s$; $D \rightarrow p, q, r$
3.	$A\rightarrow p,q,s,r$; $B\rightarrow q,s,r$ $C\rightarrow p,q,s,r$; $D\rightarrow r$

Exercise # 2

PART - I

1.	(C)	2.	(A)
3.	(C)	4.	(C)
5.	(B)	6.	(D)
7.	(C)	8.	(D)
9.	(A)	10.	(C)
11.	(C)	12.	(C)
13.	(A)	14.	(B)
15.	(D)	16.	(B)
17.	(B)	18.	(B)
19.	(A)	20.	(B)
21.	(A)		

PART - II

1.	1.5	2.	1.2
3.	66.21	4.	2.33
5.	1.5	6.	2.24
7.	0	8.	12.1
9.	1.125	10.	0.5
11.	31.5	12.	9.9
13.	5.29	14.	14.4
15.	7	16.	2.5
17.	1.25	18.	1.2
19.	0.25	20.	4.5
21.	2.5		

PART - III

1.	(A, B)	2.	(B, C)
3.	(B, D)	4.	(A, C)
5.	(A, C)	6.	(A, C)
7.	(A, D)	8.	(A, C)
9.	(A, C)	10.	(A, D)
11.	(A, C)	12.	(A, C)
13.	(A)	14.	(A, B, C)
15.	(C, D)	16.	(A, B, C, D)
17.	(A, B, C, D)	18.	(A, D)
19.	(B, D)	20.	(A, C, D)
21.	(A, B, C, D)	22.	(A, B)
23.	(A, B, C)	24.	(C, D)
25.	(A, B, C)		

PART - IV

1.	(B)	2.	(A)
3.	(B)	4.	(D)
5.	(C)	6.	(D)
7.	(B)	8.	(A)
9.	(D)		

Exercise # 3

PART - I

1.	(B)	2.	(C)
3.	(B)	4.	(D)
5.	(A)	6.	(C)
7.	(C)	8.	(C)
9.	(A, B, D)		

PART - II

1.	(1)	2.	(4)
3.	(2)	4 .	(1)
5.	(3)	6.	(2)
7.	(3)	8.	(1)
9.	(3)	10.	8.00

Reliable Ranker Problems

1. If ax + by = 1 and $cx^2 + dy^2 = 1$ have only one solution, then prove that $\frac{a^2}{c} + \frac{b^2}{d} = 1$.

2. Prove that roots of $a^2x^2 + (b^2 + a^2 - c^2)x + b^2 = 0$ are not real, if a + b > c and $|a - b| < c$.
(where a, b, c are positive real numbers)

3. If α, β are two roots of the equation $\frac{(a-x)\sqrt{a-x} + (x-b)\sqrt{x-b}}{\sqrt{a-x} + \sqrt{x-b}} = a - b,$ then show that $\alpha^2 + \beta^2 = a^2 + b^2$.

4. Let Δ^2 be the discriminant and α, β be the roots of the equation $ax^2 + bx + c = 0$. Then find equation whose roots are $2a\alpha + \Delta$ and $2a\beta - \Delta$.

5. If $V_n = \alpha^n + \beta^n$, where α, β are roots of equation $x^2 + x - 1 = 0$. Then prove that $V_n + V_{n-3} = 2V_{n-2}$ and hence evaluate V_7 (n is a whole number)

6. a, b, c, d are four distinct real numbers and they are in A.P.
If $2(a - b) + x(b - c)^2 + (c - a)^3 = 2(a - d) + (b - d)^2 + (c - d)^3$ then find the permissible values of x.

7. Find all 'm' for which $f(x) \equiv x^2 - (m - 3)x + m > 0$ for all values of 'x' in [1, 2].

8. If the roots of the equation $ax^2 - 2bx + c = 0$ are imaginary, find the number of real roots of
$4e^x + (a + c)^2 (x^3 + x) = 4b^2 x$.

9. Show that $(4a^2 - 1)\left(\frac{x^2}{x^2+1}\right)^2 + 2(4a^2 - 1)\left(\frac{x^2}{x^2+1}\right) + 4 = 0$ can't have real roots for any real 'a'.

10. If $\alpha, \beta; \beta, \gamma$ and γ, α are the roots of $a_i x^2 + b_i x + c_i = 0$; i = 1, 2, 3 then show that

$$(\alpha+\beta+\gamma)+(\alpha\beta+\beta\gamma+\alpha\gamma)+\alpha\beta\gamma = \pm\left\{\prod_{i=1}^{3}\left(\frac{a_i - b_i + c_i}{a_i}\right)\right\}^{1/2} - 1$$

11. The equations $ax^2 + bx + c = 0$ and $cx^2 + bx + a = 0$ have a common root and the difference of their other roots is one. Then show that $|ac| = |a^2 - c^2|$. Hence or otherwise find the maximum and minimum value of $\left|\frac{a}{c}\right|$.

12. If the quadratic equations $x^2 + abx + c = 0$ and $x^2 + acx + b = 0$ have a common root, then find the equation containing their other roots.

13. Suppose that $a_1 > a_2 > a_3 > a_4 > a_5 > a_6$ and

$p = a_1 + a_2 + a_3 + a_4 + a_5 + a_6$

$q = a_1a_3 + a_3a_5 + a_5a_1 + a_2a_4 + a_4a_6 + a_6a_2$

$r = a_1a_3a_5 + a_2a_4a_6$,

then show that roots of the equation $2x^3 - px^2 + qx - r = 0$ are real.

14. If $\beta + \cos^2\alpha$, $\beta + \sin^2\alpha$ are the roots of $x^2 + 2bx + c = 0$ and $\gamma + \cos^4\alpha$, $\gamma + \sin^4\alpha$ are the roots of $X^2 + 2BX + C = 0$, then prove that $b^2 - B^2 = c - C$.

15. Find the fifth degree polynomial which leaves remainder 1 when divided by $(x - 1)^3$ and remainder –1 when divided by $(x + 1)^3$?

16. If roots of the quadratic equation $x^2 - (2n + 18)x - n - 11 = 0$, $n \in$ set of integers, are rational, then find the value(s) of n.

17. Find the set of values of 'a' if $(x^2 + x)^2 + a(x^2 + x) + 4 = 0$ has

(i) all four real & distinct roots.

(ii) four roots in which only two roots are real and distinct.

(iii) all four imaginary roots.

(iv) four real roots in which only two are equal.

18. How many quadratic equations are there which are unchanged by squaring their roots ?

19. Let $P(x) = x^5 + x^2 + 1$ have zeros $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ and $Q(x) = x^2 - 2$, then find

(i) $\prod_{i=1}^{5} Q(\alpha_i)$ (ii) $\sum_{i=1}^{5} Q(\alpha_i)$ (iii) $\sum_{1 \le i < j \le 5}\sum Q(\alpha_i)\, Q(\alpha_j)$ (iv) $\sum_{i=1}^{5} Q^2(\alpha_i)$

20. If a, b, c are non–zero, unequal rational numbers then prove that the roots of the equation $(abc^2)x^2 + 3a^2cx + b^2cx - 6a^2 - ab + 2b^2 = 0$ are rational.

21. If a, b, c represents sides of a Δ then prove that equation $x^2 - (a^2 + b^2 + c^2)x + a^2b^2 + b^2c^2 + c^2a^2 = 0$ has imaginary roots.

22. Let x_1, x_2, x_3 satisfying the equation $x^3 - x^2 + \beta x + \gamma = 0$ are in G.P where $(x_1, x_2, x_3 > 0)$ then find the interval in which β and γ lie.

23. If x_1 is a root of $ax^2 + bx + c = 0$, x_2 is a root of $-ax^2 + bx + c = 0$ where $0 < x_1 < x_2$, show that the equation $ax^2 + 2bx + 2c = 0$ has a root x_3 satisfying $0 < x_1 < x_3 < x_2$.

24. Find the value of parameters 'a' for which equation $x^4 - (a + 1)x^3 + x^2 + (a + 1)x - 2 = 0$ have at least two distinct positive real roots.

25. Find the number of positive real roots of $x^4 - 4x - 1 = 0$.

26. If a, b, c are positive real numbers such that a > b > c and the quadratic equation $(a + b - 2c)x^2 + (b + c - 2a)x + (c + a - 2b) = 0$ has a root in the interval (–1, 0), then prove that the equation $x^2 + (a + c)x + 4b^2 = 0$ cannot have real roots.

27. If α, β^2 are integers, β^2 is non-zero multiple of 3 and $\alpha + i\beta$, -2α are roots of $x^3 + ax^2 + bx - 316 = 0$, a, b, $\beta \in R$, then find a, b.

Answers

4. $x^2 + 2bx + b^2 = 0$ or $x^2 + 2bx - 3b^2 + 16ac = 0$ **5.** -29

6. $x \in (-\infty, -8] \cup [16, \infty)$ **7.** $(-\infty, 10)$ **8.** 1

11. Minimum value $= \dfrac{-1+\sqrt{5}}{2}$ and maximum value $= \dfrac{1+\sqrt{5}}{2}$ **12.** $x^2 - a(b + c)x + a^2bc = 0$

15. $f(x) = \dfrac{1}{8}(3x^5 - 10x^3 + 15x)$ **16.** $n = -8$ or -11

17. (i) $a \in (-\infty, -4)$ (ii) $a \in \left(\dfrac{65}{4}, \infty\right)$ (iii) $a \in \left(-4, \dfrac{65}{4}\right)$ (iv) $a \in \phi$

18. 4 **19.** (i) -23 (ii) -10 (iii) 40 (iv) 20

22. $0 < \beta < \dfrac{1}{3}$ and $\gamma < 0$ **24.** $a \in [2\sqrt{2} - 1, \infty)$ **25.** 1 **27.** $a = 0$, $b = 63$