1. **(c)** We have

$$f\left(\frac{x+y}{3}\right)=\frac{f(x)+f(y)}{3},\ f(0)=0 \text{ and } f'(0)=3$$

$$f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to0}\frac{f\left(\frac{3x+3h}{3}\right)-f(x)}{h}$$

$$=\lim_{h\to0}\frac{\frac{f(3x)+f(3h)}{3}-\frac{f(3x)+f(0)}{3}}{h}=\lim_{h\to0}\frac{f(3h)-f(0)}{3h}=3$$

$\therefore$ $f(x) = 3x + c$, ✌ $f(0) = 0 \Rightarrow c = 0$

$\therefore$ $f(x) = 3x$

2. **(b)** $x^2+y^2=a^2 \Rightarrow 2x+2yy'=0 \Rightarrow y'=-x/y$

$\Rightarrow yy' + x = 0$

$$\Rightarrow yy''+y'^2+1=0 \Rightarrow y=-\left(\frac{1+y'^2}{y''}\right) \quad ....(i)$$

$$\therefore k=\frac{1}{a}=\left|\frac{1}{\sqrt{x^2+y^2}}\right|=\left|\frac{1}{y\sqrt{1+\frac{x^2}{y^2}}}\right|=\left|\frac{1}{y\sqrt{1+y_1^2}}\right| \quad \left[\because y'=-\frac{x}{y}\right]$$

$$=\left|\frac{-y''}{(1+y'^2)\sqrt{1+y'^2}}\right|=\frac{|y''|}{(1+y'^2)^{3/2}}$$

3. **(c)** Since f(x) is continuous at x = 0 $\therefore$ $\lim_{x\to0} f(x)=f(0)$

Take any point x = a, then at x = a

$$\lim_{x\to a} f(x)=\lim_{h\to0} f(a+h)$$

$$=\lim_{h\to0}\left[f(a)+f(h)\right] \quad [\because f(x+y)=f(x)+f(y)]$$

$$=f(a)+\lim_{h\to0} f(h)=f(a)+f(0)=f(a+0)=f(a)$$

$\therefore$ f(x) is continuous at x = a. Since x = a is any arbitrary point, therefore f(x) is continuous for all x.

4. **(a)** Let $u=\tan^{-1}\frac{2x}{1-x^2}$ ...... (i) and $v = \sin^{-1}\frac{2x}{1+x^2}$ ...... (ii)

In equation (i) put, $x = \tan\theta$

$$u = \tan^{-1}\left[\frac{2\tan\theta}{1-\tan^2\theta}\right] = \tan^{-1}(\tan 2\theta)$$

$$u = 2\theta \qquad \frac{du}{d\theta} = 2 \text{ ...... (a)}$$

In equation (ii), put $x = \tan\theta$

$$v = \sin^{-1}\left[\frac{2\tan\theta}{1+\tan^2\theta}\right] = \sin^{-1}(\sin 2\theta)$$

$$\Rightarrow v = 2\theta \qquad \frac{dv}{d\theta} = 2 \text{ .... (b)}$$

From equations (a) and (b),

$$\frac{du}{dv} = \frac{du}{d\theta}\times\frac{d\theta}{dv} = 2\times\frac{1}{2} = 1$$

Required differential coefficient will be 1.

5. **(c)** In the definition of the function, $b \neq 0,$ then f(x) will be undefined in x > 0.

✌ f(x) is continuous at x = 0, ∴ LHL = RHL = f(0)

$$\Rightarrow \lim_{\substack{x\to 0\\ x<0}} \frac{\sin(a+1)x + \sin x}{x} = \lim_{\substack{x\to 0\\ x<0}} \frac{\sqrt{x+bx^2}-\sqrt{x}}{bx^{3/2}} = c$$

$$\Rightarrow \lim_{x\to 0}\left(\frac{\sin(a+1)x}{x} + \frac{\sin x}{x}\right) = \lim_{x\to 0}\frac{\sqrt{1+bx}-1}{bx} = c$$

$$\Rightarrow (a+1)+1 = \lim_{x\to 0}\frac{(1+bx)-1}{bx(\sqrt{1+bx}+1)} = c$$

$$\Rightarrow a + 2 = \lim_{x\to 0}\frac{1}{\sqrt{1+bx}+1} = c$$

$$\Rightarrow a + 2 = \frac{1}{2} = c \;\therefore\; a = -\frac{3}{2},\ c = \frac{1}{2},\ b \neq 0$$

6. **(d)** $f(x) = [x]^2 - [x^2] = (-1)^2 - (0)^2 = 0, -1 < x < 0 \Rightarrow 0 < x^2 < 1$

$= 0 - 0 = 0,\ 0 \le x < 1$ and $= 1 - 1 = 0, 1 \le x < \sqrt{3}$

and $= 1 - 3 = -2, \sqrt{3} \le x < \sqrt{4}$

$\therefore$ From above it is clear that the function is discontinuous at $\sqrt{n}\ \forall\, n \in \mathbf{I}$ except at x = 1.

7. **(d)** For $f(x)$ to be continuous at $x = 0$, we should have

$$\lim_{\infty \to 0} f(x) = f(0) = 12(\log 4)^3$$

$$\lim_{\infty \to 0} f(x) = \lim_{x \to 0}\left(\frac{4^x - 1}{x}\right)^3 \times \frac{\left(\frac{x}{p}\right)}{\left(\sin\frac{x}{p}\right)} \cdot \frac{px^2}{\log\left(1 + \frac{1}{3}x^2\right)}$$

$$= (\log 4)^3 \cdot 1 \cdot p \cdot \lim_{x \to 0}\left(\frac{x^2}{\frac{1}{3}x^2 - \frac{1}{18}x^4 + \ldots}\right)$$

$= 3p\,(\log 4)^3 \cdot$ Hence p = 4.

8. **(c)** $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

$$\Rightarrow 3c^2 - 6c + 2 = \frac{3/8 - 0}{1/2 - 0} = \frac{3}{4}$$

$$\Rightarrow c = 1 \pm \frac{\sqrt{21}}{6} \Rightarrow c = 1 + \frac{\sqrt{21}}{6} \notin \left(0, \frac{1}{2}\right) \Rightarrow c = 1 - \frac{\sqrt{21}}{6}$$

9. **(d)** We have, $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}(\log a)^n = \sum_{n=0}^{\infty} \frac{(x \log a)^n}{n!}$

$$= e^{x \log a} = e^{\log a^x} = a^x$$

$$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{a^{-h} - 1}{-h} = \log_e a$$

$$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{a^h - 1}{h} = \log_e a$$

Since $Lf'(0) = Rf'(0)$, $\therefore$ f(x) is differentiable at x = 0

Since every differentiable function is continuous, therefore, f(x) is continuous at x = 0.

10. (a) $\lim_{x\to 0^+} \frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}} = \lim_{x\to 0^+} \frac{1-e^{-2/x}}{1+e^{-2/x}} = 1$

and $\lim_{x\to 0^-} \frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}} = \lim_{x\to 0^-} \frac{e^{2/x} - 1}{e^{2/x} + 1} = -1.$

Hence $\lim_{x\to 0} f(x)$ exists if $\lim_{x\to 0} g(x) = 0$.

If g(x) = a ≠ 0 (constant) then

$$\lim_{x\to 0+} f(x) = a \text{ and } \lim_{x\to 0-} f(x) = -a.$$

Thus $\lim_{x\to 0} f(x)$ doesn't exist in this case.

$\therefore$ $\lim_{x\to 0} f(x)$ exists in case of (b), (c) and (d) each.

11. (d) $f(x) = \max. \{x, x^3\}$

$$= \begin{cases} x \ ; & x < -1 \\ x^3; & -1 \le x \le 0 \\ x \ ; & 0 \le x \le 1 \\ x^3; & x \ge 1 \end{cases}$$

$$\therefore \quad f'(x) = \begin{cases} 1 \ ; & x < -1 \\ 3x^2; & -1 \le x \le 0 \\ 1 \ ; & 0 \le x \le 1 \\ 3x^2; & x \ge 1 \end{cases}$$

Clearly f is not differentiable at – 1, 0 and 1.

12. (b) f is continuous at $x = \pi/4$, if $\lim_{x\to\pi/4} f(x) = f(\pi/4)$.

Now, $L = \lim_{x\to\pi/4} (\sin 2x)^{\tan^2 2x}$

$$\Rightarrow \log L = \lim_{x\to\pi/4} \tan^2 2x \log\sin 2x$$

$$= \lim_{x\to\pi/4} \frac{\log \sin 2x}{\cot^2 2x} \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\pi/4} \frac{2\cot 2x}{-2\cot 2x \operatorname{cosec}^2 2x.2} = -\frac{1}{2}$$

or $L = e^{-1/2}$ $\therefore f(\pi/4) = e^{-1/2} = 1/\sqrt{e}$

13. **(a)** Given $f^{-1}(x) = g(x)$

$$\Rightarrow x = f[g(x)]$$

Diff. both side w.r.t (x)

$$\Rightarrow 1 = f'[g(x)].g'(x) \Rightarrow g'(x) = \frac{1}{f'(g(x))}$$

Given, $f'(x) = \sin x$ $f'(g(x)) = \sin[g(x)]$

$$\Rightarrow \frac{1}{f'(g(x))} = \operatorname{cosec}[g(x)]$$

Hence, $g'(x) = \operatorname{cosec}[g(x)]$

14. **(d)** $|x|$ is non-differentiable function at

$x = 0$ as L.H.D = –1 and R.H. D = 1

$$|x| = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}$$

But $\cos|h|$ is differentiable

Any combination of two such functions will be non-differentiable . Hence option (a) and (b) are ruled out.

Now, consider $\sin|x| + |x|$

$$L' = \lim_{h\to 0} \frac{\sin|-h| + |-h|}{-h}$$

$$= \lim_{h\to 0} \frac{\sin h}{-h} - 1 = -1 - 1 = -2$$

$$R' = \lim_{h\to 0} \frac{\sin|h| + |h|}{h}$$

$= \lim_{h\to 0} \frac{\sin h}{h} + 1 = 1 + 1 = 2$

Consider $\sin|x| - |x|$

$$L' = \lim_{h\to 0} \frac{\sin|-h| - |-h|}{-h} = \lim_{h\to 0} \frac{\sin h}{-h} + 1 = 0$$

$$R' = \lim_{h\to 0} \frac{\sin|h| - |h|}{h} = \lim_{h\to 0} \frac{\sin h}{h} - 1 = 0$$

Hence, $\sin|x| - |x|$ is differentiable at $x = 0$.

15. **(b)** Let f (x) = $a_n x^n + a_{n-1} x^{n-1} +$ .... + $a_1 x$ = 0

The other given equation,

$$na_n x^{n-1} + (n-1)\ a_{n-1} x^{n-2} + \ldots + a_1 = 0 = f'(x)$$

Given $a_1 \neq 0 \Rightarrow f(0) = 0$

Again f (x) has root α, $\Rightarrow f(\alpha) = 0$

f (0) = f(α)

By Rolle’s theorem,

f ′(x) = 0 has root between $(0, \alpha)$

Hence $f'(x)$ has a positive root smaller than α.

16. **(a)** If f is continuous at x = 0, then

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$

$$\Rightarrow f(0) = \lim_{x\to 0^-} f(x)$$

$$k = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} \frac{\cos\frac{\pi}{2}[0-h]}{[0-h]}$$

$$k = \lim_{h\to 0} \frac{\cos\frac{\pi}{2}[-h]}{[-h]} = \lim_{h\to 0} \frac{\cos\frac{\pi}{2}[-h-1]}{[-h-1]}$$

$$k=\lim_{h\to 0}\frac{\cos\left(-\frac{\pi}{2}\right)}{-1};k=0$$

17. (b) The denominator of the given function is always defined

Also, tan [x]π = tan n π = 0 [[x] = integer, say n]

$\therefore$ f(x) = 0 $\forall$ x

$\therefore$ f(x) is continuous and differentiable for all x.

18. (c) Put $x^n=\cos\alpha$, $y^n=\cos\beta$

$$\Rightarrow a=\frac{\sin\alpha+\sin\beta}{\cos\alpha-\cos\beta}=\frac{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}$$

$$=-\cot\left(\frac{\alpha-\beta}{2}\right)$$

$\Rightarrow 2\cot^{-1}(-a)=\alpha-\beta$

$\Rightarrow \cos^{-1}(x^n)-\cos^{-1}(y^n)=2\cot^{-1}(-a)$

$$\Rightarrow \frac{y^{n-1}}{\sqrt{1-y^{2n}}}\frac{dy}{dx}=\frac{x^{n-1}}{\sqrt{1-x^{2n}}} \Rightarrow \sqrt{\frac{1-x^{2n}}{1-y^{2n}}}\frac{dy}{dx}=\frac{x^{n-1}}{y^{n-1}}$$

19. (b) Clearly x = 8 satisfies the given equation. Assume that

f (x) = e^{x-8} + 2x – 17 = 0 has a real root α other than x = 8. We may suppose that α > 8 (the case for α < 8 is exactly similar). Applying Rolle's theorem on [8, α],

we get β ∈ (8, α), such that f ' (β) = 0.

But f ' (β) = $e^{\beta-8}$ + 2, so that $e^{\beta-8}$ = – 2 which is not possible, Hence there is no real root other than 8.

20. (d) (a) $\lim\limits_{h\to 0^+}\frac{f(a+h)-f(a)}{h}$ exist finitely

$$\therefore \lim_{h\to 0^+} f(a+h)-f(a)$$

$$=\lim_{h\to 0^+}\left(\frac{f(a+h)-f(a)}{h}\right)h=0$$

$$\Rightarrow \lim_{h\to 0^+} f(a+h) = f(a)$$

Similarly, $\lim_{h\to 0^-} f(a+h) = f(a)$

$\therefore$ f is continuous at x = a

(b) Function is not differentiable at 5x = (2n + 1) $\frac{\pi}{2}$ only, which are not in domain

(c) Let f (x) = $\frac{1}{x^2}$ and g (x) = – $\frac{1}{x^2}$,

$\lim_{x\to 0} f(x)+g(x)$ exists whatever $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ does not exist.

21. **(5)** Since f'(x) = g(x), f′ (x) = g′ (x)

Put f′ (x) = – f(x). Hence g′ (x) = –f(x)

we have h′ (x) = 2f(x) f'(x) + 2g(x) g′ (x)

= 2[f(x) g(x) + g(x) [–f(x)]] = 2 [f(x) g(x) – f(x) g(x)] = 0

$\therefore$ h(x) = C, a constant

$\therefore$ h(0) = C i.e. C = 5

h(x) = 5 for all x. Hence h (10) = 5.

22. **(3)** Let $y = \left(1+\frac{1}{x}\right)^x$

Taking logarithm of both sides, we get

$$\log y = x\left[\log\left(1+\frac{1}{x}\right)\right]$$

$$\Rightarrow \frac{1}{y}y_1(x) = \frac{x^2}{x+1}\left(-\frac{1}{x^2}\right)+\log\left(1+\frac{1}{x}\right)$$

$$= -\frac{1}{x+1}+\log\left(1+\frac{1}{x}\right) \quad \text{......... (1)}$$

Since, y (2) = (1 + 1/2)^2 = 9/4

so, y_1 (2) = (9/4) $\left(-\frac{1}{3}+\log\frac{3}{2}\right)$

Again differentiate eq (1) w.r.t (x), we get

$$\frac{y(x)y_2(x)-[y_1(x)]^2}{(y(x))^2}=\frac{1}{(1+x)^2}-\frac{1}{x(x+1)}$$

By putting $x = 2$, we get

$$\frac{y(2)y_2(2)-(y_1(2))^2}{(y(2))^2}=\frac{-1}{18}$$

Now, put value of y (2) and $y_1(2)$

$$\Rightarrow y_2\,(2) = \left(\frac{9}{4}\right)\left(-\frac{1}{3}+\log\frac{3}{2}\right)^2-\frac{1}{8}$$

$$4\left(y_2(2)+\frac{1}{8}\right)=9\left(\log\frac{3}{2}-\frac{1}{3}\right)^2$$

$\Rightarrow$ Required expression = 3

23. (4.5) $f[(\pi/2)^-]=\lim\limits_{h\to 0}\frac{1-\sin^3[(\pi/2)-h]}{3\cos^2[(\pi/2)-h]}=\lim\limits_{h\to 0}\frac{1-\cos^3 h}{3\sin^2 h}=\frac{1}{2}$

$$f[(\pi/2)^+]=\lim_{h\to 0}\frac{q[1-\sin\{(\pi/2)+h\}]}{[\pi-2\{(\pi/2)+h\}]^2}=\lim_{h\to 0}\frac{q(1-\cosh)}{4h^2}=\frac{q}{8}$$

$$\therefore p=\frac{1}{2}=\frac{q}{8}\Rightarrow p=\frac{1}{2},q=4.$$

24. (3) $\underset{x\to 0}{\text{Lt}}\dfrac{(e^x-1)^2}{\sin\left(\frac{x}{a}\right)\log\left(1+\frac{x}{4}\right)}$

$$=\underset{x\to 0}{\text{Lt}}\frac{\dfrac{(e^x-1)^2}{x}\cdot x^2}{\dfrac{x}{a}\cdot\dfrac{\sin\left(\frac{x}{a}\right)}{\left(\frac{x}{a}\right)}\cdot\dfrac{\log\left(1+\frac{x}{4}\right)}{\frac{x}{4}}\cdot\dfrac{x}{4}}$$

$\Rightarrow$ 4a = 12 $\Rightarrow$ a = 3

25. **(5)** $F'(x) = \left[f\left(\frac{x}{2}\right) . f'\left(\frac{x}{2}\right) + g\left(\frac{x}{2}\right) g'\left(\frac{x}{2}\right) \right]$

Here, $g(x) = f'(x)$

and $g'(x) = f''(x) = -f(x)$

so $F'(x) = f\left(\frac{x}{2}\right) g\left(\frac{x}{2}\right) - f\left(\frac{x}{2}\right) g\left(\frac{x}{2}\right) = 0$

$\Rightarrow$ F (x) is constant function

so F (10) = 5