CHAPTER 10

TRIGONOMETRIC EQUATION

10.1 INTRODUCTION

The equations involving trigonometric functions of one or more unknown variables are known as 'trigonometric equations'. For example $\cos\theta = 0$, $\cos^2\theta - 4\cos\theta = 1$, $\sin^2\theta + \sin\theta = 2$, $\cos^2\theta - 4\sin\theta = 1$, etc.

10.2 SOLUTION OF TRIGONOMETRIC EQUATION

A solution of a trigonometric equation is the value of the unknown variable (angle) that satisfies the equation. For example $\sin\theta = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \theta = \frac{\pi}{4}$ or $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}$.

Thus, the trigonometric equation may have infinite number of solutions.

10.2.1 Classification of Solutions of Trigonometric Equations

(i) Particular solution.
(ii) Principal solution.
(iii) General solution.

10.3 PARTICULAR SOLUTION

Any specific solution that satisfies a given trigonometric equation is called a **particular solution**. For example, sin x = ... has a particular solution $x = \frac{\pi}{3}$.

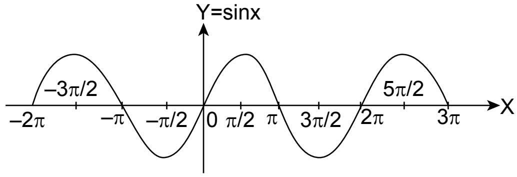
10.4 PRINCIPAL SOLUTION

The solutions of a trigonometric equation having least magnitude that is belonging to principal domain of trigonometric function are called 'principal solution'. For example sin $x = \frac{1}{2}$ has principal solution $\frac{\pi}{6}$.

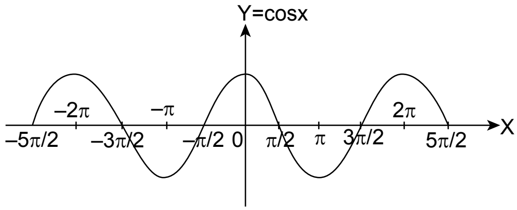
Parallely cos $x = -\frac{1}{2}$ has principal solutions $\frac{2\pi}{3}$.

The following figures represent principal domains of trigonometric functions.

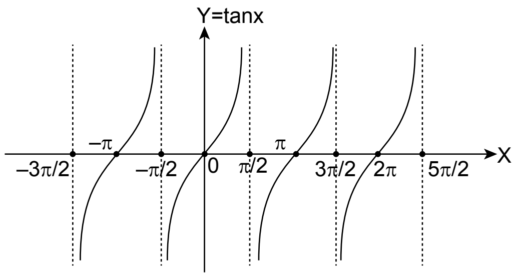
Principal Domain: $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$



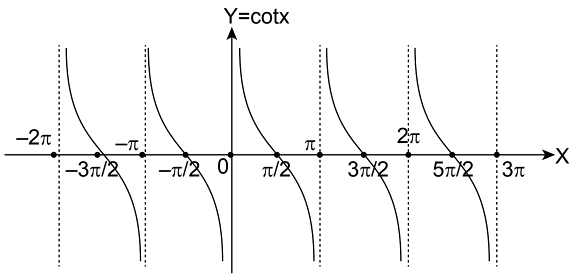
Principal Domain: $[0, \pi]$



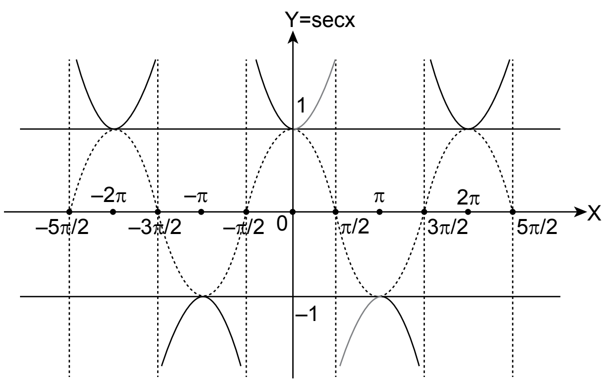
Principal Domain: $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$



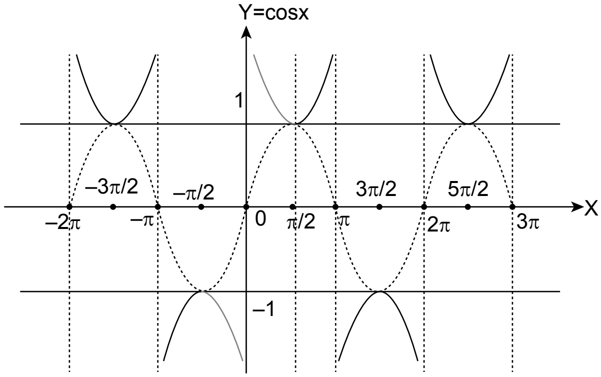
Principal Domain: $(0, \pi)$



Principal Domain: $[0, \pi] \sim \left\{\frac{\pi}{2}\right\}$



Principal Domain: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \sim \{0\}$



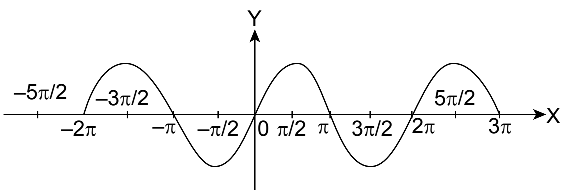
10.5 GENERAL SOLUTION

Since, trigonometric functions are periodic; a solution can be generalized by means of periodicity of the trigonometric functions. An expression, which is a function of integer n and a particular solution α representing all possible particular solutions of a trigonometric equation is called its 'general solution'. We use the following results for solving the trigonometric equations:

Result 1: $\sin\theta = 0 \quad \Leftrightarrow \quad \theta = n\pi, n \in \mathbb{Z}$.

General Solutions for Some Standard Equations

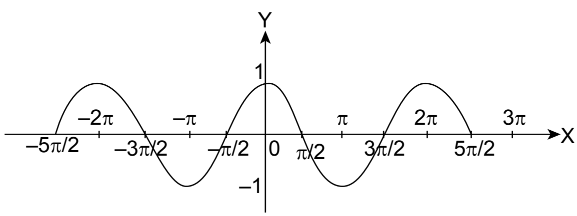
$\text{Sin}\ \theta = 0 \Rightarrow \theta = n\pi \qquad \sin\theta = 1 \Rightarrow \theta = (4n+1)\frac{\pi}{2} \qquad \sin\theta = -1, \Rightarrow \theta = (4n-1)\frac{\pi}{2}$



Result 2: $\cos\theta = 0 \quad \Leftrightarrow \quad \theta = (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}$.

General Solutions for Some Standard Equations

$\cos\theta = 0 \Rightarrow \theta = (2n+1)\frac{\pi}{2} \qquad \cos\theta = 1 \Rightarrow \theta = 2n\pi \qquad \cos\theta = -1 \Rightarrow \theta = (2n+1)\pi$

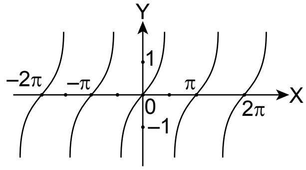


Result 3: $\tan\theta = 0 \quad \Leftrightarrow \quad \theta = n\pi, n \in \mathbb{Z}$.

General Solutions for Some Standard Equations

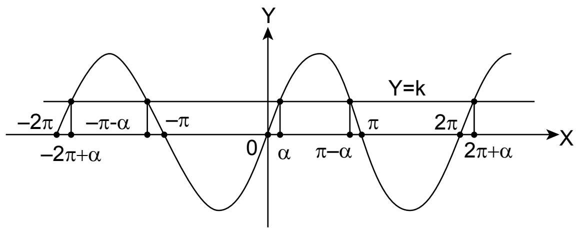
$\tan\theta = 0$

$\Rightarrow \theta = n\pi$ $\tan\theta = 1 \Rightarrow \theta = (4n+1)\dfrac{\pi}{4}$ $\tan\theta = -1, \Rightarrow \theta = (4n-1)\dfrac{\pi}{4}$

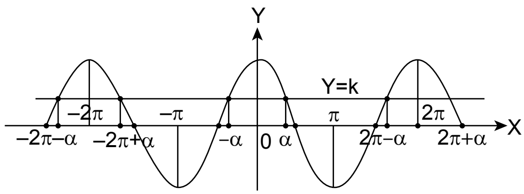


Result 4: $\sin\theta = \sin\alpha$

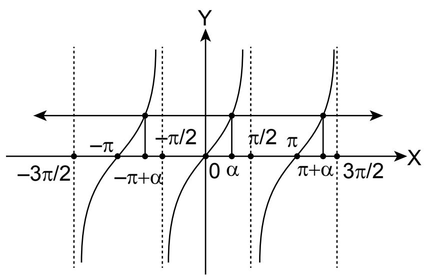
$\Leftrightarrow$ $\theta = n\pi + (-1)^n\alpha$, where $n \in \mathbb{Z}$ and α is a particular solution, preferably taken least non-negative, or that having least magnitude.



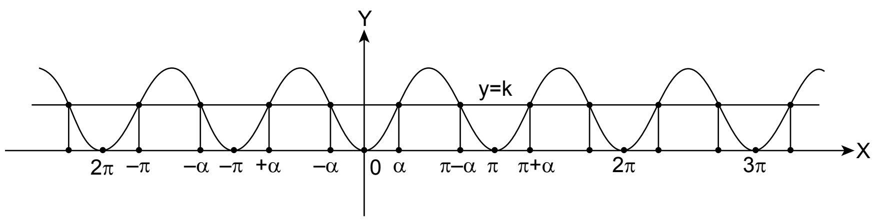
Result 5: $\cos\theta = \cos\alpha$ $\Leftrightarrow$ $\theta = 2n\pi \pm \alpha,\ n \in \mathbb{Z}$



Result 6: $\tan\theta = \tan\alpha$ $\Leftrightarrow$ $\theta = n\pi + \alpha,\ n \in \mathbb{Z}$



Result 7: $\sin^2\theta = \sin^2\alpha, \cos^2\theta = \cos^2\alpha, \tan^2\theta = \tan^2\alpha$ $\Leftrightarrow$ $\theta = n\pi \pm \alpha;\ n \in \mathbb{Z}$

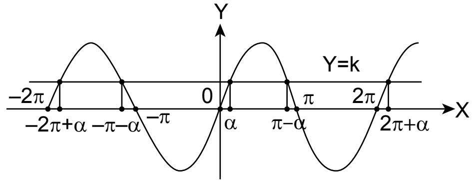


Remark:

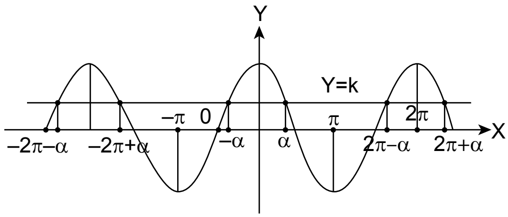
In formulae if we take any of α, the set of all possible solutions represented by general solution remains unique.

Theorem 1: $\sin\theta = k$, where $k \in [-1, 1]$ has general solution $q = n\pi + (-1)^n\alpha$

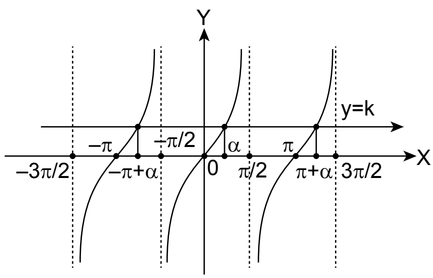
Where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ s.t. $\sin\alpha = k$



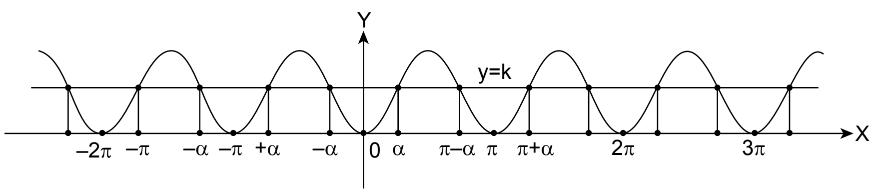
Theorem 2: $\cos\theta = k$, where $k \in [-1, 1]$ has general solution $\theta = 2n\pi \pm \alpha$; where $\alpha \in [0, \pi]$ s.t. $\cos\alpha = k$.



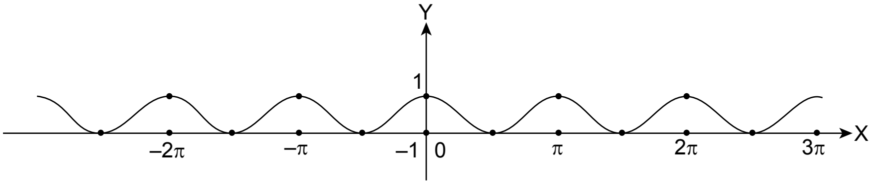
Theorem 3: $\tan\theta = k$, where $k \in \mathbb{R}$ has general solution $\theta = n\pi + \alpha$; where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ s.t. $\tan\alpha = k$.



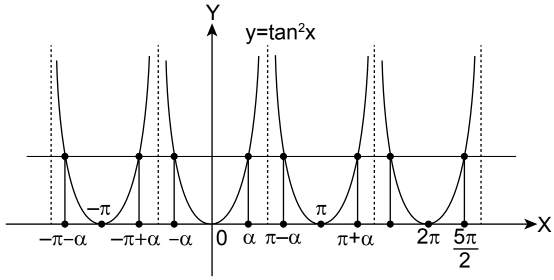
Theorem 4: $\sin^2\theta = k$, where $k \in [0, 1]$ has general solution $\theta = n\pi \pm \alpha$; where $\alpha \in \left[0, \frac{\pi}{2}\right]$ s.t. $\sin^2\alpha = k$.



Theorem 5: $\cos^2\theta = k$, where $k \in [0, 1]$ has general solution $\theta = n\pi \pm \alpha$ where $\alpha \in \left[0, \frac{\pi}{2}\right]$ s.t. $\cos^2\alpha = k$.



Theorem 6: $\tan^2\theta = k$, where $k \in [0, \infty)$ has general solution $\theta = n\pi \pm \alpha$; where $\alpha \in \left[0, \frac{\pi}{2}\right)$ s.t. $\tan^2\alpha = k$.



10.6 SUMMARY OF THE ABOVE RESULTS

1. $\sin\theta = 0 \Leftrightarrow \theta = n\pi\ ;\ n \in \mathbb{Z}$
2. $\cos\theta = 0 \Leftrightarrow \theta = (2n+1)\frac{\pi}{2};\ n \in \mathbb{Z}$
3. $\tan\theta = 0 \Leftrightarrow \theta = n\pi\ ;\ n \in \mathbb{Z}$
4. $\sin\theta = \sin\alpha \Leftrightarrow \theta = n\pi + (-1)^n\alpha\ ;\ n \in \mathbb{Z}$
5. $\cos\theta = \cos\alpha \Leftrightarrow \theta = 2n\pi \pm \alpha;\ n \in \mathbb{Z}$
6. $\tan\theta = \tan\alpha \Leftrightarrow \theta = n\pi + \alpha;\ n \in \mathbb{Z}$
7. $\sin^2\theta = \sin^2\alpha$, $\cos^2\theta = \cos^2\alpha$; $\tan^2\theta = \tan^2\alpha$; $n \in \mathbb{Z}$
8. $\sin\theta = 1 \Leftrightarrow \theta = (4n+1)\frac{\pi}{2};\ n \in \mathbb{Z}$
9. $\sin\theta = -1 \Leftrightarrow \theta = (4n+3)\frac{\pi}{2};\ n \in \mathbb{Z}$
10. $\cos\theta = 1 \Leftrightarrow \theta = 2n\pi$
11. $\cos\theta = -1 \Leftrightarrow \theta = (2n+1)\pi;\ n \in \mathbb{Z}$
12. $\sin\theta = \sin\alpha$ and $\cos\theta = \cos\alpha \Leftrightarrow \theta = 2n\pi + \alpha;\ n \in \mathbb{Z}$

Notes:

1. *The general solution should be given unless the solution is required in a specified interval or range.*
2. *α is a particular solution, preferably taken least positive or that having least magnitude.*

10.7 TYPE OF TRIGONOMETRIC EQUATIONS

Type 1: Trigonometric equations which can be solved by use of factorization

e.g., $(2\cos x - \sin x)(1 - \sin x) = \cos^2 x$ $\Rightarrow$ $(2\cos x - \sin x)(1 + \sin x) = 1 - \sin^2 x$

$\Rightarrow$ $(1 + \sin x)(2\cos x - 1) = 0$ $\Rightarrow$ $\sin x = -1$ or $\cos x = \frac{1}{2}$

$\Rightarrow$ $x = (4n+3)\frac{\pi}{2}$ or $2n\pi \pm \frac{\pi}{3}, n \in \mathbb{Z}$, are the general solutions.

Type 2: Trigonometric equations which can be solved by reducing them to quadratic equations.

e.g., $2\sin^2 x + 2\sin x = 5\cos^2 x$ $\Rightarrow$ $2\sin^2 x + 2\sin x = 5(1 - \sin^2 x)$

$\Rightarrow$ $7\sin^2 x + 2\sin x - 5 = 0$ $\Rightarrow$ $\sin x = -1$ or $\sin x = \frac{5}{7}$

$\Rightarrow$ $x = (4n+3)\frac{\pi}{2}; n \in \mathbb{Z}$ or $x = n\pi + (-1)^n \alpha; n \in \mathbb{Z}$

And $\sin\alpha = \frac{5}{7}$, are the required general solutions.

Type 3: Trigonometric equation which can be solved by transforming a sum or difference of trigonometric ratios into their product

e.g., $\cos x - \sin 3x = \cos 2x$ $\Rightarrow$ $\cos x - \cos 2x = \sin 3x$

$\Rightarrow$ $2\sin\left(\frac{3x}{2}\right)\sin\frac{x}{2} = \sin 3x = 2\sin\frac{3x}{2}\cos\frac{3x}{2}$

$\Rightarrow$ $2\sin\left(\frac{3x}{2}\right)\left[\sin\frac{x}{2} - \cos 3\frac{x}{2}\right] = 0$ $\Rightarrow$ $\sin\frac{3x}{2} = 0$

$\Rightarrow$ $\frac{3x}{2} = n\pi; n \in \mathbb{Z}$ $\Rightarrow$ $x = \frac{2n\pi}{3}; n \in \mathbb{Z}$ ....(i)

or $\sin\frac{x}{2} - \cos\frac{3x}{2} = 0$ $\Rightarrow$ $\cos\left(\frac{\pi}{2} - \frac{x}{2}\right) - \cos\frac{3x}{2} = 0$

$\Rightarrow$ $2\sin\left[\frac{\pi}{4} + \frac{x}{2}\right]\sin\left[x - \frac{\pi}{4}\right] = 0$ $\Rightarrow$ $x = m\pi + \frac{\pi}{4}$ ....(ii)

Combining equation (i) and (ii) general solutions are given by $x = \frac{2n\pi}{3}; 2n\pi - \frac{\pi}{2}; n\pi + \frac{\pi}{4}; n \in \mathbb{Z}$

Type 4: Trigonometric equations which can be solved by transforming a product of trigonometric ratios into their sum or difference. For example $\sin x . \cos 5x = \sin 4x \cos 2x$

$\Rightarrow$ $\sin 6x + \sin(-4x) = \sin 6x + \sin 2x$ $\Rightarrow$ $\sin 2x + \sin 4x = 0$ $\Rightarrow$ $2\sin(3x)\cos x = 0$

$\Rightarrow$ $x = \frac{n\pi}{3}$ or $x = (2n+1)\frac{\pi}{2}; n \in Z$

Type 5: Trigonometric equations of the form $a\sin x + b\cos x = c$, where $a, b, c \in \mathbb{R}$, can be solved by dividing both sides of the equation by $\sqrt{a^2 + b^2}$.

To solve the equation $a\cos\theta + b\sin\theta = c$, put $a = r\cos\phi$, $b = r\sin\phi$; such that $r = \sqrt{a^2 + b^2}$, $\phi = \tan^{-1}\frac{b}{a}$

i.e., take $\phi \in \left(-\frac{\pi}{2}\ \frac{\pi}{2}\right)$ such that $\tan\phi = \frac{b}{a}$.

Substituting these values in the equation we have, r cos ϕ cos θ + r sin ϕ sin θ = c

$\Rightarrow \quad \cos(\theta-\phi)=\frac{c}{r} \qquad \Rightarrow \quad \cos(\theta-\phi)=\frac{c}{\sqrt{a^2+b^2}}$

Notes:

1. *If $|c|>\sqrt{a^2+b^2}$, then the equation a cos θ + b sin θ = c has no solution.*
2. *If $|c|\le\sqrt{a^2+b^2}$, then put $\frac{|c|}{\sqrt{a^2+b^2}}$ = cos α, so that cos(θ – ϕ) = cos α.*

$\Rightarrow$ *(θ – ϕ) = 2n π ± α* $\Rightarrow$ *θ = 2n π ± α + ϕ, where n ∈ ℤ, e.g.,* $\sin x+\cos x=\sqrt{2}$.

$\Rightarrow$ *a = b = 1; Let a = r cosθ; b = r sinθ*

$\Rightarrow \quad r=\sqrt{a^2+b^2}=\sqrt{2} \qquad \therefore \quad 1=\sqrt{2}\cos\theta;\ 1=\sqrt{2}\sin\theta$

$\Rightarrow \quad \tan\theta=1 \qquad \Rightarrow \quad \theta=\tan^{-1}1 \Rightarrow \theta=\frac{\pi}{4}$

$\therefore \quad \sqrt{2}\left(\cos\left(x-\frac{\pi}{4}\right)\right)=\sqrt{2} \qquad \Rightarrow \quad \cos\left(x-\frac{\pi}{4}\right)=1 \qquad \Rightarrow \quad x=2n\pi+\frac{\pi}{4};n\in\mathbb{Z}$

3. *Trigonometric equation of the form a sinx + cosx = c can also be solved by changing sinx and cosx into their corresponding tangent of half the angle and solving for tan x/2, i.e., we substitute* $\cos x=\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ *and* $\sin x=\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$.

Type 6: **Equation of the form:** R(sin x ± cos x, sin x . cos x) = 0

Where R is a rational function of the arguments in the brackets,

Put sin x + cos x = t ......(i)

and use the following identity: $(\sin x+\cos x)^2=\sin^2 x+\cos^2 x+2\sin x\cos x=1+2\sin x\cos x$

$\Rightarrow \quad \sin x\cos x=\frac{t^2-1}{2}$ ....(ii)

Taking equation (i) and (ii) into account, we can reduce given equation into; $R(t,(t^2-1)/2)=0$.

Similarly, by the substitution(sin x – cos x) = t, we can reduce the equation of the form;

R(sin x – cos x, sin x cos x) = 0 to an equation

$R(t,(1-t^2)/2)=0$

Type 7: Trigonometric equations which can be solved by the use boundedness of the trigonometric ratios sinx and cosx e.g., $\sin\frac{5x}{4}+\cos x=2$. Now, the above equation is true if $\sin\frac{5x}{4}=1$ and cos x = 1

$\Rightarrow \quad \frac{5x}{4}=2n\pi+\frac{\pi}{2},n\in\mathbb{Z}$ and x = 2mπ , m ∈ z

$\Rightarrow \quad x=\frac{(8n+2)\pi}{5},n\in\mathbb{Z}$ ....(iii)

and x= 2mπ , m ∈ ℤ ....(iv)

Now, to find general solution of equation (i); $\frac{(8n+2)\pi}{5}=2m\pi$

$\Rightarrow \quad 8n + 2 = 10\,m \qquad \Rightarrow \quad n = \dfrac{5m-1}{4}$

If m = 1	then	n = 1
m = 5	then	n = 6
….	……	……..
….	……	……..
If $m = 4p - 3, p \in \mathbb{Z}$	then	$n = 5p - 4, p \in \mathbb{Z}$

$\therefore$ General solution of a given equation can be obtained by

$$x = \{2m\pi;\ m \in \mathbb{Z}\} \cup \left\{\frac{(8n+2)}{5}\pi; n \in \mathbb{Z}\right\} \sim \{2m\pi; m = 4p-3; p \in \mathbb{Z}\}$$

or $$x = \{2m\pi;\ m \in \mathbb{Z}\} \cup \left\{\frac{(8n+2)}{5}\pi; n \in \mathbb{Z}\right\} \sim \left\{\frac{(8n+2)}{5}\pi; n = 5p-4; p \in \mathbb{Z}\right\}.$$

Type 8: A trigonometric equation of the form: R(sin kx, cos nx, tan mx, cot ℓ(x) = 0; ℓ, m, n, then use the following formulae:

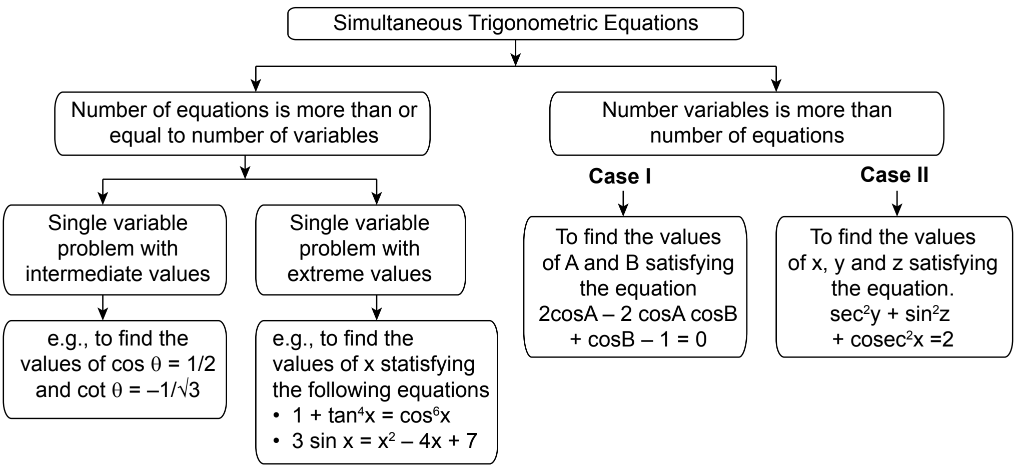
$$\sin x = \frac{2\tan x/2}{1+\tan^2 x/2}, \quad \cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2}, \quad \tan x = \frac{2\tan x/2}{1-\tan^2 x/2}, \quad \cot x = \frac{1-\tan^2 x/2}{2\tan x/2}$$

10.8 HOMOGENEOUS EQUATION IN SINX AND COSX

The equation of the form $a_0 \sin^n x + a_1 \sin^{n-1} x \cos x + a_2 \sin^{n-2} x \cos^2 x + \ldots.. + a_n \cos^n x = 0$, where a_0, a_1, ….. a_n are real numbers and the sum of the exponents in sin x and cos x in each term is equal to n, are said to be homogeneous with respect to sin x and cos x. For cos x ≠ 0, the above equation can be written as, $a_0 \tan^n x + a_1 \tan^{n-1} x + \ldots.. + a_n = 0$.

10.9 SOLVING SIMULTANEOUS EQUATIONS

Here we discuss problems related to the solution of two equations satisfied simultaneously. We may divide the problems into two categories as shown by the following diagram:



When number of equations is more than or equal to number of variables:

■ **Single variable problems with intermediate values:**

Step 1: Find the values of variable x satisfying both equations.

Step 2: Find common period of function used in both the equation, say T and obtain $x = \alpha \in (0, T]$ satisfying both the equations.

Step 3: Generalizing the value of α, we get $x = nT + \alpha$.

■ **Single variable problem with extreme values**

Step 1: When LHS and RHS of a equation have their ranges, say R_1 and R_2 in common domain and $R_1 \cap R_2 = \phi$ then the equations have no solution.

Step 2: If $R_1 \cap R_2$ have finitely many elements and the number of elements are few, then individual cases can be analyzed and solved.

Step 3: Generalizing the value of α, we get $x = nT + \alpha$.

10.9.1 More Than One Variable Problems

- Substitute one variable (say y) in terms of other variable x, i.e., eliminate y and solve as the trigonometric equations in one variables.
- Extract the linear/algebraic simultaneous equations from the given trigonometric equations and solve as simultaneous algebric equations.
- Many times you may need to make appropriate substitutions.
- **When number of variables is more than number of equations:**

  To solve an equation, involving more than one variable, definite solutions can be obtained if extreme values (range) of the functions are used.

10.9.1.1 Some important results

1. While solving a trigonometric equation, squaring the equations at any step should be avoided as far as possible. If squaring is necessary, check the solution for extraneous values.
2. Never cancel terms containing unknown terms on the two sides, which are in product. It may cause loss of the genuine solution.
3. The answer should not contain any such values of angles which make any of the terms undefined or infinite.
4. Domain should not change. If it changes, necessary corrections must be made.
5. Check that denominator is not zero at any stage, while solving equations.

10.10 TRANSCEDENTAL EQUATIONS

To solve the equation when the terms on the two side (L.H.S and R.H.S) of the equation are of different nature, e.g., trigonometric and algebraic, we use inequality method. Which is used to verify whether the given equation has any real solution or not. In this method, we follow the steps given below:

Step I: If given equation is f(x) = g(x), then let y = f(x) and y = g(x), i.e., break the equation in two parts.

Step II: Find the extreme values of both sides of equation giving range of values of y for both side. If there is any value of y satisfying both the inequalities, then there will be a real solution otherwise, there will be no real solution.

10.11 GRAPHICAL SOLUTIONS OF EQUATIONS

For solution of equation f(x) – g(x) = 0

Let α is root $\Rightarrow f(\alpha) = g(\alpha) = k(\text{say})$

$\Rightarrow$ $y = f(x)$ and $y = g(x)$

have same output for input $x = \alpha$.

$\Rightarrow$ (α, k) satisfying both the curves $y = f(x)$ and $y = g(x)$.

Solutions of equation f(x) – g(x) = 0 are abscissa (x-co-ordinate) of the point of intersection of the graph y = f(x) and y = g(*x*).

Algorithm: To solve the equation f(x) – g(x) = 0, e.g., 10sinx – x = 0

Step 1: Write the equation f(x) = g(x), i.e., sinx = x/10.

Step 2: Draw the graph of y = f(x) and y = g(x) on same x – y plane.

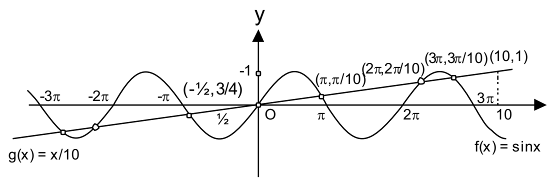
Let f(x) = sinx and g(x) = $\frac{x}{10}$

also we know that, $-1 \le \sin x \le 1$

$\therefore \quad -1 \le \frac{x}{10} \le 1$

$\Rightarrow \quad -10 \le x \le 10$

Thus sketching both the curves when x ∈ [−10, 10].



Step 3: Count the number of the points of intersection. If graphs of y = f(x) and y = g(x) cuts at one, two, three,...., no points, then number of solutions are one, two, three....., zero respectively.

From the given graph, we can conclude that f(x) = sinx and g(x) = $\frac{x}{10}$ intersect at 7 points. So number of solutions are 7.

10.12 SOLVING INEQUALITIES

To solve trigonometric inequalities including trigonometric functions, it is good to practice periodicity and monotonicity of functions. Thus, first solve the inequality for one period and then get the set of all solutions by adding numbers of the form $2n\pi$; $n \in \mathbb{Z}$ to each of the solutions obtained on that interval.

For example: Find the solution set of inequality sinx > 1/2.

Solution: When sinx = 1/2, the two values of x between 0 and 2π are $\pi/6$ and $5\pi/6$ from the grpah of y = sinx, it is obvious that between 0 and 2π.

$$\sin x > \frac{1}{2} \text{ for } \pi/6 < x < 5\pi/6$$

Hence sinx > 1/2 $\Rightarrow 2n\pi + \frac{\pi}{6} < x < 2n\pi + \frac{5\pi}{6}$

The required solution set is $\bigcup_{n \in Z} \left(2n\pi + \frac{\pi}{6}, 2n\pi + \frac{5\pi}{6}\right)$

10.12.1 Review of Some Important Trigonometric Values

1. $\sin 15° = \frac{\sqrt{3}-1}{2\sqrt{2}}$
2. $\cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}}$
3. $\tan 15° = 2 - \sqrt{3} = \cot 75°$
4. $\cot 15° = 2 + \sqrt{3} = \tan 75°$
5. $\sin\left(22\frac{1}{2}\right)^\circ = \frac{1}{2}\left(\sqrt{2-\sqrt{2}}\right)$
6. $\cos 22° = \left(22\frac{1}{2}\right)^\circ = \frac{1}{2}\left(\sqrt{2+\sqrt{2}}\right)$
7. $\tan\left(22\frac{1}{2}\right)^\circ = \sqrt{2-1}$
8. $\cot\left(22\frac{1}{2}\right)^\circ = \sqrt{2}+1$
9. $\sin 18° = \frac{\sqrt{5}-1}{4} = \cos 72°$
10. $\cos 18° = \frac{\sqrt{10+2\sqrt{5}}}{4} = \sin 72°$
11. $\sin 36° = \frac{\sqrt{10-2\sqrt{5}}}{4} = \cos 54°$
12. $\cos 36° = \frac{\sqrt{5}+1}{4} = \sin 54°$
13. $\sin 9° = \frac{\sqrt{3+\sqrt{5}} - \sqrt{5-\sqrt{5}}}{4} = \cos 81°$
14. $\cos 9° = \frac{\sqrt{3+\sqrt{5}} + \sqrt{5-\sqrt{5}}}{4} = \sin 81°$
15. $\cos 36° - \cos 72° = 1/2$
16. $\cos 36° . \cos 72° = 1/4$