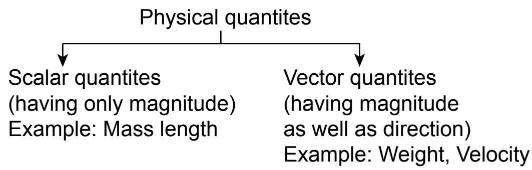
CHAPTER 28

VECTORS

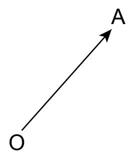
28.1 PHYSICAL QUANTITIES

A property of phenomenon, body, or substance, which has magnitude that can be expressed as a number and a reference.

Type of Physical Quantities:



Directed Line Segment: A line segment drawn in a given direction is called a directed line segment.



A directed line segment has the following three properties:

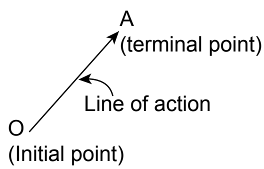
Length: OA, i.e., length of line segment OA.

Support/line of support/line of action: The line of which OA is a line segment.

Sense: The sense of directed line segment is from O to A.

Representation of a vector: A vector is represented by a directed line segment OA, where O is called initial point and, A is called terminal point of vector. Length of the line segment OA is called magnitude of vector and an arrow gives the direction of a vector.

The above vector is expressed as $\overrightarrow{OA}$.



Notation of a vector: A vector is denoted by small letters of the English alphabet under an arrow. For example, above $\overrightarrow{OA}$ can be denoted by $\vec{a}$, i.e., $\overrightarrow{OA} = \vec{a}$. $|\vec{a}|$, or simply 'a' represents the magnitude of vector called modulus of vectors.

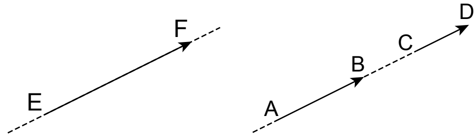
28.1.1 Equality of Two Vectors

Two vectors are said to be equal, if and only if, they have:

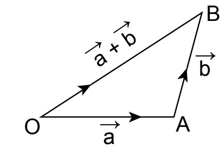
(a) equal magnitudes (i.e., same length)

(b) same direction (i.e., same or parallel support; their lines of action may be different)

(c) same sense.

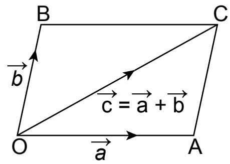


Triangle law of vector addition: If two vectors are represented by two adjacent sides of a triangle taken in the same order, then the closing side of the triangle taken in the opposite order, represents the sum of the first two vectors.



28.1.1.1 Parallelogram law of vector addition

If two vectors are represented by the two adjacent sides of a parallelogram, both in magnitude and direction, then their resultant will be given by the diagonal through the intersection of these sides (in both senses, i.e., magnitude and direction).



Remarks:

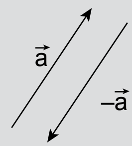
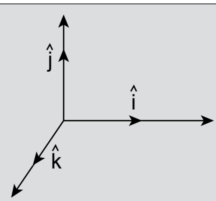
(i) *Number of line segments obtained by joining two of n points (no three lying on a line) = ${}^{n}c_2$.*

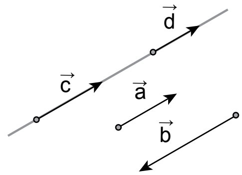
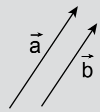
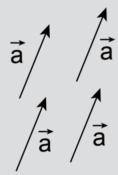
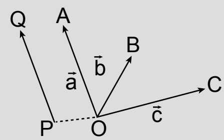
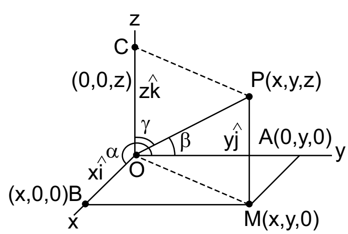
(ii) *Maximum number of vectors obtained by joining two of the n-points (no three lying on a line) = $2 \times {}^{n}c_2$.*

(iii) *Number of diagonal obtained by joining two of n-vertices of an n-sided convex polygon = $({}^{n}c_2 - n)$.*

(iv) *Maximum number of diagonal vectors obtained by joining two on n-vertices of n-sided convex polygon = $2({}^{n}c_2 - n)$.*

28.2 CLASSIFICATION OF VECTORS

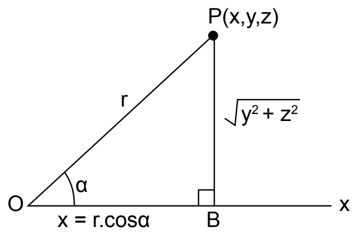
Opposite Vectors (Negative Vectors)	The negative of a vector $\vec{a}$ is defined as a vector having same magnitude that of $\vec{a}$, and the direction opposite to $\vec{a}$. It is denoted as $-\vec{a}$.	
Zero Vector (Null Vector)	A vector whose initial and terminal points are same is called a null vector. e.g., $\overrightarrow{AA}$. Such vector has zero magnitude and arbitrary (indefinite) direction. It is denoted by $\vec{O}$. $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=\overrightarrow{AA}$, or $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=\vec{O}$.	
Unit Vector	A unit vector is a vector whose magnitude is unity. We write a unit vector in the direction of $\vec{a}$ as $\hat{a}$. which is given by $\frac{\vec{a}}{\lvert\vec{a}\rvert}$. Unit vector along x-axis, y-axis, and z-axis are denoted by $\hat{i}, \hat{j}$ and $\hat{z}$, respectively.	

Collinear/Parallel Vectors	Vectors having same or parallel line of action; irrespective of their magnitude.	
Like Parallel Vectors	Two vectors having parallel line of action drawn in the same sense irrespective of their magnitude are called like parallel vectors.	
Unlike Parallel Vectors:	Two vectors having parallel line of action drawn in the opposite sense irrespective of their magnitude are called Unlike parallel vectors. Opposite vectors are unlike parallel vectors.	
Free Vectors	A vector $\vec{a}$ which can be represented by any one of the two directed line segments $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ whose lengths are equal and are in the same direction is known as a free vector. Such vectors, have freedom to have their initial point any where.	
Localized Vector	If a vector is restricted to pass through a specified point (i.e., a fixed point) then it is called localized vector. An example of a localized vector is a force, as its effect depends on the point of its application. Co-terminus vectors, position vectors etc., are examples of localized vectors.	
Co-initial Vectors	Vectors having same initial point (say, origin) are called co-initial vectors. If vectors in plane (or shape) are free vectors, then they can be shifted parallelly and can be converted to co-initial vectors having their initial points at origin.	
Position Vector	If P is a point having co-ordinates (x, y) or (x, y, z) (accordingly, P is in plane or space); then position vectors of point P is denoted by $\vec{r}$ and is given by $\overrightarrow{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$; Length of position vector $\overrightarrow{OP} = \left\|\overrightarrow{OP}\right\| = \left\|\vec{r}\right\| = r = \sqrt{x^2+y^2+z^2}$	

28.2.1 Representation of a Free Vector in Component Form

If $\overrightarrow{PQ}$ is a vector with initial point $P(x_1, y_1, z_1)$ and terminal point $Q(x_2, y_2, z_2)$, then $\overrightarrow{PQ} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$.

28.2.2 Direction cosine and Direction Ratios of Vectors



Direction of a vector $\overrightarrow{OP}$ is defined as the smallest angles, which the vector $\overrightarrow{OP}$ makes with the positive direction of co-ordinates axes.

Direction cosines of $\overrightarrow{OP}$ along x-axis = $\cos\alpha = \ell$ (denotes)

Direction cosines of $\overrightarrow{OP}$ along y-axis = $\cos\beta = m$ (denotes)

Direction cosines of $\overrightarrow{OP}$ along z-axis = $\cos\gamma = n$ (denotes)

Thus, direction cosine are $<\cos\alpha, \cos\beta, \cos\gamma> \equiv \left\langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right\rangle$; where P(x, y, z) and

$r = \sqrt{x^2 + y^2 + z^2} = \overrightarrow{OP}$.

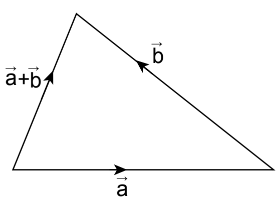
Properties of Direction cosines of $\overrightarrow{OP}$

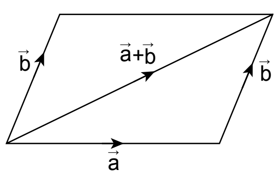
1. Direction cosines have values in [–1, 1].
2. $\ell^2 + m^2 + n^2 = 1$; where $<\ell, m, n>$ are direction cosines.
3. If $x = \ell r, y = mr, z = nr$; where $<\ell, m, n>$ are direction cosines.
4. If $\hat{r}$ = unit vector along $\vec{r}$, then $\hat{r} = \ell\hat{i} + m\hat{j} + n\hat{k}$; where $<\ell, m, n>$ are direction of $\vec{r}$.
5. Direction cosine of like parallel vectors are same, e.g., for $\vec{a}$ and $3\vec{a}$.
6. Direction cosine of unlike parallel vectors are numerically same, but opposite sign e.g., for $\vec{a}$ and $-3\vec{a}$.

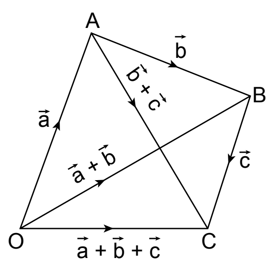
28.3 ADDITION OF VECTORS

If .. and $\vec{r}_2 = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$, then $\vec{r}_1 + \vec{r}_2 = (x_1 + x_2)\hat{i} + (y_1 + y_2)\hat{j} + (z_1 + z_2)\hat{k}$

Geometrically: $\vec{a} + \vec{b}$ is the vector given by triangle law and parallelogram law of vector addition.







28.3.2.1 Properties of vector addition

(i) **Commutative:** $\vec{a}+\vec{b}=\vec{b}+\vec{a}$

(ii) **Associative:** $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$; can be generalized for any number of vector.

(iii) **Additive Identity:** $\vec{0}$ (Null vector) is additive identity, i.e., $\vec{a}+\vec{0}=\vec{a}=\vec{0}+\vec{a}\ \forall\ \vec{a}$.

(iv) **Additive Inverse:** $-\vec{a}$ is additive inverse of $\vec{a}$, i.e., $\vec{a}+(-\vec{a})=\vec{0}=(-\vec{a})+\vec{a}$.

(v) **Triangle inequality:**

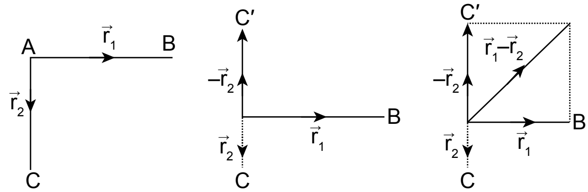
(a) $|\vec{a}+\vec{b}|\le|\vec{a}|+|\vec{b}|$ (b) $|\vec{a}+\vec{b}|\ge|\vec{a}|-|\vec{b}|$ (c) $\big||\vec{a}|-|\vec{b}|\big|\le|\vec{a}+\vec{b}|\le|\vec{a}|+|\vec{b}|$

(vi) The negative of a vector, sum, and difference of two vectors, i.e., $\pm\vec{a},\pm\vec{b},\pm(\vec{a}+\vec{b})$ all lie in same plane or parallel plane.

28.4 SUBTRACTION OF VECTORS

If $\vec{r}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$ and $\vec{r}_2=x_2\hat{i}+y_2\hat{j}+z_2\hat{k}$, then $\vec{r}_1-\vec{r}_2=(x_1-x_2)\hat{i}+(y_1-y_2)\hat{j}+(z_1-z_2)\hat{k}$.

Geometrically: Subtraction of $\vec{r}_1$ from $\vec{r}_2$ is nothing, but addition of $\vec{r}_1$ and $-\vec{r}_2$.



28.4.1 Properties of Vector Subtraction

(i) **Not commutative:** $\vec{a}-\vec{b}\ne\vec{b}-\vec{a}$, but $(\vec{a}-\vec{b})=-(\vec{b}-\vec{a})$

(ii) **Not associative:** $\vec{a}-(\vec{b}-\vec{c})\ne(\vec{a}-\vec{b})-\vec{c})$

(iii) $|\vec{a}|=|-\vec{a}|;|\vec{a}-\vec{b}|=|\vec{b}-\vec{a}|$

(iv) **Triangle inequality:** (a) $|\vec{a}-\vec{b}|\le|\vec{a}|+|\vec{b}|$; (b) $|\vec{a}-\vec{b}|\ge|\vec{a}|-|\vec{b}|$; (c) $\big||\vec{a}|-|\vec{b}|\big|\le|\vec{a}-\vec{b}|\le|\vec{a}|+|\vec{b}|$

Multiplication of a vector by a scalar λ (real number)
It is the product of scalar λ with $\vec{a}$.

$\therefore\ \lambda\vec{a}=\lambda(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})=\lambda a_1\hat{i}+\lambda a_2\hat{j}+\lambda a_3\hat{k} \Rightarrow |\lambda\vec{a}|=\lambda|\vec{a}|$, i.e., length of $\lambda\vec{a}$ is λ times that of $\vec{a}$.

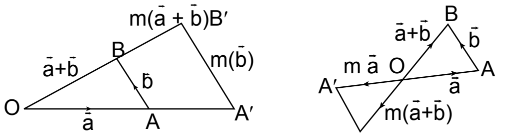
Remarks:

(i) $\hat{a}=\dfrac{\vec{a}}{|\vec{a}|}$ is a vector along $\vec{a}$ having unit length

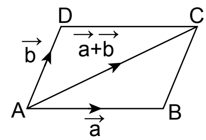
$\therefore\ \vec{b}=\pm b\left(\dfrac{\vec{a}}{|\vec{a}|}\right)$ *according as $\vec{b}$ is along, or in opposite direction to that of $\vec{a}$.*

(ii) Division of $\vec{a}$ by non-zero scalar λ, is multiplication of $\vec{a}$ by $\dfrac{1}{\lambda}$, i.e., $\dfrac{\vec{a}}{\lambda}=\dfrac{1}{\lambda}.(\vec{a})$.

(iii) $\lambda(\vec{a}+\vec{b}) = \vec{a}\lambda + \lambda\vec{b}$ *(i.e., scalar multiplication distributes over vector addition)*

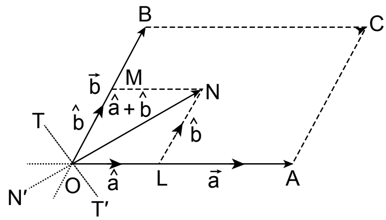


Unit vector along diagonal of a parallelogram



i.e., unit vector along $\overrightarrow{AC} = \dfrac{\vec{a}+\vec{b}}{|\vec{a}+\vec{b}|}$.

Unit vector along angle bisector of parallelogram



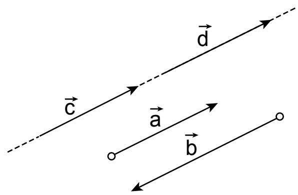
(a) Unit vector along internal angle bisector of ∠O.

= unit vector along the diagonal of rhombus OLMN of unit length = $\dfrac{\vec{a}+\vec{b}}{|\vec{a}+\vec{b}|}$ (along the internal angle bisector of ∠O.

(b) Unit vector along the internal angle bisector of ∠O outwards = $ON' = -\left(\dfrac{\vec{a}+\vec{b}}{|\vec{a}+\vec{b}|}\right)$.

(c) Unit vector along the external angle bisector at O along $OT = \dfrac{(-\hat{a}+\hat{b})}{|-\hat{a}+\hat{b}|}$.

28.5 COLLINEAR VECTORS



Vectors which are parallel to the same line are called collinear vectors irrespective of their magnitude and sense of direction.

Hence, $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are representing collinear vectors and for collinear vectors the line of action is either same or parallel.

28.5.1 Conditions for Vectors to be Collinear

Two vectors are said to be collinear if any one of the following conditions is satisfied:

(a) There exists a relation $\vec{a} = m\vec{b}$; where m is a non-zero scalar.

(b) If $\vec{a}$ and $\vec{b}$ are non-zero collinear vectors, then there exists a set of x and y other than (0, 0), such that $x\vec{a} + y\vec{b} = \vec{0}$. Here, converse is also true, i.e., if $x\vec{a} + y\vec{b} = \vec{0}$ and x, y are non-zero scalars, then $\vec{a}$ and $\vec{b}$ are collinear vectors.

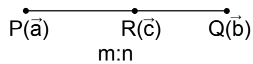
(c) For two vectors $\vec{a}$ and $\vec{b}$ to be collinear $\vec{a}\times\vec{b} = \vec{0}$, i.e., $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{0}$.

Notes:

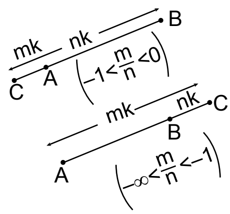
1. *If $\vec{a}$ and $\vec{b}$ are non-zero and non-collinear, then* $x\vec{a} + y\vec{b} = \vec{0} \Rightarrow x = 0, y = 0$ *as proved in the theorem as given below.*
2. *If three points* $A(\vec{a}), B(\vec{b}), C(\vec{c})$ *are collinear then* $(\vec{b} - \vec{a}) = \lambda(\vec{c} - \vec{b})$ *or equivalently* $(\vec{b} - \vec{a}) = \lambda(\vec{c} - \vec{b})$ *, i.e.,* $(\vec{b} - \vec{a})$ *and* $(\vec{c} - \vec{b})$ *are collinear vectors.*

Theorem: If $\vec{a}$ and $\vec{b}$ are two non-collinear non-zero vectors, m and n are scalars, such that $m\vec{a} + n\vec{b} = 0$, then m = 0 and n = 0.

28.6 SECTION FORMULA



Let P and Q points have their position vectors $\vec{a}$ and $\vec{b}$ respectively, then the position vector of point R dividing the line segment PQ internally in the ratios m : n is given by $\vec{c} = \dfrac{n\vec{a} + m\vec{b}}{m + n}$.



If R divides PQ externally in the ratio m : n, (or internally in the ratio-m/n)), then $\vec{c} = \dfrac{n\vec{a} - m\vec{b}}{n - m}$.

Remarks:

(i) $\dfrac{m}{n} > 0$, *then division is internal.*

(ii) $\dfrac{m}{n} < 0$, *then division is external.*

(iii) If $\dfrac{m}{n} \in (-1, 0)$, *then R lies outside PQ, near P.*

(iv) If $\dfrac{m}{n} \in (-\infty, -1)$, *then R lies outside PQ, near Q.*

(v) $\dfrac{m}{n} = 1$, *then* $\vec{c} = \dfrac{\vec{a} + \vec{b}}{2}$. *i.e., R is mid-point of PQ.*

(vi) $\dfrac{m}{n} = -1$, *then* $\overrightarrow{PR} = -\overrightarrow{RQ} \quad \Rightarrow$ *no such point R exist.*

(vii) If positions vectors of vertices A, B, C of ΔABC are respectively $\vec{a}, \vec{b}$ *and* $\vec{c}$, *then position vector of centroid of ΔABC is given by* $\overrightarrow{OG} \equiv \left(\dfrac{\vec{a} + \vec{b} + \vec{c}}{3}\right)$.

(viii) $\overrightarrow{OP} = (\vec{a}), \overrightarrow{OQ}(\vec{b})$ *and* $\overrightarrow{OR}(\vec{c})$ *lie on same plane.*

(ix) $\vec{c} = \dfrac{n\vec{a} + n\vec{b}}{n + m} \quad \Rightarrow \quad n\vec{c} + m\vec{c} = n\vec{a} + m\vec{b} \quad \Rightarrow \quad n\vec{a} + m\vec{b} - (n + m)\vec{c} = 0$ ……*(i)*

Clearly, section formula is applicable iff points P, Q, R lie on a straight line. Thus, from this fact, we have necessary and sufficient condition for three different point P, Q, R with position vector $\vec{a}, \vec{b}$ *and* $\vec{c}$ *to be collinear (i.e., lying on a straight line), there exist non-zero scalers* $\ell + m + n = 0$.

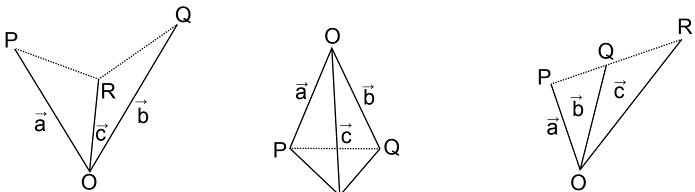
Hence, $\ell\vec{a}+m\vec{b}+n\vec{c}=\vec{0}$ *ensures coplanrity of* $\vec{a},\vec{b}$ *and* $\vec{c}$*; where as along with above the additional condition* $\ell + m + n = 0$*, ensures collinearality of point P, Q, R.*

i.e., existence of non-zero ℓ*, m, n such that* $\ell\vec{a}+m\vec{b}+n\vec{c}=\vec{0}$ *coplanarity of* $\vec{a}$*, b and* $\vec{c}$*. And*

$$\begin{cases}\ell\vec{a}+m\vec{b}+n\vec{c}=\vec{0} \text{ and} \\ \ell+m+n=0\end{cases} \Rightarrow \text{collinearity of } P, Q, R \Rightarrow \text{coplanarity of } \vec{a},\vec{b} \text{ and } \vec{c}.$$

(x) *If* $R(\vec{c})$ *divides the line joining* $P(\vec{a})$ *and* $Q(\vec{b})$ *in the ratio m:n;* $\left(\frac{n}{m}\in\mathbb{R}\sim\{0,-1\}\right)$ *then* $\vec{a},\vec{b},\vec{c}$ *lie on same plane confining the line passing through points P,Q,R and the origin. Thus if any three co-terminus (Co-initial vector) or free vectors are non-coplanar(i.e., do not lie on same or parallel plane); then terminal point of none of three vectors can divide the line segment joining the terminal point of other two vectors. Also it three co-terminus vectors having non parallel line or action are coplanar but there terminal points are non-collinear, even then none of the terminal point of three vectors can divide the line segment joining the terminal points of other two vectors.*

Thus four section formula to be valid four point P, Q and R with position vectors $\vec{a},\vec{b}$ *and* $\vec{c}$ *the position vector* $\vec{a},\vec{b},\vec{c}$ *must be coplanar and P,Q,R must be collinear. However if P,Q,R are collinear, then* $\vec{a},\vec{b},\vec{c}$ *will be coplanar. Thus for section formula to be applied for three different points P,Q,R collinearity of points P,Q,R is necessary and sufficient condition. However coplanarity of* $\vec{a},\vec{b},\vec{c}$ *is necessary condition but not sufficient.* $\vec{a},\vec{b},\vec{c}$ *are coplanar and point P,Q,R are collinear.*



$\vec{a},\vec{b},\vec{c}$ coplanar, but P,Q,R non-collinear | $\vec{a},\vec{b},\vec{c}$ non-coplanar, but P,Q,R non-collinear | $\vec{a},\vec{b},\vec{c}$ are coplanar, but and points P,Q,R are colinear

28.6.1 Collinearity of the Points

Point lying on same line are called collinear. Two points are always collinear. Thus, necessary and sufficient condition for three different points A, B and C to be collinear is that there exist three non-zero scalars x, y, z such that $x\vec{a}+y\vec{b}+z\vec{c}=\vec{o}$ and x + y + z = 0.

However, in above condition, any one scalar is zero, say x, then $y\vec{b}+z\vec{c}=\vec{o}$ and $y=-z \Rightarrow \vec{b}=\vec{c}$

$\Rightarrow$ we have points A and B, C coincident

$\Rightarrow$ $A.B(\equiv C)$ are collinear

If any two scalars are zero (say x and y), then the third are one z = 0.

$\Rightarrow$ which holds for every three vectors $\vec{a},\vec{b}$ and $\vec{c}$.

Conclusion: The necessary and sufficient condition for three point $A(\vec{a}),B(\vec{b}),C(\vec{c})$ to be collinear is that there exist three scalars x, y, z not all zeros (at most, one scalar can be zero), such that $x\vec{a}+y\vec{b}++z\vec{c}=\vec{o}$ and x + y + z = 0.

Notes:

1. *If the points $A(\vec{a}), B(\vec{b}), C(\vec{c})$ are collinear, then $\overline{AB} = \lambda\overline{BC}$ where λ is a scalar.*
2. *If three points $A(\vec{a}), B(\vec{b}), C(\vec{c})$ are collinear, then $(\vec{b} - \vec{a}) = \lambda(\vec{c} - \vec{b})$ or equivalently area of triangle ABC is zero, i.e., $(\vec{b} - \vec{a}) \times (\vec{c} - \vec{b}) = \vec{0}$.*

28.6.2 Linear Combination of Vectors

Linear combination of vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3, \ldots \vec{a}_n$ is a vector written as $\vec{r} = \lambda_1\vec{a}_1 + \lambda_2\vec{a}_2 + \lambda_3\vec{a}_3 + \ldots \lambda_n\vec{a}_n$; where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are scalars.

28.6.3 Linearly Dependent Vectors

A system of vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3, \ldots, \vec{a}_n$ is said to be linearly dependent, if there exist n scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$ (not all zero), such that $\lambda_1\vec{a}_1 + \lambda_2\vec{a}_2 + \lambda_3\vec{a}_3 + \ldots + \lambda_n\vec{a}_n = \vec{0}$ (i.e., above system is linearly dependent if one or some of them can be written as linear combination of the remaining.)

Two collinear vectors are always linearly dependent. Three co-planar vectors are always linearly dependent.

28.6.4 Linearly Independent Vectors

A system of n vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3, \ldots, \vec{a}_n$ is said to be linearly independent, if none of them can be written as the linear combination of the remaining. Therefore, mathematically it means.

If $\lambda_1\vec{a}_1 + \lambda_2\vec{a}_2 + \lambda_3\vec{a}_3 + \ldots + \lambda_n\vec{a}_n = \vec{0}$

$\Rightarrow \quad \lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$; where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are n scalars.

For example, two non-collinear vectors are always linearly independent, three non-coplanar vectors are always linearly independent.

28.6.5 Product of Two Vectors

These are of two types:

(a) Scalar Product (dot product) of two vectors:

Quantity definition: $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$; θ is the angle between $\vec{a}$ and $\vec{b}$; $0 \le \theta \le \pi$.

Geometrical interpretation: $\vec{a}.\vec{b}$ is the product of length of one vector, and the projection of other vector in the direction of the former vector, i.e., $\vec{a}.\vec{b} = |\vec{a}|\left(|\vec{b}|\cos\theta\right)$ or $|\vec{b}|\left(|\vec{a}|\cos\theta\right)$.

Remarks:

(i) If $\theta < 90^\circ \Rightarrow \vec{a}.\vec{b} > 0$; (ii) If $\theta = 90^\circ \Rightarrow \vec{a}.\vec{b} = 0$; (iii) If $\theta > 90^\circ \Rightarrow \vec{a}.\vec{b} < 0$

Properties of dot product of two vectors

(i) Dot product is commutative: $\vec{a}.\vec{b} = \vec{b}.\vec{a}$

(ii) $(\vec{a}.\vec{b})\vec{c} \ne \vec{a}(\vec{b}.\vec{c})$ in general $\because \lambda\vec{c} \ne \mu\vec{a}$

(iii) (Distributive law): Dot product distributes our vectors addition and subtraction, i.e., $\vec{a}.(\vec{b} \pm \vec{c}) = (\vec{a}.\vec{b}) \pm (\vec{a}.\vec{c})$.

(iv) $\vec{a}^2 = \vec{a}.\vec{a} = |\vec{a}|^2 = a^2$, but no other powers of a vector are defined. $\therefore\ \hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1.$

(v) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$$\vec{a}.\vec{b} = \left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right).\left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right) = a_1b_1 + a_2b_2 + a_3b_3$$

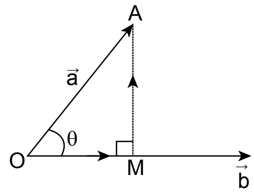
(vi) $\cos\theta = \dfrac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \dfrac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}.\sqrt{b_1^2 + b_2^2 + b_3^2}}$; i.e., $\theta = \cos^{-1}(\hat{a}.\hat{b})$

(vii) $\vec{a}.\vec{b} = 0$, therefore $\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$ (vector $\vec{a}$ and $\vec{b}$ are perpendicular to each other, provided that $\vec{a}$ and $\vec{b}$ are non-zero vectors.

(viii) $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k} = (\vec{a}.\hat{i})\hat{i} + (\vec{a}.\hat{j})\hat{j} + (\vec{a}.\hat{k})\hat{k}$

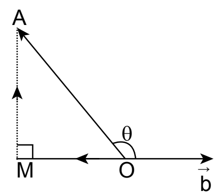
(ix) $(\vec{a} \pm \vec{b})^2 = (\vec{a} \pm \vec{b}).(\vec{a} \pm \vec{b}) = \vec{a}^2 + \vec{b}^2 \pm 2\vec{a}.\vec{b}$

Scalar projection of $\vec{a}$ on $\vec{b}$ = $|\vec{a}\cos\theta| = |\vec{a}||\cos\theta| = |\vec{a}||\vec{b}||\cos\theta| = |\vec{a}.\hat{b}|$



Similarly, scalar projection $\vec{b}$ on $\vec{a} = |\vec{b}.\hat{a}|$

Vector projection of $\vec{a}$ **on** $\vec{b}$**:** $(|\vec{a}|\cos\theta)\hat{b} = (\vec{a}.\hat{b})\hat{b}$ is a vector along, or opposite to $\vec{b}$ accordingly θ is acute or optus.

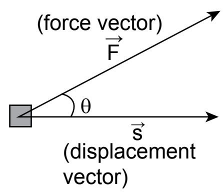


Similarly, vector projection of $\vec{b}$ on $\vec{a} = (\vec{b}.\hat{a})\hat{a}$

Scalar projection of $\vec{a}$ **perpendicular to** $\vec{b} = |\overrightarrow{MA}| = ||\vec{a}|\sin\theta| = |\vec{a} \times \hat{b}|$

Vector projection of $\vec{a}$ **perpendicular to** – (vector projection $\vec{a}$ on $\vec{b}$)

$= \vec{a} - (\vec{a}.\hat{b})\hat{b}$

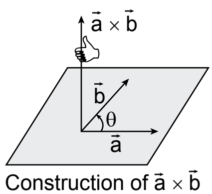


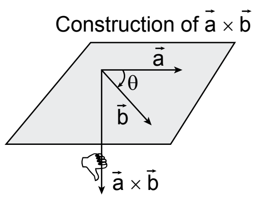
Work done:

$\therefore$ work done = $(|\vec{F}|\cos\theta) = |\vec{s}| = \vec{F}.\vec{s}$

(b) **Vector product (or cross product) of two vectors**

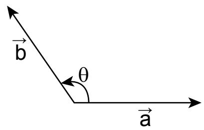
Skew product /outer product is denoted by $\vec{a} \times \vec{b}(|\vec{a}||\vec{b}|\sin\theta)$. (unit vector $\hat{n}$); $0 \le \theta \le \pi$ where direction of $\hat{n}$ is perpendicular to plane containing $\vec{a}$ and $\vec{b}$ and is directed as given by right handed thumb rule as shown in figure given below.





Magnitude of $\vec{a} \times \vec{b} = |\vec{a} \times \vec{b}| = ||\vec{a}||\vec{b}|\sin\theta.\hat{n}| = |\vec{a}||\vec{b}|\sin\theta$ as $\theta \in [0, \pi]$

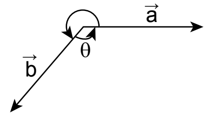
Remarks:



(i) If θ > π, then $\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$ *; Now, while evaluating* $\vec{b}\times\vec{a}, \theta \in [0,\pi]$.

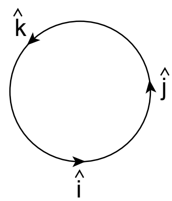
(ii) Unit vectors along $(\vec{a}\times\vec{b}) = \dfrac{\pm|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$

Where $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$; $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$



28.6.5.1 Properties of vector product

1. Anticommutative: $\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$.
2. $(m\vec{a})\times\vec{b} = m(\vec{a}\times\vec{b}) = \vec{a}\times(m\vec{b})$ (where m is a scalar).
3. If two vectors $\vec{a}$ and $\vec{b}$ are parallel, we have $\vec{a}\times\vec{b} = \vec{0}$.
4. $\vec{a}\times\vec{b} = \vec{0}$ $\Rightarrow$ $\vec{a}$ and $\vec{b}$ are parallel vectors (provided $\vec{a}$ and $\vec{b}$ are both non-zero vectors).
5. $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}$, $\hat{i}\times\hat{j} = \hat{k} = -(\hat{j}\times\hat{i})$, $\hat{j}\times\hat{k} = \hat{i} = -(\hat{k}\times\hat{j})$, $(\hat{k}\times\hat{i}) = \hat{j} = -(\hat{i}\times\hat{k})$.
6. Cross product is distributive over addition or substraction $\vec{a}\times(\vec{b}\pm\vec{c}) = \vec{a}\times\vec{b}\pm\vec{a}\times\vec{c}$. Cross product of three vectors is not associative.
7. Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$



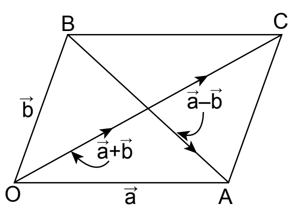
$$\Rightarrow \vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{i}(a_2b_3 - a_3b_2) + \hat{j}(a_3b_1 - a_1b_3) + \hat{k}(a_1b_2 - a_2b_1)$$

8. $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$

Remarks:

Since $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}; \theta \in [0,\pi]$ $\Rightarrow$ $\sin^{-1} = \left(\dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}\right)$ *or* $\pi - \sin^{-1} = \left(\dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}\right)$ *show that it is suggested to use dot product instead of cross product while finding the angle between two vectors.*

Geometrical interpretation: $|\vec{a}\times\vec{b}|$ represents the area of parallelogram with two adjacent sides represented by $\vec{a}$ and $\vec{b}$.



Area of Δ with two sides represented by $\vec{a}$ and $\vec{b}$ **:**

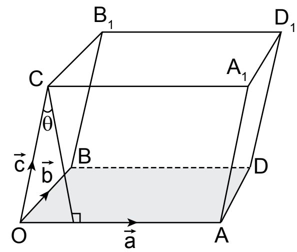
$$\frac{1}{2}|\vec{a}\times\vec{b}| = \frac{1}{4}|(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})|$$

$\Rightarrow$ $|\vec{a}\times\vec{b}| = \dfrac{1}{2}|(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})|$ = $\dfrac{1}{2}|\vec{d}_1\times\vec{d}_2|$; $\vec{d}_1$ and $\vec{d}_2$ are diagonal vector.

Scalar triple product: $\vec{a}.(\vec{b}\times\vec{c}) = \vec{b}.(\vec{c}\times\vec{a}) = \vec{c}.(\vec{a}\times\vec{b}) = [\vec{a}\,\vec{b}\,\vec{c}]$ (notation)

$\therefore$ If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$; $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$; $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ then $\left[\vec{a}\,\vec{b}\,\vec{c}\right] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Geometrical interpretation scalar triple product:



Geometrically, $[\vec{a}\,\vec{b}\,\vec{c}]$ represents the volume of above parallopiped with co-terminus edges represented by $\vec{a}, \vec{b}$ and $\vec{c}$.

Properties of scalar triple product:

(a) Dot and cross can be interchanged without changing the value of scalar triple proudct $\vec{a}.(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b}).\vec{c}$.

(b) Scalar triple proudct remains same, if cyclic order of three vectors do not changed. $\left[\vec{a}\,\vec{b}\,\vec{c}\right] = \left[\vec{b}\,\vec{c}\,\vec{a}\right] = \left[\vec{c}\,\vec{a}\,\vec{b}\right]$

(c) $\left[\vec{a}+\vec{b}+\vec{c}\,\vec{d}\right] = \left[\vec{a}\;\vec{c}\,\vec{d}\right] + \left[\vec{b}\,\vec{c}\,\vec{d}\right]$

(d) Scalar triple product vanishes when two of its vector are equal, we have $\left[\vec{a}\,\vec{a}\,\vec{b}\right] = 0$.

(e) The value of a scalar triple product, if two of its vectors are parallel, is zero, i.e., $\left[\vec{a}\,\vec{b}\,\vec{c}\right] = 0$ if $\vec{a} = \lambda\vec{b}$.

(f) For three co-planar vectors $\left[\vec{a}\,\vec{b}\,\vec{c}\right] = 0$ (even if $\left[\vec{a}\,\vec{b}\,\vec{c}\right]$ are non-zero vectors)

(g) If $\left[\vec{a}\,\vec{b}\,\vec{c}\right] = \left[\vec{d}\,\vec{a}\,\vec{b}\right] + \left[\vec{d}\,\vec{b}\,\vec{c}\right] + \left[\vec{d}\,\vec{c}\,\vec{a}\right] \Rightarrow \vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are co-planar

(h) If λ is a scalar, then $\left[\lambda\vec{a}\,\vec{b}\,\vec{c}\right] = \lambda\left[\vec{a}\,\vec{b}\,\vec{c}\right]$

(i) Volume of tetrahedron = $\frac{1}{6}\left[\vec{a}\,\vec{b}\,\vec{c}\right]$

(j) The volume of the **triangular prism (diagonally half of parallopiped)** whose adjacent sides are represented by the vectors $\vec{a}, \vec{b}, \vec{c}$ is $\frac{1}{2}\left[\vec{a}\,\vec{b}\,\vec{c}\right]$. It is composed of two similar triangles of sides a and b, two rectangles of sides a, c and b,c and rectangle having sides |a – b| and c).

Vector triple product: $\vec{a}\times(\vec{b}\times\vec{c})$ or $(\vec{a}\times\vec{b})\times\vec{c}$; however, $\vec{a}\times\vec{b}\times\vec{c}$ is meaningless.

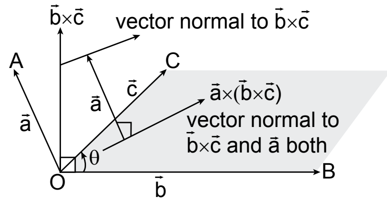
Properties of vector tipple product:

(i) $\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}.\vec{c}) - \vec{c}(\vec{a}.\vec{b})$

(ii) $(\vec{a}\times\vec{b})\times\vec{c} = \vec{b}(\vec{a}.\vec{c}) - \vec{a}(\vec{b}.\vec{c})$

(iii) $\vec{a}\times(\vec{b}\times\vec{c}) \neq (\vec{a}\times\vec{b})\times\vec{c}$; equality holds when $\vec{a}$ and $\vec{c}$ are collinear.

(iv) $\vec{a}\times(\vec{b}\times\vec{c})$ represents vector normal to plane containing $\vec{b}$ and $\vec{c}$ and also perpendicular to $\vec{a}$.



(v) If $\vec{a}$ perpendicular (plane containing and $\vec{c}$).

i.e., $\vec{a} || (\vec{b}\times\vec{c})$; then $\vec{a}\times(\vec{b}\times\vec{c}) = \vec{o}$

(vi) $\hat{i}\times(\hat{j}\times\hat{k}) = \hat{j}\times(\hat{i}\times\hat{k}) = \hat{k}\times(\hat{i}\times\hat{j}) = \vec{o}$

(vii) $\vec{a}\times(\vec{b}\times\vec{c})$ is a linear combination of those two vectors which are with in brackets.

(viii) If $\vec{r} = \vec{a}\times(\vec{b}\times\vec{c})$, then $\vec{r}$ perpendicular to $\vec{a}$ and lie in the parallel to that of $\vec{b}$ and $\vec{c}$.

28.6.6 Scalar Product of Four Vectors

$(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d})$; let $(\vec{a}\times\vec{b}) = \vec{n}$; therefore, $\vec{n}.(\vec{c}\times\vec{d}) = (\vec{n}\times\vec{c}).\vec{d} = \left((\vec{a}\times\vec{b})\times\vec{c}\right).\vec{d}$.

$$= -(\vec{c}\times(\vec{a}\times\vec{b})).\vec{d} = -((\vec{c}.\vec{b})\vec{a} - (\vec{c}.\vec{a})\,\vec{b}).\vec{d} \;=\; (\vec{c}.\vec{a})(\vec{b}.\vec{d}) - (\vec{b}.\,\vec{c})(\vec{a}.\vec{d}) = \begin{vmatrix} \vec{a}.\vec{c} & \vec{b}.\vec{c} \\ \vec{a}.\vec{d} & \vec{b}.\vec{d} \end{vmatrix}$$

It is also called as Lagrange's identity.

28.6.7 Vector Product of Four Vectors

If $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ are four vectors, the products $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$ is called vector product of four vectors.

i.e., $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a},\vec{b},\vec{d}]\vec{c} - [\vec{a},\vec{b},\vec{c}]\vec{d}$; also, $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a},\vec{c},\vec{d}]\vec{b} - [\vec{b},\vec{c},\vec{d}]\vec{a}$.

Notes:

We can look upon the above product as vector product in two ways one shown as above and other as shown below:

Let $\vec{c}\times\vec{d} = \vec{p}$, *product* $= (\vec{a}\times\vec{b})\times\vec{p} \;=\; (\vec{a}.\vec{p})\vec{b} - (\vec{p}.\vec{b})\vec{a} = [\vec{a}\,\vec{c}\,\vec{d}]\vec{b} - [\vec{c}\,\vec{d}\,\vec{b}]\vec{a}$

So, it can be defined either as linear combination of $\vec{a}$ *and* $\vec{b}$, *or as linear combination of* $\vec{c}$ *and* $\vec{d}$

Reciprocal system of vectors.

Let $\vec{a},\vec{b},\vec{c}$ be a system of three non-coplanar vectors. Then the system of vectors $\vec{a}',\vec{b}',\vec{c}'$ which satisfy $\vec{a},\vec{a}'\vec{b},\vec{b}'\vec{c},\vec{c}' = 1$ and $\vec{a}.\vec{b}' = \vec{b}.\vec{c}' = \vec{b}.\vec{a}' = \vec{c}.\vec{a}' = \vec{c}.\vec{b}' = 0$, is called the reciprocal system to the vector in term $\vec{a},\vec{b},\vec{c}$ the vector $\vec{a},\vec{b},\vec{c}$ are given by

$$\vec{a}' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}, \vec{b}' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}, \vec{c}' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

Properties of reciprocal system of vectors:

(i) $\vec{a},\vec{a}' = \vec{b},\vec{b}' = \vec{c},\vec{c}' = 1$

(ii) $\vec{a}.\vec{b}' = \vec{b}.\vec{c}' = \vec{c}.\vec{a}' = 0$

(iii) $[\vec{a}\,\vec{b}\,\vec{c}] = \dfrac{1}{[\vec{a}'\vec{b}'\vec{c}']}$

(iv) $\vec{a} = \dfrac{\vec{b}'\times\vec{c}'}{[\vec{a}'\vec{b}'\vec{c}']}$

(v) $\vec{a}.\vec{b}' = \vec{a}.\vec{c}' = \vec{b}.\vec{a}' = \vec{b}.\vec{c}' = \vec{c}.\vec{a}' = \vec{c}.\vec{b}' = 0$

(vi) $[\vec{a}\,\vec{b}\,\vec{c}]\times[\vec{a}'\vec{b}'\vec{c}'] = 1$

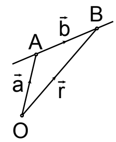
(vii) System of unit vector $\hat{i},\hat{j},\hat{k}$ is its own reciprocal $\hat{i}' = \hat{i}, \hat{j}' = \hat{j}, \hat{k}' = \hat{k}$.

(viii) The orthogonal triad of vectors $\hat{i},\hat{j},\hat{k}$ is self reciprocal.

(ix) $\vec{a}\,\vec{b}\,\vec{c}$ are non-coplanar iff $\vec{a}'\vec{b}'\vec{c}'$ are non coplanar.

Geometrical Application:

(i) **Vector equation of straight line:** A line passing through a point A with position vector $\vec{a}$ and parallel to another vector $\vec{b}$ is given by the equation $\vec{r} = \vec{a} + \lambda(\vec{b})$.



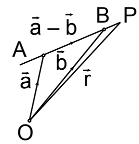
Note:

If co-ordinates of point A (x_1, y_1, z_1) and direction cosine of $\vec{b}$ is (l, m, n) respectively then the Cartesian equation of the above line can also be derived as $(x\hat{i} + y\hat{j} + z\hat{k}) = (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + \lambda(l\hat{i} + m\hat{j} + n\hat{k})$, *since i, j, k are linearly independent.*

Therefore, $(x - x_1) - \lambda\ell = 0$, $(y - y_1) - \lambda m = 0$ *and* $(z - z_1) - \lambda n = 0$

$$\Rightarrow \frac{(x - x_1)}{\ell} = \frac{(y - y_1)}{m} = \frac{(z - z_1)}{n} = \lambda$$

(ii) A line passing through two points A with position vector $\vec{a}$ and B with position vector $\vec{b}$ is given by the equation $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$; where λ is any real scalar parameter.



Note:

If co-ordinates of point A (x_1, y_1, z_1) and A (x_2, y_2, z_2). Therefore, direction ratio of line will be $(x_2 - x_1)$, $(y_2 - y_1)$ $(z_2 - z_1)$ respectively, then the Cartesian equation of the above line can also be derived as:

$$(x\hat{i} + y\hat{j} + z\hat{k}) = (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + \lambda((x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k})$$

Since i,j,k are linearly independent.

Therefore, $(x - x_1) - \lambda(x_2 - x_1) = 0$, $(y - y_1) - \lambda(y_2 - y_1) = 0$ *and* $(z - z_1) - \lambda(z_2 - z_1) = 0$

$$\Rightarrow \frac{(x - x_1)}{(x_2 - x_1)} = \frac{(y - y_1)}{(y_2 - y_1)} = \frac{(z - z_1)}{(z_2 - z_1)} = \lambda$$

Internal and external angle bisectors at a line:

The internal bisector of angle between unit vectors $\hat{a}$ and $\hat{b}$ is along the vector $\hat{a} + \hat{b}$. The external bisector is along $\hat{a} - \hat{b}$. Equation of internal and external bisectors of the line $\vec{r} = \vec{a} + \lambda\vec{b}_1$ and $\vec{r} = \vec{a} + \mu\vec{b}_2$ internally at $A(\vec{a})$ are given by $\vec{r} = \vec{a} + t\left(\frac{\vec{b}_1}{|\vec{b}_1|} \pm \frac{\vec{b}_2}{|\vec{b}_2|}\right)$.

Vector equation of a plane:

(i) The vector equation of plane passing through origin and containing $\vec{a}$ and $\vec{b}$ is $\vec{r} = \lambda_1\vec{a} + \lambda_2\vec{b}$.

$\Rightarrow \vec{r}.(\vec{a} \times \vec{b}) = 0.$

(ii) Vector equation of the plane passing through some other point $C(\vec{c})$ and co-planar with two vector $\vec{a}$ and $\vec{b}$ is $\vec{r} = \vec{c} + \lambda_1\vec{a} + \lambda_2\vec{b}$. Taking dot product with $\vec{a} \times \vec{b}$, $(\vec{r} - \vec{c}).(\vec{a} \times \vec{b}) = 0 \Rightarrow \vec{r}.(\vec{a} \times \vec{b}) = [\vec{a}\,\vec{b}\,\vec{c}]$.

(iii) Vector equation of a plane passing through three points A, B, C having position vector $\vec{a}, \vec{b}$ and $\vec{c}$ respectively.

$\overrightarrow{AB} = \vec{b} - \vec{a}; \overrightarrow{AC} = \vec{c} - \vec{a}$. Therefore, $\vec{r} = \lambda(\vec{b} - \vec{a}) + \mu(\vec{c} - \vec{a})$.

28.7 VECTOR EQUATION AND METHOD OF SOLVING

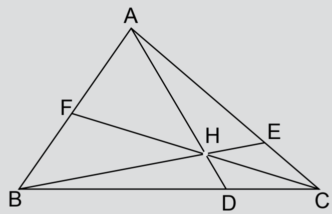
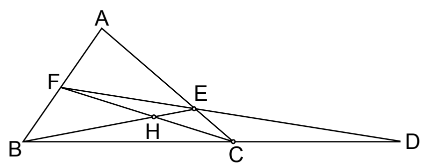
A vector equation is a relation between some unknown vector(s) and some known quantities, and the values of the unknown vectors satisfying the equation, is called the solution of equation. Solving a vector equation means, determining an unknown vector (or a number of vectors satisfying the given conditions).

Type I: $\vec{r}\times\vec{b}=\vec{a}\times\vec{b}$ $\Rightarrow$ $\vec{r}=\vec{a}+t\vec{b}$; t is any scalar.

Type II: $\vec{r}\times\vec{b}=\vec{a};\ \vec{a}\perp\vec{b}$ $\Rightarrow$ $\vec{r}=-\frac{1}{\vec{b}.\vec{b}}(\vec{a}\times\vec{b})+y\vec{b}$

Type III: $\vec{r}\times\vec{b}=\vec{c}\times\vec{b};\ \vec{r}\times\vec{a}=\vec{0};\ \vec{a}\not\perp\vec{b}$ $\Rightarrow$ $\vec{r}=\vec{c}-\left(\frac{\vec{c}.\vec{a}}{\vec{b}.\vec{a}}\right)\vec{b}$

Type IV: $k\vec{\ell}+\vec{\ell}\times\vec{a}=\vec{b};\ k\neq 0$ (scalar) $\Rightarrow$ $\vec{r}=\frac{1}{k^2+\vec{a}^2}\left[\frac{\vec{a}.\vec{b}}{k}\vec{a}+k\vec{b}+\vec{a}\times\vec{b}\right]$

Ceva's Theorem	If D, E, F are three points on the sides BC, CA, AB, respectively of a triangle ABC, such that the lines AD, BE and CF are concurrent, then $\frac{BD}{CD}.\frac{CE}{AE}.\frac{AF}{BF}=-1$ and conversely.	
Menelau's Theorem	If D, E, F are three points on the sides BC, CA, AB, respectively of a triangle ABC, such that the points D, E, F are collinear, then $\frac{BD}{CD}.\frac{CE}{AE}.\frac{AF}{BF}=1$ and conversely.	
Deasargue Theorem	If ABC, $A_1B_1C_1$ are two triangles, such that the three lines AA_1, BB_1 and CC_1 are concurrent, then the points of intersection of the three pairs of sides. BC, B_1C_1; CA, C_1A_1; AB, A_1B_1 are collinear and conversely.