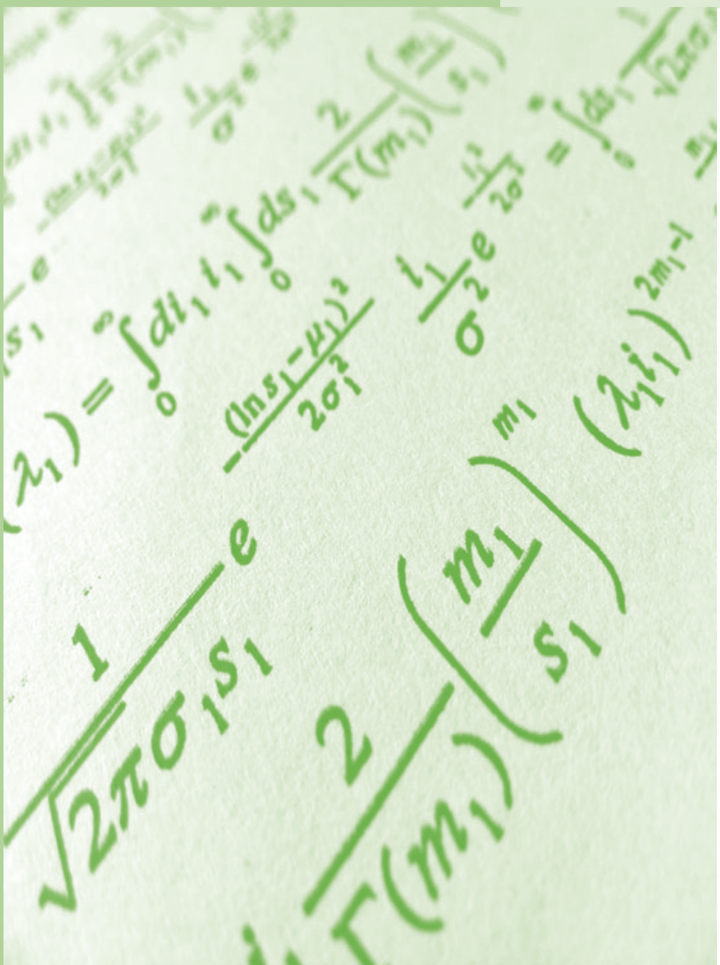
Chapter

16

Mathematical Induction and Binomial Theorem



REMEMBER

Before beginning this chapter, you should be able to:

- Understand statements, truth tables of different compound statements
- Study laws of 'algebra of statements'

KEY IDEAS

After completing this chapter, you would be able to:

- State the principle of mathematical induction
- Form a binomial expression and represent it with the help of Pascal triangle
- Apply factorial notation to binomial expressions
- Prove binomial theorem and solve problems related

INTRODUCTION

The process of mathematical induction is an indirect method which helps us to prove complex mathematical formulae that cannot be easily proved by direct methods.

For example, to prove that '$n(n + 1)$ is always divisible by 2' for n being a natural number, we can substitute $n = 1, 2, 3, \ldots$ in $n(n + 1)$, and check in each case if the result is divisible by 2. After checking, for a few of values, we can say that the formula is likely to be correct. Since, we cannot substitute all possible values of n, to prove the formula we use the principle of mathematical induction to prove the given formula.

THE PRINCIPLE OF MATHEMATICAL INDUCTION

If $P(n)$ is a statement such that,

1. $P(n)$ is true for $n = 1$.
2. $P(n)$ is true for $n = k + 1$, when it is true for $n = k$, where k is a natural number then the statement $P(n)$ is true for all natural numbers.

Let us prove some results using this principle.

EXAMPLE 16.1

Prove that $1+2+3+\cdots+n=\dfrac{n(n+1)}{2}$.

SOLUTION

Let $P(n): 1+2+\cdots+n=\dfrac{n(n+1)}{2}$ be the given statement.

Step 1: Put $n = 1$

Then, LHS = 1 and RHS $=\dfrac{1(1+1)}{2}=1$

$\therefore$ LHS = RHS $\Rightarrow$ $P(n)$ is true for $n = 1$.

Step 2: Assume that $P(n)$ is true for $n = k$.

$$\therefore \quad 1+2+3+\cdots+k=\frac{k(k+1)}{2}$$

Adding $(k + 1)$ on both sides, we get

$$1+2+3+\cdots+k+(k+1)=\frac{k(k+1)}{2}+(k+1)$$

$$=(k+1)\left(\frac{k}{2}+1\right)=\frac{(k+1)(k+2)}{2}=\frac{(k+1)(\overline{k+1}+1)}{2}$$

$\Rightarrow$ $P(n)$ is true for $n = k + 1$.

$\therefore$ By the principle of mathematical induction $P(n)$ is true for all natural numbers n.

Hence, $1+2+3+\cdots+n=\dfrac{n(n+1)}{2}$ for all $n \in N$.

EXAMPLE 16.2

Prove that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

SOLUTION

Let $P(n)$: $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ be the given statement.

Step 1: Put $n = 1$

Then, LHS = 1

RHS = $(1)^2 = 1$

$\therefore$ LHS = RHS

$\Rightarrow$ $P(n)$ is true for $n = 1$.

Step 2: Assume that $P(n)$ is true for $n = k$.

$\therefore$ $1 + 3 + 5 + \cdots + (2k - 1) = k^2$.

Adding $2k + 1$ on both sides, we get

$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = k^2 + (2k + 1) = (k + 1)^2$

$\therefore$ $1 + 3 + 5 + \cdots + (2k - 1) + (2(k + 1) - 1) = (k + 1)^2$

$\Rightarrow$ $P(n)$ is true for $n = k + 1$.

$\therefore$ By the principle of mathematical induction $P(n)$ is true for all natural numbers 'n'.

Hence, $1 + 3 + 5 + \cdots + (2n - 1) = n^2$, for all $n \in N$.

EXAMPLE 16.3

Prove that $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n \cdot (n+1) = \dfrac{n(n+1)(n+2)}{3}$.

SOLUTION

Let $P(n): 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n \cdot (n+1) = \dfrac{n(n+1)(n+2)}{3}$ be the given statement.

Step 1: Put $n = 1$

Then, LHS = $1 \cdot 2 = 2$

$$\text{RHS} = \frac{1(1+1)(1+2)}{3} = \frac{2 \times 3}{3} = 2$$

$\therefore$ LHS = RHS

$\Rightarrow$ $P(n)$ is true for $n = 1$.

Step 2: Assume that $P(n)$ is true for $n = k$.

$\therefore$ $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + k(k + 1) = \dfrac{k(k+1)(k+2)}{3}$.

Adding $(k + 1)(k + 2)$ on both sides, we get $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + k(k + 1) + (k + 1)(k + 2)$

$$= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) = (k+1)(k+2)\left(\frac{k}{3}+1\right) = \frac{(k+1)(k+2)(k+3)}{3}$$

$$\therefore \quad 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + k \cdot (k+1) + (k+1)(k+2) = \frac{(k+1)(\overline{k+1}+1)(\overline{k+1}+2)}{3}.$$

$\Rightarrow$ $P(n)$ is true for $n = k + 1$.

$\therefore$ By the principle of mathematical induction $P(n)$ is true for all natural numbers

Hence, $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n \cdot (n+1) = \dfrac{n(n+1)(n+2)}{3}$, $n \in N$.

EXAMPLE 16.4

Prove that $3^{n+1} > 3(n+1)$.

SOLUTION

Let $P(n)$: $3^{n+1} > 3(n+1)$

Step 1: Put $n = 1$

Then, $3^2 > 3(2)$

$\Rightarrow$ $p(n)$ is true for $n = 1$.

Step 2: Assume that $P(n)$ is true for $n = k$.

Then, $3^{k+1} > 3(k+1)$.

Multiplying throughout with '3'

$3^{k+1} \cdot 3 > 3(k+1) \cdot 3 = 9k + 9 = 3(k+2)$

$\Rightarrow$ $3^{\overline{k+1}+1} > 3(\overline{k+1}+1)$

$P(n)$ is true for $n = k + 1$.

$\therefore$ By the principle of mathematical induction, $P(n)$ is true for all $n \in N$.

Hence, $3^{n+1} > 3(n+1)$, $\forall\ n \in N$.

EXAMPLE 16.5

Prove that 7 is a factor of $2^{3n} - 1$ for all natural numbers n.

SOLUTION

Let $P(n)$: 7 is a factor of $2^{3n} - 1$ be the given statement

Step 1: When $n = 1$,

$2^{3(1)} - 1 = 7$ and 7 is a factor of itself.

$\therefore$ $P(n)$ is true for $n = 1$.

Step 2: Let $P(n)$ be true for $n = k$

$\Rightarrow$ 7 is a factor of $2^{3k} - 1$

$\Rightarrow$ $2^{3k} - 1 = 7M$, where $M \in N$

$\Rightarrow$ $2^{3K} = 7M + 1$. (1)

Now consider $2^{3(k+1)} - 1 = 2^{3k+3} - 1 = 2^{3k} \cdot 2^3 - 1$

$= 8(7M + 1) - 1$ (using (1)) $= 56M + 7$ (As $2^{3k} = 7m + 1$)

$\therefore$ $2^{3(k+1)} - 1 = 7(8M + 1)$.

$\Rightarrow$ 7 is a factor of $2^{3(k+1)} - 1$

$\Rightarrow$ $P(n)$ is true for $n = k + 1$.

$\therefore$ By the principle of mathematical induction, $P(n)$ is true for all natural numbers n.

Hence, 7 is a factor of $2^{3n} - 1$ for all $n \in N$.

Binomial Expression

An algebraic expression containing only two terms is called a binomial expression.

For example, $x + 2y$, $3x + 5y$, $8x - 7y$, etc.

We know that, $(a + b)^2 = a^2 + 2ab + b^2$.

$$(a + b)^3 = (a + b)\ (a + b)^2$$
$$= a^3 + 3a^2b + 3ab^2 + b^3.$$

Now using a similar approach we can arrive at the expressions for $(a + b)^4$, $(a + b)^5$, etc. However, when the index is large, this process becomes very cumbersome. Hence, we need a simpler method to arrive at the expression for $(a + b)^n$, for $n = 1, 2, 3, \ldots$.

The binomial theorem is the appropriate tool in this case. It helps us arrive at the expression for $(a + b)^n$, for any value of n, by using a few standard coefficients also known as binomial coefficients.

Now, consider the following cases in which we find the expansions when a binomial expression is raised to different powers.

$$(x + y)^1 = x + y$$
$$(x + y)^2 = x^2 + 2xy + y^2$$
$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

In the above examples, the coefficients of the variables in the expansions of the powers of the binomial expression are called binomial coefficients.

When the binomial coefficients are listed, for different values of n, we see a definite pattern being followed.

This pattern is given by the Pascal Triangle.

Pascal Triangle

This definite pattern, shown in the following figure, can be used to write the binomial expansions for higher powers such as $n = 6, 7, 8, \ldots$ so on. The binomial theorem gives us a general algebraic formula by means of which any power of a binomial expression can be expanded into a series of simpler terms.

The Exponent in the Binomial	The Coefficients of the Terms in the Expansion
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1

Before we take up the binomial theorem, let us review the concepts of factorial notation and the ${}^nC_r$ representation.

Factorial Notation and ${}^nC_r$ Representation

The factorial of n is denoted by $n!$ and is defined as $n! = 1 \times 2 \times 3 \times \cdots \times (n - 1) \times n$.

For example, $4! = 1 \times 2 \times 3 \times 4$ and $6! = 1 \times 2 \times 3 \times 4 \times 5 \times 6$

Also, $0! = 1$ and $n! = n\ (n - 1)!$

For $0 \le r \le n$, we define ${}^nC_r$ as ${}^nC_r = \dfrac{n!}{(n-r)!r!}$

For example, ${}^6C_2 = \dfrac{6!}{(6-2)!2!} = \dfrac{6\times 5\times 4!}{4!\times 2!} = 15$

also, ${}^nC_0 = {}^nC_n = 1$; ${}^nC_1 = {}^nC_{n-1} = n$ and ${}^nC_r = {}^nC_{n-r}$

For example, ${}^{10}C_2 = {}^{10}C_8$ and if ${}^nC_3 = {}^nC_5$, then $n = 3 + 5 = 8$.

Binomial Theorem

If n is a positive integer,

$(x+y)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1}\, y + {}^nC_2\, x^{n-2}\, y^2 + \cdots + {}^nC_r\, x^{n-r}\, y^r + \cdots + {}^nC_n\, y^n$.

Important Inferences from the above Expansion

1. The number of terms in the expansion is $n + 1$.
2. The exponent of x goes on decreasing by '1' from left to right and the power of y goes on increasing by '1' from left to right.
3. In each term of the expansion the sum of the exponents of x and y is equal to the exponent (n) of the binomial expression.
4. The coefficients of the terms that are equidistant from the beginning and the end have numerically equal, i.e., ${}^nC_0 = {}^nC_n$; ${}^nC_1 = {}^nC_{n-1}$; ${}^nC_2 = {}^nC_{n-2}$ and so on.
5. The general term in the expansion of $(x + y)^n$ is given by $T_{r+1} = {}^nC_r\, x^{n-r}\, y^r$.
6. On substituting '$-y$' in place of 'y' in the expansion, we get

   $(x-y)^n = {}^nC_0\, x^n - {}^nC_1\, x^{n-1}\, y + {}^nC_2\, x^{n-2}\, y^2 - {}^nC_3\, x^{n-3}\, y^3 + \cdots + (-1)^n\, {}^nC_n y^n$.

   The general term in the expansion $(x - y)^n$ is $T_{r+1} = (-1)^r\, {}^nC_r\, x^{n-r}\, y^r$.

EXAMPLE 16.6

Expand $(x + 2y)^5$.

SOLUTION

$(x+2y)^5 = {}^5C_0\, x^5 + {}^5C_1\, x^{5-1}\, (2y) + {}^5C_2\, x^{5-2}\, (2y)^2 + {}^5C_3\, x^{5-3}\, (2y)^3 + {}^5C_4\, x^{5-4}\, (2y)^4 + {}^5C_5\, (2y)^5$.

$\Rightarrow \quad (x+2y)^5 = x^5 + 5x^4\, (2y) + 10x^3 4y^2 + 10x^2 8y^3 + 5x\, 16y^4 + 2^5 y^5$

$= x^5 + 10x^4 y + 40x^3 y^2 + 80x^2 y^3 + 80xy^4 + 32y^5$.

EXAMPLE 16.7

Find the 3rd term in the expansion of $(3x - 5y)^7$.

SOLUTION

The general term in $(x - y)^n$ is $T_{r+1} = (-1)^r\, {}^nC_r\, x^{n-r} y^r$

$\therefore\ T_3 = T_{2+1} = (-1)^2\, {}^7C_2\, (3x)^{7-2}\, (5y)^2 = {}^7C_2 (3x)^5\, (5y)^2$.

Middle Terms in the Expansion of $(x + y)^n$

Depending on the nature of n, i.e., whether n is even or odd, there may exist one or two middle terms.

Case 1:

When n is an even number, then there is only one middle term in the expansion $(x + y)^n$, which is $\left(\frac{n}{2}+1\right)$th term.

Case 2:

When n is odd number, there will be two middle terms in the expansion of $(x + y)^n$, which are $\left(\frac{n+1}{2}\right)$th and $\left(\frac{n+3}{2}\right)$th terms.

EXAMPLE 16.8

Find the middle term in the expansion of $(2x + 3y)^8$.

SOLUTION

Since n is even number, $\left(\frac{8}{2}+1\right)$th term, i.e., 5th term is the middle term in $(2x + 3y)^8$.

$T_5 = T_{4+1} = {}^8C_4\,(2x)^{8-4}\,(3y)^4 = {}^8C_4\,(2x)^4\,(3y)^4$.

EXAMPLE 16.9

Find the middle terms in the expansion of $(5x - 7y)^7$.

SOLUTION

Since n is an odd number, the expansion contains two middle terms.

$\left(\frac{7+1}{2}\right)$th and $\left(\frac{7+3}{2}\right)$th terms are the two middle terms in the expansion of $(5x - 7y)^7$.

$T_4 = T_{3+1} = (-1)^3\,{}^7C_3\,(5x)^{7-3}\,(7y)^3 = -{}^7C_3\,(5x)^4\,(7y)^3$

$T_5 = T_{4+1} = (-1)^4 \cdot {}^7C_4\,(5x)^{7-4} \cdot (7y)^4 = {}^7C_4\,(5x)^3\,(7y)^4$.

Term Independent of x

In an expansion of form $\left(x^p + \frac{1}{x^q}\right)^n$, the term for which the exponent of x is 0 is said to be the term that is independent of x or a constant term.

For example, in the expansion $\left(x+\frac{1}{x}\right)^2 = x^2 + 2 + \frac{1}{x^2}$, the 2nd term is independent of 'x'.

EXAMPLE 16.10

Find the term independent of x in $\left(x + \frac{1}{x}\right)^4$.

SOLUTION

Let T_{r+1} be the independent term of x in the given expansion.

$$\therefore \quad T_{r+1} = {}^4C_r x^{4-r}\left(\frac{1}{x}\right)^r = {}^4C_r \frac{x^{4-r}}{x^r} = {}^4C_r x^{4-2r}.$$

For the term independent of x the power of x should be zero.

$\therefore$ $4 - 2r = 0$ or $r = 2$.

$\Rightarrow$ $T_{2+1} = T_3$ term, is the independent term of the expansion.

Note If r is not a positive integer, then the expansion does not contain constant term.

EXAMPLE 16.11

Find the coefficient of x^2 in $\left(x^2 + \frac{1}{x^3}\right)^6$.

SOLUTION

Let T_{r+1} be the term containing x^2.

$$T_{r+1} = {}^6C_r (x^2)^{6-r}\left(\frac{1}{x^3}\right)^r = {}^6C_r x^{12-2r}\frac{1}{x^{3r}} = {}^6C_r x^{12-5r}$$

As the coefficient of x is 2.

$12 - 5r = 2 \quad \Rightarrow \quad r = 2$

$\therefore$ Coefficient of $x^2 = {}^6C_2 = 15$.

The Greatest Coefficient in the Expansion of $(1 + x)^n$ (where n is a Positive Integer)

The coefficient of the $(r + 1)$th term in the expansion of $(1 + x)^n$ is ${}^nC_r$.

${}^nC_r$ is maximum when $r = \frac{n}{2}$ (if n is even) and

$r = \frac{n+1}{2}$ or $\frac{n-1}{2}$ (if n is odd).

EXAMPLE 16.12

Find the total number of terms in the expansion of $(2 + 3x)^{15} + (2 - 3x)^{15}$.

SOLUTION

$(2 + 3x)^{15} = {}^{15}C_0(2)^{15} + {}^{15}C_1(2)^{14}(3x)^1 + \cdots + {}^{15}C_{14}(2)^1(3x)^{14} + {}^{15}C_{15}(3x)^{15}$ and $(2 - 3x)^{15}$
$= {}^{15}C_0(2)^{15} - {}^{15}C_1(2)^{14}(3x^1) + \cdots + {}^{15}C_{14}(2)^1(3x)^{14} - {}^{15}C_{15}(3x)^{15}$.

Adding the two equations, we see that the terms in even positions get cancelled, and we get
$(2 + 3x)^{15} + (2 - 3x)^{15} = 2[{}^{15}C_0(2)^{15} + {}^{15}C_2(2)^{13}(3x)^2 + \cdots + {}^{15}C_{14}(2)^1(3x)^{14}]$

$\therefore$ Total number of terms = 8

Alternately, the number of terms in $(a + x)^n + (a - x)^n$, if n is odd is $\frac{n+1}{2}$.

Hence, in this case, the number of terms are $\frac{15+1}{2} = 8$.

EXAMPLE 16.13

If the expansion $\left(x^2 + \frac{1}{x^3}\right)^n$ is to contain an independent term, then what should be the value of n?

SOLUTION

General term, $T_{r+1} = {}^nC_r \cdot x^{n-r}\, y^r$, for $(x + y)^n$.

$\Rightarrow$ General term of $\left(x^2 + \frac{1}{x^3}\right)^n$ is ${}^nC_r \cdot x^{2n-2r} \cdot \frac{1}{x^{3r}} = {}^nC_r \cdot x^{2n-5r}$.

For a term to be independent of x, $2n - 5r$ should be equal to zero, That is, $2n - 5r = 0$.

$\Rightarrow$ $r = \frac{2}{5}n$, since r can take only integral values, n has to be a multiple of 5.

EXAMPLE 16.14

If the coefficient of x^7 in $\left(ax + \frac{1}{x}\right)^9$ and x^{-7} in $\left(bx - \frac{1}{x}\right)^9$ are equal, find the relation between a and b?

SOLUTION

For $\left(ax + \frac{1}{x}\right)^9$, $T_{r+1} = {}^9C_r(ax)^{9-r}\left(\frac{1}{x}\right)^r = {}^9C_r(a)^{9-2r}(x)^{9-2r}$ as $9 - 2r = 7$, $r = 1$.

$\therefore$ Coefficient of x^7 is ${}^9C_1(a)^{9-1} = 9(a)^8$.

Now, for $\left(bx - \frac{1}{x}\right)^9$, $T_{r+1} = {}^9C_r(bx)^{9-r}\left(-\frac{1}{x}\right)^r = {}^9C_r(b)^{9-r}(-1)^r x^{9-2r}$ as $9 - 2r = -7$, $r = 8$.

$\therefore$ Coefficient of x^{-7} is ${}^9C_8\, b^{9-8}\,(-1)^8 = 9b$

$\therefore$ $9a^8 = 9b$, i.e., $a^8 - b = 0$.

EXAMPLE 16.15

Find the term independent of 'x' in the expansion of $(1+x^2)^4\left(1+\frac{1}{x^2}\right)^4$.

SOLUTION

$$(1+x^2)^4\left(1+\frac{1}{x^2}\right)^4 = ({}^4C_0 + {}^4C_1x^2 + \cdots + {}^4C_4x^8)\times({}^4C_0 + {}^4C_1x^{-2} + \cdots + {}^4C_4x^{-8})$$

The term independent of 'x' is the term containing the coefficient,

$({}^4C_0 \cdot {}^4C_0 \cdot {}^4C_1 \cdot {}^4C_1 + \cdots + {}^4C_4 \cdot {}^4C_4)$

$= ({}^4C_0)^2 + ({}^4C_1)^2 + ({}^4C_2)^2 + ({}^4C_3)^2 + ({}^4C_4)^2$

$= 1^2 + 4^2 + 6^2 + 4^2 + 1^2 = 70$.

EXAMPLE 16.16

Find the sum of the coefficients of the terms of the expansion $(1 + x + 2x^2)^6$.

SOLUTION

Substituting $x = 1$, we have $(1 + 1 + 2)^6$, which gives us the sum of the coefficients of the terms of the expansion.

$\therefore$ Sum $= 4^6$.

EXAMPLE 16.17

Find the value of x, if the fourth term in the expansion of $\left(\frac{1}{x^2} + x^2 \cdot 2^x\right)^6$ is 160.

SOLUTION

4th term $\Rightarrow$ $T_{3+1} = {}^6C_3 \cdot \left(\frac{1}{x^2}\right) \cdot (x^2)^3 \cdot (2^x)^3$

$\therefore$ ${}^6C_3 \cdot (2^x)^3 = 160$

That is, $20 \cdot 2^{3x} = 160$

$\therefore$ $2^{3x} = 8 \Rightarrow 2^{3x} = 2^3$

$\therefore$ $x = 1$.

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. If $p(n)$ is a statement which is true for $n = 1$ and true for $(n + 1)$ then ________.
2. According to the principle of mathematical induction, when can we say that a statement $X(n)$ is true for all natural numbers n?
3. If $p(n) = n(n + 1)(n + 2)$ then highest common factor of $p(n)$, for different values of n where n is any natural number is ________.
4. Is $2^{3n} - 1$ a prime number for all natural numbers n?
5. The product of $(q - 1)$ consecutive integers where $q > 1$ is divisible by ________.
6. An algebraic expression with two terms is called a ________.
7. In Pascal triangle, each row of coefficients is bounded on both sides by ________.
8. The number of terms in the expansion of $(x + y)^n$ is ________ (where n is a positive integer).
9. In the expansion of $(x + y)^n$, if the exponent of x in second term is 10, what is the exponent of y in 11th term.
10. What is the coefficient of a term in a row of Pascal triangle if in the preceding row, the coefficient on the immediate left is 5 and on the immediate right is 10.
11. In the expansion of various powers of $(x + y)^n$, if the expansion contains 49 terms, then it is the expansion of ________.
12. In the expression of $(x + y)^{123}$, the sum of the exponents of x and y in 63rd term is ________.
13. $(n - r)! =$ ________.
14. The value of ${}^{n+1}C_r =$ ________.
15. If ${}^{n}C_r = 1$ and $n = 6$, then what may be the value(s) of r be?
16. In the expansion of $(x + y)^n$, $T_{r+1} =$ ________.
17. ${}^{7}C_2 =$ ________.
18. ${}^{1230}C_0 =$ ________.
19. The coefficient of x in the expansion of $(2x + 3)^5$ is ________ .
20. The coefficient of y^7 in the expansion of $(y + z)^7$ is ________.
21. $(x + y)^3 =$ ________.
22. The term which does not contain 'a' in the expansion of $\left(\frac{x}{a} + 6x\right)^{12}$ is ________.
23. If ${}^{12}C_r\,(4)^{12-r}\,(x)^{12-3r}$ is a constant term in an expansion, then $r =$ ________.
24. Write the first, the middle and the last terms in the expansion of $(x^2 + 1)^3$.
25. Constant term in the expansion of $(x + 3)^{16}$ is ________.
26. The sum of the first n even natural numbers is ________.
27. The sum of the first n odd natural numbers is ________.
28. The elements in the fifth row of Pascal triangle is ________.
29. If ${}^{n}C_3 = {}^{n}C_{15}$, then ${}^{20}C_n$ is ________.
30. The inequality $2^n > n$ is true for ________.

Short Answer Type Questions

Directions for questions 31 to 39: By mathematical Induction prove the following.

31. $1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1} = 2^{n-1}$, $n \in N$.
32. $a - b$ divides $a^n - b^n$, $n \in N$.
33. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}, n \in N.$
34. $2.5 + 3.8 + 4.11 + \cdots +$ upto n terms $= n(n^2 + 4n + 5)$, $n \in N$.
35. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}, n \in N.$
36. $a + (a + d) + (a + 2d) + \cdots$ upto n terms $= \dfrac{n}{2}[2a + (n-1)d]$

37. $\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\cdots+\frac{1}{(3n-1)(3n+2)}$

$=\frac{n}{6n+4}, n \in N.$

38. 9 is a factor of $4^n + 15n - 1$, $n \in N$.

39. $2 + 5 + 8 + \cdots + (3n - 1) = \frac{n(3n+1)}{2}, n \in N.$

40. Expand $\left(3x^2 + \frac{5}{y^2}\right)^6$.

41. Expand $(5x + 3y)^8$.

42. Find the middle term or terms of the expansion of $(x + 5y)^9$.

43. Find the middle term or terms of the expansion of $\left(x + \frac{1}{x}\right)^6$.

44. Find the 7th term in the expansion of $\left(5x - \frac{1}{7y}\right)^9$.

45. Prove that $2^{n+1} > 2n + 1$; $n \in N$.

Essay Type Questions

46. Find the value of $(\sqrt{3}+1)^5 - (\sqrt{3}-1)^5$.

47. Find the coefficient of x^{-5} in the expansion of $\left(2x^2 - \frac{1}{5x}\right)^8$.

48. Find the term independent of x in $\left(6x^2 - \frac{1}{7x^3}\right)^{10}$.

49. Find the coefficient of x^3 in the expansion of $\left(x^2 + \frac{1}{3x^3}\right)^4$.

50. Find the term independent of x in $\left(2x^5 + \frac{1}{3x^2}\right)^{21}$.

CONCEPT APPLICATION

Level 1

1. $n^2 + n + 1$ is a/an ______ number for all $n \in N$.
(a) even (b) odd
(c) prime (d) None of these

2. If the expansion $\left(x^3 + \frac{1}{x^2}\right)^n$ contains a term independent of x, then the value of n can be
(a) 18 (b) 20
(c) 24 (d) 22

3. $1 + 5 + 9 + \cdots + (4n - 3)$ is equal to
(a) $n(4n - 3)$ (b) $(2n - 1)$
(c) $n(2n - 1)$ (d) $(4n - 3)^2$

4. For all $n \in N$, which of the following is a factor of $2^{3n} - 1$?
(a) 3 (b) 5
(c) 7 (d) None of these

5. For what values of n is $14^n + 11^n$ divisible by 5?
(a) When n is an even positive integer
(b) For all values of n
(c) When n is a prime number
(d) When n is a odd positive integer

6. The smallest positive integer n for which $n! < \frac{(n-1)^n}{2}$ holds is
(a) 4 (b) 3
(c) 2 (d) 1

7. The third term from the end in the expansion of $\left(\frac{4x}{3y} - \frac{3y}{2x}\right)^9$ is
(a) ${}^9C_7 \frac{3^5}{2^3}\frac{y^5}{x^5}$ (b) $-{}^9C_7 \frac{3^5}{2^3}\frac{y^5}{x^5}$
(c) ${}^9C_7 \frac{3^3}{2^5}\frac{y^5}{x^3}$ (d) None of these

8. In the 8th term of $(x + y)^n$, the exponent of x is 3, then the exponent of x in 5th term is
(a) 5 (b) 4
(c) 2 (d) 6

PRACTICE QUESTIONS

9. The sum of the elements in the sixth row of Pascal triangle is

(a) 32 (b) 63
(c) 128 (d) 64

10. In $(x + y)^n - (x - y)^n$ if the number of terms is 5, then find n.

(a) 6 (b) 5
(c) 10 (d) 9

11. If the third term in the expansion of $(x + x^{\log_2^x})^6$ is 960, then the value of x is

(a) 2 (b) 3
(c) 4 (d) 8

12. Find the sum of coefficients of all the terms of the expansion $(ax + y)^n$.

(a) ${}^nC_0a^n + {}^nC_1a^{n-1}\ x^{n-1}y + {}^nC_2a^{n-2}\ x^{n-2}y^2 + \ldots + {}^nC_ny^n$

(b) ${}^nC_0a^n + {}^nC_1a^{n-1} + {}^nC_2a^{n-2} + \ldots + {}^nC_n$

(c) 2^n

(d) None of these

13. If the sum of the coefficients in the expansion $(4ax - 1 - 3a^2x^2)^{10}$ is 0, then the value of a can be

(a) 2 (b) 4
(c) 1 (d) 7

14. Find the coefficient of x^4 in the expansion of $\left(2x^2 + \dfrac{3}{x^3}\right)^7$.

(a) ${}^7C_2\ 2^5\ 3^3$ (b) ${}^7C_2\ 2^5\ 3^2$
(c) ${}^7C_2\ 3^5\ 2^2$ (d) ${}^7C_3\ 2^5\ 3^2$

15. $n^2 - n + 1$ is an odd number for all

(a) $n > 1$ (b) $n > 2$
(c) $n \geq 1$ (d) $n \geq 5$

16. $7^{n+1} + 3^{n+1}$ is divisible by

(a) 10 for all natural numbers n.
(b) 10 for odd natural numbers n
(c) 10 for even natural numbers n.
(d) None of these

17. For $n \in N$, $2^{3n} + 1$ is divisible by

(a) 3^{n+11} (b) 3^{n-11}
(c) 3^{n+1} (d) 3^{n+111}

18. $2^n - 1$ gives the set of all odd natural numbers for all $n \in N$. Comment on the given statement.

(a) True for all values of n
(b) False
(c) True for only odd values of n
(d) True for only prime values of n

19. In the 5th term of $(x + y)^n$, the exponent of y is 4, then the exponent of y in the 8th term is

(a) 1 (b) 7
(c) 5 (d) 9

20. If the coefficients of 6th and 5th terms of expansion $(1 + x)^n$ are in the ratio 7 : 5, then find the value of n.

(a) 11 (b) 12
(c) 10 (d) 9

21. The third term from the end in the expansion of $(3x - 2y)^{15}$ is

(a) $-{}^{15}C_5 3^{13}2^2x^{13}y^2$ (b) ${}^{15}C_5 3^{13}2^2x^{13}y^2$

(c) ${}^{15}C_5 3^2 2^{13}x^2y^{13}$ (d) $-{}^{15}C_5 3^2 2^{13}x^2y^{13}$

22. Find the sixth term in the expansion of $\left(2x^2 - \dfrac{3}{7x^3}\right)^{11}$.

(a) $-{}^{11}C_5\dfrac{2^6 3^5}{7^5}x^3$ (b) ${}^{11}C_5\dfrac{2^6 3^5}{7^5}x^{-3}$

(c) $-{}^{11}C_5\dfrac{2^6 3^5}{7^5}x^{-3}$ (d) None of these

23. The term independent of x in the expansion of $\left(x^3 - \dfrac{1}{x^2}\right)^{10}$ is

(a) ${}^{10}C_6$ (b) ${}^{10}C_7$

(c) ${}^{10}C_9$ (d) ${}^{-10}C_6$

24. Which term is the constant term in the expansion of $\left(2x - \dfrac{1}{3x}\right)^6$?

(a) 2nd term (b) 3rd term
(c) 4th term (d) 5th term

25. The sum of the coefficients in the expansion of $(x + y)^7$ is

(a) 119 (b) 64
(c) 256 (d) 128

26. The number of terms which are not radicals in the expansion $(\sqrt{7}+4)^6+(\sqrt{7}-4)^6$, after simplification is

(a) 6 (b) 5
(c) 4 (d) 3

27. The coefficient of x^4 in the expansion of $\left(4x^2+\frac{3}{x}\right)^8$ is

(a) ${}^8C_5 12^5$ (b) ${}^8C_4 12^4$
(c) ${}^8C_3 12^3$ (d) ${}^8C_6 12^6$

28. In the expansion of $(a + b)^n$, the coefficients of 15th and 11th terms are equal. Find the number of terms in the expansion.

(a) 26 (b) 25
(c) 20 (d) 24

29. The number of terms in the expansion of $[(2x + 3y)^4 (4x - 6y)^4]^9$ is

(a) 36 (b) 37
(c) 10 (d) 40

30. If sum of the coefficients of the first two odd terms of the expansion $(x + y)^n$ is 16, then find n.

(a) 10 (b) 8
(c) 7 (d) 6

Level 2

31. The number of rational terms in the expansion of $\left(x^{\frac{1}{5}}+y^{\frac{1}{10}}\right)^{45}$ is

(a) 5 (b) 6
(c) 4 (d) 7

32. The remainder when $9^{49} + 7^{49}$ is divided by 64 is

(a) 24 (b) 8
(c) 16 (d) 38

33. If $p(n) = (n - 2)(n - 1)\,n(n + 1)(n + 2)$, then greatest number which divides $p(n)$ for all $n \in N$ is

(a) 12 (b) 24
(c) 120 (d) None of these

34. For $n \in N$, $a^{2n-1} + b^{2n-1}$ is divisible by

(a) $a + b$ (b) $(a + b)^2$
(c) $a^3 + b^3$ (d) $a^2 + b^2$

35. Find the coefficient of the independent term in the expansion of $\left(x^{\frac{1}{2}}+7x^{-\frac{1}{3}}\right)^{10}$.

(a) ${}^{10}C_4 7^4$ (b) ${}^{10}C_6 7^6$
(c) ${}^{10}C_6 7^5$ (d) ${}^{10}C_4 7^7$

36. Find the term which has the exponent of x as 8 in the expansion of $\left(x^{\frac{5}{2}}-\frac{3}{x^3\sqrt{x}}\right)^{10}$.

(a) T_2 (b) T_3
(c) T_4 (d) Does not exist

37. The greatest number which divides $25^n - 24n - 1$ for all $n \in N$ is

(a) 24 (b) 578
(c) 27 (d) 576

38. If three consecutive coefficients in the expansion of $(1 + x)^n$, where n is a natural number are 36, 84 and 126 respectively, then n is

(a) 8
(b) 9
(c) 10
(d) Cannot be determined

39. Find the value of k for which the term independent of x in $\left(x^2+\frac{k}{x}\right)^{12}$ is 7920.

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$
(c) $\sqrt{2}$ (d) 2

40. Find the coefficient of x^7 in the expansion of $\left(7x+\frac{2}{x^2}\right)^{13}$.

(a) $78 \times 8^8 \times 4$
(b) $78 \times 7^6 \times 4^2$
(c) $78 \times 7^{11} \times 4$
(d) $78 \times 7^{11} \times 4^2$

41. The value of $(\sqrt{5}+2)^6+(\sqrt{5}-2)^6$ is
(a) a positive integer.
(b) a negative integer.
(c) an irrational number.
(d) a rational number but not an integer.

42. The ratio of the coefficients of x^4 to that of the term independent of x in the expansion of $\left(x^2+\frac{9}{x^2}\right)^{18}$ is
(a) 1 : 6 (b) 3 : 8
(c) 1 : 10 (d) 1 : 8

43. $\sum_{r=2}^{16} {}^{16}C_r =$
(a) $2^{15}-15$ (b) $2^{16}-16$
(c) $2^{16}-17$ (d) $2^{17}-17$

44. Number of non-zero terms in the expansion of $(5\sqrt{5}x+\sqrt{7})^6+(5\sqrt{5}x-\sqrt{7})^6$ is
(a) 4 (b) 10
(c) 12 (d) 14

45. Find the value of $(98)^4$ by using the binomial theorem.
(a) 92236846 (b) 92236816
(c) 92236886 (d) 92236806

46. If the number of terms in the expansion $(2x+y)^n-(2x-y)^n$ is 8, then the value of n is ______. (where n is odd)
(a) 17 (b) 19
(c) 15 (d) 13

47. If the expansion $\left(2x^5+\frac{1}{3x^4}\right)^n$ contains a term independent of x, then the value of n can be ______.
(a) 6 (b) 18
(c) 3 (d) 12

48. Find the sum of the coefficients in the expansion of $\left(5x^6-\frac{4}{x^9}\right)^{10}$.
(a) 5^{10} (b) 1
(c) 4^{10} (d) 0

49. $49^n+16n-1$ is divisible by ______ $(n \in N)$.
(a) 64 (b) 28
(c) 48 (d) 54

50. Find the coefficient of x^{11} in the expansion of $(1+2x+x^2)^6$.
(a) 1 (b) 2
(c) 6 (d) 12

Level 3

51. The number of irrational terms in the expansion of $\left(x^{\frac{2}{3}}+y^{\frac{1}{4}}\right)^{81}$ is __________.
(a) 70 (b) 12
(c) 75 (d) 13

52. Find the independent term in the expansion of $\left(x^4+\frac{3}{8x^3\sqrt{x}}\right)^{15}$.
(a) ${}^{15}C_4\left(\frac{3}{8}\right)^{16}$ (b) ${}^{15}C_{12}\left(\frac{3}{8}\right)^{4}$
(c) ${}^{15}C_8\left(\frac{3}{8}\right)^{8}$ (d) ${}^{15}C_7\left(\frac{3}{8}\right)^{15}$

53. $\sum_{r=1}^{30} r\frac{{}^{30}C_r}{{}^{30}C_{r-1}} =$
(a) 930 (b) 465
(c) 310 (d) 630

54. For all $n \in N$, $41^n-40n-1$ is divisible by
(a) 41 (b) 40
(c) 300 (d) 500

55. If m and n are the coefficients of x^{a^2} and x^{b^2} respectively in $(1+x)^{a^2+b^2}$, then
(a) $n=2m$
(b) $m+n=0$
(c) $2n=m$
(d) $m=n$

56. For each $n \in N$, $5^{3n} - 1$ is divisible by _____.

(a) 115 (b) 124

(c) 5 (d) 6

57. In the expansion $(6 + 9x)^5$ the coefficient of x^3 is ______.

(a) $2^2 \times 3^8$

(b) $2^4 \times 3^7$

(c) $2^3 \times 3^8 \times 5$

(d) $2^4 \times 3^7 \times 5$

58. In the expansion of $(x + y)^n$, the coefficients of the 17th and the 13th terms are equal. Find the number of terms in the expansion.

(a) 18 (b) 22

(c) 28 (d) 29

59. Find the coefficient of x^{-2} in the expansion of $\left(x^2 + \dfrac{4}{x^5}\right)^6$.

(a) 240 (b) 150

(c) 100 (d) 180

60. In the 10th term of $(x + y)^n$, the exponent of x is 3, then the exponent of x in the 7th term is ______.

(a) 3 (b) 6

(c) 5 (d) 7

61. $\sqrt{20}\{(\sqrt{20} + 1)^{100} - (\sqrt{20} - 1)^{100}\}$ is a/an _____.

(a) irrational number (b) whole number

(c) negative number (d) None of these

62. If $x = -\,{}^nC_1 + {}^nC_2\,(2) - {}^nC_3\,(2)^2 + \ldots$ (where n is odd), then $x =$ _____.

(a) 1 (b) -1

(c) 0 (d) 12

63. Find the independent term in the expansion of $\left(5x^2 - \dfrac{1}{x^4}\right)^6$.

(a) 8250 (b) 8560

(c) 9250 (d) 9375

64. The sum of the first three coefficients in the expansion $\left(x + \dfrac{1}{y}\right)^n$ is 22. Find the value of n.

(a) 8 (b) 7

(c) 6 (d) 5

65. If ${}^{12}C_0$ ${}^{12}C_1, \ldots, {}^{12}C_{12}$ are the binomial coefficients of the expansion $(1 + x)^{12}$, then ${}^{12}C_0 - {}^{12}C_1 + {}^{12}C_2 - {}^{12}C_3 + \cdots + {}^{12}C_{12} =$ ______.

(a) 4096 (b) 1024

(c) 0 (d) -1024

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. $p(n)$ is true for all natural numbers.
2. When $x(n)$ is true for $n = 1$ and also true for $n + 1$.
3. $3! = 6$
4. Not a prime number
5. $(q - 1)!$
6. binomial
7. 1
8. $n + 1$
9. 10
10. 15
11. $(x + y)^{48}$
12. 123
13. $1 \cdot 2 \cdot 3 \ldots (n - r - 1) \cdot (n - r)$
14. $\dfrac{(n+1)!}{(n-r+1)!\,r!}$
15. 0 or 6
16. ${}^nC_r\, x^{n-r}y^r$
17. 21
18. 1
19. 810
20. 1
21. $x^3 + 3x^2y + 3xy^2 + y^3$
22. t_{13}
23. 4
24. $3x^4$; $3x^2$
25. 3^{16}
26. $n(n + 1)$
27. n^2
28. 1, 5, 10, 10, 5, 1
29. 190
30. all integers

Short Answer Type Questions

40. $$(3x^2)^6 + 6(3x^2)^5\frac{5}{y^2} + 15(3x^2)^4\left(\frac{5}{y^2}\right)^2 + 20(3x^2)^3\left(\frac{5}{y^2}\right)^3 + 15(3x^2)^2\left(\frac{5}{y^2}\right)^4 + 6(3x^2)\left(\frac{5}{y^2}\right)^5 + \left(\frac{5}{y^2}\right)^6.$$
41. $(5x)^8 + 8(5x)^7(3y) + 28(5x)^6(3y)^2 + 56(5x)^5(3y)^3 + 70(5x)^4(3y)^4 + 56(5x)^3(3y)^5 + 28(5x)^2(3y)^6 + 8(5x)(3y)^7 + (3y)^7$
42. ${}^9C_4\, 5^4x^5y^4$ and ${}^9C_5\, 5^5x^4y^5$
43. ${}^6C_3$
44. ${}^9C_6(5x)^3\left(\dfrac{1}{7y}\right)^6$

Essay Type Questions

46. 152
47. $\dfrac{-16}{5^7}$
48. ${}^{10}C_4\,\dfrac{6^6}{7^4}$
49. $\dfrac{4}{3}$
50. ${}^{21}C_{15}\,\dfrac{2^6}{3^{15}}$

CONCEPT APPLICATION

Level 1

1. (b)	**2.** (b)	**3.** (c)	**4.** (c)	**5.** (d)	**6.** (a)	**7.** (b)	**8.** (d)	**9.** (d)	**10.** (d)
11. (a)	**12.** (b)	**13.** (c)	**14.** (b)	**15.** (c)	**16.** (c)	**17.** (c)	**18.** (b)	**19.** (b)	**20.** (a)
21. (d)	**22.** (c)	**23.** (a)	**24.** (c)	**25.** (d)	**26.** (c)	**27.** (b)	**28.** (b)	**29.** (b)	**30.** (d)

Level 2

31. (a)	**32.** (c)	**33.** (c)	**34.** (a)	**35.** (b)	**36.** (d)	**37.** (d)	**38.** (b)	**39.** (c)	**40.** (c)
41. (a)	**42.** (c)	**43.** (c)	**44.** (a)	**45.** (b)	**46.** (c)	**47.** (b)	**48.** (b)	**49.** (a)	**50.** (d)

Level 3

51. (c)	**52.** (c)	**53.** (b)	**54.** (b)	**55.** (d)	**56.** (b)	**57.** (c)	**58.** (d)	**59.** (a)	**60.** (b)
61. (b)	**62.** (b)	**63.** (d)	**64.** (c)	**65.** (c)					

CONCEPT APPLICATION

Level 1

1. Substitute different natural numbers for 'n' in the given expression.
2. Use the formula, $r = \dfrac{np}{p+q}$ to find the independent term in the expansion of $\left(ax^p + \dfrac{b}{x^q}\right)^n$.
3. For $n = 2$, find the sum of first two terms and check for which option this sum is obtained.
4. For $n = 1, 2, 3$ and 4, find the value of $2^{3n} - 1$ and obtain a general conclusion.
5. Evaluate the given expression for $n = 1, 2, 3, 4$ and 5.
6. Among the options, identify the value that satisfies the given inequality.
7. The rth term from the end in the expansion $(x + y)^n$ is $(n - r + 2)$th term from the beginning.
8. The exponent of x of the terms in the expansion of $(x + y)^n$ decreases as we go from left to right.
9. The sum of the elements in the nth row of Pascal triangle is 2^n.
10. $(x + y)^n - (x - y)^n$ has $\dfrac{n}{2}$ terms when n is even and $\dfrac{n+1}{2}$ terms when n is odd.
11. Use $T_{r+1} = {}^nC_r x^{n-r} y^r$.
13. $(x + y + z)^n$ has ${}^{n+2}C_2$ terms.
14. To find the coefficient of x^k in the expansion of $\left(ax^p + \dfrac{b}{x^q}\right)^n$ use the formula $r = \dfrac{np-k}{p+q}$.
15. For each choice, check if the given expression is divisible by it for the given values of n.
16. Substitute $n = 1, 2, 3, 4, \ldots$ in the given expression.
17. In the given expression substitute different values of n and then identify the factor.
18. Substitute $n = 1, 2, 3, 4, \ldots$ in the given expression.
19. As we go from left to right, the exponent of y in the expansion of $(x + y)^n$ increases.
20. Use $T_{r+1} = {}^nC_r x^{n-r} y^r$.
21. Use $T_{r+1} = {}^nC_r x^{n-r} y^r$.
22. Use $T_{r+1} = {}^nC_r x^{n-r} y^r$.
23. To find the independent term in the expansion of $\left(ax^p + \dfrac{b}{x^q}\right)^n$ use the formula $r = \dfrac{np}{p+q}$.
24. To find the independent term in the expansion of $\left(ax^p + \dfrac{b}{x^q}\right)^n$ use the formula $r = \dfrac{np}{p+q}$.
25. Put $x = 1$ and $y = 1$ in the given expression.
26. Use the binomial expansion of $(x + y)^n + (x - y)^n$.
27. Use the formula $r = \dfrac{np-k}{p+q}$ in the expansion of $\left(ax^p + \dfrac{b}{x^q}\right)^n$ to find the coefficient of x^k.
28. Use $T_{r+1} = n_{c_r} x^{n-r} y^r$.
29. The number of terms in the expansion of $(x + y)^n$ is $n + 1$.
30. Put $y = 1$ and proceed.

Level 2

31. Use $T_{r+1} = {}^nC_r x^{n-r} y^r$. Find the number of values of r for which both r and as $(n - r)$ are integers.
32. (i) $(8 + 1)^{49} + (8 - 1)^{49}$.
    (ii) Use the binomial expansion $(x + y)^n + (x - y)^n$ and simplify.
33. The product of n consecutive integers is always divisible by $n!$
34. Substitute $n = 1, 2, 3, 4, \ldots$ and verify from the options.

35. To find the independent term, in the expansion of $\left(ax^p + \frac{b}{x^q}\right)^n$. Use the formula $r = \frac{np}{p+q}$.

36. To find the coefficient of x^k in the expansion of $\left(ax^p + \frac{b}{x^q}\right)^n$. Use the formula $r = \frac{np-k}{p+q}$.

37. Substitute $n = 1, 2, 3, 4 \ldots$ in the given expression.

38. Let the three consecutive coefficients in the expansion $(1 + x)^n$ be ${}^nC_r$, ${}^nC_{r+1}$ and ${}^nC_{r+2}$.

39. To find the independent term in the expansion of $\left(ax^p + \frac{b}{x^q}\right)^n$, use the formula $r = \frac{np}{p+q}$.

40. To find the coefficient of x^k in the expansion $\left(ax^p + \frac{b}{x^q}\right)^n$, use $r = \frac{np-k}{p+q}$.

41. $(x + a)^n + (x - a)^n = 2({}^nC_0\, x^n + {}^nC_2\, x^{n-2}\, a^2 + {}^nC_4\, x^{x-4}\, y^4 + \ldots)$.

42. In the expansion of $\left(ax^p + \frac{b}{x^q}\right)^n$ to find

(i) the independent term, use the formula $r = \frac{np}{p+q}$

and

(ii) the coefficient of x^k, use the formula $r = \frac{np-k}{p+q}$.

43. Use ${}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n$.

44. The number of terms in the expansion of $(x + y)^n + (x - y)^n$ is $\frac{n}{2} + 1$ when n is even and $\frac{n+1}{2}$ when n is odd.

45. $(98)^4 = (100 - 2)^4$. Expand using binomial theorem.

46. The number of terms in the expansion $(x + y)^n - (x - y)^n$ is $\frac{n+1}{2}$, when n is odd.

The given number of terms = 8

$\therefore \quad \frac{n+1}{2} = 8$

$n + 1 = 16 \quad \Rightarrow \quad n = 15.$

47. In the expansion $\left(ax^p + \frac{b}{x^q}\right)^n$, the independent term is T_{r+1} where $r = \frac{np}{p+q}$.

$r = \frac{5n}{5+4} = \frac{5n}{9}$. Since r is are integer, 'n' must be a multiple of 9.

From the options, the value of n can be 18.

48. The given expression is $\left(5x^6 - \frac{4}{x^9}\right)^{10}$,

Put $x = 1$, so we get the sum of the coefficients, i.e., $(5 - 4)^{10} = 1^{10} = 1$.

49. Put $n = 1$, $49 + 16 - 1 = 64$.

$\therefore$ It is divisible by 64.

50. $(1 + 2x + x^2)^6 = [(1 + x)^2]^6 = (1 + x)^{12}$.

Coefficient of x^{11} is ${}^{12}C_{11} = 12$.

Level 3

54. Substitute $n = 1, 2, 3, 4, \ldots$ in the given expression and verify from the options.

55. (i) Use ${}^nC_r = {}^nC_{n-r}$

(ii) $m = (a^2 + b^2)c_{a^2}x^{a^2}$ and

$n = (a^2 + b^2)c_{b^2}x^{b^2}$

(iii) Find.

56. $5^{3n} - 1 = (5^3)^n - 1 = (125)^n - 1$

$= (125 - 1)\ ((5^3)^{n-1} + (5^3)^{n-2} + \cdots + 1)$

$= 124m$ (where 'm' is some positive integer).

$\therefore\ 5^{3n} - 1$ is always divisible by 124.

57. Given expression is $(6 + 9x)^5$

$T_{3+1} = {}^5C_3(6)^2(9x)^3 = 2 \times 5 \times (2 \times 3)^2 \times 3^6 x^3$.

$\therefore$ The required coefficient $= 2^3 \times 3^8 \times 5$.

58. The coefficient of the 17th term is ${}^nC_{16}$.

The coefficient of the 13th term is ${}^nC_{12}$.

Given that the coefficients are equal.

${}^nC_{16} = {}^nC_{12}$

$n = 16 + 12 = 28.$

$\therefore$ The number of terms in the expansion is 28 + 1 = 29.

59. In the expansion $\left(ax^p + \frac{b}{x^q}\right)^n$, the term containing x^k is T_{r+1} where $r = \frac{np - k}{p + q}$.

Here, $r = \frac{12+2}{5+2} = \frac{14}{4} = 2.$

$$T_{2+1} = T_3 = {}^6C_2(x^2)^4\left(\frac{4}{x^5}\right)^2 = {}^6C_2(16)x^{-2}$$

$= (15)\ (16)\ x^{-2}$

$\therefore$ The required coefficient = 240.

60. The given expansion is $(x + y)^n$

$T_{10} = T_{9+1} = {}^nC_9(x)^{n-9}(y)^9$

Given $n - 9 = 3$

$n = 12$

$T_7 = T_{6+1} = {}^{12}C_6\ (x)^6\ (y)^6$

$\therefore$ The exponent of x is 6.

61. Let $x = \sqrt{20}$ and $n = 100$

$= x((x + 1)^{100} - (x - 1)^{100})$

$= x[{}^nC_1x^{n-1} + {}^nC_3x^{n-3} + \ldots]$

$= 2[{}^nC_1x^n + {}^nC_3x^{n-2} + {}^nC_5x^{n-4} + \ldots]$

Now $n = 100$, $n - 2$, $n - 4$, … are all even number.

But $x = \sqrt{20}$

$\therefore$ $x^n \cdot x^{n-2}$, x^{n-4}, are all integers and ${}^nC_1$, ${}^nC_3$, …, ${}^nC_5$ are all integers.

Hence, the given expression is a whole number.

62. Given $x = -{}^nC_1 + {}^nC_2\ (2)\ \ldots$

$2x = -{}^nC_1\ (2) + {}^nC_2\ (2)^2 - {}^nC_3\ (2)^3 + \ldots$

$1 + 2x = 1 - {}^nC_1\ (2) + {}^nC_2\ (2)^2 - {}^nC_3\ (2)^3 + \ldots$

$1 + 2x = (1 - 2)^n$

$1 + 2x = (-1)^n$

$1 + 2x = -1$ (given n is odd)

$2x = -2$

$x = -1.$

63. In the expansion $\left(ax^p + \frac{b}{x^q}\right)^n$, the independent term is T_{r+1}, where $r = \frac{np}{p+q}$.

Here, $r = \frac{6 \times 2}{6} = 2.$

The independent term is T_3.

$$T_3 = T_{2+1} = {}^6C_2(5x^2)^4\left(\frac{-1}{x^4}\right)^2$$

$= 5^4 \times 15 = 9375.$

64. Given that ${}^nC_0 + {}^nC_1 + {}^nC_2 = 22$

$\Rightarrow\ 1 + n + \frac{n(n-1)}{2} = 22$

$\Rightarrow\ 2 + 2n + n^2 - n = 44$

$\Rightarrow\ n^2 + n - 42 = 0$

$\Rightarrow\ (n + 7)(n - 6) = 0$

$\Rightarrow\ n = -7$ or $n = 6.$

65. We know that

$(1 + x)^{12} = {}^{12}C_0 + {}^{12}C_1\ x + {}^{12}C_2\ x^2 + {}^{12}C_3\ x^3 + \cdots + {}^{12}C_{12}\ x^{12}$

Put $x = -1$

$0 = {}^{12}C_0 - {}^{12}C_1 + {}^{12}C_2 - {}^{12}C_3 + \cdots + {}^{12}C_{12}.$