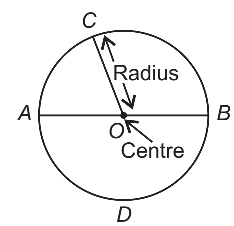
6 Plane Geometry: Circle

KEY FACTS

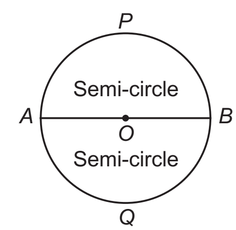
A. Definitions

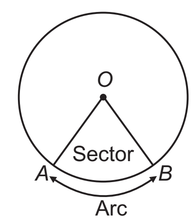
1.
- The paths (locus) traced out by a moving point, at a fixed distance from a fixed point is called a **circle.**



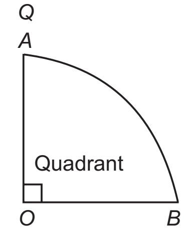
- The path so traced out is called the **circumference** (abbreviation ⊙ce), the fixed point is called the **centre** and the fixed distance is called the **radius.**

  In the given Fig., $O \rightarrow$ centre; $OC \rightarrow$ radius; $ACBD \rightarrow$ circumference; $AB \rightarrow$ diameter (2 × radius)

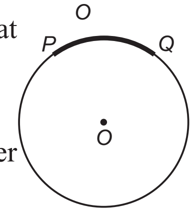




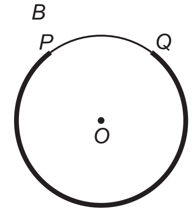
2.
- A diameter divides a circle into **two equal parts**, each part being a **semi-circle** *i.e., APB* and *AQB* are semi-circles.
- The part of a circle enclosed by any two radii of a circle is called **sector**, *i.e., AOB*
- A part of the circumference is called an **arc**, *i.e.,* $\overset{\frown}{AB}$.



- A **quadrant** is one-fourth of a circle, where the two bounding radii are at rt. $\angle s$ to each other.
- Any two points on a circle divide the circle into two parts. The smaller part is called the **minor arc** and the larger part is called the **major arc.**

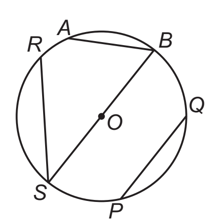


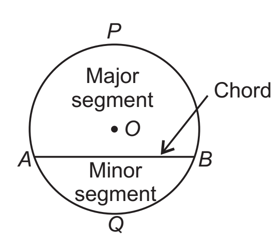
PQ → minor arc



PQ → major arc

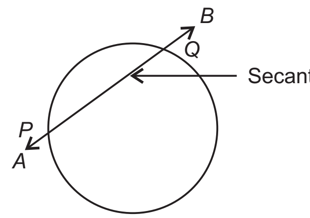
3.
- A line segment whose end points lie on the circle is called a **chord.** *AB, PQ, RS* are all chords.
- The chord passing through the centre of the circle is the **longest chord** and is the **diameter** of the circle, *e.g. BS* is the diameter.



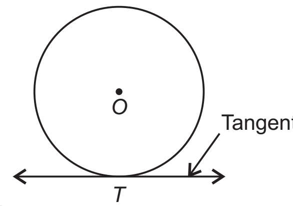


- A chord divides a circle into two regions called segments of the circle. The **larger part,** containing the centre *i.e., APB* in the given figure is called the **major segment** and the **smaller part** not containing the centre, *i.e., AQB* is called the **minor segment.**

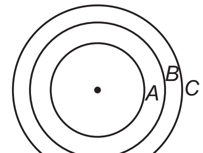
4. • A line intersecting a circle in two distinct points is called a **secant.** Secant *AB* intersects the given circle in points *A* and *B*.



• A line which intersects the circle in exactly one point is called a **tangent.** The point of intersection, *T*, is called the **point of contact** or the **point of tangency.**

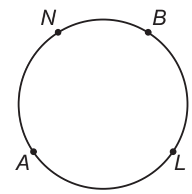


5. • Circles having the same centre are called **concentric circles.**

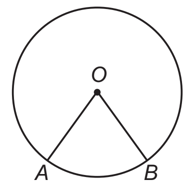


• Circles with equal radii are called **congruent circles.**

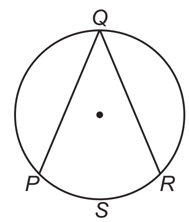
• Points lying on the same circle are called **concyclic points.** *A, L, B* and *N* are concyclic points.



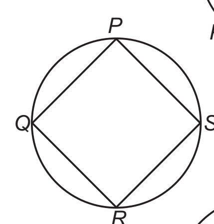
6. • **Central angle:** An angle formed at the centre of the circle is called the central angle. $\angle AOB$ is the central angle.



• When two chords have a common end point, then the angle included between these two chords at the common point is called the **inscribed angle.** $\angle PQR$ is inscribed by the arc *PSR*.



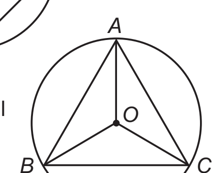
7. • A quadrilateral whose all four vertices lies on a circle is called a **cyclic quadrilateral.**



Cyclic quadrilateral

• A circle which passes through all the three vertices of a triangle is called a **circumcircle**. The **circumcentre** is always equidistant from the vertices of the triangle.

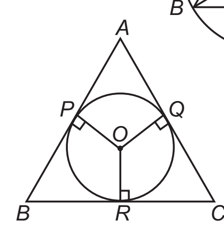
$OA = OB = OC$



Circumcircle

• A circle which touches all the three sides of a triangle, *i.e.,* all the three sides of the triangle are tangents to the circle is called an **incircle.** **Incentre** is always equidistant from the sides of a triangle.

$OP = OQ = OR$.



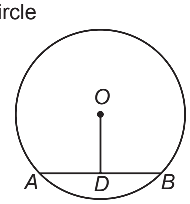
Incircle

THEOREMS

I. CHORD PROPERTIES

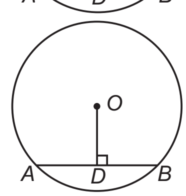
Theorem 1. ***A straight line, drawn from the centre of a circle perpendicular to the chord bisects the chord.***

If $OD \perp AB$, then $AB = 2\,AD = 2\,BD$.



Theorem 2. ***The line joining the centre of the circle to the mid-point of the chord is perpendicular to the chord.***

Given, $AD = DB$, then $OD \perp AB$.



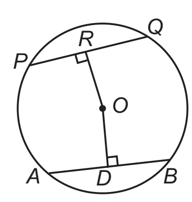
Theorem 3. ***The perpendicular bisectors of two chords of a circle intersect at its centre.***

Theorem 4. ***The perpendicular bisectors of a chord of a circle always passes through the centre.***

Theorem 5. ***One and only one circle can be drawn through three points not lying in the same straight line.***

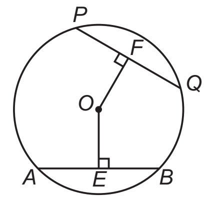
Theorem 6. ***Equal chords of a circle are (or of congruent circles) equidistant from the centre.***

$AB = PQ \Rightarrow OD = OR$



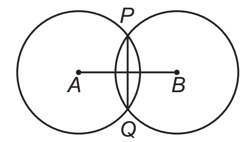
Theorem 7. ***Chords which are equidistant from the centre in a circle (or congruent circles) are equal.***

$OE = OF \Rightarrow AB = PQ.$



Theorem 8. ***The angular bisector of the angle between two equal chords of a circle passes through the centre.***

Theorem 9. ***If two circles intersect, then the line joining their centres is the perpendicular bisector of the common chord. AB is the perpendicular bisector of PQ.***

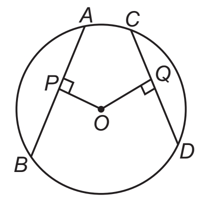


Theorem 10. ***If any two chords of a circle, the one which is greater is nearer to the circle.***

$AB > CD \Rightarrow OP < OQ$

Conversely, of any two chords of a circle, the nearer to the center is greater.

$OP < OQ \Rightarrow AB > CD.$



SOLVED EXAMPLES

We now take up some examples to illustrate the properties and results discussed so far.

Ex. 1. The distance between two points *A* and *B* is 3 cm. *A* circle of radius 1.7 cm is drawn to pass through these points. Find the distance of *AB* from the centre of the circle.

Sol. Let O be the centre of the circle of radius 1.7 cm which is drawn to pass through A and B. From O draw $OD \perp AB$. Then OD is the required distance.

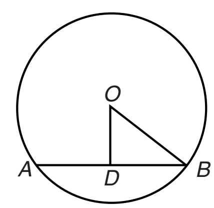
$\therefore \quad AD = DB = 1.5$ cm (*perp. from centre bisects chord*)

$\therefore$ In rt. $\angle d\ \Delta ODB$.

$$OD^2 = OB^2 - DB^2 \quad \text{(Pythagoras Th.)}$$

$$= (1{\cdot}7)^2 - (1{\cdot}5)^2 = 2{\cdot}89 - 2.25 = 0{\cdot}64$$

$\therefore \quad OD = \sqrt{0.64} = \mathbf{0{\cdot}8\ cm.}$

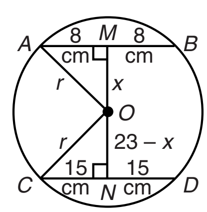


Ex. 2. *AB* and *CD* are two parallel chords of a circle such that *AB* = 16 cm and *CD* = 30 cm. If the chords are on the opposite sides of the centre and the distance between them is 23 cm, find the radius of the circle.

Sol. Let O be the centre of the circle and radius r cm. Draw $OM \perp AB$ and $ON \perp CD$.

Then, MON is a straight line and

$$AM = \frac{1}{2}AB = 8 \text{ cm and } CN = \frac{1}{2}CD = 15 \text{ cm.}$$



Let $OM = x$ cm. Then, $ON = (23 - x)$ cm.

Join OA and OC. Then $OA = OC = r$ cm.

In right $\Delta\ OMA$, $OA^2 = AM^2 + OM^2 \Rightarrow r^2 = 8^2 + x^2$ ...(*i*)

In right ΔONC, $OC^2 = CN^2 + ON^2 \Rightarrow r^2 = 15^2 + (23 - x)^2$ ...(*ii*)

From (*i*) and (*ii*), we have $8^2 + x^2 = 15^2 + (23 - x)^2$

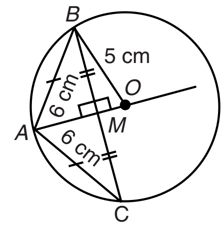
$$\Rightarrow 64 + x^2 = 225 + 529 - 46x + x^2 \Rightarrow 46x = 754 - 64 \Rightarrow 46x = 690 \Rightarrow x = \frac{690}{46} = 15 \text{ cm.}$$

$\therefore$ From (*i*), $r^2 = 8^2 + 15^2 = 64 + 225 = 289 \Rightarrow r = \sqrt{289} = 17$

Hence, the radius of the circle is **17 cm.**

Ex. 3. In a circle of radius 5 cm, *AB* and *AC* are two chords such that *AB* = *AC* = 6 cm. Find the length of the chord *BC*.

Sol. Since, the angular bisector of the angle between two equal chords of a circle passes through the centre therefore, *AO* and so *AM* is the bisector of $\angle BAC$ and also is perpendicular bisector of chord *BC*.



$\therefore$ $\angle AMB = 90°$ and $BM = MC$

Let $OM = x$. Then $AM = 5 - x$

In right $\Delta\, AMB$, $AB^2 = AM^2 + MB^2$ (Pythagoras Theorem)

$\Rightarrow$ $6^2 = (5-x)^2 + BM^2$

$\Rightarrow$ $BM^2 = 36 - (5-x)^2$ ...(*i*)

In right $\Delta\, OMB$, $BO^2 = BM^2 + MO^2 \Rightarrow 5^2 = BM^2 + x^2 \Rightarrow BM^2 = 25 - x^2$ ...(*ii*)

$\therefore$ From (*i*) and (*ii*), we have $36 - (5-x)^2 = 25 - x^2$

$\Rightarrow$ $36 - (25 + x^2 - 10x) = 25 - x^2$

$\Rightarrow$ $11 + 10x = 25 \Rightarrow 10x = 25 - 11 = 14 \Rightarrow x = \dfrac{14}{10} = 1.4$ cm

$\therefore$ From (*ii*), $BM^2 = 25 - x^2 = 25 - (1{\cdot}4)^2 = 25 - 1{\cdot}96 = 23{\cdot}04 \Rightarrow BM = \sqrt{23.04} = 4{\cdot}8$ cm

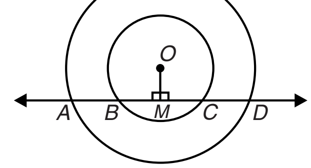
Hence, length of the chord $BC = 2\,BM = 2 \times 4{\cdot}8 =$ **9·6 cm.**

Ex. 4. If a line *l* intersects two concentric circles at points *A, B, C* and *D* as shown in the figure, prove that *AB = CD*.

OR

Prove that two concentric circles intercept equal portions on any straight line that cuts them.

Sol. Let *O* be the centre of the two concentric circles and *OM* the perpendicular from *O* to the line *l*. *AD* is the chord of the larger circle and *BC*, the chord of the smaller circle.



Since, perpendicular from the centre to a chord bisects the chord, therefore,

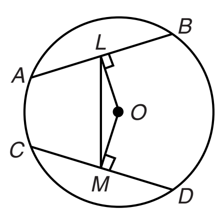
$AM = MD$ ...(*i*)

and $BM = MC$ ...(*ii*)

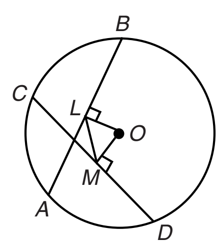
(*i*) and (*ii*) gives, $AM - BM = MD - MC \Rightarrow$ ***AB = CD.***

Ex. 5. Prove that the line joining the mid-points of two equal chords of a circle makes equal angles with the chords.

Sol. Let *AB* and *CD* be the two equal chords of a circle with centre *O*. Let *L* and *M* be the mid-points of *AB* and *CD* respectively.



(*a*)



(*b*)

Then $OL \perp AB$ and $OM \perp CD$ (*The line joining centre to the mid-point of a chord is perp. to the chord.*)

Also, $AB = CD \therefore OL = OM$ (*Equal chords are equidistant from the centre*)

$\therefore$ In $\Delta\, OLM$, $\angle OLM = \angle OML$ (*In a Δ, angles opp. equal sides are equal*) ...(*i*)

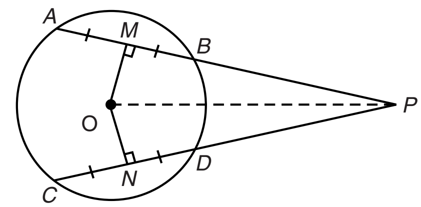
$\therefore$ $\angle OLA = 90°$ and $\angle OMC = 90°$

$\Rightarrow$ $\angle ALM = 90° - \angle OLM$ and $\angle CML = 90° - \angle OML$ ...(*ii*)

From (*i*) and (*ii*) it follows that $\boldsymbol{\angle ALM = \angle CML}$.

Ex. 6. Two equal chords *AB* and *CD* of a circle with centre *O*, when produced meet at a point *P* outside the circle. Prove that (*i*) *PB* = *PD* and (*ii*) *PA* = *PC*.

Sol. Draw $OM \perp AB$, $ON \perp CD$ and join OP. Then



$$AM = BM = \frac{1}{2}\ AB$$

and $$CN = DN = \frac{1}{2}\ CD$$

(*The perpendicular from the centre of a circle bisects the chord.*)

But $AB = CD$ (*given*)

$\Rightarrow$ $\frac{1}{2}\ AB = \frac{1}{2}\ CD \quad \Rightarrow \quad AM = BM = CN = DN$ ...(*i*)

Also, $OM = ON$ (*Equal chords are equidistant from the centre*)

In right Δ*s* *OMP* and *ONP*, we have

$OM = ON$, $\angle OMP = \angle ONP$ (each 90°) and OP is common

$\therefore$ $\Delta\ OMP \cong \Delta\ ONP$ (*RHS*)

$\therefore$ $MP = NP$ (*c.p.c.t.*) ...(*ii*)

$\therefore$ From (*i*) and (*ii*), we get

$MP - BM = NP - DN \quad \Rightarrow \quad$ ***PB* = *PD*.**

and $MP + AM = NP + CN \quad \Rightarrow \quad$ ***PA* = *PC*.**

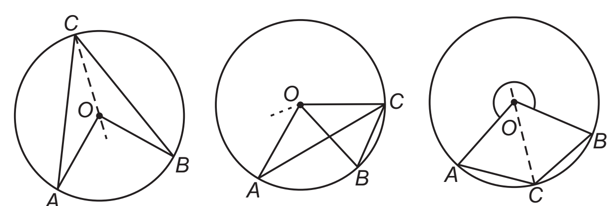
II. THEOREMS ON ARCS AND ANGLES

Theorem 11. ***The angle subtended at the centre by an arc of a circle is double the angle which this are subtends at any point on the remaining part of the circle.***

In all these cases,

$\angle AOB = 2\ \angle ACB$

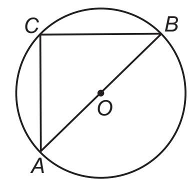
Note in diagram (*iii*) Reflex $\angle AOB = 2\ \angle ACB$.



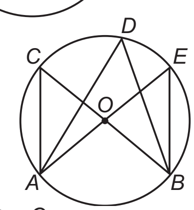
(*i*) (*ii*) (*iii*)

Theorem 12. ***Angle in a semicircle is a right angle.***

$\angle ACB = 90°$

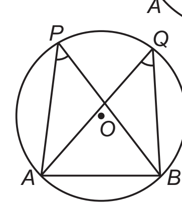


Theorem 13. ***If an arc of a circle subtends a right angle at any point on the remaining part of the circle, then the arc is a semi-circle.***



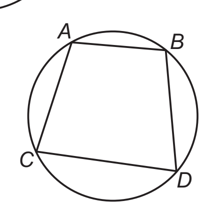
Theorem 14. ***Angles in the same segment of a circle are equal.***

$\angle ACB = \angle ADB = \angle AEB$.



Theorem 15. ***If the straight line joining two points subtends equal angles at two other points on the same side of it, then the four points lie on a circle,* i.e., *are concyclic.***

$\angle APB = \angle AQB \Rightarrow$ Points *A*, *P*, *Q*, *B* lie on the same circle.

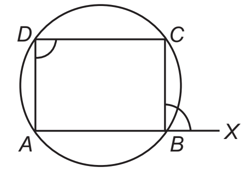


Theorem 16. ***The opposite angles of a cyclic quadrilateral are supplementary.***

$\angle ACD + \angle ABD = 180°$, $\angle CAB + \angle CDB = 180°$. Conversely, if a pair of opposite angles of a quadrilateral are supplementary, then the qudrilateral is cyclic.

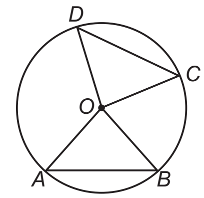
Theorem 17. ***If the side of a cyclic quadrilateral is produced, the exterior angle so formed is equal to the interior opposite angle, e.g., $\angle CBX = \angle ADC$.***

$\therefore$ If a parallelogram is inscribed in a circle, it is always a rectangle.

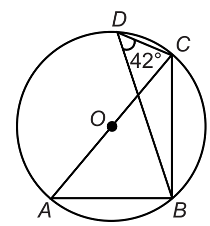


Theorem 18. ***Equal chords (or equal arcs) of a circle (or congruent circles) subtend equal angles at the centre.***

$AB = CD$ (or $\overline{AB} = \overline{CD}$) $\Rightarrow$ $\angle AOB = \angle COD$.



Ex. 7. In a given circle *ABCD, O* is the centre and $\angle BDC = 42^\circ$. Calculate the $\angle ACB$.



Sol. *AOC* is a diameter since *O* is the centre of the circle.

$\therefore$ $\angle ABC = 90^\circ$ (*$\angle$ in a semi-circle*)

$\angle BAC = \angle BDC = 42^\circ$ (*$\angle$s in the same segment*)

$\therefore$ In Δ *ABC*,

$42^\circ + 90^\circ + \angle ACB = 180^\circ$ (*$\angle$ sum of Δ*)

$\therefore$ $\angle ACB = 180^\circ - 132^\circ =$ **48°.**

Ex. 8. (*i*) In Fig. (*i*), *O* is the centre of the circle and the measure of arc *ABC* is 110°. Using the above results, find $\angle ADC$ and $\angle ABC$.

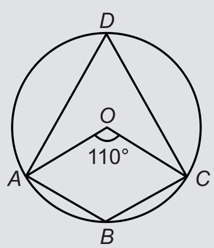


Fig. (*i*)

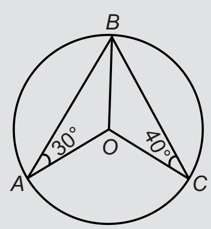


Fig. (*ii*)

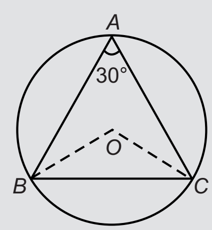


Fig. (*iii*)

(*ii*) In Fig. (*ii*), calculate the measure of $\angle AOC$.

(*iii*) In Fig. (*iii*), *ABC* is a triangle in which $\angle BAC = 30^\circ$. Show that *BC* is equal to the radius of the circum-circle of ΔABC, whose centre is *O*.

Sol. (*i*) Arc *AC* subtends $\angle AOC = 110^\circ$ at the centre and $\angle ADC$ at the remaining part of the circumference,

$\therefore$ $$\angle ADC = \frac{1}{2}\angle AOC = \frac{1}{2}\times 110^\circ = \mathbf{55^\circ}$$

Now, reflex $\angle AOC = 360^\circ - 110^\circ = 250^\circ$

Major arc *ADC* subtends reflex $\angle AOC$ at the centre and $\angle ABC$ at the remaining part of the circumference

$\therefore$ $$\angle ABC = \frac{1}{2}(\text{reflex } \angle AOC) = \frac{1}{2}\times 250^\circ = \mathbf{125^\circ}$$

Alternatively $\angle ABC + \angle ADC = 180^\circ$ (*opp. $\angle$s of a cyclic quad. are supplementary*)

$\Rightarrow$ $\angle ABC + 55^\circ = 180^\circ$ $\Rightarrow$ $\angle ABC = 180^\circ - 55^\circ =$ **125°**

(*ii*) In Δ *AOB*, $OA = OB$ (*radii of the same circle*)

$\therefore$ $\angle OBA = \angle OAB = 30^\circ$ (*angles opposite equal sides*)

Similarly, in Δ *BOC*, $\because$ $OB = OC$ $\therefore$ $\angle OBC = \angle OCB = 40^\circ$

$\therefore$ $\angle ABC = \angle OBA + \angle OBC = 30^\circ + 40^\circ = 70^\circ$

Now, arc *AC* subtends $\angle AOC$ at the centre and $\angle ABC$ at the remaining circumference.

$\therefore$ $\angle AOC = 2\angle ABC = 2 \times 70^\circ =$ **140°**

(*iii*) Arc *BC* subtends $\angle BOC$ at the centre and $\angle BAC$ at the remaining circumference.

$\therefore$ $\angle BOC = 2\ \angle BAC = 2 \times 30^\circ = 60^\circ$

$\therefore$ In ΔBOC, $\angle OBC + \angle OCB = 180^\circ - \angle BOC = 180^\circ - 60^\circ = 120^\circ$

But $\angle OBC = \angle OCB$ ($\because$ *OB = OC, being radii*)

$\therefore$ $\angle OBC = \angle OCB = \frac{1}{2} \times 120^\circ = 60^\circ$

$\therefore$ ΔOBC is equilateral (each $\angle = 60^\circ$) $\Rightarrow BC = OB = OC$

$\Rightarrow$ BC is equal to the radius of the circumcircle.

Ex. 9. (*i*) In Fig. (*i*), *O* is the centre of the circle. The angle subtended by the arc *BCD* at the centre is 140°. *BC* is produced to *P*. Determine $\angle BAD$ and $\angle DCB$, and $\angle DCP$.

(*ii*) In Fig. (*ii*), *C* is a point on the minor arc *AB* of the circle with centre *O*. Given $\angle ACB = x^o$ and $\angle AOB = y^o$, express *y* in terms of *x*. Calculate *x*, if *ACBO* is a parallelogram.

(*iii*) In Fig. (*iii*), *AB* is a diameter of a circle with centre *O* and radius *OD* is perpendicular to *AB*. If *C* is any point on arc *DB*, find $\angle BAD$, $\angle ACD$.

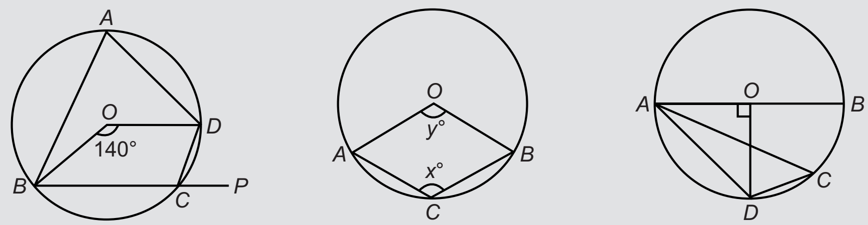


Fig. (*i*) **Fig. (*ii*)** **Fig. (*iii*)**

Sol. (*i*) $\angle BAD = \frac{1}{2}\angle BOD = \frac{1}{2} \times 140^\circ = \mathbf{70^\circ}$

(*angle at the centre by arc BCD = twice angle at the remaining circumference*)

Now, arc *BAD* makes reflex $\angle BOD = (360^\circ - 140^\circ) = 220^\circ$ at the centre, and $\angle BCD$ at a point *C* on the remaining circumference.

$\therefore$ $\angle BCD = \frac{1}{2}$ (reflex $\angle BOD$) $= \frac{1}{2} \times 220^\circ = \mathbf{110^\circ}$.

Also, $\angle BCD + \angle DCP = 180^\circ$ (Linear pair) $\Rightarrow \angle DCP = 180^\circ - \angle BCD = 180^\circ - 110^\circ = \mathbf{70^\circ}$.

(*ii*) Major arc *AB* subtends reflex $\angle AOB$ at the centre and $\angle ACB = x^\circ$ at a point *C* on the remaining circumference

$\therefore$ reflex $\angle AOB = 2\angle ACB$

$\Rightarrow$ $360^\circ - y = 2x \Rightarrow y = 360 - 2x$ ...(*i*)

If *ACBO* is a parallelogram, then

$x^\circ = y^\circ$ (*opp. $\angle$s of a ||gm*)

$\Rightarrow$ $x = y \Rightarrow x = 360 - 2x$ [*From* (*i*)]

$\Rightarrow$ $3x = 360 \Rightarrow \boldsymbol{x = 120^\circ}$.

(*iii*) Arc *BD* makes $\angle BOD$ at the centre and $\angle BAD$ at point *A* on remaining circumference.

$\therefore$ $\angle BAD = \frac{1}{2}\angle BOD = \frac{1}{2} \times 90^\circ = \mathbf{45^\circ}$.

Also, arc *AD* makes $\angle AOD$ at the centre and $\angle ACD$ at point *C* on the remaining circumference.

$\therefore$ $\angle ACD = \frac{1}{2}\angle AOD = \frac{1}{2} \times 90^\circ = \mathbf{45^\circ}$.

Thus, $\angle BAD = \angle ACD = 45^\circ$.

Ex. 10. (*i*) In Fig. (*i*), find the value of the angles *x* and *y*.

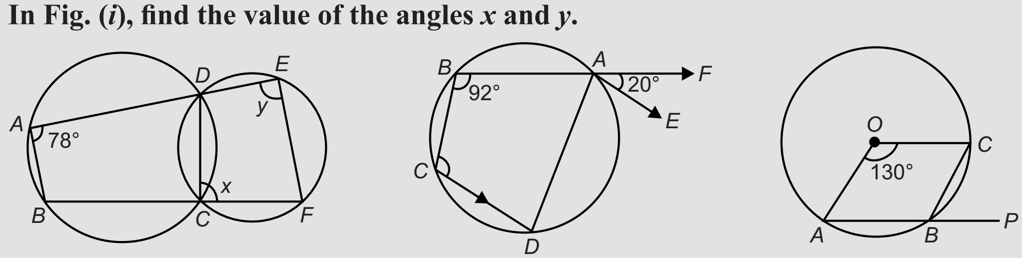


Fig. (*i*) **Fig. (*ii*)** **Fig. (*iii*)**

(*ii*) In Fig. (*ii*), *ABCD* is a cyclic quadrilateral. *AE* is drawn parallel to *CD* and *BA* is produced to *F*. If $\angle ABC = 92°$, $\angle FAE = 20°$, find $\angle BCD$.

(*iii*) In Fig. (*iii*), *O* is the centre of the circle. Arc *ABC* subtends an angle of 130º at the centre *O*. *AB* is extended up to *P*. Find $\angle PBC$.

Sol. (*i*) Side *BC* of cyclic quad. *ABCD* is produced to *F*.

$\therefore \quad \angle DCF = \angle BAD$ (*ext.* ∠ = *int. opp.* ∠)

$\Rightarrow \quad$ $\boldsymbol{x = 78°}$

In cyclic quad. *DCFE*, $x + y = 180°$ (*opp.* ∠*s of a cyclic quad. are supplementary*)

$\Rightarrow \quad 78° + y = 180° \Rightarrow y = 180° - 78° = \mathbf{102°}$

(*ii*) In cyclic quad. *ABCD*, $\angle ADC + \angle ABC = 180°$ (sum of opp. ∠*s* = 180º)

$\Rightarrow \quad \angle ADC + 92° = 180° \Rightarrow \angle ADC = 180° - 92° = 88°$

Now, $AE \parallel CD$ and *AD* cuts them

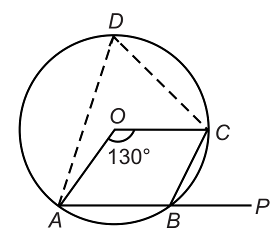
$\therefore \quad \angle EAD = \angle ADC = 88°$ (*alternate* ∠*s*)

$\therefore \quad \angle FAD = 20° + 88° = 108°$

So, $\quad \angle BCD = \angle FAD = \mathbf{108°}$. (*In cyclic quad. ext.* ∠ = *int. opp.* ∠)

(*iii*) Let *D* be any point on the major arc *AC*.

Then, $\quad \angle ADC = \dfrac{1}{2}\angle AOC = \dfrac{1}{2}\times 130° = 65°$



(*Angle subtended by an arc at the centre = twice the angle by the arc at the remaining circumference*)

Now, *ABCD* is a cyclic quadrilateral whose side *AB* is produced to any point *P*.

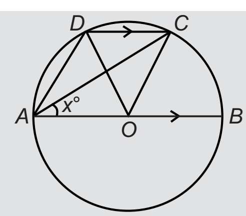
$\therefore$ ext. $\angle PBC$ = int. opp. $\angle ADC \Rightarrow \angle PBC = \mathbf{65°}$.

Ex. 11. In the figure given, *AB* is a diameter of the circle with centre *O* and $CD \parallel BA$. If $\angle CAB = x$, find the value of

(*i*) $\angle COB$

(*ii*) $\angle DOC$

(*iii*) $\angle DAC$ (*iv*) $\angle ADC$.



Sol. (*i*) $\angle COB = 2\angle CAB = \mathbf{2x°}$ (*angle at the centre* = 2 × *angle at the remaining part of the circumference*)

(*ii*) $\quad \angle OCD = \angle COB = \mathbf{2x°}$ (*alternate* ∠*s*, $DC \parallel AB$)

$OD = OC$ (*radii of the same circle*)

$\Rightarrow \quad \angle OCD = \angle ODC$

$\Rightarrow \quad \angle ODC = 2x°$

$\therefore \quad$ In ΔDOC, $\angle DOC = 180° - (2x° + 2x°) = \mathbf{180° - 4x°}$ (∠*sum prop. of a* Δ)

(*iii*) $\quad \angle DAC = \dfrac{1}{2}\angle DOC = \dfrac{1}{2}(180 - 4x)°$

(*angle made by arc DC at the centre = Twice the angle at the remaining part of the circumference*)

$= (90 - 2x)°$

(*iv*) In ΔADC, $\quad \angle ACD = \angle CAB = x°$ (*alt* ∠*s*; $DC \parallel AB$)

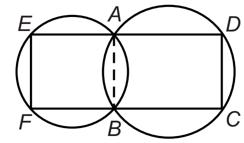
$\therefore \quad \angle ADC = 180° - (x° + 90° - 2x°) = \mathbf{(90 + x)°}$. (∠*sum prop. of a* Δ)

Ex. 12. Two circles *ABCD, ABFE* intersect at *A* and *B*. *EAD* and *FBC* are straight lines. Prove that *EF* is parallel to *DC*.

Sol. Given. Circles *ABCD*, *ABFE* intersecting at *A* and *B*, and *EAD* and *FBC* are st. lines.

To prove that $EF \parallel DC$

Construction. Join AB



Proof. $\angle DAB = \angle EFB$ (*ext.* $\angle$ *of cyclic quad.*)

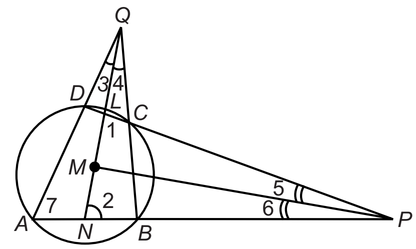
$\angle DAB + \angle BCD = 2$ rt. $\angle s$ (*opp.* $\angle s$ *of cyclic quad.*)

$\therefore$ $\angle EFB + \angle BCD = 2$ rt. $\angle s$ ($\because \angle EFB = \angle DAB$)

But $\angle s$ *EFB* and *BCD* are int. $\angle s$ on the same side of the transversal *FC*.

$\therefore$ $EF \parallel DC$.

Ex. 13. Prove that the bisectors of the angles formed by producing opposite sides of a cyclic quadrilateral (provided they are not parallel) intersect at right angle.



Sol. Given. Cyclic quad. *ABCD* whose opp. sides when produced intersect at *P* and *Q*. *PM* and *QN* are angular bisectors of $\angle s$ *P* and *Q* respectively, and meet each other at *M*. Let *QN* intersect *CD* at *L*.

To prove that $\angle QMP = 90°$

Proof. $\angle 1 = \angle LCQ + \angle CQL$ (*ext.* $\angle$ *of a* Δ *= sum of int opp* $\angle s$)

But $\angle LCQ = \angle BAD$ (*ext.* $\angle$ *of cyclic quad ABCD*)

and $\angle CQL = \angle LQD$ (*QL is bisector of* $\angle DQC$*, given*)

$\therefore$ $\angle 1 = \angle BAD + \angle LQD = \angle 7 + \angle 3$

Now, $\angle 2 = \angle AQN + \angle NAQ$ (*ext.* $\angle$ *of* Δ *QAN*)

$= \angle 3 + \angle 7$

$\Rightarrow$ $\angle 1 = \angle 2$

Now, in Δs *PLM* and *PNM*, we have

$\angle 1 = \angle 2$ (*proved above*)

$\angle 5 = \angle 6$ (*given, PM bisects* $\angle NPL$)

PM is common

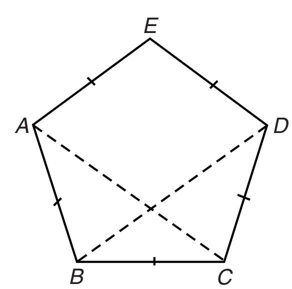
$\therefore$ $\Delta PLM \cong \Delta PNM$ (*AAS*)

$\Rightarrow$ $\angle PML = \angle PMN$ (*c.p.c.t.*)

But $\angle PML + \angle PMN = 180°$ (*LMN is a st. line*)

$\therefore$ $\angle PML = \angle PMN = 90°$, *i.e.*, $\boldsymbol{QM \perp PM}$.

Ex. 14. Prove that any four vertices of a regular pentagon are concyclic.



Sol. Let *ABCDE* be a regular pentagon.

Join *AC* and *BD*

In Δs *ABC* and *BCD*, we have

$AB = DC$ (*sides of a regular pentagon*)

$\angle ABC = \angle BCD$ (*angles of a regular pentagon*)

BC is common

$\therefore$ $\Delta ABC \cong \Delta BCD$ (*SAS*)

$\Rightarrow$ $\angle BAC = \angle BDC$ (*c.p.c.t.*)

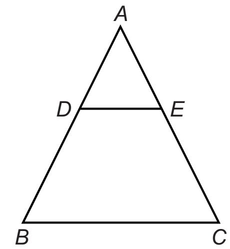
$\Rightarrow$ *A*, *B*, *C*, *D* are cyclic (*Since* $\angle BAC$ *and* $\angle BDC$ *are angles subtended by BC on the same side of it.*)

Hence, **any four vertices of a regular pentagon are concyclic.**

Ex. 15. *D* and *E* are points on equal sides of *AB* and *AC* of an isosceles Δ *ABC* such that *AD = AE*. Prove that *B, C, E, D* are concyclic.

Sol. Given. Isos. ΔABC in which $AB = AC$, *D* and *E* are points on *AB* and *AC* respectively such that $AD = AE$.

To prove that points *B*, *C*, *E*, *D* are concyclic



Proof. In Δ *ABC*, $AB = AC \Rightarrow \angle B = \angle C$ ...(1)

In Δ *ADE*, $AD = AE \Rightarrow \angle ADE = \angle AED$ ...(2)

Now, In Δ *ABC*, $\angle A + \angle B + \angle C = 180°$ ($\angle$ *sum of a* Δ)

In Δ *ADE*, $\angle A + \angle ADE + \angle AED = 180°$

$\therefore$ $\angle A + \angle B + \angle C = \angle A + \angle ADE + \angle AED$

$\Rightarrow$ $\angle B + \angle C = \angle ADE + \angle AED$

$\Rightarrow$ $2\angle B = 2\angle AED \Rightarrow \angle AED = \angle B$ (*From (1) and (2)*)

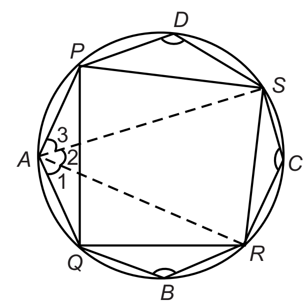
Now, $\angle AED + \angle CED = 180°$ (*AEC is a st. line*)

$\Rightarrow$ $\angle B + \angle CED = 180°$ ($\because \angle AED = \angle B$ *proved above*)

$\Rightarrow$ quad. *BCED* is a cyclic quad, *i.e.*, pts. *B*, *C*, *E*, *D* are concylic.

(*If sum of opp $\angle$s of a quad is 180° it is a cyclic quad.*)

Ex. 16. Prove that the sum of the angles in the four segments exterior to a cyclic quadrilateral is equal to 6 right angles.



Sol. *PQRS* is a cyclic quad. and angles $\angle A$, $\angle B$, $\angle C$ and $\angle D$ are angles in the four exterior segments.

Join *AR* and *AS.*

In cyclic quad. *AQBR,* $\angle 1 + \angle B = 180°$

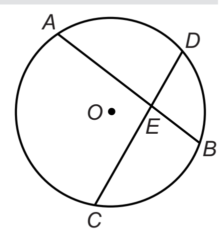
In cyclic quad. *ARCS,* $\angle 2 + \angle C = 180°$

In cyclic quad. *APDS,* $\angle 3 + \angle D = 180°$

$\therefore \angle 1 + \angle 2 + \angle 3 + \angle B + \angle C + \angle D = 180° + 180° + 180° = 6$ rt. $\angle$**s**

$\Rightarrow \angle A + \angle B + \angle C + \angle D =$ **6 rt. $\angle$s.**

Ex. 17. In the given figure two equal chords *AB* and *CD* of a circle with centre *O,* intersect each other at *E.* Prove that *AD = CB.*



Sol. We have

chord *AB* = chord *CD* (*given*)

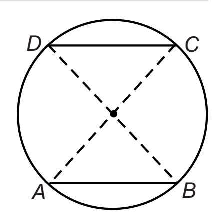
$\Rightarrow$ minor arc *AB* = minor arc *CD*

$\Rightarrow$ $\overset{\frown}{AB} = \overset{\frown}{CD}$

$\Rightarrow$ $\overset{\frown}{AB} - \overset{\frown}{BD} = \overset{\frown}{CD} - \overset{\frown}{BD}$

$\Rightarrow$ $\overset{\frown}{AD} = \overset{\frown}{CB} \Rightarrow AD = CB.$ (*In equal circles chords of the equal arcs are also equal.*)

Ex. 18. *A, B, C, D are four consecutive points on a circle such that AB = CD. Prove that AC = BD.*



Sol. $AB = CD$ (*given*)

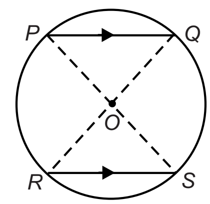
$\Rightarrow$ $\overset{\frown}{AB} = \overset{\frown}{CD}$ (*In equal circles equal chords cut off equal arcs*)

$\Rightarrow$ $\overset{\frown}{AB} + \overset{\frown}{BC} = \overset{\frown}{BC} + \overset{\frown}{CD}$

$\Rightarrow$ arc *ABC* = arc *BCD*

$\Rightarrow$ chord *AC* = chord *BD* $\Rightarrow AC = BD$. **Hence proved.**

Ex. 19. In the given Fig. *O* is the centre of the circle, chord *PQ* is parallel and equal to chord *RS* and *QR* is the diameter. Prove that arc *PR* = arc *QS.*



Sol. Join *PS.*

Now, $\angle POR = 2\angle PQR$ ($\angle$ *at the centre in twice the* $\angle$ *at the remaining circumference*)

$\angle SOQ = 2\angle SRQ$ ($\angle$ *at the centre is twice the* $\angle$ *at the remaining circumference*)

But $\angle PQR = \angle SRQ$ (*Alternate* $\angle$*s, PQ* || *RS*)

$\therefore$ $\angle POR = \angle SOQ$

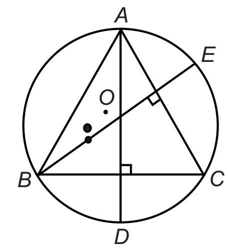
$\Rightarrow$ Arc PR = Arc QS (*In the same circle, arcs subtending equal angles at the centre are equal.*)

Hence, proved.

Note. It is clear otherwise also.

$\because$ $PQR = SRQ$ $\therefore$ arc PR = arc QS.

Ex. 20. In ΔABC, the perpendiculars from vertices A and B on their opposite sides meet (when produced) the circum-circle of ΔABC at points D and E respectively. Prove that arc CD = arc CE.



Sol. $\angle CAD = 90° - \angle C$ $\because AD \perp BC$

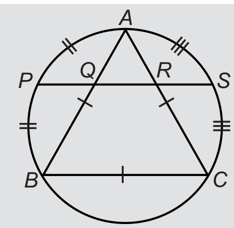
$\angle CBE = 90° - \angle C$ $\because BE \perp AC$

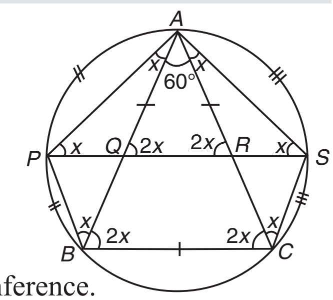
$\Rightarrow \angle CAD = \angle CBE \Rightarrow \frac{1}{2}\angle COD = \frac{1}{2}\angle COE$

$\Rightarrow \angle COD = \angle COE$

$\Rightarrow$ arc CD = arc CE.

Ex. 21. In the given Fig., ΔABC is equilateral, P and S are mid-points of arcs AB and AC. Prove that $PQ = QR = RS$.





Sol. Chord AB = Chord AC (sides of an equilateral Δ)

$\Rightarrow$ Arc APB = Arc ASC

Also, given, Arc AP = Arc PB = $\frac{1}{2}$ Arc APB

Arc AS = Arc SC = $\frac{1}{2}$ Arc ASC

$\Rightarrow$ Arc AP = Arc PB = Arc AS = Arc SC

Now Arc APB subtends $\angle ACB$ and Arc ASC subtends angle $\angle ABC$ on the circumference.

$\because$ Arc APB = Arc ASC $\Rightarrow \angle ACB = \angle ABC = 2x$ (say)

Now Arc PB = Arc SC $\Rightarrow \angle PAB = \angle CAS = \frac{1}{2} \times 2x = x.$

$PS \parallel BC$, $\angle AQR = \angle ARQ = 2x$

$\therefore$ $\angle APQ = \angle AQR - \angle PAQ = 2x - x = x$ (ext. $\angle$ property of a Δ)

Similarly $\angle ASR = x$.

Now, ΔABC is equilateral $\Rightarrow 2x = 60°$ $\Rightarrow x = 30°$

$\Rightarrow$ $\angle AQR = \angle ARQ = 2x = 60° \Rightarrow \Delta AQR$ is equilateral.

Also, $\angle PAQ = \angle APQ \Rightarrow AQ = QP$ (In ΔAQP) ...(*i*)

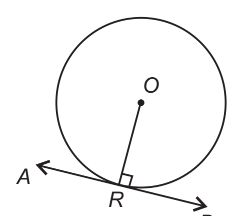
$\angle RAS = \angle SRA \Rightarrow AR = RS$ (In ΔARS) ...(*ii*)

$\Delta\, AQR$ being equilateral $AQ = AR = QR$ ...(*iii*)

$\therefore$ From (*i*), (*ii*) and (*iii*),

PQ = QR = RS.

III. THEOREMS ON TANGENTS AND SECANTS



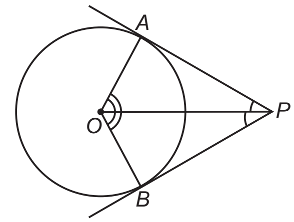
Theorem 19. ***The tangent at any point of a circle is perpendicular to the radius through the point of contact. OR $\perp$ AB.***

Theorem 20. ***If two tangents are drawn to a circle from an external point then,***

(*i*) ***the tangents are equal in length, i.e., PA = PB***

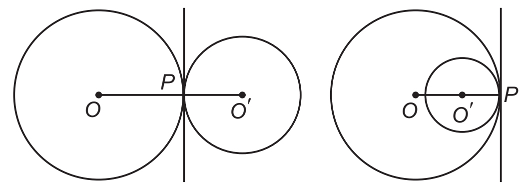
(*ii*) ***the tangents subtend equal angles at the centre of the circle, i.e.,*** $\angle POA = \angle POB$

(*iii*) ***the tangents are equally inclined to the line joining the point and the centre of the circle, i.e.,*** $\angle APO = \angle BPO$



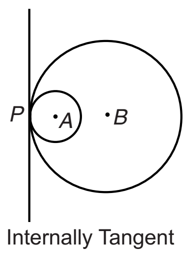
(*iv*) ***the angle between the tangents is supplementary of the angle that they subtend at the centre, i.e.,*** $\angle AOB + \angle APB = 180°$.

Theorem 21. ***If two circles touch each other, the point of contact lies on the straight line joining the centres of the two circles.***

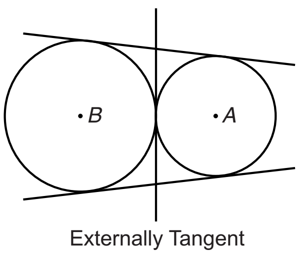


Touching or tangent circles and common tangents

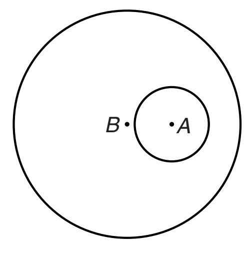
Definitions. Two circles are **tangent** if they are tangent to the same line at the same point. The two circles are also said to touch each other.



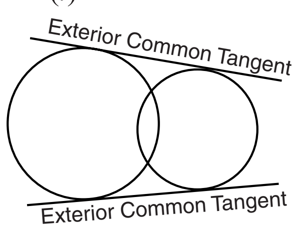
(*i*)



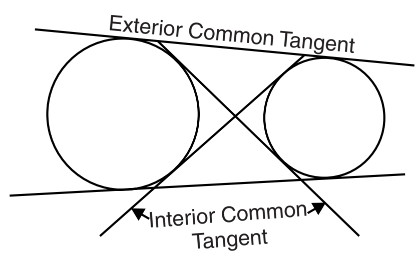
(*ii*)



(*iii*)



(*iv*)



(*v*)

If the circles are internally tangent, then there is just *one line tangent* to both of them Fig. (*i*). If the circles are externally tangent, then there are *three lines tangent* to both circles. Fig. (*ii*)

If one circle is contained in the interior of another, then there is no line that is tangent to both circles. Fig. (*iii*)

If the circles intersect in two points. Then there are *two lines tangent* to both circles Fig. (*iv*).

If the two circles do not intersect, then there are *four* lines that are tangent to both circles. Fig. (*v*)

If two circles are coplanar, and their centres are on the same side of their common tangent, then they are *internally* **tangent**, as in Fig. (*ii*).

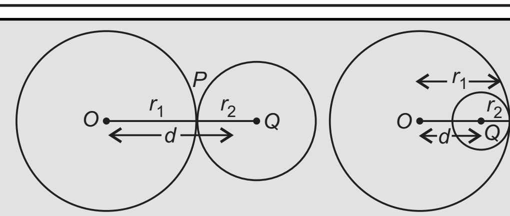
If two circles are coplanar, and their centres are on opposite sides of their common tangent, then they are *externally* **tangent** as in Fig. (*iii*).

If a line is tangent to each of two circles it is called a **common tangent** to two circles. It is called an **exterior** (or **direct**) **common tangent,** if the circles lie on the same side of it, as in Fig. (*v*) and it is called an **interior** (or **transverse**) **common tangent,** if the circles lie on opposite sides of it, as in Fig. (*v*).

Note. Two circles touch if the distance (*d*) between their centres is equal to the sum of their radii (external contact) or equal to the difference of their radii (internal contact).

i.e., $d = r_1 + r_2$, **if the circles touch externally;**

$d = r_1 - r_2$, **if the circles touch internally.**



SOLVED EXAMPLES

Ex. 22. (*a*) In Fig. (*i*), the tangent to a circle of radius 1.5 cm from an external point P, is 2 cm long. Calculate the distance of P from the nearest point of the circumference.

(*b*) In Fig. (*ii*) from an external point *P*, tangents *PA* and *PB* are drawn to circle *O*. *CD* is tangent to the circle at *E*. If *AP* = 16 cm, find the perimeter of Δ *PCD*.

(*c*) In Fig. (*iii*), there are two concentric circles of radii 3 cm and 5 cm respectively. Find the length of the chord of the outer circle which touches the inner circle.

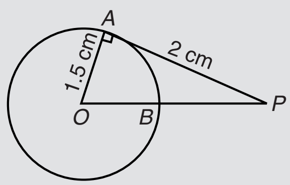


Fig. (*i*)

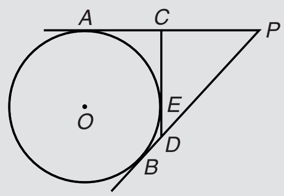


Fig. (*ii*)

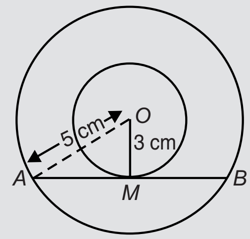


Fig. (*iii*)

Sol. (*a*) *PB* is the required distance

$\angle OAP = 90°$ (∠ *between tangent and radius through the pt. of contact.*)

In right ΔOAP, $OP^2 = OA^2 + AP^2$ (*Pythagoras*)

$= (1{\cdot}5)^2 + (2)^2 = 2{\cdot}25 + 4 = 6{\cdot}25$ cm^2

$\Rightarrow$ $OP = \sqrt{6.25}$ cm $= 2{\cdot}5$ cm

$\therefore$ $PB = OP - OB = 2{\cdot}5$ cm $- 1{\cdot}5$ cm $=$ **1 cm.**

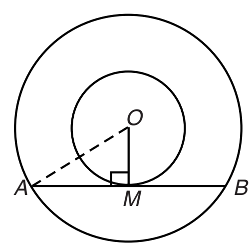
(*b*) $CE = CA$ and $DE = DB$ (*tangents to the circle from external points C and D*) ...(*i*)

Perimeter of $\Delta PCD = PC + CD + PD = PC + (CE + ED) + PD$

$= (PC + CA) + (DB + PD)$ [Using (*i*)]

$= PA + PB = 16$ cm $+ 16$ cm $=$ **32 cm.**

(*c*) Let *O* be the centre of the two concentric circles and let *AB* be the chord of the outer circle which touches the inner circle at *M*.



Then, $OM \perp AB$ (*Tangent* ⊥ *radius through the pt. of contact*)

Also, $AM = MB \Rightarrow AB = 2\,AM$ (⊥ *from centre bisects chord*)

$OA = 5$ cm (*radius of the outer circle*)

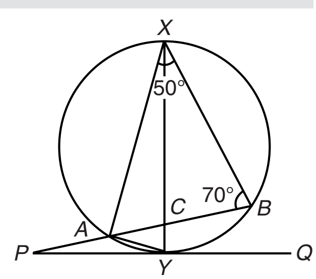
$OM = 3$ cm (*radius of the inner circle*)

Now, in right Δ *OMA*, $AM^2 = OA^2 - OM^2$ (*Pythagoras*)

$\Rightarrow$ $AM^2 = 5^2 - 3^2 = 25 - 9 = 16$ cm^2 $\Rightarrow$ AM $= \sqrt{16} = 4$ cm

$\therefore$ $AB = 2\,AM = 2 \times 4 =$ **8 cm.**

Ex. 23. In the adjoining Fig., *XY* is a diameter of the circle, *PQ* is a tangent to the circle at *Y*. Given that $\angle AXB = 50°$ and $\angle ABX = 70°$, calculate $\angle BAY$ and $\angle APY$.



Sol. In ΔAXB,

$\angle XAB + \angle AXB + \angle ABX = 180°$

$\Rightarrow$ $\angle XAB = 180° - (50° + 70°) = 180° - 120° = 60°$.

XY being the diameter of the circle.

$\Rightarrow$ $\angle XAY = 90°$ (∠ *in a semi circle*)

$\therefore$ $\angle BAY = \angle XAY - \angle XAB = 90° - 60° =$ **30°**.

Now $\angle BXY = \angle BAY = 30°$ (∠*s in the same segment of the circle*)

$\therefore$ $\angle ACX = \angle BXC + \angle CBX = 30° + 70° = 100°$(*ext.* ∠ = *sum of int. opp.* ∠*s in* ΔBXC)

Also, $\angle XYP = 90°$ (*radius through the point of contact is perpendicular to the tangent*)

For ΔCPY,

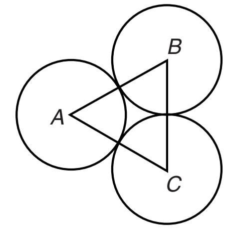
$\angle ACX = \angle APY + \angle CYP$ (*ext.* ∠ = *sum of int. opp.* ∠*s*)

$\Rightarrow$ $100° = \angle APY + 90°$

$\Rightarrow$ $\boldsymbol{\angle APY = 10°}$

Ex. 24. Three circles have the centres at *A, B, C* and each circle touches the other two externally. If *AB* = 5 cm, *BC* = 7 cm, *CA* = 6 cm, find the radii of the three circles.

Sol. Let the radii of the circles with centres *A*, *B*, *C* be *x* cm, *y* cm and *z* cm respectively. Then



$$AB = x + y = 5 \quad ...(i)$$
$$BC = y + z = 7 \quad ...(ii)$$
$$CA = z + x = 6 \quad ...(iii)$$

Adding, $2(x + y + z) = 18$

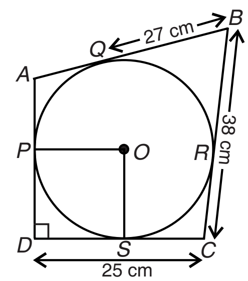
$\therefore \quad x + y + z = 9 \quad ...(iv)$

Subtracting each equation in turn from (*iv*), we obtain

$$z = 9 - 5 = 4,$$
$$x = 9 - 7 = 2, y = 9 - 6 = \mathbf{3}.$$

Ex. 25. In the given figure a circle is inscribed in quadrilateral *ABCD*. If *BC* = 38 cm, *BQ* = 27 cm, *DC* = 25 cm and *AD* ⊥ *DC*, find the radius of the circle.

Sol. Let the sides *AD*, *AB*, *BC* and *CD* touch the circle at point *P*, *Q*, *R* and *S* respectively. Since tangent to a circle is perpendicular to the radius through the point of contact.



$\therefore$ $OP \perp AD$ and $OS \perp DC$. Also $AD \perp DC$ (given)

$\therefore$ *OPDS* is a square.

$BR = BQ = 27$ cm (tangents from an external point to a circle are equal in length)

$\therefore$ $CR = BC - BR = (38 - 27)$ cm = 11 cm

Similarly, $CS = CR = 11$ cm

$\therefore$ $DS = DC - CS = (25 - 11)$ cm = 14 cm

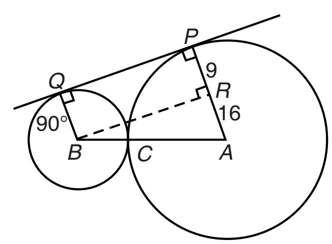
$\therefore$ Radius of circle = $OP = DS$ = **14 cm.** ($\because$ *OPDS is a square*)

Ex. 26. Two circles of radii 25 cm and 9 cm touch each other externally. Find the length of the direct common tangent.

Sol. Let the two circles with centres *A*, *B* and of radii 25 cm and 9 cm touch each other externally at point *C*. Then, $AB = AC + CB = (25 + 9)$ cm = 34 cm.

Let *PQ* be the direct common tangent. $\therefore$ $BQ \perp PQ$ and $AP \perp PQ$.

Draw $BR \perp AP$. Then *BRPQ* is a rectangle.



(*Tangent ⊥ radius at the point of contact*)

In ΔABR, $AB^2 = AR^2 + BR^2$ (*Pythagoras' Theorem*)

$\Rightarrow \quad (34)^2 = (16)^2 + BR^2$

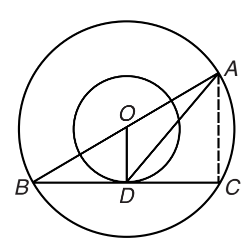
$\Rightarrow \quad (BR)^2 = 1156 - 256 = 900$

$\Rightarrow \quad BR = \sqrt{900}$ cm = 30 cm

$\therefore \quad PQ = BR =$ **30 cm.**

Ex. 27. The radii of two concentric circles are 13 cm and 8 cm respectively. *AB* is a diameter of the bigger circle. *BD* is a tangent to the smaller circle touching it at *D*. Find the length of *AD*.

Sol. Let the line *BD* intersect the bigger circle at *C*. Join *AC*. Then, in the smaller circle.



$OD \perp BD$ (*radius ⊥ tangent at the point of contact*)

$\Rightarrow$ $OD \perp BC \Rightarrow BD = DC$ (*BC is the chord of the bigger circle and perpendicular from the centre of the circle to a chord bisects the chord*)

$\Rightarrow$ *D* is the mid-point of *BC*

Also, given *O* is the mid-point of *AB* (*AB is the diameter of the bigger circle*)

$\therefore$ In ΔBAC, *O* is the mid-point of *AB* and *D* is the mid-point of *BC*.

$\therefore \quad OD = \frac{1}{2} AC$ (*segment joining the mid-points of any two sides of a triangle is half the third side*)

$\Rightarrow \quad AC = 2\ OD \Rightarrow AC = 2 \times 8 = 16$ cm

In right $\Delta\ OBD$

$$OD^2 + BD^2 = OB^2 \Rightarrow BD = \sqrt{OB^2 - OD^2} = \sqrt{(13)^2 - 8^2} = \sqrt{169 - 64} = \sqrt{105}$$

$\therefore \quad DC = BD = \sqrt{105}$

Now $AD^2 = AC^2 + DC^2$

$\Rightarrow \quad AD^2 = 16^2 + (\sqrt{105})^2 = 256 + 105$

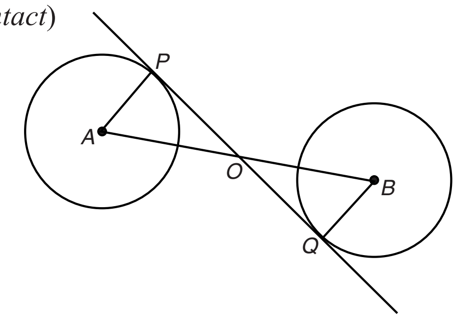
$\Rightarrow \quad AD^2 = 361 \Rightarrow AD = \sqrt{361} = $ **19 cm.**

Ex. 28. *PQ* is a transverse common tangent to the circles with centres *A* and *B* touching them at *P* and *Q* respectively: Prove that $\frac{AP}{BQ} = \frac{AO}{BO}$ where *O* is the point of intersection of the common tangent and the line joining the centres.

Sol. *PQ* is the tangent to the circle with centre *A* at point *P* and to the circle with centre *B* at point *Q*.

$\therefore \quad \angle APQ = 90°$ (*radius* $\perp$ *tangent at the point of contact*)

Similarly $\angle BQP = 90°$



$\therefore$ In $\Delta\ APO$ and BQO

$\angle APO = \angle BQO$ (*each = 90°*)

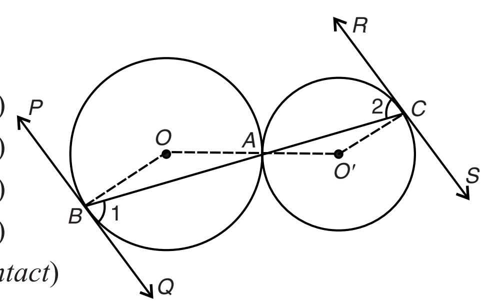
$\angle AOP = \angle BOQ$ (*vert. opp.* $\angle$*s*)

$\therefore \quad \Delta\ APO \sim \Delta\ BQO$ (*AA similarity*)

$\Rightarrow \quad \frac{AP}{BQ} = \frac{AO}{BO}$.

Ex. 29. In the given figure, two circles with centres *O* and *O′* touch externally at a point *A*. A line through *A* is drawn to intersect these circles in *B* and *C*. Prove that the tangents at *B* and *C* are parallel.

Sol. The two circles with centres *O* and *O′* touch externally at *A*. Line through *A* intersects the circles at *B* and *C*. Tangents *PBQ* and *RCS* are drawn. We have to prove *PBQ* || *RCS*.



Join *O* and *O′* to *A*, *B* and *C*

$OA = OB$ (*radii of the same circle*)

$\Rightarrow \quad \angle OBA = \angle OAB$ (*angles opp. equal sides are equal*)

$\Rightarrow \quad \angle OBA = \angle O'AC$ ($\angle OAB = \angle O'AC$, *vert. opp.* $\angle$*s*)

$\Rightarrow \quad \angle OBA = \angle O'CA$ ($\angle O'AC = \angle O'CA$, $\because O'C = O'A$)

Also, $OBQ = O'CR$ (*each = 90°, radius* $\perp$ *tangent at the point of contact*)

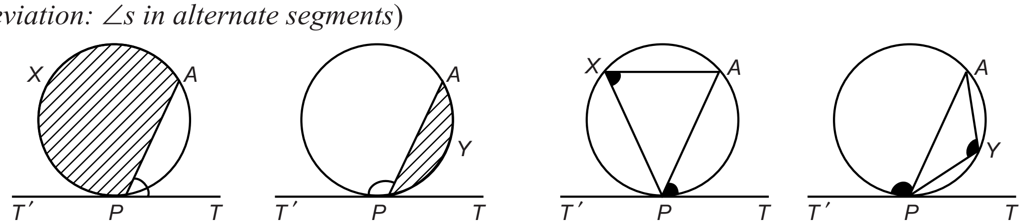
$\Rightarrow \quad \angle OBA + \angle 1 = \angle O'CA + \angle 2$

$\Rightarrow \quad \angle 1 = \angle 2$ ($\because \angle OBA = O'CA$)

$\Rightarrow \quad PBQ \parallel RCS$ (*alt.* $\angle$*s are equal, BAC is the transversal*)

IV. ALTERNATE SEGMENT THEOREM

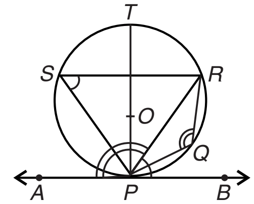
The alternate segment property: *T′PT* is a tangent to a circle at the point *P* and *PA* is a chord of contact. $\angle APT$ and the segment *AXP* lie on opposite sides of the chord of contact. Therefore, when dealing with $\angle APT$, segment *AXP* is called the alternate segment and any angle *AXP* is called the angle in the alternate segment. (*Abbreviation:* $\angle$*s in alternate segments*)



Theorem 22. ***If a st. line touches a circle and from the point of contact, a chord is drawn, the angles between the tangent and the chord are respectively equal to the angles in the alternate segments.***

(**Abbreviation:** $\angle$ *s in alternate segments*)

e.g., $\angle RPB = \angle PSR, \angle APR = \angle PQR$



SOLVED EXAMPLES

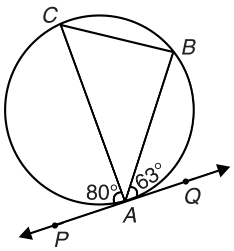
Ex. 30. In the given figure, line *PQ* touches the circle at *A*. If $\angle PAC$ = 80°, and $\angle QAB$ = 63°, calculate the angles of ΔABC.

Sol. $\angle ACB = \angle QAB = \mathbf{63°}$ ($\angle$*s in alternate segments*)

$\angle ABC = \angle PAC = \mathbf{80°}$ ($\angle$*s in alternate segments*)

Now, in ΔABC, $\angle CAB + \angle ABC + \angle ACB = 180°$ ($\angle$ *sum of a* Δ)

$\Rightarrow \angle CAB + 80° + 63° = 180° \Rightarrow \angle CAB = 180° - 143° = \mathbf{37°}$



Ex. 31. *PA* and *PB* are two tangents of a circle. $\angle APB$ = 50° and chord *AC* is drawn parallel to *PB*. Find by calculation the angles of ΔABC.

Sol. $PA = PB$ (*Tangents from external point of a circle are equal*)

$\Rightarrow \angle PAB = \angle PBA = x$ (*say*)

In ΔPAB, $x + x + 50° = 180°$

$\Rightarrow 2x = 130° \Rightarrow x = 65°$

$\therefore \angle ACB = \angle PBA = x = \mathbf{65°}$ ($\angle$ *s in alternate segments*)

Now, $\angle CAP + \angle APB = 180°$ (*co-int* $\angle$*s, AC* || *PB*)

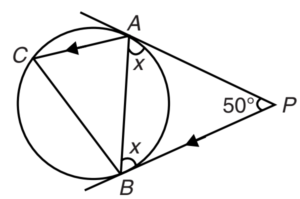
$\Rightarrow \angle CAB + x + 50° = 180° \Rightarrow \angle CAB + 65° + 50° = 180°$

$\Rightarrow \angle CAB = 180° - 115° = \mathbf{65°}$

$\therefore \angle CBA = 180° - (\angle ACB + \angle CAB) = 180° - (65° + 65°)$

$= 180° - 130° = \mathbf{50°}$

Hence, the angles of ΔABC are 65°, 65°, 50°.



Ex. 32. *PQ* and *PR* are two equal chords of a circle. Show that *SPT,* a tangent at *P* is parallel to *QR*.

OR

***P* is the mid-pt. of arc *QPR* of a circle. Show that the tangent at *P* is parallel to the chord *QR*.**

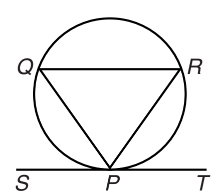
Sol. $PQ = PR$ (*Given*)

$\therefore \angle PRQ = \angle PQR$ ($\angle$*s opp. equal sides in a* Δ)

But $\angle RPT = \angle PQR$ ($\angle$*s in alternate segments*)

$\therefore \angle PRQ = \angle RPT$

But these are alternate angles $\therefore$ *SPT* || *QR*. **Hence proved.**



Ex. 33. In the given figure, *SAT* is the tangent to the circumcircle of *a* Δ *ABC* at the vertex *A*. A line parallel to *SAT* intersects *AB* and *AC* at the points *D* and *E* respectively. Prove that $\Delta ABC \sim \Delta AED$, and $AB \times AD = AE \times AC$.

Sol. In the given figure we have

$\angle SAD = \angle ADE$ (*alternate* $\angle$*s*)

$\angle TAE = \angle AED$ (*alternate* $\angle$*s*)

Also, $\angle SAD = \angle ACB$ ($\angle$*s in alternate segments*)

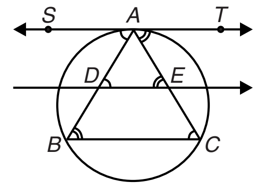
$\angle TAE = \angle ABC$ ($\angle$*s in alternate segments*)

$\therefore$ $\angle ADE = \angle ACB$ and $\angle AED = \angle ABC$

$\Rightarrow$ $\Delta ADE \sim \Delta ABC$ (*AAA similarity*)

$\Rightarrow$ $\dfrac{AD}{AC} = \dfrac{AE}{AB}$ (*corr. sides proportional*)

$\Rightarrow$ $\boldsymbol{AB \times AD = AE \times AC.}$ **Proved.**



Ex. 34. Two lines *ABC* and *ADE* are intersected by two parallel lines in *B, D* and *C, E* respectively. Prove that the circumcircles of Δ *ABD* and Δ *ACE* touch each other at *A*.

Sol. Draw TA tangent to the circumcircle of Δ ACE

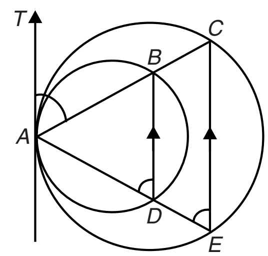
Now, $\angle TAC = \angle AEC$ ($\angle$*s in alternate segments*)

$\angle AEC = \angle ADB$ (*Corr.* $\angle$*s, BD* || *CE*)

$\therefore$ $\angle TAC = \angle ADB$

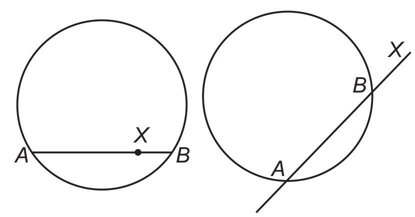
or $\angle TAB = \angle ADB$

Since $\angle ADB$ is an angle in the alternate segment, therefore, *TA* is a tangent to the circumcircle of Δ *ADB* also. Thus, *TA* is a tangent to both the circles at the same point *A*, hence, the two circles touch each other at *A*. **Proved.**



V. SEGMENTS OF A CHORD

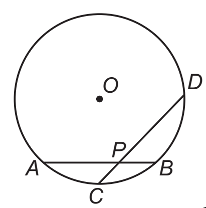
Definition: If *AB* is any chord of a circle, and if *X* is any point either on *AB* or *AB* produced, then *AX* and *BX* are called the segments of the chord formed by the point of division *X*.



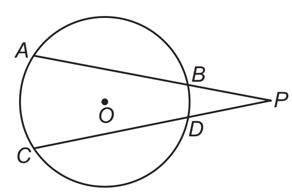
Theorem:

Theorem 23. ***If two chords of a circle intersect internally or externally, then the product of the lengths of their segments are equal.***

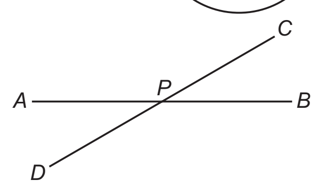
(*i*) When two chords *AB* and *CD* of a circle with centre *O,* intersect at a point *P* inside the circle, then ***AP* . *PB* = *CP* . *PD***



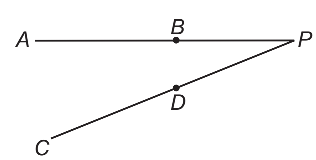
(*ii*) Two chords *AB* and *CD* of a circle, when produced intersect at a point *P* outside the circle, then ***PA.PB* = *PC.PD.***



Theorem 24. ***Conversely, if two straight lines AB and CD cut each other either both internally or both externally at P so that PA . PB = PC . PD, then the four points A, B, C, D lie on a circle.***

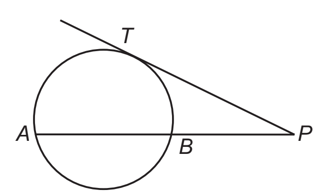


Theorem 25. ***If a chord and a tangent intersect externally, then the product of the lengths of the segments of the chord is equal to the square of the length of the tangent from the point of contact to the point of intersection.***



e.g., when, chord *AB* and tangent *TP* of a circle intersect at a point *P* outside the circle, then $\boldsymbol{PA \cdot PB = PT^2}$.

Theorem 26. **Conversely,** ***if from any point P on a line AB produced another straight line is drawn and a point T is taken on it such that*** $\boldsymbol{PA \cdot PB = PT^2}$***, then the line PT is a tangent to the circle which passes through A, B, T.***



SOLVED EXAMPLES

Ex. 35. In the given figure, chords *AB* and *CD* intersect at a point *P*. If *CP* = 6, *CD* = 9 and *AB* = 19, what are the lengths of *AP* and *PB*?

Sol. $CP = 6, \quad DP = CD - CP = 9 - 6 = 3$

Let $AP = x, \quad \text{then} \quad PB = AB - AP = 19 - x$

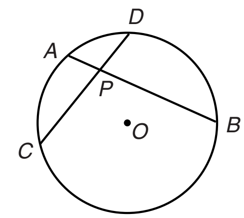
Now $AP \,.\, PB = CP \,.\, DP$

$\therefore \quad x(19 - x) = 6 \times 3 \Rightarrow 19x - x^2 = 18$

$\Rightarrow \quad x^2 - 19x + 18 = 0 \Rightarrow (x - 1)(x - 18) = 0$

$\Rightarrow \quad x = \mathbf{1, 18}$

Hence, the lengths of *AP* and *PB* are **1 unit** and **18 units.**



Ex. 36. In the given figure, *P* is outside the circle and secants *PCA* and *PDB* intersect the circle at *C* and *A, D* and *B* respectively. If *PA* = 24, *CA* = 16 and *DB* = 26, find *PB*.

Sol. Let $PD = x$

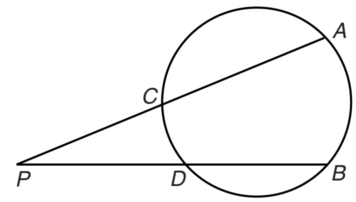
$PD \times PB = PC \times PA \Rightarrow x(x + 26) = 8 \times 24$

$\Rightarrow \quad x^2 + 26x - 192 = 0$

$\Rightarrow \quad (x + 32)(x - 6) = 0$

$\Rightarrow \quad x = 6$, taking only positive value of x.

$\Rightarrow \quad PB = 6 + 26 = \mathbf{32}.$

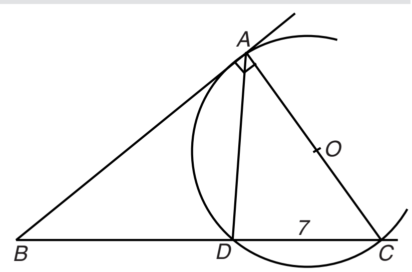


Ex. 37. In the adjoining figure, the angle *A* of the triangle *ABC* is a right angle. The circle on *AC* as diameter cuts *BC* at *D*. If *BD* = 9, and *DC* = 7, calculate the length of *AB*.

Sol. Since *OA* is a radius and $\angle BAC = 90°$, therefore *BA* is a tangent to the circle, by the tangent-radius property.

Hence, $BA^2 = BD \times BC = 9 \times 16 = 144$

$\Rightarrow \quad BA = \mathbf{12}.$



Ex. 38. If the diagonals of the quadrilateral *ABCD* cut at *O*, and if *OA* = 3 cm, *OB* = 9 cm, *AC* = 15 cm, *BD* = 13 cm, prove that *ABCD* is cyclic.

Sol. $OC = AC - OA = 15 - 3 = 12$ cm

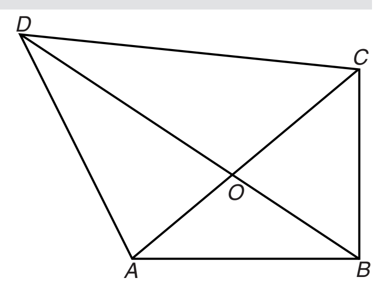
$OD = BD - BO = 13 - 9 = 4$ cm

$\therefore \quad AO \,.\, OC = 3 \times 12 = 36 \text{ cm}^2$

$BO \,.\, OD = 9 \times 4 = 36 \text{ cm}^2$

$\Rightarrow \quad AO \,.\, OC = BO \,.\, OD$

Hence, ***ABCD* is a cyclic quadrilateral.**

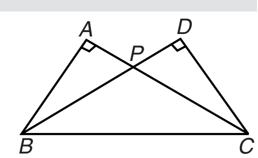


Ex. 39. *ABC* and *DBC* are two right triangles with common hypotenuse *BC* and with their sides *AC* and *DB* intersecting at *P*. Prove that *AP* . *PC* = *BP* . *PD*.

Sol. $\therefore \quad \angle BAC = \angle BDC$ (*each = 1 rt.* $\angle$)

$\therefore$ Pts. *B*, *A*, *D*, *C* are concyclic (*$\angle$s on the same side of segment BC are equal*)

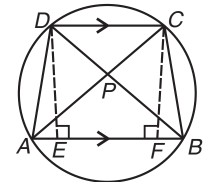
$\therefore \quad AP \,.\, PC = BP \,.\, PD$ (*product of segments of intersecting chords of a circle*)



Ex. 40. In a trapezium *ABCD, AB* || *CD* and *AD* = *BC*. If *P* is the point of intersection of the diagonals *AC* and *BD*, prove that *PA* × *PC* = *PB* × *PD*.

Sol. Draw $DE \perp AB$ and $CF \perp AB$

In Δ*s DEA* and *CFB*, we have



$AD = BC$ (*Given*)

$\angle DEA = \angle CFB$ (*each* = 90°)

$DE = CF$ (*Distance between two parallels*)

$\therefore$ $\Delta DEA \cong \Delta CFB$ (*R.H.S.*)

$\Rightarrow$ $\angle DAE = \angle CBF$ (*c.p.c.t.*) ...(*i*)

Now $\angle D + \angle B = \angle ADC + \angle CBA = \angle ADC + \angle CBF$

$= \angle ADC + \angle DAE$ (*From* (*i*))

$= 180°$ (*DC* || *AB, Sum of co-int.* $\angle s$ = 180°)

$\Rightarrow$ Opposite angles of trapezium *ABCD* are supplementary.

$\Rightarrow$ *ABCD* is a cyclic quadrilateral.

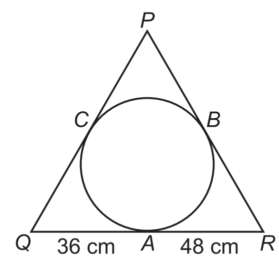
Thus *AC* and *BD* are two chords of the circle circumscribring the trapezium such that they intersect at *P*.

Hence, ***PA* × *PC* = *PB* × *PD***.

Ex. 41. The radius of the incircle of a triangle is 24 cm. The segments into which one side is divided by the points of contact are 36 cm and 48 cm. Find the lengths of the other two sides of the triangle.

Sol. Let the sides *QR*, *PR* and *QP* touch the incircle in points *A*, *B* and *C* respectively.

Suppose *QR* is divided by point *A* into segments *QA* and *AR* measuring 36 cm and 48 cm respectively.



$\because$ *AQ* and *QC* are tangents to the incircle from point touching it at points *A* and *C* respectively and lengths of tangents from the same external point are equal,

$QC = QA = 36$ cm

Similarly, $RB = RA = 48$ cm.

Let $PC = PB = x$ cm. Also let $QR = a$, $PR = b$, $PQ = c$.

Then, $a = (36 + 48)$ cm, $b = (x + 48)$ cm, $c = (x + 36)$ cm

$$\therefore \text{ Semi-perimeter } (s) = \frac{1}{2}(a+b+c)$$

$$= \frac{(36+48+x+48+x+36)\text{ cm}}{2} = (x+84)\text{ cm}$$

$(s - a) = x$ cm, $(s - b) = 36$ cm, $(s - c) = 48$ cm

$$\therefore \text{ Area of } \Delta PQR = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{(x+84)\,.\,x.\,36\,.\,48}\,.\text{ cm}^2$$

$$\therefore \text{ In radius } r = \frac{\Delta}{s} = \frac{\text{Area of } \Delta PQR}{s} = \frac{\sqrt{(x+84)\,.\,x\,.\,36\,.\,48}}{x+84} = 24\sqrt{\frac{3x}{x+84}}$$

$$\because\ r = 24,\ \therefore\ 24 = 24\sqrt{\frac{3x}{x+84}} \Rightarrow \sqrt{\frac{3x}{x+84}} = 1$$

$\Rightarrow$ $3x = x + 84$ $\Rightarrow$ $2x = 84 \Rightarrow x = 42$

$\therefore$ $b = (x + 48)$ cm = **90 cm**

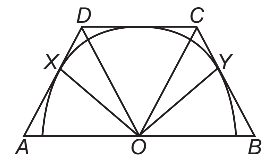
$\therefore$ $c = (x + 36)$ cm = **78 cm.**

Ex. 42. In a quadrilateral *ABCD*, a circle with centre at the mid-point of *AB* touches the sides *BC, CD* and *AD*. Show that $AB^2 = 4AD\ BC$.

Sol. Let *O* be the mid-point of *AB*. Let *X* and *Y* be respectively the points of contact of *AD* and *BC* with the circle. Then,

$OA = OB$ (*O* is mid-point of *AB*)

$OX = OY$ (radii)



$\Rightarrow \angle OAD = \angle OBC$ …(*i*)

$\because$ *AD* and *DC* are tangets,

$\angle ADC = 2\ \angle ADO$ (Tangents are equally inclined to the line joining the centre and pt. of contact of tangents) ...(*ii*)

Similarly, $\angle BCD = 2\ \angle BCO$ ...(*iii*)

Now $\angle OAD + \angle ADC + \angle DCB + \angle OBC = 180°$

$= 2\ (\angle OAD + \angle ADO + \angle AOD)$ (Area of ΔOAD)

$\Rightarrow \angle OAD + 2\ \angle ADO + 2\ \angle BCO + \angle OAD = 2\ (\angle OAD + \angle ADO + \angle AOD)$ (using (*i*), (*ii*) and (*iii*))

$\Rightarrow \angle BCO = \angle AOD$

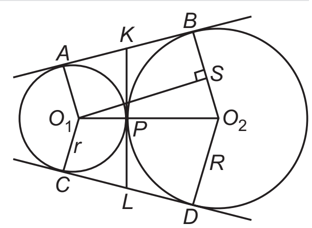
Now $OA = OB,\ \angle OAD = \angle OBC,\ \angle BCO = \angle AOD \Rightarrow \Delta\, AOD \cong \Delta\, BCO$

$$\Rightarrow \frac{AO}{BC} = \frac{AD}{BO} \Rightarrow AO\ .\ BO \quad \Rightarrow AD\ .\ BC$$

$$\Rightarrow \frac{1}{2}AB . \frac{1}{2}AB = AD\ .\ BC$$

$\Rightarrow \boldsymbol{AB^2 = 4\ AD\ .\ BC.}$

Ex. 43. Two circles with radii *r* and *R* are externally tangent at a point *P*. Determine the length of the segment cut from the common tangent through *P* by the other common tangents.



Sol. Without loss of generality, we may assume that $r \le R$. Let the circle with radius *r* have centre O_1 and the circle with radius *R* have centre O_2. Let *P* be their point of tangency. Let the common external tangents meet the circles at *A, B, C* and *D,* as in the diagram. Let the internal common tangent meet the external common tangents at *K* and *L*.

Let *S* be the point on O_2B such that $O_1S \perp O_2B$. Then $O_1S = AB$ and $O_2S = R - r$. Also

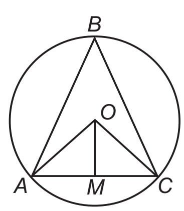
$$O_1S = \sqrt{(O_1O_2)^2 - (O_2S)^2} = \sqrt{(R+r)^2 - (R-r)^2} = 2\sqrt{Rr}.$$

Thus, $KP = \frac{1}{2}AB^* = \sqrt{Rr}.$ Similarly, since $CD = AB = 2\sqrt{Rr},$ we have (* Since $KP = KA = KB$)

$PL = \sqrt{Rr},$ which implies that $KL = 2\sqrt{Rr}.$

PRACTICE SHEET

Level-1



1. In the given figure, *O* is the centre of the circle. *OA* = 3 cm, *AC* = 3 cm and $OM \perp AC$. What is $\angle ABC$ equal to?

(*a*) 60° (*b*) 45°
(*c*) 30° (*d*) None of these

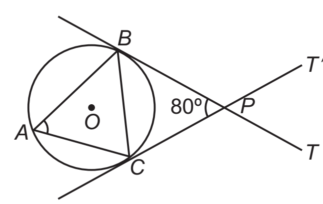
(*CDS 2011*)

2. *AC* is the diameter of the circumcircle of the cyclic quadrilateral *ABCD*. If $\angle BDC = 42°$, then what is $\angle ACB$ equal to?

(*a*) 42° (*b*) 45° (*c*) 48° (*d*) 58°

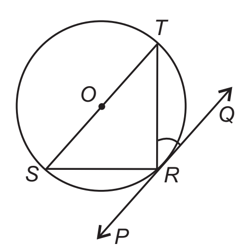
3. If *A, B, C,* are three consecutive points on the arc of a semi-circle such that the angles subtended by the chords *AB* and *AC* at the centre *O* are 60° and 100° respectively. Then $\angle BAC$ is equal to

(*a*) 20° (*b*) 50° (*c*) 80° (*d*) 200°



4. In the given figure, *BT* and *CT′* are two tangents at points *B* and *C* on the circle and $\angle BPC = 80°$. Then $\angle A$ is

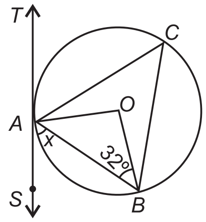
(*a*) 80° (*b*) 60°
(*c*) 50° (*d*) 40°



5. In the given figure, *ST* is the diameter of the circle with centre *O*, *PQ* is the tangent at point *R*. If $\angle TRQ = 40°$, find $\angle RTS$.

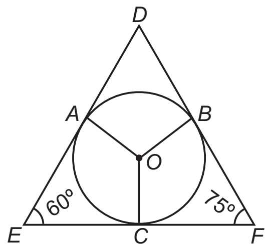
(*a*) 40° (*b*) 50°
(*c*) 60° (*d*) 30°

6. In the given figure, *TAS* is a tangent to the circle at the point *A*. If $\angle OBA = 32°$, what is the value of x?



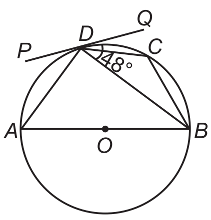
(*a*) 64° (*b*) 40°
(*c*) 58° (*d*) 50°

7. In a triangle *DEF*, *O* is the centre of the incircle *ABC*. $\angle DEF = 60°$, $\angle DFE = 75°$. Find $\angle AOB$



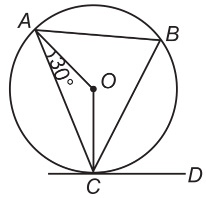
(*a*) 75°
(*b*) 45°
(*c*) 135°
(*d*) cannot be determined.

8. In the adjoining figure, *O* is the centre of the circle and *AB* is the diameter. Tangent *PQ* touches the circle at *D*. $\angle BDQ = 48°$. Find the ratio of $\angle DBA : \angle DCB$.



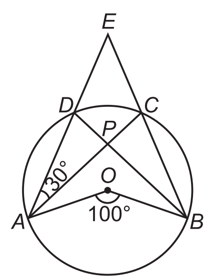
(*a*) 22/7 (*b*) 7/22
(*c*) 7/12 (*d*) can't be determined.

9. In the given diagram, *O* is the centre of the circle and *CD* is a tangent. $\angle CAB$ and $\angle ACD$ are supplementary to each other. $\angle OAC = 30°$. Find the value of $\angle OCB$?



(*a*) 30° (*b*) 20°
(*c*) 60° (*d*) None of these

10. In the given figure, *O* is the centre of the circle. *AC* and *BD* intersect at *P*. If $\angle AOB = 100°$ and $\angle DAP = 30°$, what is $\angle APB$?

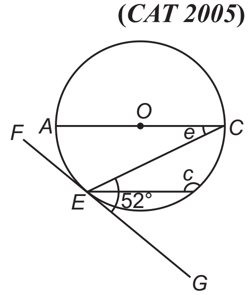


(*a*) 77° (*b*) 80°
(*c*) 85° (*d*) 90° **(*CDS 2010*)**

11. What is the distance (in cm) between two parallel chords of lengths 32 cm and 24 cm in a circle of radius 20 cm?

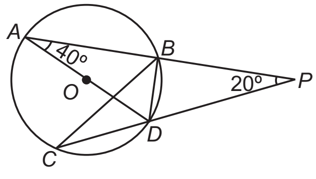
(*a*) 1 or 7 (*b*) 2 or 14 (*c*) 3 or 21 (*d*) 4 or 28
(*CAT 2005*)

12. In the given figure, *O* is the centre of the circle and *AC* the diameter. The line *FEG* is tangent to the circle at point *E*. If $\angle GEC = 52°$, find the value of $\angle e + \angle c$.



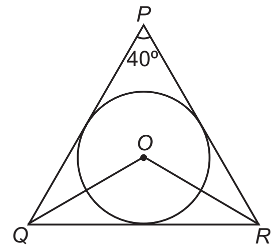
(*a*) 154° (*b*) 156°
(*c*) 166° (*d*) 180° **(*CAT 2011*)**

13. *PBA* and *PDC* are two secants. *AD* is the diameter of the circle with centre at *O*. $\angle A = 40°$, $\angle P = 20°$. Find the measure of $\angle DBC$.



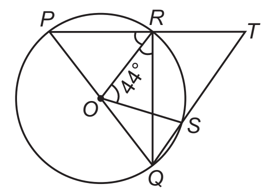
(*a*) 30° (*b*) 45°
(*c*) 50° (*d*) 40°

14. In the given figure, *O* is the centre of incircle for Δ *PQR*. Find $\angle QOR$ if $\angle QPR = 40°$.



(*a*) 140° (*b*) 110°
(*c*) 80° (*d*) 120°

15. In the given figure, *PQ* is the diameter of the circle whose centre is at *O*. If $\angle ROS = 44°$ and *OR* is the bisector of $\angle PRQ$, then what is the value of $\angle RTS$?



(*a*) 46° (*b*) 64°
(*c*) 69° (*d*) None of these **(*CDS 2010*)**

Level-2

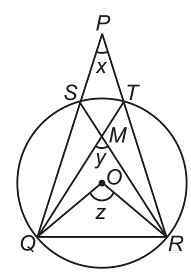
16. *ACB* is a tangent to a circle at *C* . *CD* and *CE* are chords such that $\angle ACE > \angle ACD$. If $\angle ACD = \angle BCE = 50°$, then

(*a*) $CD = CE$
(*b*) *ED* is not parallel to *AB*
(*c*) *ED* passes through the centre of the circle
(*d*) Δ *CDE* is right angled triangle.

17. Two circles intersect each other at *O* and *P*. *AB* is a common tangent to the circles. Then the angles subtended by the line segment *AB* at *O* and *P* are:

(*a*) complementary (*b*) supplementary
(*c*) equal (*d*) None of these

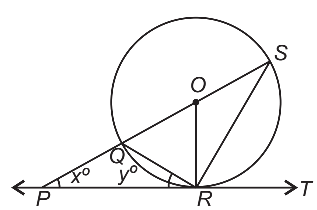
18. *O* is the centre of the given circle. Then $\angle x + \angle y$ equals



(*a*) $2\angle z$ (*b*) $\frac{\angle z}{2}$

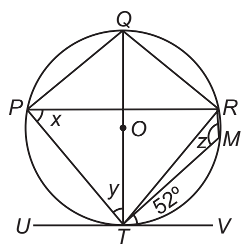
(*c*) $\angle z$ (*d*) None of these

19. In the given figure, *PT* touches the circle whose centre is *O*, at *R*. Diameter *SQ* when produced meets *PT* at *P*. If $\angle SPR = x°$, $\angle QRP = y°$, then $x + 2y =$



(*a*) 100° (*b*) 120°
(*c*) 80° (*d*) 90°

20. In the given figure, *O* is the centre of the circle. The line *UTV* is a tangent to the circle at *T*. $\angle VTR = 52°$ and Δ *PTR* is an isosceles triangle such that $TP = TR$. What is $\angle x + \angle y + \angle z$ equal to?



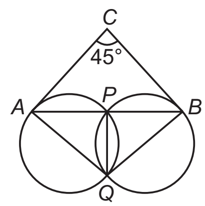
(*a*) 175° (*b*) 208°
(*c*) 218° (*d*) 250° **(*CDS 2009*)**

21. *A, B, C, D* are four distinct points on a circle whose centre is at *O*. If $\angle OBD - \angle CDB = \angle CBD - \angle ODB$, then what is $\angle A$ equal to?

(*a*) 45° (*b*) 60° (*c*) 120° (*d*) 135°

(*CDS 2009*)

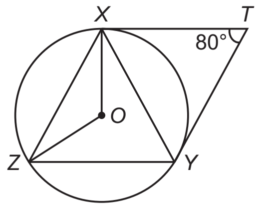
22. *PQ* is a common chord of two circles. *APB* is a secant line joining points *A* and *B* on the two circles. Two tangents *AC* and *BC* are drawn. If $\angle ACB = 45°$, then what is $\angle AQB$ equal to?



(*a*) 75° (*b*) 90°

(*c*) 120° (*d*) 135° **(*CDS 2009*)**

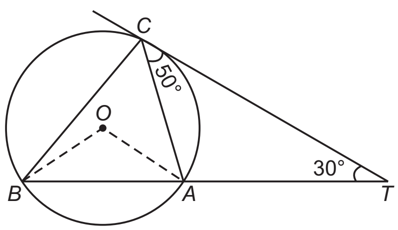
23. In the given figure, *O* is the centre of the circumcircle of Δ *XYZ*. Tangents at *X* and *Y* intersect at *T*. $\angle XTY = 80°$, what is the value of $\angle ZXY$



(*a*) 20° (*b*) 40°

(*c*) 60° (*d*) 80° **(*CDS 2007*)**

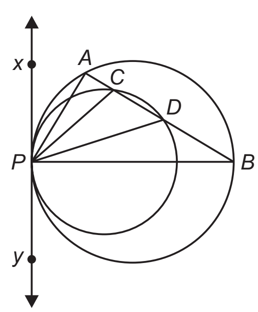
24. In the figure given below (not drawn to scale) *A, B* and *C* are three points on a circle with centre *O*. The chord *BA* is extended to a point *T* such that *CT* becomes a tangent to the circle at point *C*. If $\angle ATC = 30°$ and $\angle ACT = 50°$, then $\angle BOA$ is



(*a*) 100° (*b*) 150° (*c*) 80°

(*d*) not possible to determine **(*CAT 2003*)**

25. Two circles touch internally at point *P* and a chord *AB* of the circle of longer radius intersects the other circle in *C* and *D*. Which of the following holds good?



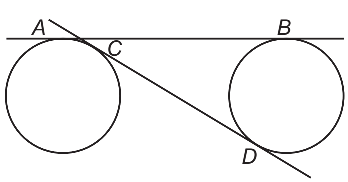
(*a*) $\angle CPA = \angle DPB$

(*b*) $2\angle CPA = \angle CPD$

(*c*) $\angle APX = \angle ADP$

(*d*) $\angle BPY = \angle CPD + \angle CPA$

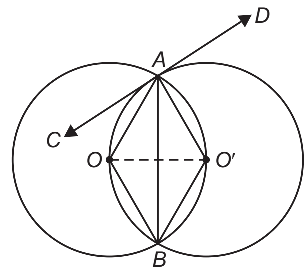
26. If two equal circles of radius 5 cm have two common tangents *AB* and *CD* which touch the circle on *A, C* and *B, D* respectively and if *CD* = 24 cm, find the length of *AB*.



(*a*) 27 cm (*b*) 25 cm

(*c*) 26 cm (*d*) 30 cm

27. Two circles *C* (*O*, *r*) and *C* (*O*′, *r*′) intersect at two points *A* and *B*. *O* lies on *C* (*O*′, *r*′). A tangents *CD* is drawn to the circle *C* (*O*′, *r*′) at *A*. Then,



(*a*) $\angle OAC = \angle OAB$

(*b*) $\angle OAB = \angle AO'O$

(*c*) $\angle AO'B = \angle AOB$

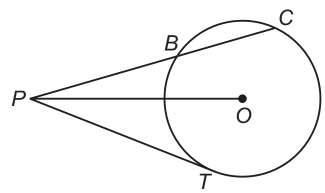
(*d*) None of these

28. *ABC* is an equilateral triangle inscribed in a circle with *AB* = 5 cm. Let the bisector of angle *A* meet *BC* in *X* and the circle in Y. What is the value of *AX.AY* ?

(*a*) 16 cm^2 (*b*) 20 cm^2 (*c*) 25 cm^2 (*d*) 30 cm^2

(*CDS 2011*)

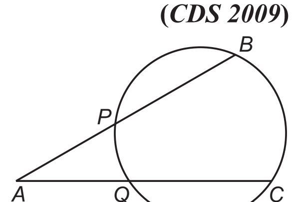
29. In the given figure, *PT* is a tangent to a circle of radius 6 cm. If *P* is at a distance of 10 cm from the centre *O* and *PB* = 5 cm, then what is the length of chord *BC*?



(*a*) 7.8 cm (*b*) 8 cm (*c*) 8.4 cm (*d*) 9 cm

(*CDS 2009*)

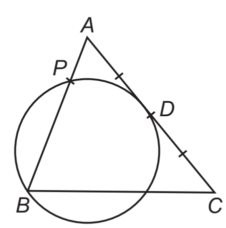
30. In the given figure, *AP* = 3 cm, *PB* = 5 cm, *AQ* = 2 cm and *QC* = *x*. What is the value of *x*?



(*a*) 6 cm (*b*) 8 cm

(*c*) 10 cm (*d*) 12 cm

31. In a Δ*ABC*, *AB* = *AC*. A circle through *B* touches *AC* at *D* and intersects *AB* at *P*. If *D* is the mid-point of *AC*, which one of the following is correct?



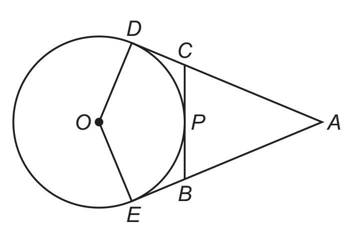
(*a*) *AB* = 2*AP*

(*b*) *AB* = 3*AP*

(*c*) *AB* = 4*AP*

(*d*) 2*AB* = 5*AP* **(*CDS 2007*)**

32. In the given circle, *O* is the centre of the circle and *AD, AE* are the two tangents. *BC* is also a tangent, then:



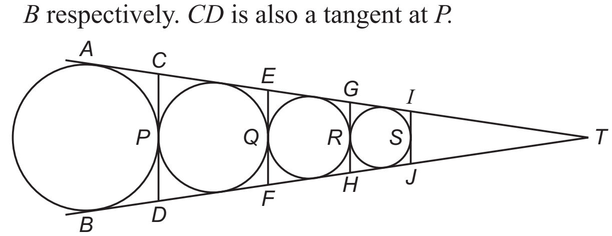
(*a*) *AC* + *AB* = *BC*

(*b*) 3*AE* = *AB* + *BC* + *AC*

(*c*) *AB* + *BC* + *AC* = 4*AE*

(*d*) 2*AE* = *AB* + *BC* + *AC*

33. In the given figure, *AT* and *BT* are the two tangents at *A* and *B* respectively. *CD* is also a tangent at *P*.



There are some more circles touching each other and the tangents *AT* and *BT* also. Which one of the following is true?

(*a*) $PC + CT = PD + DT$ (*b*) $RG + GT = RH + HT$

(*c*) $PC + QE = CE$ (*d*) All of these

34. Two circles with radii '*a*' and '*b*' respectively touch each other externally. Let '*c*' be the radius of a circle that touches these two circles as well as a common tangent to the circles. Then,

(*a*) $\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}} = \frac{1}{\sqrt{c}}$ (*b*) $\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}} = \frac{-1}{\sqrt{c}}$

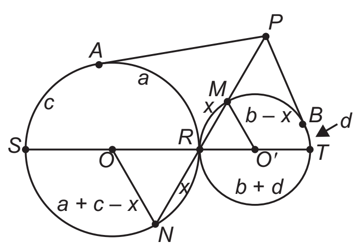
(*c*) $\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} = \frac{1}{\sqrt{c}}$ (*d*) None of these

35. Triangle *PAB* is formed by three tangents to circle *O* and $\angle APB = 40°$. Then $\angle AOB$ equals

(*a*) 45° (*b*) 60° (*c*) 70° (*d*) 55°

Level-3

36. In the given figure, *PA* is tangent to semi-circle *SAR*. *PB* is tangent to semi-circle *RBT*; *SRT* is a straight line, the lengths of the arcs are indicated in the figure Angle *APB* is measured by



(*a*) $\frac{1}{2}(a-b)$ (*b*) $a+b$ (*c*) $\frac{1}{2}(a+b)$ (*d*) $(a-b)$

37. Let *ABCD* be a quadrilateral in which *AB* is parallel to *CD* and perpendicular to *AD*, $AB = 3\ CD$, the area of quadrilateral is 4 sq. unit. If a circle can be drawn touching all the sides of the quadrilateral, then the radius of the circle is

(*a*) 2 units (*b*) $\sqrt{3}$ units (*c*) $\frac{\sqrt{3}}{2}$ units (*d*) $2\sqrt{3}$ units

(*RMO 2006*)

38. Two fixed circles in a plane intersect in points *P* and *Q*. A variable line through *P* meets the circles again in *A* and *B*. Prove that the angle *AQB* is of constant measure.

39. Let *A* be one of the two points of intersection of two circles with centre *X, Y* respectively. The tangents at *A* to the two circles meet the circles again at *B, C*. Let the point *P* be located so that *PXAY* is a parallelogram. Show that *P* is also the circumcentre of ΔABC.

40. Let *ABC* be a triangle and a circle *C'* be drawn lying inside the triangle, touching its incircle *C* externally and also touching the two sides *AB* and *AC*. Show that the ratio of the radii of the two circles *C'* and *C* is equal to $\tan^2\left(\frac{\pi-2}{4}\right)$.

41. Three circles touch each other externally and all the three touch a line. If two of them are equal and the third has radius 4 cm. Find the radius of the equal circles.

42. *ABC* is an equilateral triangle inscribed in a circle. *P* is any point on the minor arc *BC*. Prove that $PA = PB + PC$.

43. Let ΔABC be equilateral. On side *AB* produced, we choose a point *P* such that *A* lies between *P* and *B*. We now denote α as the lengths of sides of ΔABC; r_1 as the radius of incircle of ΔPAC and r_2 as the exradius of ΔPBC with respect to side *BC*. Then prove that $r_1 + r_2$ equals $\frac{\alpha\sqrt{3}}{2}$.

(*Austrain Polish Mathematics Comptt.*)

44. Let *ABCD* be a cyclic quadrilateral and let *P* and *Q* be points on the sides *AB* and *AD* respectively, such that $AP = CD$ and $AQ = BC$. Let *M* be the point of intersection of *AC* and *PQ*. Then, show that *M* is the midpoint of *PQ*.

(*Australian Mathematical Olympiad*)

45. Two disjoint circles C_1 and C_2 with centers O_1 and O_2 are given. A common exterior tangent touches circles C_1 and C_2 at *A* and *B* respectively and O_1O_2 intersects circles C_1 and C_2 at points *C* and *D* respectively. Prove that:

(*a*) the points *A, B, C* and *D* are concyclic

(*b*) the straight lines *AC* and *BD* are perpendicular.

46. *ABC* is a triangle with $\angle A > \angle C$ and *D* is a point on *BC* such that $\angle BAD = \angle ACB$. The perpendicular bisectors of *AD* and *DC* intersect in the point *E*. Prove that $\angle BAE = 90°$.

47. Points *D* and *E* are given on the sides *AB* and *AC* of ΔABC in such a way that $DE \parallel BC$ and tangent to the incircle of ΔABC. Prove that $DE \leq \frac{1}{8}(AB + BC + CA)$

(*Italian Selection Test*)

48. Two circles intersect each other in points *M* and *N*. An arbitrary point *A* of the first circle, which is not *M* or *N*, is connected with *M* and *N*, and the straight lines *AM* and *AN* intersect the second circle again in the points *B* and *C*. Prove that the tangent to the first circle at *A* is parallel to the straight line *BC*. **(*Swiss Mathematical Test*)**

ANSWERS

1. (*c*)	**2.** (*c*)	**3.** (*a*)	**4.** (*c*)	**5.** (*b*)	**6.** (*c*)	**7.** (*c*)	**8.** (*b*)	**9.** (*a*)	**10.** (*b*)
11. (*d*)	**12.** (*c*)	**13.** (*a*)	**14.** (*b*)	**15.** (*d*)	**16.** (*a*)	**17.** (*b*)	**18.** (*c*)	**19.** (*d*)	**20.** (*c*)
21. (*b*)	**22.** (*d*)	**23.** (*d*)	**24.** (*a*)	**25.** (*a*)	**26.** (*c*)	**27.** (*a*)	**28.** (*c*)	**29.** (*a*)	**30.** (*c*)
31. (*c*)	**32.** (*d*)	**33.** (*d*)	**34.** (*c*)	**35.** (*c*)	**36.** (*b*)	**37.** (*c*)			

HINTS AND SOLUTIONS

1. Given $OA = 3$ cm $\Rightarrow OC = 3$ cm (*radii of the circle*)

Also $AC = 3$ cm $\Rightarrow OA = OC = AC \Rightarrow \Delta AOC$ is equilateral

$\Rightarrow \angle AOC = 60°$.

$\Rightarrow \angle ABC = \frac{1}{2} \times \angle AOC = 30°$ (*Angle subtended by an arc of a circle at the centre is double the angle subtended by it at any point on the remaining part of the circle.*)

2. $\angle CAB = \angle CDB = 42°$

($\angle$*s in the same segment are equal*)

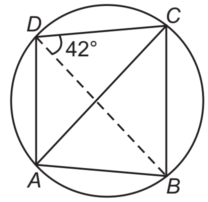
$\angle ABC = 90°$

($\angle$ *in a semicircle = 90°*)

$\therefore$ In ΔABC,

$\angle ACB = 180° – (\angle CAB + \angle ABC)$

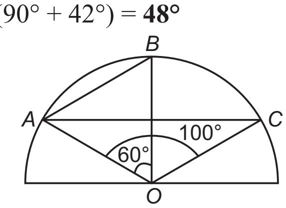
$= 180° – (90° + 42°) = \mathbf{48°}$



3. $\angle BOC = \angle AOC – \angle AOB$

$= 100° – 60° = 40°$

$\angle BAC = \frac{1}{2} \times \angle BOC = 20°$



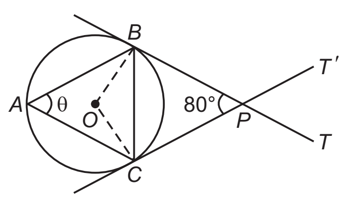
(*Angle subtended by an arc at the centre is double the angle subtended by it at any point on the remaining part of circle.*)

4. Join BO and OC.

In quadrilateral $BOCP$,

$\angle OBP = \angle OCP = 90°$

(*Tangent at any point of a circle is perpendicular to the radius through the point of contact*)



$\angle BPC = 80°$

$\Rightarrow \angle BOC = 360° – (180° + 80°) = 100°$

$\Rightarrow \angle BAC = \theta = \frac{1}{2} \times \angle BOC = 50°$

5. $\angle TSR = \angle TRQ = 40°$ (*Angles in alternate segment are equal*)

ST being the diameter, $\angle SRT = 90°$ (*Angle in a semi-circle*)

$\therefore \angle RTS = 180° – (\angle TSR + \angle SRT)$

$= 180° – (130°) = \mathbf{50°.}$

6. $OA = OB \Rightarrow \angle OBA = \angle OAB = 32°$ (*Isosceles Δ property*)

$\angle AOB = 180° – (\angle OBA + \angle OAB) = 180° – 64° = 116°$

$\Rightarrow \angle ACB = \frac{1}{2} \times \angle AOB) = \frac{1}{2} \times 116° = 58°$

Also, $\angle BAS = x = \angle ACB = 58°$(*Angle in alternate segment are equal*)

7. In ΔDEF, $\angle EDF = 180° – (60° + 75°)$

$= 180° – 135° = 45°$

$\angle OAD = \angle OBD = 90°$ (*Tangents DE and DF are perpendicular to radii OA and OB respectively at A and B*)

$\therefore$ In quad . $DAOB$, $\angle AOB = 360° – (90° + 90° + 45°)$

$= 360° – 225° = 135°$.

8. $\angle DAB = \angle BDQ = 48°$ (*Angles in alternate segment are equal*)

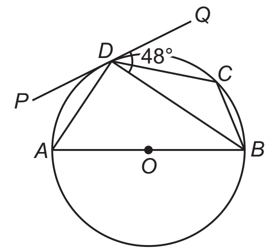
$\angle ADB = 90°$

(*Angle in a semi-circle*)

$\therefore \angle ABD = 180° – (\angle DAB + \angle ADB)$

$= 180° – (48° + 90°) = 42°$

$\therefore \angle DCB = 180° – \angle DAB = 180° – 48° = 132°$



(*opp.* $\angle$*s of a cyclic quad are supp.*)

$\therefore \frac{\angle DBA}{\angle DCB} = \frac{42}{132} = \mathbf{\frac{7}{22}}$.

9. $\angle OCD = 90°$

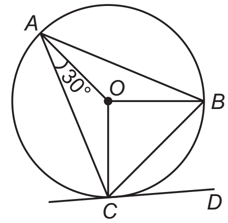
(*Tangent CD* $\perp$ *Radius OC*)

$\angle OCA = \angle OAC = 30°$

(*OA = OC, radii*)

$\angle ACD = \angle OCD + \angle OCA = 90° + 30°$

$= 120°$



$\angle BAC = 180° – 120° = 60°$

(Given $\angle ACD$ and $\angle BAC$ are supp.)

$\Rightarrow \angle BCD = \angle BAC = 60°$

(*Angles in alternate segment are equal*)

$\therefore \angle OCB = \angle OCD – \angle BCD = 90° – 60° = \mathbf{30°.}$

10. $\angle ADB = \frac{1}{2} \times \angle AOB = 50°$

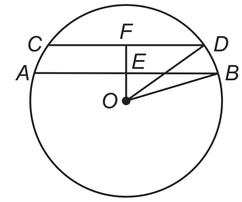
In ΔDPA, $\angle ADP + \angle DAP + \angle DPA = 180°$

$\Rightarrow \angle DPA = 180° – (50° + 30°) = 100°$

Also, DPB being a straight line,

$\angle APB = 180° – \angle DPA = 180° – 100° = \mathbf{80°.}$

11. Let AB and CD be chords of lengths 32 cm and 24 cm in a circle with centre O, on the same side of the centre.



Then $OA = OD = 20$ cm

Let the perpendicular from the centre intersect the chords AB and CD at E and F respectively. Then, E and F are the midpoints of AB and CD respectively. (Perpendicular from the centre of the circle to a chord bisects the chord.)

Now in rt. ΔOFD, $OF = \sqrt{OD^2 – FD^2} = \sqrt{20^2 – 12^2}$

$\Rightarrow \quad OF = \sqrt{400 – 144} = \sqrt{256} = 16$ cm

In rt. ΔOEB, $OE = \sqrt{OB^2 – EB^2} = \sqrt{20^2 – 16^2}$

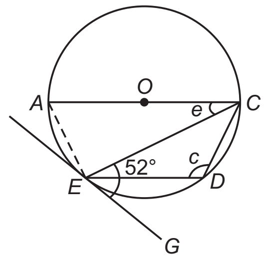
$= \sqrt{400 – 256} = \sqrt{144} = \mathbf{12\ cm.}$

$\therefore$ Required distance $= EF = OF – OE = (16 – 12)$ cm $= 4$ cm.

So the option containing value 4 is correct. The other required distance is 28 cm when the chords lie on the opposite side of centre O.

12. $\angle OAE = \angle GEC = 52°$

(*Angles in alternate segments are equal*)



$\angle AEC = 90°$

(*Angle in a semi circle*)

$\therefore$ In $\Delta\ ACE$, $e = \angle ACE$

$= 180° – (90° + 52°) = 38°$

ACDE is a cyclic quadrilateral

$\Rightarrow c = \angle EDC = 180° – \angle CAE = 180° – 52° = 128°$

(*opp. ∠s of a cyclic quad. are supp.*)

$\therefore c + e = 128° + 38° =$ **166°.**

13. In $\Delta\ ADP$, ext. $\angle ADC$ = Int. $\angle s$ $(\angle A + \angle P)$

$= 40° + 20° = 60°.$

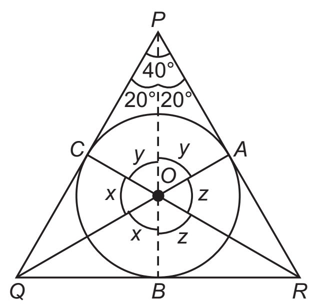
$\angle ABC = \angle ADC = 60°$ (*Angles in the same segment*)

$\because$ *AD* is the diameter, $\angle ABD = 90°$

$\therefore \angle DBC = \angle ABD – \angle ABC = 90° – 60° =$ **30°.**

14. *PO* is joined.

Since the circle is the incircle for $\Delta\ ABC$, *PO, QO, RO* are the angle bisector of $\angle P$, $\angle Q$ and $\angle R$ respectively.



$\angle CPO = \angle APO$

$= \frac{1}{2} \times 40° = 20°$

Also, $OA \perp PR$, $OC \perp PQ$, $OB \perp QR$

(*Radii ⊥ tangent at point of contact*)

$\Rightarrow \angle OAP = 90°$

$\Rightarrow \angle AOP = y = 180° – (90° + 20°) = 70°$

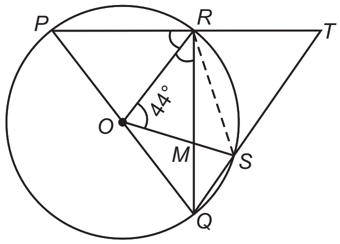
Now, $y + y + x + x + z + z = 360°$

$\Rightarrow 2y + 2(x + z) = 360°$

$\Rightarrow 2(x + z) = 360° – 2y = 360° – 140° = 220°$

$\Rightarrow x + z = 110°$ $\Rightarrow \angle QOR =$ **110°.**

15. Since *OR* is the bisector of $\angle PRQ$



$\angle PRO = \angle ORQ = 45°$

($\because$ *∠PRQ = 90° ∠ in a semicircle*)

Also, $OP = OR$ (radii)

$\therefore \angle OPR = \angle ORP = 45°$

In $\Delta\ ORS$, $OR = OS \Rightarrow \angle ORS = \angle OSR = \frac{180° – 44°}{2} = 68°$

$\therefore \angle MRS = 68° – 45° = 23°$

$\Rightarrow \angle PRS = 90° + 23° = 113°$

$\angle PRS + \angle PQS = 180°$

$\Rightarrow \angle PQS = 180° – \angle PRS = 180° – 113° = 67°$

(*opp. ∠s of cyclic quad. PQSR*)

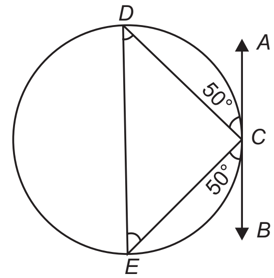
In $\Delta\ PTQ$, $\angle PTQ = 180° – (\angle QPT + \angle PQT)$

$= 180° – (45° + 67°) = 68°$

$\Rightarrow \angle RTS = \angle PTQ =$ **68°.**

16. $\angle ACD = \angle CED = 50°$

(*Alternate Segment Theorem*)



$\angle BCE = \angle CDE = 50°$

$\Rightarrow \angle D = \angle E = 50° \Rightarrow CD = CE$

Also, $\angle ACD = \angle D = 50°$, but these are alternate interior angles

$\Rightarrow ED \parallel AB$

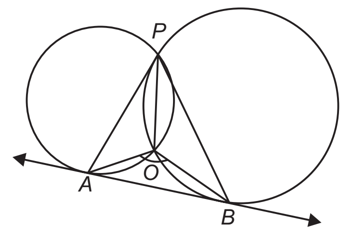
$\angle DCE = 180° – (50° + 50°) = 80°$

$\therefore \Delta CDE$ is an acute angled triangle.

17. $\angle OAB = \angle OPA$

$\angle OBA = \angle OPB$

{*Angles in alternate segment are equal.*}



$\therefore \angle OAB + \angle OBA$

$= \angle OPA + \angle OPB$

$= \angle APB.$

In ΔAOB, $\angle AOB$

$= 180° – (\angle OAB + \angle OBA) = 180° – \angle APB$

$\Rightarrow \angle AOB + \angle APB = 180°.$

i.e., $\angle s$ *AOB* and *APB* are supplementary

18. $\angle QSR = \angle QTR = \frac{1}{2} \times \angle QOR = \frac{z}{2}$

(*Angle subtended by an arc of a circle at the centre is double the angle subtended by it at any point on the remaining part of the circle*)

$\angle PSM = \angle PTM = 180° – \frac{z}{2}$ (Straight line)

Also, $\angle SMT = \angle QMR = y$ (vert. opp. $\angle s$)

$\therefore$ In quad. *PSMT*, $\angle SMT + \angle PTM + \angle TPS + \angle PSM = 360°$

$\Rightarrow y + 180° – \frac{z}{2} + x + 180° – \frac{z}{2} = 360°$

$\Rightarrow$ $\boldsymbol{x + y = z}$**.**

19. $\angle QSR = \angle QRP = y°$ (*Angles in alternate segment are equal*)

Also, $\angle QRS = 90°$ (*Angle in a semi-circle*)

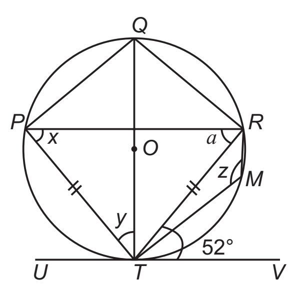
$\angle PRS = \angle PRQ + \angle QRS = y° + 90°$

In $\Delta\ PRS$, $\angle SPR + \angle PRS + \angle PSR = 180°$

$\Rightarrow x° + y° + 90° + y° = 180°$

$\Rightarrow x + 2y =$ **90°.**

20. $x = \angle VTR = 52°$ (*Angles in alternate segment are equal*)



$x + z = 180°$

(*PTMR is a cyclic quad.*)

$\Rightarrow z = 180° – x = 180° – 52°$

$= 128°.$

In $\Delta\ PTR$, $PT = TR \Rightarrow a = x$

$= 52°$

$\Rightarrow \angle PTU = a = 52°$ (*Angles in alternate segment are equal*)

$\angle QTU = y + \angle PTU$

$\Rightarrow y + \angle PTU = 90°$ ($\because \angle QTU = 90°$, rad. $OT \perp$ tangent UV)

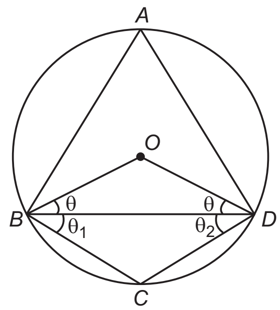
$\Rightarrow y = 90° - 52° = 38°$

$\therefore x + y + z = 52° + 38° + 128° =$ **218°.**

21. Given, $\angle OBD - \angle CDB$

$= \angle CBD - \angle ODB$

$\Rightarrow \angle OBD + \angle ODB = \angle CBD + \angle CDB$ ...(*i*)



$\because$ $OB = OD$ (radii)

$\angle OBD = \angle ODB = \theta$ (say)

Let $\angle CBD = \theta_1$, $\angle CDB = \theta_2$

Then putting these value in eqn (*i*), we have

$\theta + \theta = \theta_1 + \theta_2 \Rightarrow 2\theta = \theta_1 + \theta_2$ ...(*ii*)

Also, $\angle BOD = 180° - 2\theta$

$\Rightarrow$ Reflex $\angle BOD = 360° - (180° - 2\theta)$

$\Rightarrow \angle BCD = \frac{1}{2} \times$ Reflex $\times \angle BOD$

$= \frac{1}{2} \times [360° - (180° - 2\theta)]$ (*Angle subtended at centre by an arc = 2 × Angle subtended at any point on remaining part of the circle*)

Also $\angle BCD = 180° - (\theta_1 + \theta_2)$

$\therefore 180° - (\theta_1 + \theta_2) = \frac{360° - (180° - 2\theta)}{2}$

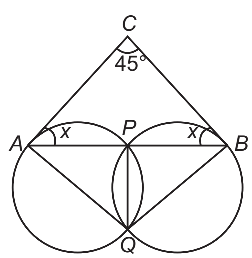
$\Rightarrow 180° - 2\theta = 90° + \theta$ ($\because \theta_1 + \theta_2 = 2\theta$)

$\Rightarrow 3\theta = 90° \Rightarrow \theta = 30°$

$\therefore \angle BOD = 180° - 60° = 120°$

$\Rightarrow \angle BAD = \frac{1}{2} \times \angle BOD =$ **60°.**

22. Since the tangents drawn on the two given circles, from the same external point are equal, $CA = CB$



$\Rightarrow \angle CAB = \angle CBA = x$ (say)

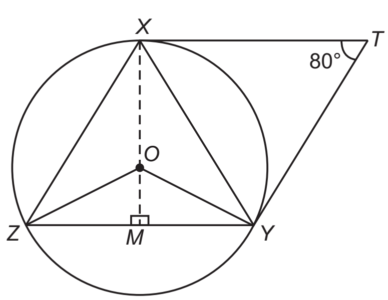
In Δ CAB, $45° + x + x = 180°$

$\Rightarrow 2x = 135° \Rightarrow x = 67\frac{1°}{2}$

$\angle AQP = \angle BQP = x = 67\frac{1°}{2}$ (*Alternate Segment Theorem*)

$\therefore \angle AQB = \angle AQP + \angle BQP = 67\frac{1°}{2} + 67\frac{1°}{2} =$ **135°.**

23. Given, $\angle XTY = 80°$



$TX = TY$ (*Tangents from the same external point are equal*)

$\Rightarrow \angle TXY = \angle TYX$

$= \frac{1}{2}(180° - \angle XTY)$

$= \frac{1}{2}(180° - 80°) = 50°$

$OX \perp XT$ (*radii* $\perp$ *tangent at point of contact*)

$\Rightarrow \angle OXT = 90° \Rightarrow \angle OXY = \angle OXT - \angle TXY = 90° - 50° = 40°$

Also, $OM \perp ZY$

$\therefore$ In ΔXMY, $\angle XYM = 180° - (\angle XMY + \angle MXY)$

$= 180° - (90° + 40°) = 50°$

Also, by alternate segment theorem,

$\angle XZY = \angle TXY = 50°$

$\therefore$ In ΔXZY, $\angle X = 180° - (\angle XZY + \angle XYZ)$

$= 180° - (50° + 50°) =$ **80°.**

24. In Δ ATC, $\angle CAT = 180° - (\angle ACT + \angle ATC)$

$= 180° - (50° + 30°) = 100°$.

Also, $\angle CBA = \angle ACT = 50°$ (Alternate Segment Theorem)

$\therefore$ ext $\angle CAT$ = int. opp. $\angle s$ ($\angle CBA + \angle BCA$)

$\Rightarrow 100° = 50° + \angle BCA \Rightarrow \angle BCA = 50°$

$\Rightarrow \angle BOA = 2 \times \angle BCA =$ **100°.**

25. In the bigger circle, $\angle APX = \angle ABP$

In the smaller circle, $\angle CPX = \angle PDC$

{*Angles in alternate segment are equal.*}

$\Rightarrow \angle APX + \angle CPA = \angle CPX = \angle PDC$

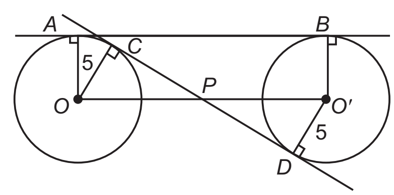
$\Rightarrow \angle ABP + \angle CPA = \angle PDC$ ($\because \angle APX = \angle ABP$)

$\Rightarrow \angle ABP + \angle CPA = \angle DBP + \angle DPB$ (*ext.* $\angle$ *theorem in* Δ PDB)

$= \angle ABP + \angle DPB$

$\Rightarrow$ $\boldsymbol{\angle CPA = \angle DPB.}$

26. Hint



Refer to diagram: $\Delta OCP \cong \Delta O'BP \Rightarrow OP = O'P$

Let $CP = x$. Then, $PD = 24 - x$

$OP^2 = 5^2 + x^2$, $O'P^2 = 5^2 + (24 - x)^2$

$OP^2 = O'P^2 \Rightarrow 25 + x^2 = 25 + (24 - x)^2 \Rightarrow x = 12$

$\therefore OP^2 = 5^2 + 12^2 \Rightarrow OP = 13 = O'P$

$\therefore AB = OO' = OP + PO' = 13 + 13 =$ **26 cm.**

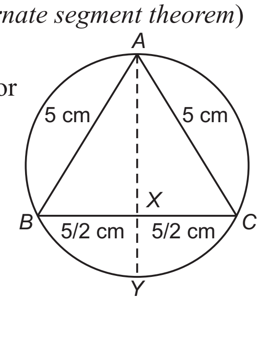
27. In Δ AOB, $OA = OB \Rightarrow \angle OBA = \angle OAB$

(*Isosceles* Δ *property*)

Also, $\angle OAC = \angle OBA$ (*Alternate segment theorem*)

$\Rightarrow \angle OAC = \angle OAB$.

28. $\because$ In an equilateral Δ; angle bisector AX bisects the base BC at X.



$\therefore BX = CX = \frac{5}{2}$ cm

$AX = \sqrt{5^2 - (5/2)^2}$

$= \sqrt{25 - \frac{25}{4}} = \sqrt{\frac{75}{4}} = \frac{5\sqrt{3}}{2}$

AY and *BC* being the chords of the circle,

$AX \,.\, XY = BX \,.\, XC$

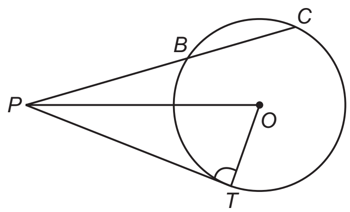
$\Rightarrow \frac{5\sqrt{3}}{2} \cdot XY = \frac{5}{2} \cdot \frac{5}{2}$

$\Rightarrow XY = \frac{5}{2\sqrt{3}}$

$\therefore AX \,.\, AY = \frac{5\sqrt{3}}{2} \cdot \left(\frac{5\sqrt{3}}{2} + \frac{5}{2\sqrt{3}}\right)$

$= \frac{75}{4} + \frac{25}{4} = \frac{100}{4}$ = **25 cm².**

29. Given $PO = 10$ cm, radius $OT = 6$ cm, $PB = 5$ cm



In rt. Δ *OTP*, ($\angle OTP = 90°$ *radius* $OT \perp$ *tangent PT*)

$PT = \sqrt{OP^2 - OT^2}$

$= \sqrt{100 - 36} = \sqrt{64} = 8$

$\therefore$ By Tangent – Secant Theorem

$PT^2 = PB \times PC$

$\Rightarrow \quad 8^2 = 5 \times (BC + PB)$

$\Rightarrow \quad 64 = 5\,(BC + 5) \Rightarrow 5BC = 39$

$\Rightarrow \quad BC$ = **7.8 cm.**

30. If two chords *PB* and *QC* intersect externally at a point *A*, then

$AB \times AP = AC \times AQ$

$\Rightarrow 8 \times 3 = (2 + x) \times 2$

$\Rightarrow 2 + x = 12 \Rightarrow x$ = **10 cm.**

31. Using the tangent-secant theorem, we have

$AB \times AP = AD^2 = \left(\frac{AC}{2}\right)^2 \quad (\because AD = DC)$

$\Rightarrow AB \times AP = \frac{1}{4} AC^2 = \frac{1}{4} AB^2 \quad (\because AB = AC)$

$\Rightarrow AB$ = **4 *AP*.**

32. Since the lengths of the tangents from the same external point are equal, $CD = CP$ and $BP = BE$.

Also, $AE = AD$

Now $AD = AC + CD = AC + CP$ ...(*i*)

$AE = AB + BE = AB + BP$ ...(*ii*)

$\therefore$ Adding eqns. (*i*) and (*ii*), we get

$AD + AE = AC + CP + AB + BP \quad (\because AD = AE)$

$\Rightarrow \quad 2AE = AC + AB + BC.$

33. Since the lengths of the tangents from the same external point are equal, $AT = BT$

$AC = PC$ and $BD = DP$

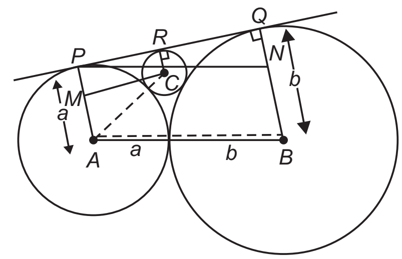
$\therefore AT = BT \Rightarrow TC + CA = TD + DB$

$\Rightarrow TC + PC = TD + PD$

Hence option (*a*) is true.

Similarly, we can prove the relations in option (*b*) and (*c*) for other circles also.

34. Let the centres of the three circles with radii *a, b, c* be *A, B* and *C* respectively. Let the common tangent touch the three circles at points, *P, Q* and *R* respectively.



Since radius $\perp$ tangent at point of contact

$\angle APR = \angle CRQ = \angle BQR = 90°$

Draw a line $CM \parallel PR$ meeting AP in *M*. Then, $\angle AMC = 90°$

$\therefore CM = PR$ and $MP = CR$ and $AM = AP - MP = a - c$ and $AC = a + c$

($\therefore$ In rt Δ*AMC*, $MC = \sqrt{AC^2 - AM^2} = \sqrt{(a+c)^2 - (a-c)^2}$

$= 2\sqrt{ac}$)

Similarly we can show that $RQ = 2\sqrt{bc}$

$\Rightarrow PQ = PR + RQ = 2\sqrt{ac} + 2\sqrt{bc}$ ...(*i*)

Also, draw a line from $P \parallel AB$ meeting *BQ* in *N*. Then,

$PN = AB = a + b,$

$QN = BQ - BN = b - a$

In rt. Δ *PQN*, $PQ = \sqrt{PN^2 - QN^2}$

$= \sqrt{(a+b)^2 - (a-b)^2} = 4ab$

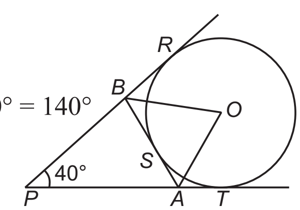
$\Rightarrow PQ = 2\sqrt{ab}$ ...(*ii*)

$\therefore$ From (*i*) and (*ii*)

$2\sqrt{ac} + 2\sqrt{bc} = 2\sqrt{ab}$

$\Rightarrow \boldsymbol{\frac{1}{\sqrt{c}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}}$**.**

35. $\angle P = 40°$



$\therefore \angle PAB + \angle PBA = 180° - 40° = 140°$

$\angle TAS = 180° - \angle PAB$

$\angle RBS = 180° - \angle PBA$

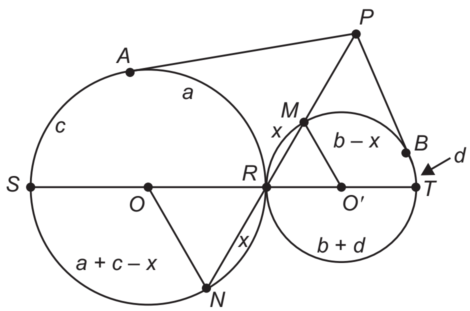
$\therefore \angle TAS + \angle RBS = 360° - (\angle PAB + \angle PBA) = 360° - 140° = 220°$

Since *OA* and *OB* bisect angles *TAS* and *RBS* respectively.

$\angle OAS + \angle OBS = \frac{1}{2} \times 220° = 110°$

$\therefore \angle AOB = 180° - 110°$ = **70°.**

36. First, draw the line connecting *P* and *R* and denote its other inter-sections with the circles by *M* and *N*; see accompanying figure. The arcs *MR* and *NR* contain the same number of degrees; so we may denote each arc by *x*. To verify this, note that we have two isosceles triangle with a base angle of one equal to a base angle of the other. $\therefore \angle NOR = \angle MOR$.



$\angle APR = \frac{1}{2}\ \{(c + a + c - x) - a\} = \frac{1}{2}\ \{2c - x\}$

$\angle BPR = \frac{1}{2}\ \{b + d + d - (b - x)\} = \frac{1}{2}\ \{2d + x\}$

and the sum of angles *APR* and *BPR* is

$\angle BPA = c + d$

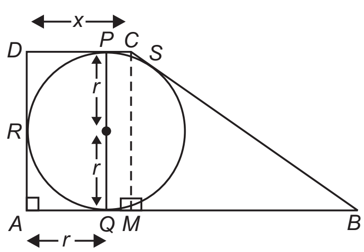
The desired angle is

$360° - \angle BPA = 360° - (c + d)$

$= (180° - c) + (180° - d)$

$= a + b.$

37. Let the radius of the circle drawn inside the quadrilateral *ABCD* be *r*.



$\because AB \parallel CD, \therefore ABCD$ is a trapezium.

Let $CD = x$, then

$AB = 3\ CD = 3x$

Draw a perpendicular *CM* from *C* on *AB*.

Area of a trapezium = $\frac{1}{2}$ (sum of parallel sides) × height

$\Rightarrow \frac{1}{2} \times (AB + CD) \times AD = 4$ (Area = 4 sq. units, given)

$\Rightarrow \frac{1}{2} \times (3x + x) \times 2r = 4$

$\Rightarrow 4xr = 4 \Rightarrow x = \frac{1}{r}$

As all the sides *i.e., AB*, *BC, CD* and *DA* touch the incircle,

$DC + AB = DP + PC + AQ + QB$

$= DR + CS + AR + SB$ $\left[\begin{array}{l}\textit{Using, tangents from the}\\ \textit{same external point are equal.}\end{array}\right]$

$= DR + AR + CS + SB$

$= AD + BC$

$\Rightarrow x + 3x = 2r + \sqrt{(2r)^2 + (2x)^2}$

($\because \angle M = 90°$, $CM = PQ = 2r$, $MB = AB - CD = 2x$)

$\Rightarrow 4x = 2r + 2\sqrt{r^2 + x^2}$

$\Rightarrow 2x - r = \sqrt{r^2 + x^2}$

On squaring both the sides, we have

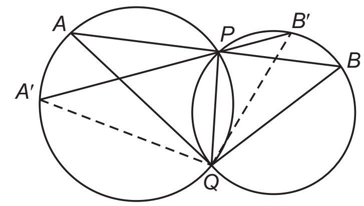
$4x^2 - 4rx + r^2 = r^2 + x^2$

$\Rightarrow 3x^2 = 4xr$

$\Rightarrow \frac{3}{r^2} = \frac{4}{r} \times r$ $\left(\because\ x = \frac{1}{r}\right)$

$\Rightarrow r^2 = \frac{3}{4} \quad \Rightarrow r = \frac{\sqrt{3}}{2}$ **units.**

38. Let the line *A′B′* be another line through *P* meeting the circles in *A′* and *B′*.



Given, *APB* is a line through *P* meeting the circle in *A* and *B* respectively.

$\angle PAQ = \angle PA'Q$ {Angles in the same segment are equal}

$\angle PBQ = \angle PB'Q$

Now in Δ *AQB*,

$\angle AQB = 180° - (\angle QAB + \angle QBA)$

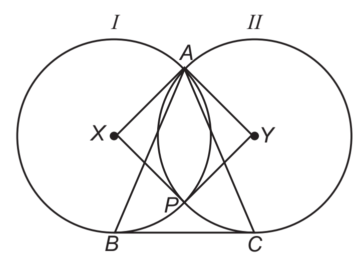
$= 180° - (\angle PAQ + \angle QBP)$

$= 180° - (\angle PA'Q + \angle PB'Q)$

$= \angle A'Q\,B'$

∴ ∠*AQB* is the same for all lines *APB*. Thus, ∠*AQB* is a constant angle.

39. To prove that *P* is the circumcentre of Δ*ABC,* we shall show that *PX* and *PY* are the perpendicular bisector of *AB* and *AC* respectively.



Since *AB* is tangent to circle II and *YA* is the radius of circle II.

$YA \perp AB$

Also, *PXAY* is a parallelogram ⇒ $AY \parallel XP$

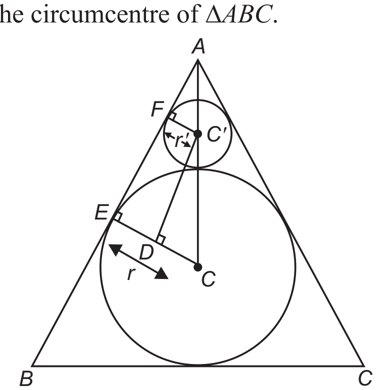
$\therefore AY \perp AB$ and $AY \parallel XP \Rightarrow XP \perp AB$

Since *X* is the centre of circle *I* and *AB* is a chord of circle *I,* and $XP \perp AB \Rightarrow XP$ bisects $AB \Rightarrow XP$ is the perpendicular bisector of *AB*

Similarly, we can show that *YP* is the perpendicular bisector of *AC.*

Since the perpendicular bisector of sides *AB* and *AC* of Δ *ABC* meet at *P, P* is the circumcentre of Δ*ABC*.

40. Let *C* be the incentre, *r* the inradius and *E* the point of contact of the incircle with *AB*. Let *C′* be the centre of the circle touching *AB, AC* and the incircle, *r′* the radius of this circle and *F* its point of contact with *AB*. Since *AB* and *AC* both touch



this circle, its centre must also lie on AC.

From C' draw $C'D \perp CE$. Then, in $\Delta C'CD$

$CD = r - r'$

$CC' = r + r'$

$\angle CDC' \Rightarrow \pi/2$ and $\angle DC'C = \angle EAC = A/2$

In $\Delta DCC' \Rightarrow \sin A/2 = \dfrac{CD}{CC'} = \dfrac{r - r'}{r + r'}$

$\Rightarrow \cos(\pi/2 - A/2) = \dfrac{r - r'}{r + r'}$

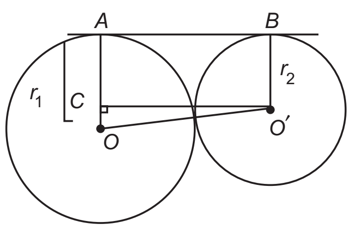
$\Rightarrow \dfrac{r - r'}{r + r'} = \cos\theta \quad$ where $\theta = \dfrac{\pi - A}{2}$

$\Rightarrow \dfrac{r}{r'} = \dfrac{1 - \cos\theta}{1 + \cos\theta}$ (on applying componendo and dividendo)

$\Rightarrow \dfrac{r}{r'} = \dfrac{2\sin^2\theta/2}{2\cos^2\theta/2} = \tan^2\theta/2$

$\Rightarrow \dfrac{r}{r'} = \mathbf{\tan^2 \dfrac{\pi - A}{4}}$.

41. Consider the condition: what is the length of the common tangent when two circles of radii r_1 and r_2 touch externally?

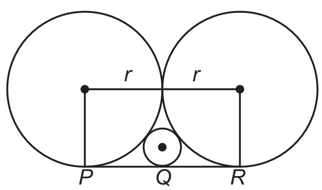


Here AB (the common tangent)

$= O'C = \sqrt{OO'^2 - OC^2}$

$= \sqrt{(r_1 + r_2)^2 - (r_1 - r_2)^2}$

$= \sqrt{4r_1 r_2} = 2\sqrt{r_1 r_2}$



Therefore, according to the given figure, PR is the length of the common tangent to circle of radii r and 4.

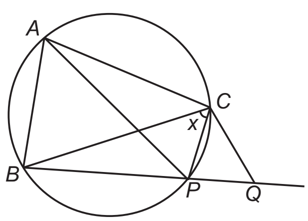
$\therefore PQ = 2\sqrt{4r} = 4\sqrt{r}$

$QR = 2\sqrt{4r} = 4\sqrt{r}$

$\because PR = PQ + QR$

$\therefore 2r = 4\sqrt{r} + 4\sqrt{r} \Rightarrow r = 4\sqrt{r} \Rightarrow r^2 = 16r \Rightarrow r =$ **16 cm.**

42. Given, ABC is an equilateral triangle and P is a point on the minor arc BC.



$\angle ABC = \angle BAC = \angle BCA = 60°$

Let $\angle BCP = x$

Produce BP to Q such that $PQ = PC$. Join CQ.

$\angle CPQ$ is the external angle of the cyclic quadrilateral $ABPC$.

$\because \angle CPQ = \angle BAC = 60°$.

$\because PC = PQ$, and $\angle CPQ = 60°$, therefore ΔCPQ is equilateral.

Consider the triangles ACP and BCQ.

$\angle ACP = 60 + x$, $\angle BCQ = 60 + x$

Now is Δs ACP and BCQ

$\angle ACP = \angle BCQ = 60 + x$ (Proved)

$\angle CAP = \angle CBP$ ($\angle CBQ$) (Angles in the same segment PC)

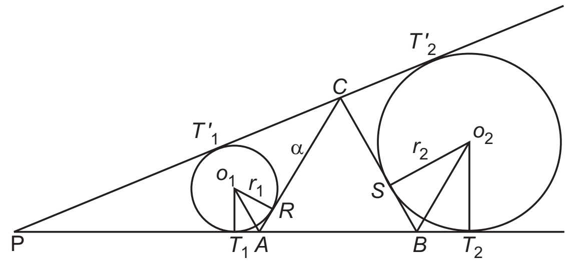
$AC = BC$ (sides of equilateral ΔABC)

$\therefore \Delta ACP \cong \Delta BCQ$ (ASA)

$\Rightarrow AP = BQ \Rightarrow AP = BP + PQ \Rightarrow \mathbf{AP = BP + PC}$

($\because PC = PQ$)

43. Let O_1 be the centre of the in-circle of ΔPAC and O_2 the centre of the circle which touches the triangle PBC on side BC. Let the tangents from P on these two circles touch them at points T_1, T_1' and T_2, T_2' respectively.



Looking at the figure, we see that $\angle T_1O_1R = 60°$ since each of $\angle s$ O_1T_1A and O_1RA being $= 90°$, it is the supplement of $\angle T_1 AR = 120°$ (as an exterior angle for ΔABC). Hence, $\angle AO_1 R = 30°$. Similarly, we obtain $\angle BO_2S = 30°$.

Since tangents drawn to a circle from an external point are equal, we have

$T_1 T_2 = T_1 A + AB + BT_2 = RA + AB + SB$

$= r_1 \tan 30° + \alpha + r_2 \tan 30° = \dfrac{r_1 + r_2}{\sqrt{3}} + \alpha,$

and

$T'_1 T'_2 = T'_1 C + CT'_2 = CR + CS = (\alpha - RA) + (\alpha - SB)$

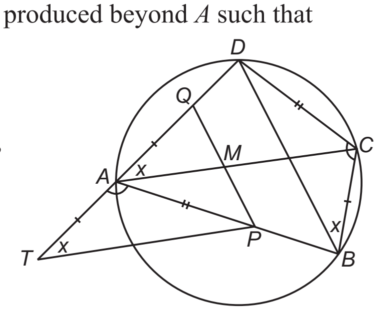
$= 2\alpha - \dfrac{r_1 + r_2}{\sqrt{3}}.$

Since common external tangents to two circles are equal, $T_1 T_2 = T'_1T'_2$. Hence,

$$\frac{r_1 + r_2}{\sqrt{3}} + \alpha = 2\alpha - \frac{r_1 + r_2}{\sqrt{3}},$$

Hence we find that, $r_1 + r_2 = \dfrac{\alpha\sqrt{3}}{2}$.

44. Let T be a point on AD produced beyond A such that $AT = BC$.



Since $AT = BC$, $AP = CD$ and $\angle TAP = \angle TAB = \angle BCD$, we get $\Delta ATP \equiv \Delta CBD$, so that

$\angle ATP = \angle CBD$.

Since $\angle CBD = \angle CAD$, we have

$\angle ATP = \angle CAD$.

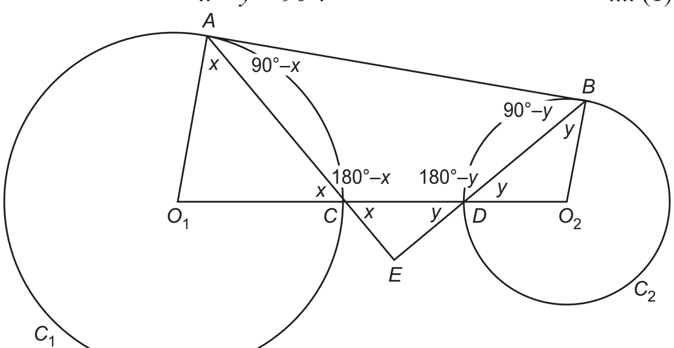
Thus, $TP \parallel AC$; that is, $TP \parallel AM$.

Hence, we get $PM : MQ = TA : AQ = BC : AQ = 1 : 1$.

Therefore, $PM = MQ$.

45. (*a*) Letting the base angles in isosceles triangles AO_1C and BO_2D be x and y, respectively, the sum of the angles in quadrilateral $ABDC$ is
$(90° – x) + (90° – y) + (180° – y) + (180° – x) = 360°$, and we have

$$x + y = 90°. \qquad .... (1)$$



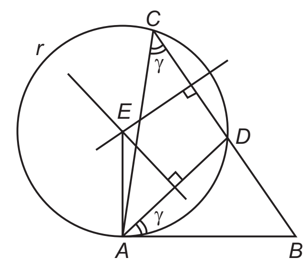
Hence, in $ABDC$, the angles at A and D add up to
$(90° – x) + (180° – y) = 270° – (x + y) = 270° – 90° =$ **180°**
and thus, $ABDC$ is cyclic. This proves (*a*).

(*b*) Let AC and BD when produced intersect at E. It follows from equation (1) that in triangle CED the angles at C and D add up to 90°. Thus, CED is a right-angled triangle with the right angle at E and AC and BD are in fact perpendicular.

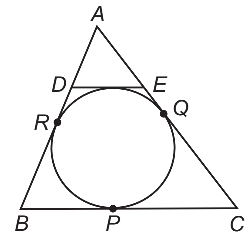
46. ∵ The perpendicular bisector of AD and DC intersect in point E

E is the circumcentre of ΔADC.

Since $\angle DAB = \angle ACD$ we have that AB is tangent to the circumcircle at A,
(Alternate Segment Theroem)
∵ radius $EA \perp$ tangent AB at point of contact A,
∴ $\angle BAE = 90°$.



47. We set $BC = a$, $CA = b$, $AB = c$, and $2s = a + b + c$. Let the incircle touch BC, CA, AB at P, Q, R, respectively. Since DE is parallel to BC, we have $\Delta ADE \sim \Delta ABC$. Thus,



$$\frac{AD + DE + AE}{AB + BC + AC} = \frac{DE}{BC} = \frac{DE}{a}.$$

Since $AD + DE + AE = AR + AQ = b + c – a$, we have

$$\frac{b + c - a}{a + b + c} = \frac{DE}{a};$$

whence, $DE = \dfrac{a\,(b + c - a)}{a + b + c}$. Then

$$\frac{1}{8}\ (AB + BC + CA) - DE$$

$$= \frac{a + b + c}{8} - \frac{a\,(b + c - a)}{a + b + c} = \frac{(a + b + c)^2 - 8a\,(b + c - a)}{8\,(a + b + c)}$$

$$= \frac{(b + c)^2 - 6a\,(b + c) + 9a^2}{8\,(a + b + c)} = \frac{(b + c - 3a)^2}{8\,(a + b + c)} \geq \mathbf{0.}$$

Thus, $\mathbf{\frac{1}{8}\ (AB + BC + CA) \geq DE.}$

48. Let AT be the tangent to the first circle at A. Then,

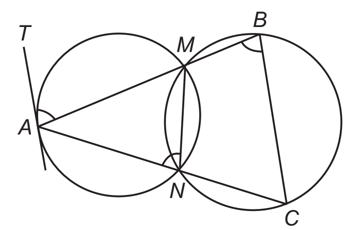
$\angle TAM = \angle ANM$

(*Angles in alternate segment are equal*)

$\Rightarrow \angle ANM = \angle MBC$

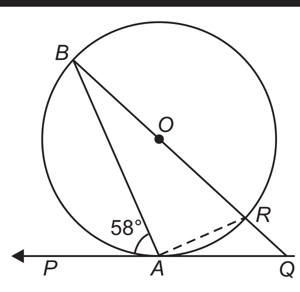
(*ext.* ∠ *= int. opp.* ∠ *in a cyclic quad.*)

we have $\angle TAB = \angle ABC \Rightarrow AT \parallel BC$. (alt. ∠*s*)



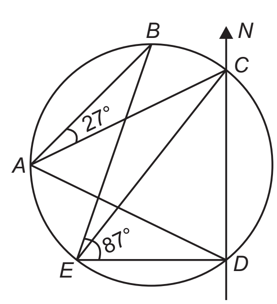
SELF ASSESSMENT SHEET

1. In the given figure, O is the centre of the circle. PQ is the tangent to the circle at A. If $\angle PAB = 58°$, then $\angle AQB$ equals



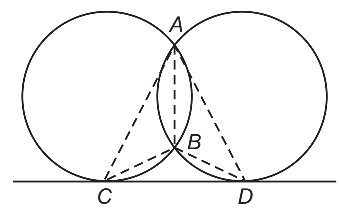
(*a*) 32°
(*b*) 26°
(*c*) 44°
(*d*) None of these

2. A, B, C, D and E are points on a circle. Point C is due north of point D and point E is due west of point D. $\angle CAB = 27°$. The angle of elevation of point B from point E is 87°. The angle of elevation of point B from point D is



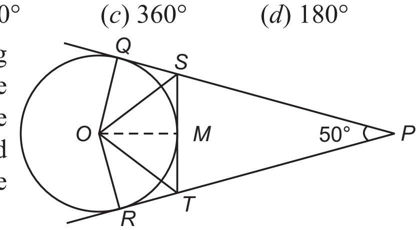
(*a*) 60° (*b*) 33°
(*c*) 63° (*d*) 24°

3. In the given figure, CD is a direct common tangent to two circles intersecting each other at A and B. Then, $\angle CAD + \angle CBD$ equals



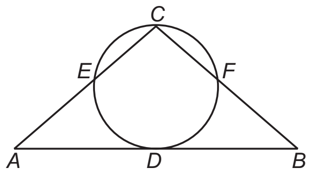
(*a*) 120° (*b*) 90° (*c*) 360° (*d*) 180°

4. In the adjoining figure 'O' is the centre of the circle and PQ, PR and ST are the three tangents.



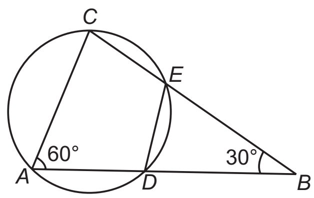
$\angle QPR = 50°$, then $\angle SOT$ equals
(*a*) 35° (*b*) 65° (*c*) 45° (*d*) 50°

5. ABC is an isosceles triangle. A circle is such that it passes through vertex C and AB acts as a tangent at D for the same circle. AC and BC intersect the circle at E and F respectively. $AC = BC = 4$ cm and $AB = 6$ cm. Also D is the mid-point of AB. What is the ratio of $EC : (AE + AD)$?



(*a*) 1 : 2 (*b*) 1 : 3 (*c*) 2 : 5 (*d*) None of these

6. In the given figure, $ADEC$ is a cyclic quadrilateral. CE and AD are extended to meet at B. $\angle CAD = 60°$ and $\angle CBA = 30°$. $BD = 6$ cm and $CE = 5\sqrt{3}$ cm. What is the ratio $AC : AD$?



(*a*) $\frac{3}{4}$ (*b*) $\frac{4}{5}$

(*c*) $\frac{2\sqrt{3}}{5}$ (*d*) cannot be determind.

7. Two circles cut each other at A and B. A straight line CAD meets the circles at C and D. If the tangents at C and D intersect at E, prove that C, E, D, B lie on a circle.

8. AB, BC, AD and DF are four straight lines and their points of intersection A, B, C, D, E and F form four Δs ADF, CDE, EBF and ABC. Show that the circumcircles of 4 Δs intersect at the same point.

ANSWERS

1. (*b*) **2.** (*c*) **3.** (*d*) **4.** (*b*) **5.** (*b*) **6.** (*a*)

HINTS AND SOLUTIONS

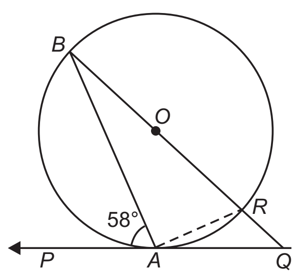
1. $\angle BAR = 90°$ (*Angle in a semicircle*)

$\angle ARB = \angle PAB = 58°$ (*Angles in alternate segments are equal*)

$\angle ABQ = 180° - (\angle BAR + \angle ARB)$ (*$\angle$Sum property of a Δ*)

$= 180° - (90° + 58°)$

$= 180° - 148° = 32°$



$\angle QAR = \angle ABR = \angle ABQ = 32°$ (*Angles in alternate segments are equal*)

$\angle AQB = 180° - (\angle ABQ + \angle BAQ)$

$= 180° - (32° + \angle BAR + \angle RAQ)$

$= 180° - (32° + 90° + 32°)$

$= 180° - 154° =$ **26°.**

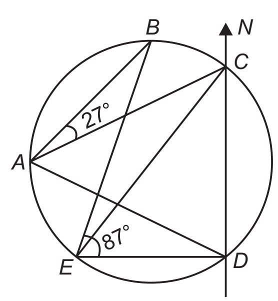
2. The required angle is $\angle BDE$.

C is north of D and E is west of D

$\Rightarrow \angle CDE = 90° \Rightarrow EC$ is the diameter of the circle

($\because$ *Angle in a semicircle is a rt.* $\angle$)

$\therefore \angle EBC = 90°$ (*Angle in a semicircle*)



$\angle BEC = \angle BAC = 27°$ (*Angle in the same segment are equal*)

$\therefore$ In Δ EBC, $\angle ECB = 180° - (\angle EBC + \angle BEC)$

$= 180° - (90° + 27°) = 63°$. (*Angle sum property of a* Δ)

$\Rightarrow \angle BDE = \angle BCE = 63°$ (*Angles in the same segment are equal*)

3. $\angle CAB = \angle BCD$, $\angle DAB = \angle BDC$ $\}$ (*Angles in alternate segments are equal*)

$\therefore \angle CAD = \angle CAB + \angle DAB = \angle BCD + \angle BDC$

Now, $\angle CAD + \angle CBD = \angle BCD + \angle BDC + \angle CBD = 180°$ (*Angle sum property in* Δ *BDC*)

4. $\angle ROQ = 180° - 50° = 130°$ ($\because \angle OQP + \angle ORP + \angle OPR + \angle ROQ = 360°$ and $\angle OQP = \angle ORP = 90°$)

$RT = TM$, $QS = SM$ (*Tangents to a circle from the same external point are equal*)

Also, $OQ = OM = OR$ (*Radii of the given circle*)

$\therefore \angle ROT = \angle TOM$ and $\angle MOS = \angle SOQ$.

($\because$ *Tangents from the an external point subtend equal angles at the centre*)

$\Rightarrow \angle SOT = \angle SOM + \angle TOM = \frac{1}{2}\angle QOM + \frac{1}{2}\angle ROM$

$\therefore \angle SOT = \frac{1}{2}\angle ROQ = \frac{1}{2} \times 130° =$ **65°.**

5. Here AC and BC are the secants of the circle and AB is the tangent at D.

$\therefore AE \times AC = AD^2 \Rightarrow AE \times 4 = (3)^2 \Rightarrow AE = 9/4$

$\therefore CE = AC - AE = 4 - \frac{9}{4} = \frac{7}{4}$

$\therefore CE : (AE + AD) = \frac{7}{4} : \left(\frac{9}{4} + 3\right) = \frac{7}{4} : \frac{21}{4} =$ **1 : 3.**

6. $\angle CED = 120°$ ($\because$ *CEDA is a cyclic quad.*)

$\Rightarrow \angle BED = 60°$

$\therefore$ In Δ EDB, $\angle EDB = 90°$

$\therefore \frac{BD}{BE} = \cos 30° \Rightarrow \frac{6}{BE} = \frac{\sqrt{3}}{2} \Rightarrow BE = 4\sqrt{3}$ cm.

$BC = BE + CE = 4\sqrt{3} + 5\sqrt{3} = 9\sqrt{3}$ cm

$\because$ AB and CB are secants of the given circle,

$$BD \times BA = BE \times EC$$

$\Rightarrow$ $6 \times BA = 4\sqrt{3} \times 9\sqrt{3}$

$\Rightarrow$ $BA = 18$ cm.

$\because$ $\angle ACB = 90°$, ΔABC is a rt. $\angle d$ Δ

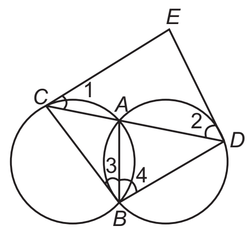
$\Rightarrow AC^2 = \sqrt{AB^2 - BC^2} = \sqrt{18^2 - (9\sqrt{3})^2}$

$= \sqrt{324 - 243} = \sqrt{81} = 9$ cm.

$\therefore$ $AD = AB - BD = 12$ cm.

$\therefore$ $AC : AD = 9 : 12 =$ **3 : 4.**

7. Join A and B, B and C, B and D.



In Δ CDE,

$\angle 1 + \angle 2 + \angle CED = 180°$ ...(1)

$\because$ CE is a tangent to the circle CBA at point C,

$\angle CBA$ is an angle in the alternate segment.

$\therefore$ $\angle 1 = \angle 3$ ...(2)

$\angle 2 = \angle 4$ ...(3)

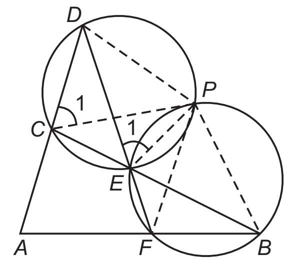
From (1), (2) and (3) we have

$\angle 3 + \angle 4 + \angle CED = 180°$

$\Rightarrow \angle CBD + \angle CED = 180°$

$\Rightarrow$ C, B, D, E are concyclic

8. Let us take the circumcircles of Δ DCE and Δ EBF meet at point P.



We have to now show that the circumcircles of Δ ADF and ΔABC also pass through P, *i.e., ADPF* and $ABPC$ are cylic quadrilaterals.

$\angle DCP = \angle DEP$ ...(*i*)

(*Angles in the same segment are equal*)

Also, $\angle DEP = \angle FBP$ ...(*ii*)

($\because$ *FBPE is a cyclic quadrilateral, ext* $\angle$ = *int. opp.* $\angle$)

$\therefore$ (*i*) and (*ii*) $\Rightarrow \angle DCP = \angle FBP = \angle ABP$. *i.e.,*

ext $\angle$ = int. opp. $\angle$ of quad. $ABPC$

$\Rightarrow$ $ABPC$ is a cyclic quadrilateral.

For cyclic quadrilateral $CDPE$, int opp. $\angle CDP$ = ext $\angle PEB$

Also, $\angle PEB = \angle PFB$ (*Angles in the same segment*)

$\therefore$ $\angle CDP = \angle PFB$ $\Rightarrow \angle ADP = \angle PFB$

$\Rightarrow$ int. opp. $\angle$ = ext $\angle$ in cyclic quad. $ADPF$.

$\therefore$ The circumcircles of Δs ADF, CDE, EFB and ABC intersect at point P.