CONTINUITY& DIFFERENTIABILITY

* A function f is said to be continuous at x = a if

Left hand limit = Right hand limit = value of the function at x = a

i.e. $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$

i.e. $\lim_{h\to 0} f(a-h) = \lim_{h\to 0} f(a+h) = f(a)$.

* A function is said to be differentiable at x = a

if $Lf'(a) = Rf'(a)$ i.e $\lim_{h\to 0}\frac{f(a-h)-f(a)}{-h} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

(i) $\frac{d}{dx}(x^n) = n\,x^{n-1}$, $\frac{d}{dx}\left(\frac{1}{x^n}\right) = -\frac{n}{x^{n+1}}$, $\frac{d}{dx}\left(\sqrt{x}\right) = -\frac{1}{2\sqrt{x}}$

(ii) $\frac{d}{dx}(x) = 1$

(iii) $\frac{d}{dx}(c) = 0, \forall c \in R$

(iv) $\frac{d}{dx}(a^x) = a^x \log a$, $a > 0, a \neq 1$.

(v) $\frac{d}{dx}(e^x) = e^x$.

(vi) $\frac{d}{dx}(\log_a x) =, \frac{1}{x \log a}$ $a > 0, a \neq 1$, x

(vii) $\frac{d}{dx}(\log x) = \frac{1}{x}$, $x > 0$

(viii) $\frac{d}{dx}(\log_a |x|) = \frac{1}{x \log a}$, $a > 0, a \neq 1, x \neq 0$

(ix) $\frac{d}{dx}(\log |x|) = \frac{1}{x}$, $x \neq 0$

(x) $\frac{d}{dx}(\sin x) = \cos x, \forall x \in R$.

(xi) $\frac{d}{dx}(\cos x) = -\sin x, \forall x \in R$.

(xii) $\frac{d}{dx}(\tan x) = \sec^2 x, \forall x \in R$.

(xiii) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x, \forall x \in R$.

(xiv) $\frac{d}{dx}(\sec x) = \sec x \tan x, \forall x \in R$.

(xv) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x, \forall x \in R$.

(xvi) $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$.

(xvii) $\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$.

(xviii) $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}, \forall x \in R$

(xix) $\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}, \forall x \in R$.

(xx) $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$.

(xxi) $\frac{d}{dx}(\operatorname{cosec}^{-1}x) = -\frac{1}{|x|\sqrt{x^2-1}}$.

(xxii) $\frac{d}{dx}(|x|) = \frac{x}{|x|}, x \neq 0$

(xxiii) $\frac{d}{dx}(ku) = k\frac{du}{dx}$

(xxiv) $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$

(xxv) $\frac{d}{dx}(u.v) = u\frac{dv}{dx} + v\frac{du}{dx}$

(xxvi) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

SOME ILUSTRATIONS :

****Q**. If $f(x) = \begin{cases} 3ax+b, & \text{if } x > 1 \\ 11 & \text{if } x = 1 \\ 5ax-2b, & \text{if } x < 1 \end{cases}$, continuous at x = 1, find the values of a and b.

Sol. $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$.........(i)

$\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0}\left[5a(1-h)-2b\right] = 5a-2b$

$\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0}\left[3a(1+h)+b\right] = 3a+b$

$f(1) = 11$

From (i) $3a+b = 5a-2b = 11$ and solution is $a = 3, b = 2$

Q. Find the relationship between a and b so that the function defined by $f(x) = \begin{cases} ax+1 & ,\text{if } x \le 3 \\ bx+3 & ,\text{if } x > 3 \end{cases}$ is continuous at x = 3.

Sol. $\because$ f(x) is cont. at x = 3 $\Rightarrow \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3)$.........(i)

$\lim_{x\to 3^-} f(x) = \lim_{h\to 0} f(3-h) = \lim_{h\to 0}\left[a(3-h)+1\right] = 3a+1$

$\lim_{x\to 3^+} f(x) = \lim_{h\to 0} f(3+h) = \lim_{h\to 0}\left[b(3+h)+3\right] = 3b+3$

$f(3) = 3a+1$

From (i) $3a+1 = 3b+3 = 3a+1 \quad \Rightarrow 3a+1 = 3b+3$

$\Rightarrow$ $\mathbf{3a-3b=2}$ is the required relation between a and b

**If $y = (\log_e x)^x + x^{\log_e x}$ find $\frac{dy}{dx}$.

Sol. $y = (\log_e x)^x + x^{\log_e x} = e^{\log\{(\log_e x)^x\}} + e^{\log\{x^{\log_e x}\}}$

$= e^{x\log\{(\log_e x)\}} + e^{\log_e x.\log_e x}$

$\frac{dy}{dx} = e^{x\log\{(\log_e x)\}}\left[x.\frac{1}{\log x}.\frac{1}{x} + \log(\log x).1\right] + e^{\log_e x.\log_e x}\left[\frac{\log x}{x} + \frac{\log x}{x}\right]$

$= (\log_e x)^x\left[\frac{1}{\log x} + \log(\log x)\right] + x^{\log_e x}\left[2\frac{\log x}{x}\right]$

**If $x = a(\theta - \sin\theta)$, $y = a(1+\cos\theta)$, find $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{2}$

Sol. $x = a(\theta - \sin\theta) \quad \Rightarrow \frac{dx}{d\theta} = a(1-\cos\theta)$

$y = a(1+\cos\theta) \quad \Rightarrow \frac{dy}{d\theta} = a(-\sin\theta)$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a(-\sin\theta)}{a(1-\cos\theta)} = -\frac{2\sin\theta/2.\cos\theta s\theta}{2\sin^2\theta/2} = -\cot\frac{\theta}{2}$

$\frac{d^2y}{dx^2} = \text{cosec}^2\frac{\theta}{2}.\frac{1}{2}.\frac{d\theta}{dx} = \frac{1}{2}\text{cosec}^2\frac{\theta}{2}.\frac{1}{a(1-\cos\theta)}$

$\left(\frac{d^2y}{dx^2}\right)_{\theta=\frac{\pi}{2}} = \frac{1}{2}\text{cosec}^2\frac{\pi}{4}.\frac{1}{a\left(1-\cos\frac{\pi}{2}\right)} = \frac{1}{2}.2.\frac{1}{a} = \frac{1}{a}$

** If $y=\sin\left(m\sin^{-1}x\right)$, prove that $\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+m^2y=0$

Sol. $y=\sin\left(m\sin^{-1}x\right) \Rightarrow \frac{dy}{dx}=\cos\left(m\sin^{-1}x\right).\frac{m}{\sqrt{1-x^2}}$

$\Rightarrow \sqrt{1-x^2}\frac{dy}{dx}=m\cos\left(m\sin^{-1}x\right)$

Again diff.w.r.t.x, $\sqrt{1-x^2}\frac{d^2y}{dx^2}+\frac{dy}{dx}\left(\frac{-2x}{\sqrt{1-x^2}}\right)=-m\sin\left(m\sin^{-1}x\right).\frac{m}{\sqrt{1-x^2}}$

$\Rightarrow \left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=-m^2\sin\left(m\sin^{-1}x\right)=-m^2y$

$\Rightarrow \left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+m^2y=0$

SHORT ANSWER TYPE QUESTIONS

1. Discuss the continuity of the function $f(x)=\begin{cases}2x-3, & \text{if } x<2\\ 5x-9, & \text{if } x\geq 2\end{cases}$.

2. Discuss the continuity of the greatest integer function f (x) = [x] at integral points.

3. Discuss the continuity of the identity function f (x) = x.

4. Discuss the continuity of a polynomial function.

5. Find the points of discontinuity of the function f defined by $f(x)=\begin{cases}3, & \text{if } 0\leq x\leq 1\\ 4, & \text{if } 1<x<3\\ 5, & \text{if } 3\leq x\leq 10\end{cases}$

6. Find the number of points at which the function $f(x)=\frac{9-x^2}{9x-x^3}$ is discontinuous.

7. Discuss the continuity of $f(x)=\begin{cases}x^{10}-1, & \text{if } x\leq 1\\ x^2, & \text{if } x>1\end{cases}$, at x = 1.

8. Discuss the continuity of modulus function f(x)= |x – 2|.

9. Discuss the continuity of the function f(x) is defined as $f(x)=\begin{cases}\frac{x}{\sqrt{x^2}}, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$ at x = 0.

10. Find the value of k for which $f(x)=\begin{cases}\frac{\sin 2x}{5x}, & x\neq 0\\ k, & x=0\end{cases}$ is continuous at x = 0.

11. Discuss the continuity of the function $f(x)=\begin{cases}\frac{x}{\sqrt{x^2}}, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$ at x = 0.

12. Find the value of k for which $f(x)=\begin{cases}\frac{1-\cos 4x}{2x^2}, & x\neq 0\\ k, & x=0\end{cases}$ is continuous at x = 0.

13. The value of k for which $f(x) = \begin{cases} \frac{kx}{|x|}, \text{if } x < 0 \\ 3, \text{if } x \geq 0 \end{cases}$ is continuous at x = 0 is :

14. Discuss the differentiability of the greatest integer function defined by f(x) = [x], 0 < x < 3 at x =1.

15. Discuss the differentiability of the function f(x) = |x – 2| at x = 2.

16. Find : $\frac{d}{dx}\left[\sin^2\left(\sqrt{\cos x}\right)\right]$

17. Find : $\frac{d}{dx}\left[\log\sin\sqrt{x^2+1}\right]$

18. Find : $\frac{d}{dx}\left[2^{-x}\right]$

19. Find : $\frac{d}{dx}\left[e^{1+\log_e x}\right]$

20. Find : $\frac{d}{dx}\left[2^{\cos^2 x}\right]$

21. Find : $\frac{d}{dx}\left[\log_e \tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right]$

22. Find : $\frac{d}{dx}\left[\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)\right]$

23. Find : $\frac{d}{dx}\left[\sin^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)\right]$

24. Find : $\frac{d}{dx}\left[\tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right)\right], \text{where } 0 < x < \frac{\pi}{4}$

25. Find : $\frac{d}{dx}\left[\sin^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right)\right]$

26. Find : $\frac{d}{dx}\left[x^{\sin x}\right]$

27. Find : $\frac{d}{dx}\left[x^{x^x}\right]$

ANSWERS

1. Continuous for all real values of x

2. Continuous everywhere

3. Continuous everywhere

4. Continuous everywhere

5. 1, 3

6. Exactly at two points

7. Continuous at x = 1

8. Continuous everywhere

9. Discontinuous at x = 0

10. $\frac{2}{5}$

11. Discontinuous at x = 0

12. 4

13. k = –3

14. not differentiable at x = 1

15. not differentiable at x = 2

16. $-\frac{2\sin x.\sin(\sqrt{\cos x}).\cos(\sqrt{\cos x})}{2(\sqrt{\cos x})}$

17. $\frac{x\cos\sqrt{x^2+1}}{\sqrt{x^2+1}.\sin\sqrt{x^2+1}}$

18. $-\frac{1}{2^x}\log 2$

19. e

20. $-2^{\cos^2 x}.\log 2.\sin 2x$

21. sec x

22. $\frac{1}{2(1+x^2)}$

23. $-\frac{1}{1+x^2}$

24. $\frac{1}{2}$

25. 1

26. $x^{\sin x}\left(\cos x.\log_e x+\frac{\sin x}{x}\right)$

27. $x^{x^x}.x^x\left[(1+\log x)\log x+\frac{1}{x}\right]$

LONG ANSWER TYPE QUESTIONS

1. Find the value of k for which f(x) = $\begin{cases} \dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & -1 \le x < 0 \\ \dfrac{2x+1}{x-1}, & 0 \le x \le 1 \end{cases}$ is continuous at x = 0.

2. Find the value of k for which f(x) = $\begin{cases} \dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & -1 \le x < 0 \\ \dfrac{2x+1}{x-1}, & 0 \le x \le 1 \end{cases}$ is continuous at x = 0 .

3. Find the value of k for which f(x) = $\begin{cases} \dfrac{(x+3)^2-36}{x-3}, & \text{if } x \ne 0 \\ k, & \text{if } x = 0 \end{cases}$ is continuous at x = 3 .

4. Find the value of k for which f(x) = $\begin{cases} kx+1, & \text{if } x \le \pi \\ \cos x, & \text{if } x > \pi \end{cases}$ is continuous at x = π .

5. Find the value of k for which f(x) = $\begin{cases} \dfrac{\sin 5x}{x^2+2x}, & \text{if } x \ne 0 \\ k+1, & \text{if } x = 0 \end{cases}$ is continuous at x = 0 .

6. If f(x) = $\begin{cases} 3ax+b, & \text{if } x > 1 \\ 11 & \text{if } x = 1 \\ 5ax-2b, & \text{if } x < 1 \end{cases}$, continuous at x = 1,find the values of a and b.

7. Determine a, b, c so that f(x) = $\begin{cases} \dfrac{\sin(a+1)x+\sin x}{x}, & x < 0 \\ c, & x = 0 \\ \dfrac{\sqrt{x+bx^2}-\sqrt{x}}{bx^{3/2}}, & x > 0 \end{cases}$ is continuous at x = 0.

8. If f(x) = $\begin{cases} \dfrac{k\cos x}{\pi-2x}, & x \ne \dfrac{\pi}{2} \\ 3, & x = \dfrac{\pi}{2} \end{cases}$, is continuous at x = $\frac{\pi}{2}$, find k.

9. Show that the function f defined by f(x) = $\begin{cases} 3x-2, & 0 < x \le 1 \\ 2x^2-x, & 1 < x \le 2 \\ 5x-4, & x > 2 \end{cases}$ is continuous at x = 2 but not differentiable .

10. Find the relationship between a and b so that the function defined by

f(x) = $\begin{cases} ax+1, & \text{if } x \le 3 \\ bx+3, & \text{if } x > 3 \end{cases}$ is continuous at x = 3.

11. For what value of λ the function f(x) = $\begin{cases} \lambda(x^2-2x), & \text{if } x \le 0 \\ 4x+1, & \text{if } x > 0 \end{cases}$ is continuous at x = 0.

12. If $f(x) = \begin{cases} \dfrac{x-4}{|x-4|} + a, & \text{if } x < 4 \\ a + b & \text{if } x = 4 \\ \dfrac{x-4}{|x-4|} + b & \text{if } x > 4 \end{cases}$ is continuous at x = 4, find a, b.

13. If the function $f(x) = \begin{cases} x^2 + ax + b, & \text{if } 0 \le x < 2 \\ 3x + 2, & \text{if } 2 \le x \le 4 \\ 2ax + 5b, & \text{if } 4 < x \le 8 \end{cases}$ is continuous on [0 , 8], find the value of a & b.

14. If $f(x) = \begin{cases} \dfrac{1-\sin^3 x}{3\cos^2 x}, & \text{if } x < \dfrac{\pi}{2} \\ a & \text{if } x = \dfrac{\pi}{2} \\ \dfrac{b(1-\sin x)}{(\pi - 2x)^2} & \text{if } x > \dfrac{\pi}{2} \end{cases}$ is continuous at $x = \frac{\pi}{2}$, find a, b.

15. Discuss the continuity of $f(x) = |x-1| + |x-2|$ at x = 1 & x = 2.

16. If $y = (\log_e x)^x + x^{\log_e x}$ find $\dfrac{dy}{dx}$.

17. If $x = a(\theta - \sin\theta)$, $y = a(1 + \cos\theta)$, find $\dfrac{d^2y}{dx^2}$ at $\theta = \dfrac{\pi}{2}$

18. If $x = a\left(\cos\theta + \log\tan\dfrac{\theta}{2}\right)$ and $y = a\sin\theta$ find $\dfrac{dy}{dx}$ at $\theta = \dfrac{\pi}{4}$.

19. If $y = \sin(m\sin^{-1} x)$, prove that $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + m^2 y = 0$

20. If $x = \sqrt{a^{\sin^{-1} t}}$, $y = \sqrt{a^{\cos^{-1} t}}$, show that $\dfrac{dy}{dx} = -\dfrac{y}{x}$.

21. If $y = \left(x + \sqrt{x^2 + a^2}\right)^n$, prove that $\dfrac{dy}{dx} = \dfrac{ny}{\sqrt{x^2 + a^2}}$

22. If $\log_e \sqrt{x^2 + y^2} = \tan^{-1}\left(\dfrac{y}{x}\right)$, prove that $\dfrac{dy}{dx} = \dfrac{x+y}{x-y}$.

23. If $x^m . y^n = (x+y)^{m+n}$, prove that $\dfrac{dy}{dx} = \dfrac{y}{x}$

24. If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, $-1 < x < 1$, prove that $\dfrac{dy}{dx} = -\dfrac{1}{(1+x)^2}$

25. If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, show that $\dfrac{dy}{dx} = \sqrt{\dfrac{1-y^2}{1-x^2}}$

26. If $y = \sqrt{x^2+1} - \log\left(\dfrac{1}{x} + \sqrt{1 + \dfrac{1}{x^2}}\right)$, find $\dfrac{dy}{dx}$.

27. If $x = \alpha\sin 2t(1 + \cos 2t)$ and $y = \beta\cos 2t(1 - \cos 2t)$, show that $\dfrac{dy}{dx} = \dfrac{\beta}{\alpha}\tan t$.

28. If $y = \sqrt{x+1} - \sqrt{x-1}$, prove that $(x^2-1)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - \dfrac{1}{4}y = 0$

29. If $y = \sqrt{x + \sqrt{x + \sqrt{x + ...\infty}}}$, then find $\frac{dy}{dx}$.

30. If $(\cos x)^y = (\sin y)^x$, then find $\frac{dy}{dx}$.

LONG ANSWER TYPE QUESTIONS
ANSWERS

1. $k = -\frac{1}{2}$ **2.** $k = -1$ **3.** $k = 12$

4. $k = -\frac{2}{\pi}$ **5.** $k = \frac{3}{2}$ **6.** $a = 3, b = 2$

7. $a = -3/2,\ c = 1/2$, b is any non-zero real number **8.** $k = 6$

10. $\mathbf{3a - 3b = 2}$ is the relation between a and b

11. there is no value of λ for which f(x) is contonuous at 0.

12. $a = 1, b = -1$ **13.** $a = 3, b = -2$ **14.** $a = \frac{1}{2}, b = 4$

15. continuous at x = 1 & x = 2. **16.** $(\log x)^x\left[\frac{1}{\log x} + \log(\log x)\right] + x^{\log x}\left[2\frac{\log x}{x}\right]$

17. $\frac{1}{a}$ **18.** 1 **26.** $\frac{\sqrt{x^2+1}}{x}$

29. $\frac{1}{2y-1}$ **30.** $\frac{\log\sin y + y\tan x}{(\log\cos x - x\cot y)}$