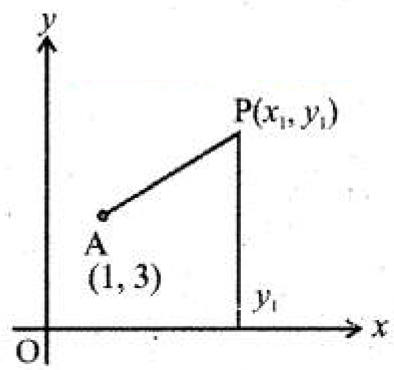
Chapter – 3

Analytical Geometry

Ex 3.1

Question 1.
Find the locus of a point which is equidistant from (1, 3) and x-axis.

Solution:



Let P(x_1, y_1) be any point on the locus.
Let A be the point (1, 3)
The distance from the x-axis on the moving pint P(x_1, y_1) is y_1.
Given that AP = y_1

$AP^2 = y_1^2$

$$(x_1 - 1)^2 + (y_1 - 3)^2 = y_1^2$$

$$x_1^2 - 2x_1 + 1 + y_1^2 - 6y_1 + 9 = y_1^2$$

∴ The locus of the point (x_1, y_1) is $x^2 - 2x - 6y + 10 = 0$

Question 2.
A point moves so that it is always at a distance of 4 units from the point (3, -2).

Solution:
Let P(x_1, y_1) be any point on the locus.
Let A be the point (3, -2)
Given that PA = 4
$PA^2 = 16$

$$(x_1 - 3)^2 + (y_1 + 2)^2 = 16$$
$$x_1^2 - 6x_1 + 9 + y_1^2 + 4y_1 + 4 = 16$$
$$x_1^2 + y_1^2 - 6x_1 + 4y_1 - 3 = 0$$

∴ The locus of the point (x_1, y_1) is $x^2 + y^2 - 6x + 4y - 3 = 0$

Question 3.

If the distance of a point from the points (2, 1) and (1, 2) are in the ratio 2 : 1, then find the locus of the point.

Solution:

Let $P(x_1, y_1)$ be any point on the locus.

Let A(2, 1) and B(1, 2) be the given point.

Given that PA : PB = 2 : 1

i.e., $\frac{PA}{PB} = \frac{2}{1}$

PA = 2PB

$PA^2 = 4PB^2$

$$(x_1 - 2)^2 + (y_1 - 1)^2 = 4[(x_1 - 1)^2 + (y_1 - 2)^2]$$
$$x_1^2 - 4x_1 + 4 + y_1^2 - 2y_1 + 1 = 4[x_1^2 - 2x_1 + 1 + y_1^2 - 4y_1 + 4]$$
$$x_1^2 + y_1^2 - 4x_1 - 2y_1 + 5 = 4x_1^2 - 8x_1 + 4y_1^2 - 16y_1 + 20$$
$$-3x_1^2 - 3y_1^2 + 4x_1 + 14y_1 - 15 = 0$$
$$\therefore 3x_1^2 + 3y_1^2 - 4x_1 - 14y_1 + 15 = 0$$

∴ The locus of the point (x_1, y_1) is $3x^2 + 3y^2 - 4x - 14y + 15 = 0$

Question 4.

Find a point on the x-axis which is equidistant from the points (7, -6) and (3, 4).

Solution:

Let $P(x_1, 0)$ be any point on the x-axis.

Let A(7, -6) and B(3, 4) be the given points.

Given that PA = PB

$PA^2 = PB^2$

$$(x_1 - 7)^2 + (0 + 6)^2 = (x_1 - 3)^2 + (0 - 4)^2$$
$$x_1^2 - 14x_1 + 49 + 36 = x_1^2 - 6x_1 + 9 + 16$$
$$-14x_1 + 6x_1 = 25 - 85$$
$$-8x_1 = -60$$
$$x_1 = \frac{-60}{-8} = \frac{15}{2}$$

∴ The required point is $\left(\frac{15}{2}, 0\right)$

Question 5.
If A(-1, 1) and B(2, 3) are two fixed points, then find the locus of a point P so that the area of triangle APB = 8 sq. units.

Solution:
Let the point $P(x_1, y_1)$.
Fixed points are A(-1, 1) and B(2, 3).
Given area (formed by these points) of the triangle APB = 8
⇒ $1/2\ [x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] = 8$
⇒ $1/2\ [x_1(1 - 3) + (-1)(3 - y_1) + 2(y_1 - 1)] = 8$
⇒ $1/2\ [-2x_1 - 3 + y_1 + 2y_1 - 2] = 8$
⇒ $1/2\ [-2x_1 + 3y_1 - 5] = 8$
⇒ $-2x_1 + 3y_1 - 5 = 16$
⇒ $-2x_1 + 3y_1 - 21 = 0$
⇒ $2x_1 - 3y_1 + 21 = 0$
∴ The locus of the point $P(x_1, y_1)$ is $2x - 3y + 21 = 0$.

Ex 3.2

Question 1.
Find the angle between the lines whose slopes are 1/2 and 3.

Solution:
Given that $m_1 = 1/2$ and $m_2 = 3$.
Let θ be the angle between the lines then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$= \left|\frac{\frac{1}{2} - 3}{1 + \frac{1}{2} \times 3}\right| = \left|\frac{\frac{1-6}{2}}{1 + \frac{3}{2}}\right| = \left|\frac{-\frac{5}{2}}{\frac{5}{2}}\right| = |-1|$$

$\tan\theta = 1$

$\tan\theta = \tan 45°$

$\therefore \theta = 45°$

Question 2.

Find the distance of the point (4, 1) from the line $3x - 4y + 12 = 0$.

Solution:

The length of perpendicular from a point (x_1, y_1) to the line $ax + by + c = 0$ is

$$d = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$

$\therefore$ The distance of the point (4, 1) to the line $3x - 4y + 12 = 0$ is

[Here $(x_1, y_1) = (4, 1)$, $a = 3$, $b = -4$, $c = 12$]

$$d = \left|\frac{3(4) - 4(1) + 12}{\sqrt{3^2 + (-4)^2}}\right|$$

$$= \left|\frac{12 - 4 + 12}{\sqrt{9 + 16}}\right| = \left|\frac{20}{5}\right| = 4 \text{ units}$$

Question 3.

Show that the straight lines $x + y - 4 = 0$, $3x + 2 = 0$ and $3x - 3y + 16 = 0$ are concurrent.

Solution:

The lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_3x + b_3y + c_3 = 0$ are concurrent if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

The given lines x + y – 4 = 0, 3x + 0y + 2 = 0, 3x – 3y + 16 = 0

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -4 \\ 3 & 0 & 2 \\ 3 & -3 & 16 \end{vmatrix}$$

= 1(0 + 6) – 1(48 – 6) – 4(-9 – 0)

= 6 – (42) + 36

= 42 – 42

= 0

The given lines are concurrent.

Question 4.
Find the value of 'a' for which the straight lines 3x + 4y = 13; 2x – 7y = -1 and ax – y – 14 = 0 are concurrent.

Solution:
The lines 3x + 4y = 13, 2x – 7y = -1 and ax – y – 14 = 0 are concurrent.

$$\begin{vmatrix} 3 & 4 & -13 \\ 2 & -7 & 1 \\ a & -1 & -14 \end{vmatrix} = 0$$

⇒ 3(98 + 1) – 4(-28 – a) – 13(-2 + 7a) = 0

⇒ 3(99) + 112 + 4a + 26 – 91a = 0

⇒ 297 + 112 + 26 + 4a – 91a = 0

⇒ 435 – 87a = 0

⇒ -87a = -435

⇒ a = $\frac{-435}{-87}$ = 5

Question 5.
A manufacturer produces 80 TV sets at a cost of ₹ 2,20,000 and 125 TV sets at a cost of ₹ 2,87,500. Assuming the cost curve to be linear, find the linear expression of the given information. Also, estimate the cost of 95 TV sets.

Solution:
Let x represent the TV sets, and y represent the cost.

TV (x)	Cost (y)
80 (x_1)	2,20,000
125 (x_2)	2,87,500

The equation of straight line expressing the given information as a linear equation in x and y is

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$\frac{y-2,20,000}{2,87,500-2,20,000} = \frac{x-80}{125-80}$$

$$\frac{y-2,20,000}{67,500} = \frac{x-80}{45}$$

$$\frac{y-2,20,000}{1,500} = \frac{x-80}{1}$$

1(y – 2,20,000) = (x – 80)1500
y – 2,20,000 = 1500x – 80 × 1500
y = 1500x – 1,20,000 + 2,20,000
y = 1500x + 1,00,000 which is the required linear expression.
When x = 95,
y = 1,500 × 95 + 1,00,000
= 1,42,500 + 1,00,000
= 2,42,500
∴ The cost of 95 TV sets is ₹ 2,42,500.

Ex 3.3

Question 1.
If the equation $ax^2 + 5xy - 6y^2 + 12x + 5y + c = 0$ represents a pair of perpendicular straight lines, find a and c.

Solution:
Comparing $ax^2 + 5xy - 6y^2 + 12x + 5y + c = 0$ with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$
We get a = a, 2h = 5, (or) h = 5/2, b = -6, 2g = 12 (or) g = 6, 2f = 5 (or) f = 5/2, c = c
Condition for pair of straight lines to be perpendicular is a + b = 0
a + (-6) = 0
a = 6
Next to find c. Condition for the given equation to represent a pair of straight lines is

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

$$\begin{vmatrix} 6 & \frac{5}{2} & 6 \\ \frac{5}{2} & -6 & \frac{5}{2} \\ 6 & \frac{5}{2} & c \end{vmatrix} = 0$$

$$\begin{vmatrix} 0 & 0 & 6-c \\ \frac{5}{2} & -6 & \frac{5}{2} \\ 6 & \frac{5}{2} & c \end{vmatrix} = 0$$

$R_1 \to R_1 - R_3$
Expanding along first row we get 0 – 0 + (6 – c) [25/4 + 36] = 0
(6-c) [25/4 + 36] = 0
6 – c = 0
6 = c (or) c = 6

Question 2.
Show that the equation $12x^2 - 10xy + 2y^2 + 14x - 5y + 2 = 0$ represents a pair of straight lines and also find the separate equations of the straight lines.

Solution:

Comparing $12x^2 - 10xy + 2y^2 + 14x - 5y + 2 = 0$ with $ax^2 + 2hxy + by^2 + 2gh + 2fy + c = 0$

We get a = 12, 2h = -10, (or) h = -5, b = 2, 2g = 14 (or) g = 7, 2f = -5 (or) $f = -\frac{5}{2}$, c = 2

Condition for the given equation to represent a pair of straight lines is

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 12 & -5 & 7 \\ -5 & 2 & \frac{-5}{2} \\ 7 & \frac{-5}{2} & 2 \end{vmatrix}$$

$$= \frac{1}{2} \times \frac{1}{2} \begin{vmatrix} 12 & -5 & 7 \\ -10 & 4 & -5 \\ 14 & -5 & 4 \end{vmatrix} \begin{matrix} R_2 \to 2R_2 \\ R_3 \to 2R_3 \end{matrix}$$

$= \frac{1}{4}$ [12(16 – 25) + 5(-40 + 70) + 7(50 – 56)]

$= \frac{1}{4}$ [12(-9) + 5(30) + 7(-6)]

$= \frac{1}{4}$ [-108 + 150 – 42]

$= \frac{1}{4}$ [0]

= 0

∴ The given equation represents a pair of straight lines.

Consider $12x^2 - 10xy + 2y^2 = 2[6x^2 - 5xy + y^2] = 2[(3x - y)(2x - y)] = (6x - 2y)(2x - y)$

Let the separate equations be 6x – 2y + l = 0, 2x – y + m = 0

To find l, m

Let $12x^2 - 10xy + 2y^2 + 14x - 5y + 2 = (6x - 2y + l)(2x - y + m)$ .... (1)

Equating coefficient of y on both sides of (1) we get

$2l + 6m = 14$ (or) $l + 3m = 7$ .... (2)

Equating coefficient of x on both sides of (1) we get

$-l - 2m = -5$ .... (3)

(2) + (3) $\Rightarrow m = 2$

Using $m = 2$ in (2) we get
$l + 3(2) = 7$
$l = 7 - 6$
$l = 1$
∴ The separate equations are $6x - 2y + 1 = 0$, $2x - y + 2 = 0$.

Question 3.
Show that the pair of straight lines $4x^2 + 12xy + 9y^2 - 6x - 9y + 2 = 0$ represents two parallel straight lines and also find the separate equations of the straight lines.

Solution:
The given equation is $4x^2 + 12xy + 9y^2 - 6x - 9y + 2 = 0$
Here $a = 4$, $2h = 12$, (or) $h = 6$ and $b = 9$
$h^2 - ab = 6^2 - 4 \times 9 = 36 - 36 = 0$
∴ The given equation represents a pair of parallel straight lines
Consider $4x^2 + 12xy + 9y^2 = (2x)^2 + 12xy + (3y)^2$
$= (2x)^2 + 2(2x)(3y) + (3y)^2$
$= (2x + 3y)^2$
Here we have repeated factors.
Now consider, $4x^2 + 12xy + 9y^2 - 6x - 9y + 2 = 0$
$(2x + 3y)^2 - 3(2x + 3y) + 2 = 0$
$t^2 - 3t + 2 = 0$ where $t = 2x + 3y$
$(t - 1)(t - 2) = 0$
$(2x + 3y - 1)(2x + 3y - 2) = 0$
∴ Separate equations are $2x + 3y - 1 = 0$, $2x + 3y - 2 = 0$

Question 4.
Find the angle between the pair of straight lines $3x^2 - 5xy - 2y^2 + 17x + y + 10 = 0$.

Solution:
The given equation is $3x^2 - 5xy - 2y^2 + 17x + y + 10 = 0$
Here a = 3, 2h = -5, b = -2
If θ is the angle between the given straight lines then

$$\theta = \tan^{-1}\left[\left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|\right] = \tan^{-1}\left[\left|\frac{2\sqrt{\left(\frac{-5}{2}\right)^2 - 3(-2)}}{3+(-2)}\right|\right]$$

$$= \tan^{-1}\left[\left|\frac{2\sqrt{\frac{25}{4}+6}}{1}\right|\right] = \tan^{-1}\left[\left|2\sqrt{\frac{25+24}{4}}\right|\right]$$

$$= \tan^{-1}\left[\left|2\times\sqrt{\frac{49}{4}}\right|\right] = \tan^{-1}\left[\left|2\times\frac{7}{2}\right|\right] = \tan^{-1}(7)$$

Ex 3.4

Question 1.
Find the equation of the following circles having
(i) the centre (3, 5) and radius 5 units.
(ii) the centre (0, 0) and radius 2 units.

Solution:
(i) Equation of the circle is $(x - h)^2 + (y - k)^2 = r^2$
Centre (h, k) = (3, 5) and radius r = 5
∴ Equation of the circle is $(x - 3)^2 + (y - 5)^2 = 5^2$
⇒ $x^2 - 6x + 9 + y^2 - 10y + 25 = 25$
⇒ $x^2 + y^2 - 6x - 10y + 9 = 0$

(ii) Equation of the circle when centre origin (0, 0) and radius r is $x^2 + y^2 = r^2$
⇒ $x^2 + y^2 = 2^2$
⇒ $x^2 + y^2 = 4$
⇒ $x^2 + y^2 - 4 = 0$

Question 2.
Find the centre and radius of the circle
(i) $x^2 + y^2 = 16$
(ii) $x^2 + y^2 - 22x - 4y + 25 = 0$

(iii) $5x^2 + 5y^2 + 4x - 8y - 16 = 0$
(iv) $(x + 2)(x - 5) + (y - 2)(y - 1) = 0$

Solution:
(i) $x^2 + y^2 = 16$
$\Rightarrow x^2 + y^2 = 4^2$
This is a circle whose centre is origin (0, 0), radius 4.

(ii) Comparing $x^2 + y^2 - 22x - 4y + 25 = 0$ with general equation of circle $x^2 + y^2 + 2gx + 2fy + c = 0$
We get $2g = -22$, $2f = -4$, $c = 25$
$g = -11$, $f = -2$, $c = 25$
Centre = $(-g, -f) = (11, 2)$

$$\text{Radius} = \sqrt{g^2 + f^2 - c}$$
$$= \sqrt{(-11)^2 + (-2)^2 - 25}$$
$$= \sqrt{121 + 4 - 25} = \sqrt{100} = 10$$

(iii) $5x^2 + 5y^2 + 4x - 8y - 16 = 0$
To make coefficient of x^2 unity, divide the equation by 5 we get,

$$x^2 + y^2 + \frac{4}{5}x - \frac{8}{5}y - \frac{16}{5} = 0$$

Comparing the above equation with $x^2 + y^2 + 2gx + 2fy + c = 0$ we get,

$$2g = \frac{4}{5}, 2f = -\frac{8}{5}, c = \frac{-16}{5}$$

$$\therefore g = \frac{2}{5}, f = -\frac{4}{5}, c = \frac{-16}{5}$$

$$\text{Centre} = (-g, -f) = \left(\frac{-2}{5}, \frac{4}{5}\right)$$

$$\text{Radius} = \sqrt{g^2 + f^2 - c^2} = \sqrt{\left(\frac{2}{5}\right)^2 + \left(\frac{-4}{5}\right)^2 - \left(\frac{-16}{5}\right)}$$

$$= \sqrt{\frac{4}{25} + \frac{16}{25} + \frac{16}{5}} = \sqrt{\frac{4 + 16 + 16 \times 5}{25}} = \sqrt{\frac{20 + 80}{25}} = \sqrt{\frac{100}{25}} = \sqrt{4} = 2$$

(iv) Equation of the circle is $(x + 2)(x - 5) + (y - 2)(y - 1) = 0$
$x^2 - 3x - 10 + y^2 - 3y + 2 = 0$
$x^2 + y^2 - 3x - 3y - 8 = 0$
Comparing this with $x^2 + y^2 + 2gx + 2fy + c = 0$
We get $2g = -3, 2f = -3, c = -8$

$g = \frac{-3}{2}, f = \frac{-3}{2}, c = -8$

Centre $(-g, -f) = \left(\frac{3}{2}, \frac{3}{2}\right)$

$$\text{Radius} = \sqrt{g^2 + f^2 - c^2} = \sqrt{\frac{9}{4} + \frac{9}{4} + 8} = \sqrt{\frac{18}{4} + 8}$$

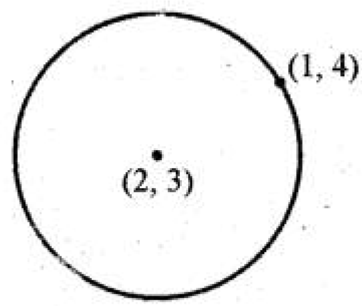
$$= \sqrt{\frac{9}{2} + 8} = \sqrt{\frac{9 + 16}{2}} = \sqrt{\frac{25}{2}} = \frac{5}{\sqrt{2}}$$

Question 3.
Find the equation of the circle whose centre is (-3, -2) and having circumference 16π.

Solution:
Circumference, $2\pi r = 16\pi$
$\Rightarrow 2r = 16$
$\Rightarrow r = 8$
Equation of the circle when centre and radius are known is $(x - h)^2 + (y - k)^2 = r^2$
$\Rightarrow (x + 3)^2 + (y + 2)^2 = 8^2$
$\Rightarrow x^2 + 6x + 9 + y^2 + 4y + 4 = 64$
$\Rightarrow x^2 + y^2 + 6x + 4y + 13 = 64$
$\Rightarrow x^2 + y^2 + 6x + 4y - 51 = 0$

Question 4.
Find the equation of the circle whose centre is (2, 3) and which passes through (1, 4).

Solution:



Centre (h, k) = (2, 3)

$$\text{Radius} = \sqrt{(1-2)^2 + (4-3)^2}$$
$$= \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

Equation of the circle with centre (h, k) and radius r is $(x - h)^2 + (y - k)^2 = r^2$

$\Rightarrow (x - 2)^2 + (y - 3)^2 = (\sqrt{2})^2$

$\Rightarrow x^2 - 4x + 4 + y^2 - 6y + 9 = 2$

$\Rightarrow x^2 + y^2 - 4x - 6y + 11 = 0$

Question 5.

Find the equation of the circle passing through the points (0, 1), (4, 3) and (1, -1).

Solution:

Let the required of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ ……… (1)

It passes through (0, 1)

$0 + 1 + 2g(0) + 2f(1) + c = 0$

$1 + 2f + c = 0$

$2f + c = -1$ …….. (2)

Again the circle (1) passes through (4, 3)

$4^2 + 3^2 + 2g(4) + 2f(3) + c = 0$

$16 + 9 + 8g + 6f + c = 0$

$8g + 6f + c = -25$ …….. (3)

Again the circle (1) passes through (1, -1)

$1^2 + (-1)^2 + 2g(1) + 2f(-1) + c = 0$

$1 + 1 + 2g - 2f + c = 0$

$2g - 2f + c = -2$ ……… (4)

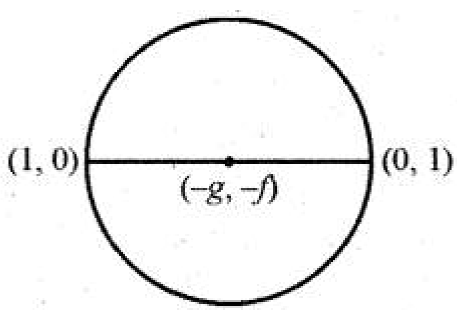
$8g + 6f + c = -25$

(4) × 4 subtracting we get, $8g - 8f + 4c = -8$

$14f - 3c = -17$ ………. (5)

14f – 3c = -17
(2) × 3 ⇒ 6f + 3c = -3
Adding we get 20f = -20
f = -1
Using f = -1 in (2) we get, 2(-1) + c = -1
c = -1 + 2
c = 1
Using f = -1, c = 1 in (3) we get
8g + 6(-1)+1 = -25
8g – 6 + 1 = -25
8g – 5 = -25
8g = -20
$g = \frac{-20}{8} = \frac{-5}{2}$
using $g = \frac{-5}{2}$, f = -1, c = 1 in (1) we get the equation of the circle.
$x^2 + y^2 + 2(\frac{-5}{2})x + 2(-1)y + 1 = 0$
$x^2 + y^2 - 5x - 2y + 1 = 0$

Question 6.
Find the equation of the circle on the line joining the points (1, 0), (0, 1), and having its centre on the line x + y = 1.

Solution:


Let the equation of the circle be
$x^2 + y^2 + 2gx + 2fy + c = 0$ ……… (1)
The circle passes through (1, 0)
$1^2 + 0^2 + 2g(1) + 2f(0) + c = 0$
1 + 2g + c = 0
2g + c = 1 …….. (2)
Again the circle (1) passes through (0, 1)

$0^2 + 1^2 + 2g(0) + 2f(1) + c = 0$
$1 + 2f + c = 0$
$2f + c = -1$ .... (3)
(2) – (3) gives $2g - 2f = 0$ (or) $g - f = 0$ .... (4)
Given that the centre of the circle $(-g, -f)$ lies on the line $x + y = 1$
$-g - f = 1$ .... (5)

(4) + (5) gives $-2f = 1 \Rightarrow f = -\frac{1}{2}$

Using $f = -\frac{1}{2}$ in (5) we get

$-g - (-\frac{1}{2}) = 1$

$-g = 1 - -\frac{1}{2} = \frac{1}{2}$

$g = -\frac{1}{2}$

Using $g = -\frac{1}{2}$ in (2) we get

$2(-\frac{1}{2}) + c = -1$

$-1 + c = -1$

$c = 0$

using $g = -\frac{1}{2}$, $f = -\frac{1}{2}$, $c = 0$ in (1) we get the equation of the circle,

$x^2 + y^2 + 2(-\frac{1}{2})x + 2(-\frac{1}{2})y + 0 = 0$

$x^2 + y^2 - x - y = 0$

Question 7.
If the lines $x + y = 6$ and $x + 2y = 4$ are diameters of the circle, and the circle passes through the point (2, 6) then find its equation.

Solution:
To get coordinates of centre we should solve the equations of the diameters $x + y = 6$, $x + 2y = 4$.
$x + y = 6$ .... (1)
$x + 2y = 4$ .... (2)
(1) – (2) $\Rightarrow -y = 2$
$y = -2$
Using $y = -2$ in (1) we get $x - 2 = 6$
$x = 8$
Centre is (8, -2) the circle passes through the point (2, 6).

$$\therefore \text{ Radius} = \sqrt{(8-2)^2 + (-2-6)^2}$$
$$= \sqrt{6^2 + (-8)^2} = \sqrt{36+64} = \sqrt{100} = 10$$

Equation of the circle with centre (h, k) and radius r is $(x - h)^2 + (y - k)^2 = r^2$
$\Rightarrow (x - 8)^2 + (y + 2)^2 = 10^2$
$\Rightarrow x^2 + y^2 - 16x + 4y + 64 + 4 = 100$
$\Rightarrow x^2 + y^2 - 16x + 4y - 32 = 0$

Question 8.
Find the equation of the circle having (4, 7) and (-2, 5) as the extremities of a diameter.

Solution:
The equation of the circle when entremities (x_1, y_1) and (x_2, y_2) are given is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$
$\Rightarrow (x - 4)(x + 2) + (y - 7)(y - 5) = 0$
$\Rightarrow x^2 - 2x - 8 + y^2 - 12y + 35 = 0$
$\Rightarrow x^2 + y^2 - 2x - 12y + 27 = 0$

Question 9.
Find the Cartesian equation of the circle whose parametric equations are $x = 3 \cos \theta, y = 3 \sin \theta, 0 \leq \theta \leq 2\pi$.

Solution:
Given $x = 3 \cos \theta, y = 3 \sin \theta$
Now $x^2 + y^2 = 9 \cos^2\theta + 9 \sin^2\theta$
$x^2 + y^2 = 9 (\cos^2\theta + \sin^2\theta)$
$x^2 + y^2 = 9$ which is the Cartesian equation of the required circle.

Ex 3.5

Question 1.
Find the equation of the tangent to the circle $x^2 + y^2 - 4x + 4y - 8 = 0$ at (-2, -2).

Solution:
The equation of the tangent to the circle $x^2 + y^2 - 4x + 4y - 8 = 0$ at (x_1, y_1) is

$xx_1 + yy_1 - 4\frac{(x+x_1)}{2} + 4\frac{(y+y_1)}{2} - 8 = 0$

Here $(x_1, y_1) = (-2, -2)$

$\Rightarrow x(-2) + y(-2) - 2(x - 2) + 2(y - 2) - 8 = 0$

$\Rightarrow -2x - 2y - 2x + 4 + 2y - 4 - 8 = 0$

$\Rightarrow -4x - 8 = 0$

$\Rightarrow x + 2 = 0$

Question 2.
Determine whether the points P(1, 0), Q(2, 1) and R(2, 3) lie outside the circle, on the circle or inside the circle $x^2 + y^2 - 4x - 6y + 9 = 0$.

Solution:
The equation of the circle is $x^2 + y^2 - 4x - 6y + 9 = 0$

$PT^2 = x_1^2 + y_1^2 - 4x_1 - 6y_1 + 9$

At P(1, 0), $PT^2 = 1 + 0 - 4 - 0 + 9 = 6 > 0$

At Q(2, 1), $PT^2 = 4 + 1 - 8 - 6 + 9 = 0$

At R(2, 3), $PT^2 = 4 + 9 - 8 - 18 + 9 = -4 < 0$

The point P lies outside the circle.

The point Q lies on the circle.

The point R lies inside the circle.

Question 3.
Find the length of the tangent from (1, 2) to the circle $x^2 + y^2 - 2x + 4y + 9 = 0$.

Solution:
The length of the tangent from (x_1, y_1) to the circle $x^2 + y^2 - 2x + 4y + 9 = 0$ is

$\sqrt{x_1^2 + y_1^2 - 2x_1 + 4y_1 + 9}$

Length of the tangent from (1, 2) = $\sqrt{1^2 + 2^2 - 2(1) + 4(2) + 9}$

= $\sqrt{1 + 4 - 2 + 8 + 9}$

= $\sqrt{20}$

= $\sqrt{4 \times 5}$

= $2\sqrt{5}$ units

Question 4.
Find the value of P if the line 3x + 4y – P = 0 is a tangent to the circle $x^2 + y^2 = 16$.

Solution:
The condition for a line y = mx + c to be a tangent to the circle $x^2 + y^2 = a^2$ is $c^2 = a^2 (1 + m^2)$
Equation of the line is 3x + 4y – P = 0
Equation of the circle is $x^2 + y^2 = 16$

4y = -3x + P

y = $\frac{-3}{4}x + \frac{P}{4}$

$\therefore$ m = $\frac{-3}{4}$, c = $\frac{P}{4}$

Equation of the circle is $x^2 + y^2 = 16$

$\therefore a^2 = 16$

Condition for tangency we have $c^2 = a^2(1 + m^2)$

$\Rightarrow \left(\frac{P}{4}\right)^2 = 16\left(1 + \frac{9}{16}\right)$

$\Rightarrow \frac{P^2}{16} = 16\left(\frac{25}{16}\right)$

$\Rightarrow P^2 = 16 \times 25$

$\Rightarrow P = \pm\sqrt{16}\sqrt{25}$

$\Rightarrow P = \pm 4 \times 5$

$\Rightarrow P = \pm 20$

Ex 3.6

Question 1.
Find the equation of the parabola whose focus is the point F(-1, -2) and the directrix is the line 4x – 3y + 2 = 0.

Solution:
F(-1, -2)
l : 4x – 3y + 2 = 0
Let P(x, y) be any point on the parabola.
FP = PM
⇒ $FP^2 = PM^2$

$$\Rightarrow (x+1)^2 + (y+2)^2 = \left[\frac{4x-3y+2}{\sqrt{4^2+(-3)^2}}\right]^2$$

$$\Rightarrow x^2 + 2x + 1 + y^2 + 4y + 4 = \frac{16x^2+9y^2+4-24xy+16x-12y}{(16+9)}$$

$$\Rightarrow 25(x^2 + y^2 + 2x + 4y + 5) = 16x^2 + 9y^2 - 24xy + 16x - 12y + 4$$

$$\Rightarrow (25-16)x^2 + (25-9)y^2 + 24xy + (50-16)x + (100+12)y + 125 - 4 = 0$$

$$\Rightarrow 9x^2 + 16y^2 + 24xy + 34x + 112y + 121 = 0$$

Question 2.
The parabola y^2 = kx passes through the point (4, -2). Find its latus rectum and focus.

Solution:
$y^2 = kx$ passes through (4, -2)
$(-2)^2 = k(4)$
⇒ 4 = 4k
⇒ k = 1

$y^2 = x = 4(\frac{1}{4})x$

$a = \frac{1}{4}$

Equation of LR is x = a or x – a = 0

i.e., $x = \frac{1}{4}$

⇒ 4x = 1

⇒ 4x – 1 = 0

Focus (a, 0) = $(\frac{1}{4}, 0)$

Question 3.
Find the vertex, focus, axis, directrix, and the length of the latus rectum of the parabola $y^2 - 8y - 8x + 24 = 0$.

Solution:
$y^2 - 8y - 8x + 24 = 0$
$\Rightarrow y^2 - 8y - 4^2 = 8x - 24 + 4^2$
$\Rightarrow (y - 4)^2 = 8x - 8$
$\Rightarrow (y - 4)^2 = 8(x - 1)$
$\Rightarrow (y - 4)^2 = 4(2)(x - 1)$
$\therefore a = 2$
$Y^2 = 4(2)X$ where X = x – 1 and Y = y – 4

	X, Y coordinates	x, y coordinates
Vertex (0, 0)	X = 0 Y = 0	$x - 1 = 0$, $x = 1$; $y - 4 = 0$, $y = 4$; (1, 4)
Focus (a, 0)	X = 2 Y = 0	$x - 1 = 2$, $x = 2 + 1 = 3$; $y - 4 = 0$, $y = 4$; (3, 4)
Axis x-axis	Y = 0	$y - 4 = 0$; $y = 4$
Directrix $x + a = 0$	X + 2 = 0	$x - 1 + 2 = 0$, $x + 1 = 0$; $x = -1$
Length of Latus rectum	4a = 8	

Question 4.
Find the co-ordinates of the focus, vertex, equation of the directrix, axis and

the length of latus rectum of the parabola (a) $y^2 = 20x$, (b) $x^2 = 8y$, (c) $x^2 = -16y$

Solution:
(a) $y^2 = 20x$
$y^2 = 4(5)x$
$\therefore a = 5$

Vertex	(0, 0)	(0, 0)
Focus	$(a, 0)$	(5, 0)
Axis	x-axis	$y = 0$
Directrix	$x + a = 0$	$x + 5 = 0$
Length of Latus rectum	$4a$	20

(b) $x^2 = 8y = 4(2)y$
$\therefore a = 2$

Vertex	(0, 0)	(0, 0)
Focus	$(0, a)$	(0, 2)
Axis	y-axis	$x = 0$
Directrix	$y + a = 0$	$y + 2 = 0$
Length of Latus rectum	$4a$	8

(c) $x^2 = -16y = -4(4)y$
$\therefore a = 4$

Vertex	(0, 0)	(0, 0)
Focus	$(0, -a)$	(0, –4)
Axis	y-axis	$x = 0$
Directrix	$y - a = 0$	$y - 4 = 0$
Length of Latus rectum	$4a$	16

Question 5.
The average variable cost of the monthly output of x tonnes of a firm producing a valuable metal is ₹ $1/5\ x^2 - 6x + 100$. Show that the average

variable cost curve is a parabola. Also, find the output and the average cost at the vertex of the parabola.

Solution:
Let output be x and average variable cost = y
$y = 1/5 x^2 - 6x + 100$
$\Rightarrow 5y = x^2 - 30x + 500$
$\Rightarrow x^2 - 30x + 225 = 5y - 500 + 225$
$\Rightarrow (x - 15)^2 = 5y - 275$
$\Rightarrow (x - 15)^2 = 5(y - 55)$ which is of the form $X^2 = 4(5/4)Y$
∴ Y average variable cost curve is a parabola
Vertex (0, 0)
$x - 15 = 0; y - 55 = 0$
$x = 15; y = 55$
At the vertex, output is 15 tonnes and average cost is ₹ 55.

Question 6.
The profit ₹ y accumulated in thousand in x months is given by $y = -x^2 + 10x - 15$. Find the best time to end the project.

Solution:
$y = -x^2 + 10x - 15$
$\Rightarrow y = -[x^2 - 10x + 5^2 - 5^2 + 15]$
$\Rightarrow y = -[(x - 5)^2 - 10]$
$\Rightarrow y = 10 - (x - 5)^2$
$\Rightarrow (x - 5)^2 = -(y - 10)$
This is a parabola which is open downwards.
Vertex is the maximum point.
∴ Profit is maximum when $x - 5 = 0$ (or) $x = 5$ months.
After that profit gradually reduces.
∴ The best time to end the project is after 5 months.

Ex 3.7

Question 1.
If m_1 and m_2 are the slopes of the pair of lines given by $ax^2 + 2hxy + by^2 = 0$, then the value of $m_1 + m_2$ is:

(a) $\frac{2h}{b}$
(b) $-\frac{2h}{b}$
(c) $\frac{2h}{a}$
(d) $-\frac{2h}{a}$

Answer:
(b) $-\frac{2h}{b}$

Question 2.
The angle between the pair of straight lines $x^2 - 7xy + 4y^2 = 0$ is:
(a) $\tan^{-1}\left(\frac{1}{3}\right)$
(b) $\tan^{-1}\left(\frac{1}{2}\right)$
(c) $\tan^{-1}\left(\frac{\sqrt{33}}{5}\right)$
(d) $\tan^{-1}\left(\frac{5}{\sqrt{33}}\right)$

Answer:
(c) $\tan^{-1}\left(\frac{\sqrt{33}}{5}\right)$

Hint:
$x^2 - 7xy + 4y^2 = 0$
Here 2h = -7, a = 1, b = 4

$$\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right| = \left|\frac{2\sqrt{\left(\frac{7}{2}\right)^2 - 4}}{1+4}\right| = \left|\frac{2\sqrt{\frac{49-16}{4}}}{5}\right| = \left|\frac{\sqrt{33}}{5}\right|$$

$$\therefore \theta = \tan^{-1}\left(\frac{\sqrt{33}}{5}\right)$$

Question 3.
If the lines 2x – 3y – 5 = 0 and 3x – 4y – 7 = 0 are the diameters of a circle, then its centre is:
(a) (-1, 1)
(b) (1, 1)

(c) (1, -1)
(d) (-1, -1)

Answer:
(c) (1, -1)

Hint:
To get centre we must solve the given equations
$2x - 3y - 5 = 0$ …….(1)
$3x - 4y - 7 = 0$ ………(2)
$(1) \times 3 \Rightarrow 6x - 9y = 15$
$(2) \times 2 \Rightarrow 6x - 8y = 14$

Subtracting, $-y = 1 \Rightarrow y = -1$
Using $y = -1$ in (1) we get
$2x + 3 - 5 = 0$
$\Rightarrow 2x = 2$
$\Rightarrow x = 1$

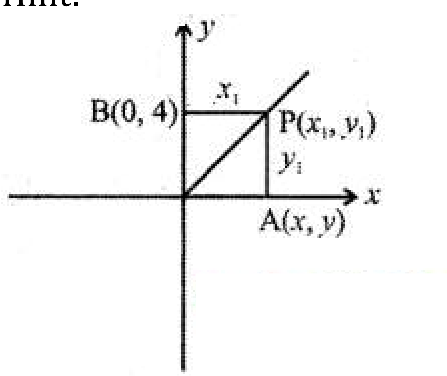
Question 4.
The x-intercept of the straight line $3x + 2y - 1 = 0$ is
(a) 3
(b) 2
(c) 1/3
(d) 1/2

Answer:
(c) 1/3
Hint:
To get x-intercept put $y = 0$ in $3x + 2y - 1 = 0$ we get
$3x - 1 = 0$
$x = 1/3$

Question 5.
The slope of the line 7x + 5y – 8 = 0 is:
(a) $\frac{7}{5}$
(b) $-\frac{7}{5}$
(c) $\frac{5}{7}$
(d) $-\frac{5}{7}$

Answer:
(b) $-\frac{7}{5}$
Hint:
Slope of 7x + 5y – 8 = 0 is = $\frac{-x \text{ coefficient}}{y \text{ coefficient}} = -\frac{7}{5}$

Question 6.
The locus of the point P which moves such that P is at equidistance from their coordinate axes is:
(a) y = $\frac{1}{x}$
(b) y = -x
(c) y = x
(d) y = $\frac{-1}{x}$

Answer:
(c) y = x
Hint:


Given PA = PB

$y_1 = x_1$
∴ Locus is y = x

Question 7.
The locus of the point P which moves such that P is always at equidistance from the line x + 2y + 7 = 0:
(a) x + 2y + 2 = 0
(b) x – 2y + 1 = 0
(c) 2x – y + 2 = 0
(d) 3x + y + 1 = 0

Answer:
(a) x + 2y + 2 = 0
Hint:
Locus is line parallel to line x + 2y + 7 = 0 which is x + 2y + 2 = 0

Question 8.
If $kx^2 + 3xy - 2y^2 = 0$ represent a pair of lines which are perpendicular then k is equal to:
(a) $\frac{1}{2}$
(b) $-\frac{1}{2}$
(c) 2
(d) -2

Answer:
(c) 2
Hint:
Here a = k, b = -2
Condition for perpendicular is
a + b = 0
⇒ k – 2 = 0
⇒ k = 2

Question 9.
(1, -2) is the centre of the circle $x^2 + y^2 + ax + by - 4 = 0$, then its radius:
(a) 3
(b) 2
(c) 4

(d) 1

Answer:
(a) 3
Hint:
Given centre (-g, -f) = (1, -2)
From the given equation c = -4
Radius = $\sqrt{g^2 + f^2 - c} = \sqrt{1 + 4 - (-4)} = \sqrt{9}$ = 3

Question 10.
The length of the tangent from (4, 5) to the circle $x^2 + y^2 = 16$ is:
(a) 4
(b) 5
(c) 16
(d) 25

Answer:
(b) 5
Hint:

Length of the tangent from (x_1, y_1) to the circle $x^2 + y^2 = 16$ is
$\sqrt{x_1^2 + y_1^2 - 16} = 5$

Question 11.
The focus of the parabola $x^2 = 16y$ is:
(a) (4, 0)
(b) (-4, 0)
(c) (0, 4)
(d) (0, -4)

Answer:
(c) (0, 4)
Hint:
$x^2 = 16y$
Here 4a = 16 ⇒ a = 4
Focus is (0, a) = (0, 4)

Question 12.
Length of the latus rectum of the parabola $y^2 = -25x$:
(a) 25
(b) -5
(c) 5
(d) -25

Answer:
(a) 25
Hint:
$y^2 = -25a$
Here $4a = 25$ which is the length of the latus rectum.

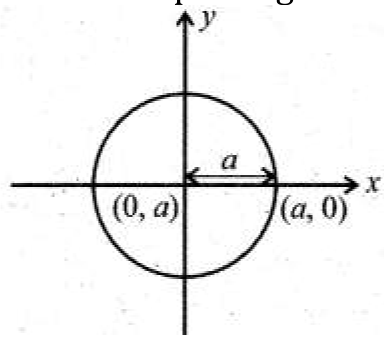
Question 13.
The centre of the circle $x^2 + y^2 - 2x + 2y - 9 = 0$ is:
(a) (1, 1)
(b) (-1, 1)
(c) (-1, 1)
(d) (1, -1)

Answer:
(d) (1, -1)
Hint:
$2g = -2, 2f = 2$
$g = -1, f = 1$
Centre = (-g, -f) = (1, -1)

Question 14.
The equation of the circle with centre on the x axis and passing through the origin is:
(a) $x^2 - 2ax + y^2 = 0$
(b) $y^2 - 2ay + x^2 = 0$
(c) $x^2 + y^2 = a^2$
(d) $x^2 - 2ay + y^2 = 0$

Answer:
(a) $x^2 - 2ax + y^2 = 0$
Hint:
Let the centre on the x-axis as (a, 0).

This circle passing through the origin so the radius



Now centre (h, k) = (a, 0)
Radius = a
Equation of the circle is $(x – a)^2 + (y – 0)^2 = a^2$
$\Rightarrow x^2 – 2ax + a^2 + y^2 = a^2$
$\Rightarrow x^2 – 2ax + y^2 = 0$

Question 15.
If the centre of the circle is (-a, -b) and radius is $\sqrt{a^2 - b^2}$ then the equation of circle is:
(a) $x^2 + y^2 + 2ax + 2by + 2b^2 = 0$
(b) $x^2 + y^2 + 2ax + 2by – 2b^2 = 0$
(c) $x^2 + y^2 – 2ax – 2by – 2b^2 = 0$
(d) $x^2 + y^2 – 2ax – 2by + 2b^2 = 0$

Answer:
(a) $x^2 + y^2 + 2ax + 2by + 2b^2 = 0$
Hint:
Equation of the circle is $(x – h)^2 + (y – k)^2 = r^2$
$\Rightarrow (x + a)^2 + (y + b)^2 = a^2 – b^2$
$\Rightarrow x^2 + y^2 + 2ax + 2by + a^2 + b^2 = a^2 – b^2$
$\Rightarrow x^2 + y^2 + 2ax + 2by + 2b^2 = 0$

Question 16.
Combined equation of co-ordinate axes is:
(a) $x^2 – y^2 = 0$
(b) $x^2 + y^2 = 0$
(c) $xy = c$
(d) $xy = 0$

Answer:
(d) $xy = 0$
Hint:
Equation of x-axis is $y = 0$
Equation of y-axis is $x = 0$
Combine equation is $xy = 0$

Question 17.
$ax^2 + 4xy + 2y^2 = 0$ represents a pair of parallel lines then 'a' is:
(a) 2
(b) -2
(c) 4
(d) -4

Answer:
(a) 2
Hint:
Here $a = 0, h = 2, b = 2$
Condition for pair of parallel lines is $b^2 - ab = 0$
$4 - a(2) = 0$
$\Rightarrow -2a = -4$
$\Rightarrow a = 2$

Question 18.
In the equation of the circle $x^2 + y^2 = 16$ then v intercept is (are):
(a) 4
(b) 16
(c) ± 4
(d) ± 16

Answer:
(c) ± 4
Hint:
To get y-intercept put $x = 0$ in the circle equation we get
$0 + y^2 = 16$
$\therefore y = \pm 4$

Question 19.
If the perimeter of the circle is 8π units and centre is (2, 2) then the equation

of the circle is:
(a) $(x - 2)^2 + (y - 2)^2 = 4$
(b) $(x - 2)^2 + (y - 2)^2 = 16$
(c) $(x - 4)^2 + (y - 4)^2 = 16$
(d) $x^2 + y^2 = 4$

Answer:
(c) $(x - 2)^2 + (y - 2)^2 = 16$
Hint:
Perimeter, $2\pi r = 8\pi$
$r = 4$
Centre is (2, 2)
Equation of the circle is $(x - 2)^2 + (y - 2)^2 = 4^2 = 16$

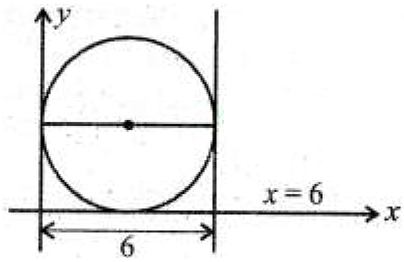
Question 20.
The equation of the circle with centre (3, -4) and touches the x-axis is:
(a) $(x - 3)^2 + (y - 4)^2 = 4$
(b) $(x - 3)^2 + (y + 4)^2 = 16$
(c) $(x - 3)^2 + (y - 4)^2 = 16$
(d) $x^2 + y^2 = 16$

Answer:
(b) $(x - 3)^2 + (y + 4)^2 = 16$
Hint:
Centre (3, -4).
It touches the x-axis.
The absolute value of y-coordinate is the radius, i.e., radius = 4.
Equation is $(x - 3)^2 + (y + 4)^2 = 16$

Question 21.
If the circle touches the x-axis, y-axis, and the line $x = 6$ then the length of the diameter of the circle is:
(a) 6
(b) 3
(c) 12
(d) 4

Answer:
(a) 6

Hint:

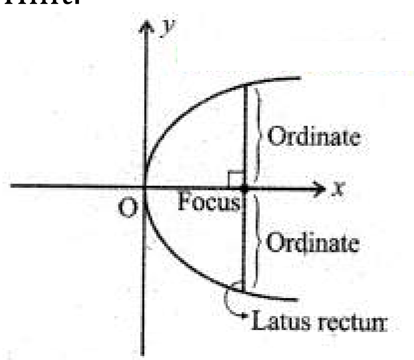


Question 22.
The eccentricity of the parabola is:
(a) 3
(b) 2
(c) 0
(d) 1

Answer:
(d) 1

Question 23.
The double ordinate passing through the focus is:
(a) focal chord
(b) latus rectum
(c) directrix
(d) axis

Answer:
(b) latus rectum
Hint:



Question 24.
The distance between directrix and focus of a parabola $y^2 = 4ax$ is:
(a) a

(b) 2a
(c) 4a
(d) 3a

Answer:
(b) 2a

Question 25.
The equation of directrix of the parabola $y^2 = -x$ is:
(a) $4x + 1 = 0$
(b) $4x - 1 = 0$
(c) $x - 1 = 0$
(d) $x + 4 = 0$

Answer:
(b) $4x - 1 = 0$
Hint:
$y^2 = -x$.
It is a parabola open leftwards.
Here $4a = 1 \Rightarrow a = 1/4$
Equation of directrix is $x = a$.
i.e., $x = 1/4$ (or) $4x - 1 = 0$