7. ROTATIONAL MECHANICS

1. INTRODUCTION

In this chapter we will be studying the kinematics and dynamics of a solid body in two kinds of motion. The first kind of motion of a solid body is rotation about a stationary axis, also called pure rotation. The second kind of motion of a solid is the plane motion wherein the center of mass of the solid body moves in a certain stationary plane while the angular velocity of the body remains permanently perpendicular to that plane. Here the body executes pure rotation about an axis passing through the center of mass and the center of mass itself translates in a stationary plane in the given reference frame. The axis through the center of mass is always perpendicular to the stationary plane. We will also learn about the inertia property in rotational motion, and the quantities torque and angular momentum which are rotational analogue of force and linear momentum respectively. The law of conservation of angular momentum is an important tool in the study of motion of solid bodies.

2. BASIC CONCEPT OF A RIGID BODY

A solid is considered to have structural rigidity and resists change in shape, size and density. A rigid body is a solid body which has no deformation, i.e. the shape and size of the body remains constant during its motion and interaction with other bodies. This means that the separation between any two points of a rigid body remains constant in time regardless of the kind of motion it executes and the forces exerted on it by surrounding bodies or a field of force.

A metal cylinder rolling on a surface is an example of a rigid body as shown in Fig. 7.1.

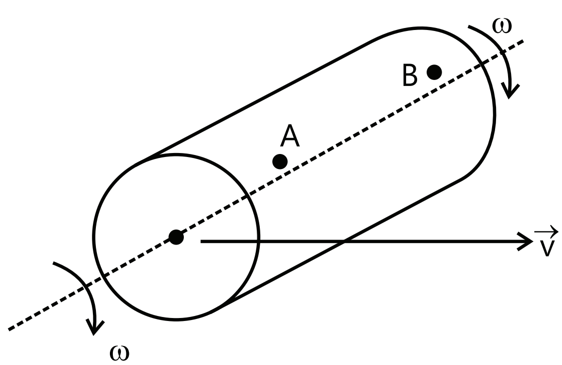


Figure 7.1: Metal cylinder rolling on a surface is a rigid body system. Relative distance between points A and B do not change.

Let velocities of points P and Q of a rigid body with respect to a reference frame be V_P and V_Q as shown in the Fig. 7.2.

As the body is rigid, the length PQ should not change during the motion of the body, i.e. the relative velocity between P and Q along the line joining P and Q should be zero i.e. velocity of approach or separation is zero. Let x-axis be along PQ, then

$\vec{V}_{QP}$ = relative velocity of Q with respect to P

$$\vec{V}_{QP} = (V_Q\cos\theta_2\,\hat{i} + V_Q\sin\theta_2\,\hat{j}) - (V_P\cos\theta_1\,\hat{i} - V_P\sin\theta_1\,\hat{j})$$

$$\vec{V}_{QP} = (V_Q\cos\theta_2 - V_P\cos\theta_1)\,\hat{i} + (V_P\sin\theta_1 + V_Q\sin\theta_2)\,\hat{j}$$

Now $V_P \cos\theta_1 = V_Q \cos\theta_2$

(Since velocity of separation is 0)

$\vec{V}_{QP} = (V_P \sin\theta_1 + V_Q \sin\theta_2)\hat{j}$ (which is perpendicular to line PQ).

Hence, we can conclude that for each and every pair of particles in a rigid body, relative motion between the two points in the pair will be perpendicular to the line joining the two points.

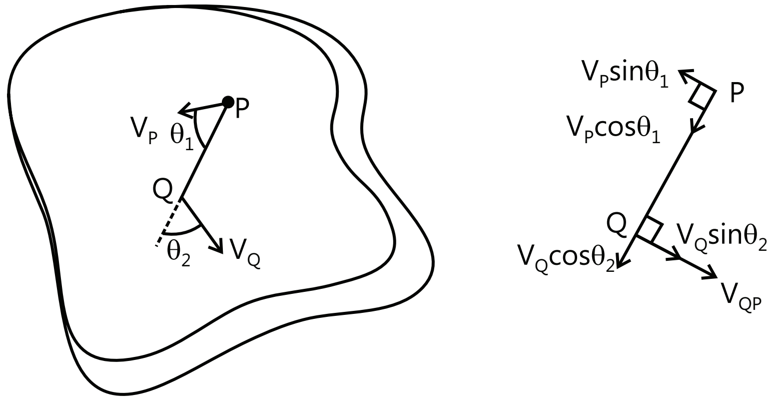


Figure 7.2: Relative velocity between two points of a rigid body

PLANCESS CONCEPTS

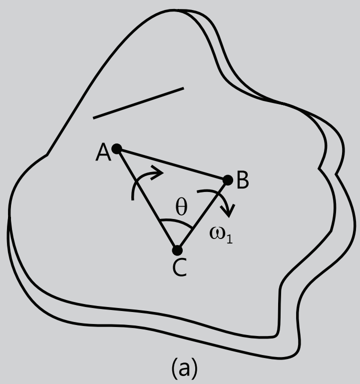


Figure 7.3: (a) Angular velocity of A and B w.r.t. C is ω_1 (b) Angular velocity of A and C w.r.t. B is ω_1

Suppose A, B, C are points of a rigid system hence during any motion the lengths of sides AB, BC, and CA will not change, and thus the angle between them will not change, and so they all must rotate through the same angle. Hence all the sides rotate by the same rate. Or we can say that each point is having the same angular velocity with respect to any other point on the rigid body.

Neeraj Toshniwal (JEE 2009 AIR 21)

3. MOTION OF A RIGID BODY

We will study the dynamics of three kinds of motion of a rigid body.

(a) Pure Translational motion

(b) Pure Rotational Motion

(c) Combined Translational and Rotational motion

Let us briefly discuss the characteristics of these three types of motion of a rigid body.

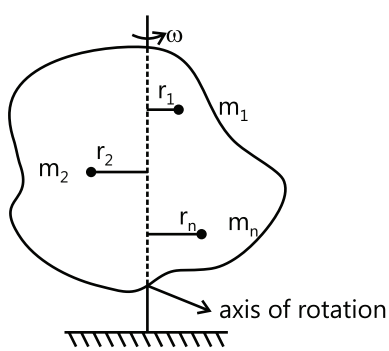


Figure 7.4: Body in pure rotational motion.

3.1 Pure Translational motion

A rigid body is said to be in pure translational motion if any straight line fixed to it remains parallel to its initial orientation all the time. E.g. a car moving along a straight horizontal stretch of a road. In this kind of motion, the displacement of each and every particle of the rigid body is the same during any time interval. All the points of the rigid body have the same velocity and acceleration at any instant. Thus to study the translational motion of a rigid body, it is enough to study the motion of an individual point belonging to that rigid body i.e. the dynamics of a point.

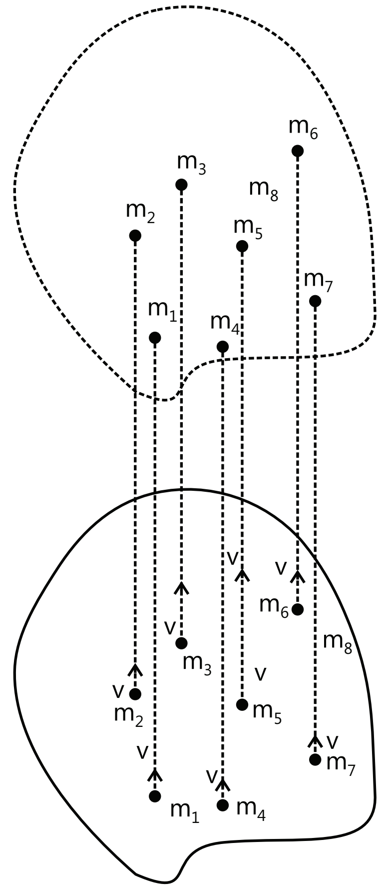


Figure 7.5: Body in pure translational motion.

3.2 Pure Rotational Motion

Suppose a rigid body of any arbitrary shape rotates about an axis which is stationary in a given reference frame. In this kind of motion every point of the body moves in a circle whose center lies on the axis of rotation at the foot of the perpendicular from the particle to this axis, and radius of the circle is equal to the perpendicular distance of the point from this axis. Every point of the rigid body moves through the same angle during a particular time interval. Such a motion is called pure rotational motion. Each particle has same instantaneous angular velocity (since the body is rigid) and different particles move in circles of different radii, the planes of all these circles are parallel to each other. Particles moving in smaller circles have less linear velocity and those moving in bigger circles have large linear velocity at the same instant.

In the Fig. 7.4 particles of mass m_1, m_2, m_3..... have linear velocities v_1, v_2, v_3....

If ω is the instantaneous angular velocity of the rigid body, then

$$v_1 = \omega r_1, v_2 = \omega r_2, v_3 = \omega r_3 \; ...... , \; v_n = \omega r_n$$

3.3 Combined Translational and Rotational Motion

A rigid body is said to be in combined translational and rotational motion if the body performs pure rotation about an axis and at the same time the axis translates with respect to a reference frame. In other words there is a reference frame K' which is rigidly fixed to the axis of rotation, such that the body performs pure rotation in the K' frame. The K' frame in turn is in pure translational motion with respect to a reference frame K. So to describe the motion of the rigid body in the K frame, the translational motion of K' frame is super-imposed on the pure rotational motion of the body in the K' frame.

Illustration 1: A body is moving down into a well through a rope passing over a fixed pulley of radius 10 cm. Assume that there is no slipping between rope and pulley. Calculate the angular velocity and angular acceleration of the pulley at an instant when the body is going down at a speed of 20 cm s^{-1} and has an acceleration of 4.0 m s^{-2}. **(JEE MAIN)**

Sol: Since the rope does not slip on the pulley, the linear speed and linear acceleration of the rim of the pulley will be equal to the speed and acceleration of the body respectively.

Therefore, the angular velocity of the pulley is

$$\omega = \frac{\text{linear velocity of rim}}{\text{radius of rim}} = \frac{20 \text{ cm s}^{-1}}{10 \text{ cm}} = 2 \text{ rad s}^{-1}$$

And the angular acceleration of the pulley is

$$\alpha = \frac{\text{linear acceleration of rim}}{\text{radius of rim}} = \frac{4.0 \text{ ms}^{-2}}{10 \text{cm}} = 40 \text{ rad s}^{-2}$$

4. ROTATIONAL KINEMATICS

Suppose a rigid body performing pure rotational motion about an axis of rotation rotates by an angle $\Delta\theta$ in a time interval Δt. The instantaneous angular velocity ω, is defined as,

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \qquad \text{...(i)}$$

Similarly, the instantaneous angular acceleration α is defined as,

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \qquad \text{...(ii)}$$

The relations between linear distance s, linear velocity v and linear acceleration a, and the corresponding angular variables describing circular motion θ, ω, and α respectively are given as:

$$s = r\theta; \quad v = r\omega; \quad a_t = r\alpha \qquad \text{...(iii)}$$

Here the subscript t along with a in the expression for acceleration signifies that this is the tangential component of linear acceleration.

If a body rotates with uniform angular acceleration,

$$\omega = \omega_0 + \alpha t\ ;\ \theta = \omega_0 t + \frac{1}{2}\alpha t^2;\ \omega^2 = \omega_0^2 + 2\alpha\theta \qquad \text{...(iv)}$$

where ω_0 is initial angular velocity.

The equations for angular displacement, angular velocity and angular acceleration are similar to the corresponding equations of linear motion.

Illustration 2: A disc starts rotating with constant angular acceleration of $\pi/2$ rad s^{-2} about a fixed axis perpendicular to its plane and through its center. Calculate

(a) The angular velocity of the disc after 4 s

(b) The angular displacement of the disc after 4s and

(c) Number of turns accomplished by the disc in 4 s. **(JEE MAIN)**

Sol: Use the first and second equations of angular motion with constant angular acceleration.

Here $\alpha = \frac{\pi}{2}$ rad s^{-2}; $\omega_0 = 0$; $t = 4$ s;

(a) $\omega(4\ s) = 0 + \left(\frac{\pi}{2}\text{rad s}^{-2}\right) \times 4\ s = 2\pi \text{ rad s}^{-1}$

(b) $\theta_{(4s)} = 0 + \frac{1}{2}\left(\frac{\pi}{2}\text{rad s}^{2}\right) \times (16s^2) = 4\pi$ rad

(c) $\Rightarrow n \times 2\pi \text{ rad} = 4\pi \text{ rad} \Rightarrow n = 2.$

PLANCESS CONCEPTS

For variable angular acceleration we should proceed with differential equation $\frac{d\omega}{dt} = \alpha$

Akshat Kharaya (JEE 2009 AIR 235)

5. MOMENT OF INERTIA

Before discussing the dynamics of rigid body motion let us study about an important property of a rigid body called Moment of Inertia which is indispensable in understanding its dynamics.

Physical Significance of Moment of Inertia: As the name suggests, moment of inertia is the measure of the rotational inertia property of a rigid body, the rotational analog of mass in translational motion. "It is the property of the rigid body by virtue of which it opposes any change in its state of uniform rotational motion." The moment of inertia of a rigid body depends on its mass, on the location and orientation of the axis of rotation and on the shape and size of the body or in other words on the distribution of the mass of the body with respect to the axis of rotation. SI units of moment of inertia is Kg-m^2. Moment of inertia about a particular axis of rotation is a scalar positive quantity.

Definition: Moment of inertia of a system of n particles about an axis is defined as:

$$I = m_1r_1^2 + m_2r_2^2 + ...+ m_nr_n^2 \text{ i.e. } I = \sum_{i=1}^{n} m_i r_i^2 \quad ...(i)$$

where, r_i is the perpendicular distance of ith particle of mass m_i from the axis of rotation.

For a continuous rigid body, the moment of inertia can be calculated as:

$$I = \int r^2(dm) \quad ...(ii)$$

where dm is the mass of an infinitesimal element of the body at a perpendicular distance r from the axis of rotation.

Moment of inertia depends on:

(a) Mass of the rigid body.

(b) Shape and size of the rigid body.

(c) Location and orientation of the axis of rotation.

PLANCESS CONCEPTS

Moment of inertia does not change if the mass:

(i) Is shifted parallel to the axis of rotation because r_i does not change.

(ii) Is rotated about the axis of rotation in a circular path because r_i does not change.

Chinmay S Purandare (JEE 2012 AIR 698)

Illustration 3: Two particles having masses m_1 & m_2 are situated in a plane perpendicular to line AB at a distance of r_1 and r_2 respectively as shown.

(i) Find the moment of inertia of the system about axis AB?

(ii) Find the moment of inertia of the system about an axis passing though m_1 and perpendicular to the line joining m_1 and m_2.

(iii) Find the moment of inertia of the system about an axis passing through m_1 and m_2.

(iv) Find moment of inertia about an axis passing though center of mass and perpendicular to line joining m_1 and m_2 . **(JEE MAIN)**

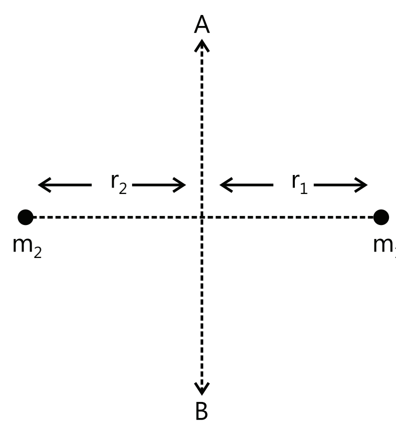


Figure 7.6

Sol: Use the formula for moment of inertia of a system of n particles. Find the distance of center of mass from m_1.

(i) Moment of inertia of particle on right is $I_1 = m_1r_1^2$

Moment of inertia of particle on left is $I_2 = m_2 r_2^2$

Moment of inertia of the system about AB is $I = I_1 + I_2 = m_1 r_1^2 + m_2 r_2^2$

(ii) Moment of inertia of particle on right is $I_1 = 0$

Moment of inertia of particle on left is $I_2 = m_2 (r_1 + r_2)^2$

Moment of inertia of the system about axis is $I = I_1 + I_2 = 0 + m_2 (r_1 + r_2)^2$

(iii) Moment of inertia of particle on right is $I_1 = 0$

Moment of inertia of particle of left is $I_2 = 0$

Moment of inertia of the system about axis is $I = I_1 + I_2 = 0 + 0$

(iv) of system $r_{cm} = m_2 \left(\frac{r_1 + r_2}{m_1 + m_2} \right)$ = Distance of center mass from mass m_1

Distance of center of mass from mass $m_2 = m_1 \left(\frac{r_1 + r_2}{m_1 + m_2} \right)$

So moment of inertia about center of mass = $I_{cm} = m_1 \left(m_2 \frac{r_1 + r_2}{m_1 + m_2} \right)^2 + m_2 \left(m_1 \frac{r_1 + r_2}{m_1 + m_2} \right)^2$

$$I_{cm} = \frac{m_1 m_2}{m_1 + m_2} (r_1 + r_2)^2 .$$

Illustration 4: Three particles each of mass m, are situated at the vertices of an equilateral triangle PQR of side a as shown in the Fig 7.7. Calculate the moment of inertia of the system about

(i) The line PX perpendicular to PQ in the plane of PQR.

(ii) One of the sides of the triangle PQR

(iii) About an axis passing through the centroid and perpendicular to plane of the triangle PQR. **(JEE MAIN)**

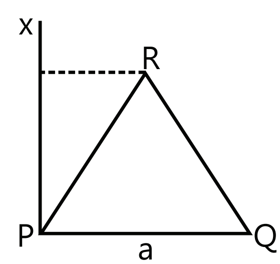


Figure 7.7

Sol: Use the formula for moment of inertia of a system of n particles.

(i) Perpendicular distance of P from PX = 0; perpendicular distance of Q from PX = a perpendicular distance of R from PX = a/2. Thus, the moment of inertia of the particle at P is 0, that of particle Q is ma^2, and of the particle at R is $m(a/2)^2$.

The moment of inertia of the three particle system about PX is $I = 0 + ma^2 + m(a/2)^2 = \frac{5ma^2}{4}$

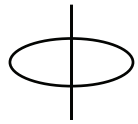
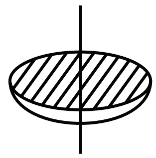
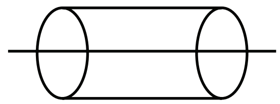
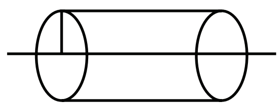
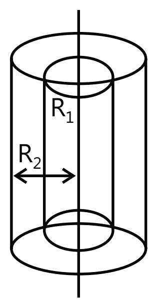
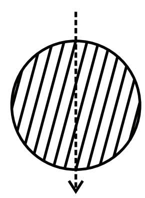
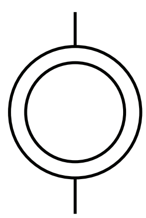
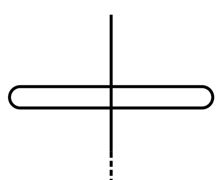
Note that the particles on the axis do not contribute to the moment of inertia.

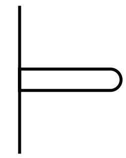
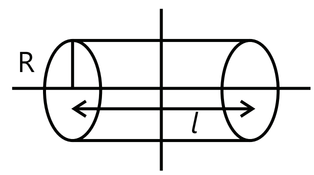
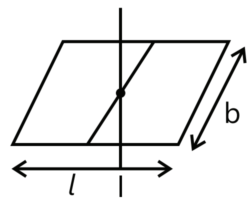
(ii) Moment of inertia about the side PR = mass of particle Q × square of perpendicular distance of Q from side PR,

$$I_{PR} = m \left(\frac{\sqrt{3}}{2} a \right)^2 = \frac{3ma^2}{4}$$

(iii) Distance of centroid from each of the particles is $\frac{a}{\sqrt{3}}$, so moment of inertia about an axis passing through the centroid and perpendicular to the plane of triangle PQR = $I_C = 3m \left(\frac{a}{\sqrt{3}} \right)^2 = ma^2$

Table 7.1: Formulae of MOI of symmetric bodies

S. No.	Body, mass M	Axis	Figure	I	K(Radius of Gyration)
1.	Ring or loop of radius R	Through its center and perpendicular to its plane		MR^2	R
2.	Disc, radius R	Perpendicular to its plane through its center		$\frac{MR^2}{2}$	$\frac{R}{\sqrt{2}}$
3.	Hollow cylinder, radius R	Axis of cylinder		MR^2	R
4.	Solid cylinder, radius R	Axis of cylinder		$\frac{MR^2}{2}$	$\frac{R}{\sqrt{2}}$
5.	Thick walled cylinder,	Axis of cylinder		$\frac{M\left(R_1^2+R_2^2\right)}{2}$	$\frac{\sqrt{R_1^2+R_2^2}}{2}$
6.	Solid sphere, radius R	Diameter		$\frac{2}{5}MR^2$	$\sqrt{\frac{2}{5}}R$
7.	Spherical shell radius, R	Diameter		$\frac{2MR^2}{3}$	$\sqrt{\frac{2}{3}}R$
8.	Thin rod, length L	Perpendicular to rod at middle point		$\frac{ML^2}{12}$	$\frac{L}{2\sqrt{3}}$

9.	Thin rod, length L	Perpendicular to rod at one end		$\frac{ML^2}{3}$	$\frac{L}{\sqrt{3}}$
10.	Solid cylinder, length l	Through center and perpendicular to length		$\frac{MR^2}{4}+\frac{Ml^2}{12}$	$\sqrt{\frac{R^2}{4}+\frac{l^2}{12}}$
11.	Rectangular sheet, length l and breadth b	Through center and perpendicular to plane		$\frac{M(l^2+b^2)}{12}$	$\sqrt{\frac{l^2+b^2}{12}}$

PLANCESS CONCEPTS

While deriving the MOI *of* any rigid body the element chosen should be such that:

Either perpendicular distance of axis from each point of the element is same or the moment of inertia of the element about the axis of rotation is known.

Nitin Chandrol (JEE 2012 AIR 134)

5.1 Theorems on Moment of Inertia

1. Theorem of Parallel Axes: This theorem is very useful in cases when the moment of inertia about an axis z_C passing through the center of mass (C.O.M) of the rigid body is known, and we sought to find the moment of inertia about any other axis z which is parallel to the axis z_C as shown in Fig. 7.8. The moment of inertia of the rigid body about axis z is equal to the sum of the moment of inertia about axis z_C and the product of the mass m of the body by the square of perpendicular distance between the two axes. If the moment of inertia of the rigid body about axis z_C is I_C, then the moment of inertia I of this body about any parallel axis z , is given by $I = I_C + Md^2$ ...(i)

where d is the perpendicular distance between the two axes.

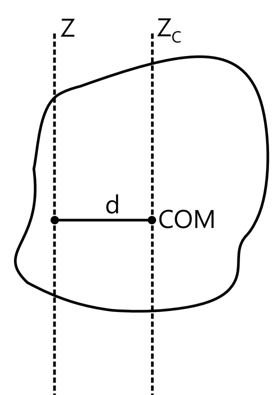


Figure 7.8: Parallel axes

Illustration 5: Find the moment of inertia of a uniform sphere of mass m and radius R about a tangent if the sphere is (i) solid (ii) hollow **(JEE MAIN)**

Sol: We know the formula for moment of inertia of sphere about an axis passing through its center. Use the parallel axes theorem to find the moment of inertia about the tangent.

(i) Using parallel axis theorem

$I = I_C + md^2$

For solid sphere

$I_C = \frac{2}{5}mR^2$, d = R ; $\quad I = \frac{7}{5}mR^2$

(ii) Using parallel axis theorem

$I = I_C + md^2$

For hollow sphere

$I_C = \frac{2}{3}mR^2$, d = R ; $I = \frac{5}{3}mR^2$

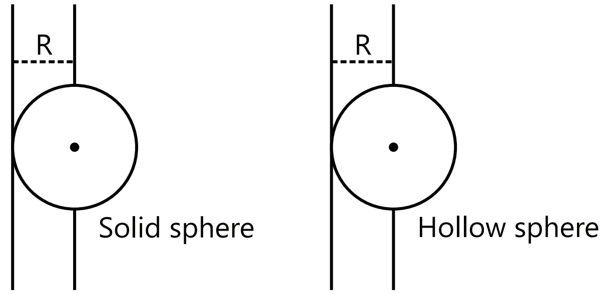


Figure 7.9

Illustration 6: Find the moment of inertia of the two uniform joint roads having mass m each about point P as shown in Fig 7.10. Use parallel axis theorem. **(JEE MAIN)**

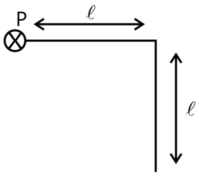


Figure 7.10

Sol: We know the formulae for moment of inertia of rod about the axes passing through its center and through one of its ends and perpendicular to it. Use the parallel axes theorem to find the moment of inertia about the point P.

Moment of inertia of rod 1 about axis P, $I_1 = \frac{m\ell^2}{3}$

Moment of inertia of rod 2 about axis p, $I_2 = \frac{m\ell^2}{12} + m\left(\sqrt{5}\frac{\ell}{2}\right)^2$

So moment of inertia of a system about axis p; $I = \frac{5m\ell^2}{3}$

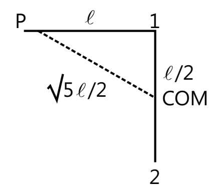


Figure 7.11

2. Theorem of Perpendicular Axes: This theorem is applicable only in case of two dimensional rigid body or planar lamina as shown in Fig. 7.12. Let the lamina lie in the x-y plane and I_x and I_y be the moment of inertia of the lamina about x and y axes respectively then the moment of inertia about z-axis perpendicular to the plane of the lamina and passing through the point of intersection of x and y axes is given as:

$I_z = I_x + I_y$ ...(ii)

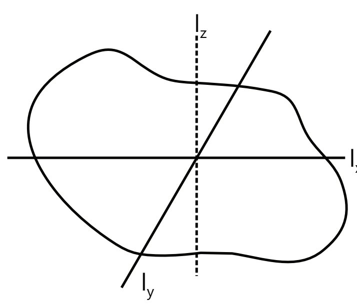


Figure 7.12: Perpendicular axes

Illustration 7: Find the moment of inertia of a half-disc about an axis perpendicular to the plane and passing through its center of mass. Mass of this disc is M and radius is R. **(JEE MAIN)**

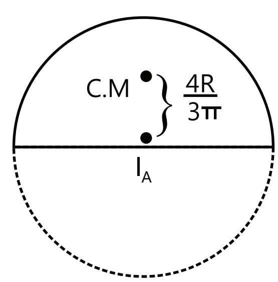


Figure 7.13

Sol: We know the formula for the moment of inertia of the half disc about a perpendicular axis through the center A. Use the parallel axes theorem to find the moment of inertia about a perpendicular axis through the center of mass.

The COM of half disc will be at distance $4R/3\pi$ from the center A. Let moment of inertia of half disc about a perpendicular axis passing through A be I_A.

First we fill the remaining half with same density to get a full disc of mass 2M.

The moment of inertia about center A of full disc will be $2I_A$,

So, $I_A = \frac{2MR^2}{2\times 2} = \frac{MR^2}{2}$; $I_A = I_{CM} + M \times \left(\frac{4R}{3\pi}\right)^2$; $I_{CM} = \frac{MR^2}{2} - M\times\left(\frac{4R}{3\pi}\right)^2$

Illustration 8: Calculate the moment of inertia of a uniform disc of mass M and radius R about a diameter.

(JEE MAIN)

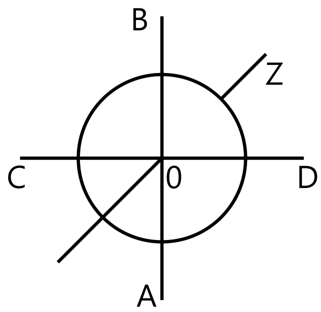


Figure 7.14

Sol: For a uniform disc all diameters are equivalent, i.e. moment of inertia about any diameter will be equal to that about any other diameter. We know the formula for moment of inertia of disc about axis perpendicular to its plane and passing through its center. Use the perpendicular axes theorem to find the moment of inertia about a diameter.

Let AB and CD be two mutually perpendicular diameters of the disc. Take them as x and y axes and the line perpendicular to the plane of the disc through the center as the Z – axis. The moment of inertia of the ring about the Z – axis is $I = \frac{1}{2}MR^2$. As the disc is uniform, all of its diameters are equivalent and so $I_x = I_y$

From perpendicular an axis theorem $I_z = I_x + I_y$; hence $I_x = \frac{I_z}{2} = \frac{MR^2}{4}$

Illustration 9: In the Fig 7.15 shown find the moment of inertia of square plate having mass m and sides a about axis 2 passing through point C (center of mass) and in the plane of plate. **(JEE MAIN)**

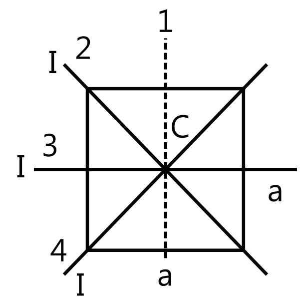


Figure 7.15

Sol: For uniform square plate axes 2 and 4 along diagonals are equivalent and axes 1 and 3 are equivalent. Suppose I_c is the moment of inertia about the axis perpendicular to the plane of plate and passing through the center C. Use perpendicular axes theorem to prove that the axes 1 and 2 are also equivalent.

Using perpendicular axes theorem $I_C = I_4 + I_2 = I' + I' = 2I'$ .... (i)

Using perpendicular axes theorem $I_C = I_3 + I_1 = I + I = 2I$ ....(ii)

From (i) and (ii) we get $I' = I$

$I_c = 2I = \frac{ma^2}{6} \Rightarrow I' = \frac{ma^2}{12}$

5.2 Radius of Gyration

The radius of gyration of a rigid body about an axis z is equal to the radius of a ring whose mass is equal to the mass of the rigid body, and the moment of inertia of the ring about an axis passing through its center and perpendicular to its plane is equal to the moment of inertia of the rigid body about the axis z. Radius of gyration can also be defined as the perpendicular distance from the axis of rotation where all mass of the rigid body can be assumed to be concentrated when the rigid body is performing pure rotation to get the equation of motion of the body. Thus, the radius of gyration is the ''equivalent distance'' of the rigid body from the axis of rotation.

$I = MK^2$

I = Moment of inertia of the rigid body about an axis

M = Mass of the rigid body

K = Radius of gyration about the same axis

or $K = \sqrt{\frac{I}{M}}$ ...(iii)

Length K is the property of the rigid body which depends upon the shape and size of the body and on the orientation and location of the axis of rotation. S.I. Unit of K is meter.

Illustration 10: Find the radius of gyration of a hollow uniform sphere of radius R about its tangent. **(JEE MAIN)**

Sol: Use the formula for radius of gyration.

Moment of inertia of a hollow sphere about a tangent = $\frac{5}{3}MR^2$

$$MK^2 = \frac{5}{3}MR^2 \Rightarrow K = \sqrt{\frac{5}{3}}R$$

5.3 Moment of Inertia of a Body Having a Cavity

If we know the moments of inertia of different parts of a rigid body about the same axis, then the moment of inertia of the entire body can be calculated by simply adding the moments of inertia of the different parts (about the same axis) i.e. moment of inertia is an additive quantity. This principle can be used to calculate the moment of inertia of a body having hollow spaces by first assuming the hollow spaces to be filled with same density as that of the body and evaluating the moment of inertia of the whole body about the given axis and then add the moments on inertia of the hollow spaces about the same axis considering them to have negative mass.

Illustration 11: A uniform disc of radius R has a round disc of radius R/3 cut as shown in Fig 7.16. The mass of the disc equals M. Find the moment of inertia of such a disc relative to the axis passing through geometrical center of original disc and perpendicular to the plane of the disc. **(JEE ADVANCED)**

Sol: Consider the whole disc without the cavity. The cavity can be thought of as a negative mass of same density as disc. We know the formula for moment of inertia of uniform disc about axis perpendicular to its plane and passing through its center. Find the moment of inertia of cavity (negative mass) about the perpendicular axis passing through center of whole disc. The moment of inertia of disc with cavity is the sum of the moment of inertia of whole disc and the moment of inertia of cavity (negative).

Let the mass per unit area of the material of disc be σ. Now the empty space can be considered as having density $-\sigma$

Now $I_0 = I_\sigma + I_{-\sigma}$

$$I_\sigma = (\sigma\pi R^2)R^2/2 = \text{MI of } \sigma \text{ about O} = MR^2/2$$

$$I_{-\sigma} = \frac{-\sigma\pi(R/3)^2(R/3)^2}{2} + [-\sigma\pi(R/3)^2](2R/3)^2$$

$$= \text{M.I of} -\sigma \text{ About O} = -MR^2/18$$

$$I_0 = MR^2/2 - MR^2/18$$

$$I_0 = \frac{4}{9}MR^2$$

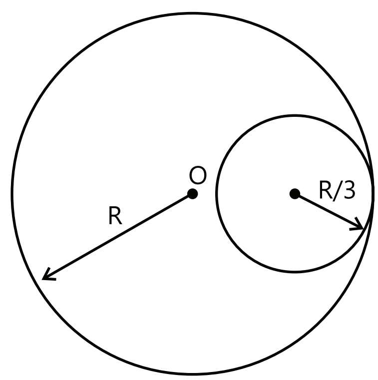


Figure 7.16

6 TORQUE

6.1 Torque About a Point

Torque of force F relative to a point O is defined as

$$\vec{\tau} = \vec{r} \times \vec{F} \qquad ...(x)$$

where $\vec{F}$ = force applied to a point on a body

$\vec{r}$ = position vector of the point of application of force relative to the point O in a chosen reference frame about which we want to determine the torque (see Fig. 7.17).

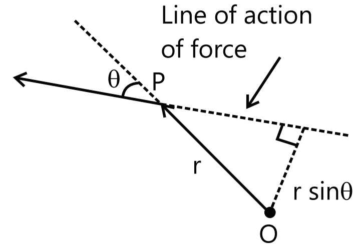


Figure 7.17: Torque of a force

Torque is a vector quantity and its direction is given by the right hand rule for cross product of vectors.

Magnitude of torque $|\vec{\tau}| = r F \sin\theta = r_{\perp}F = rF_{\perp}$

where θ is the angle between the force $\vec{F}$ and the position vector $\vec{r}$ of point of application.

$r_{\perp} = r \sin\theta$ = perpendicular distance of line of action of force from point O.

$F_{\perp} = F \sin\theta$ = component of $\vec{F}$ perpendicular to $\vec{r}$

SI unit of torque is N-m.

Illustration 12: Find the torque about point O and A. **(JEE MAIN)**

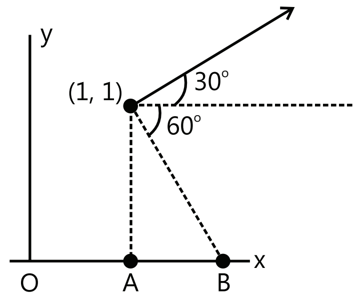


Figure 7.18

Sol: Express the position vector of A relative to O in terms of unit vectors $\hat{i}$ and $\hat{j}$. Force is given in terms of unit vectors $\hat{i}$ and $\hat{j}$.

Torque about point O, $\vec{\tau} = \vec{r_0} \times \vec{F}$, $\vec{r_0} = \hat{i} + \hat{j}$, $\vec{F} = 5\sqrt{3}\,\hat{i} + 5\hat{j}$

$\vec{\tau} = (\hat{i} + \hat{j}) \times (5\sqrt{3}\hat{i} + 5\hat{j}) = 5(1-\sqrt{3})\hat{k}$

Torque about point A, $\vec{\tau} = \vec{r_a} \times \vec{F}$, $\vec{r_a} = \hat{j}$, $\vec{F} = 5\sqrt{3}\,\hat{i} + 5\hat{j}$

$\vec{\tau} = \hat{j} \times (5\sqrt{3}\hat{i} + 5\hat{j}) = -5\sqrt{3}\hat{k}$

Illustration 13: A particle of mass m is released in vertical plane from a point on the x – axis, it falls vertically along the y – axis. Find the torque τ about origin? **(JEE MAIN)**

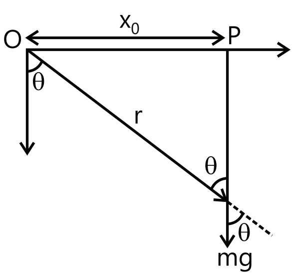


Figure 7.19

Sol: Torque is produced by the force of gravity. This will be equal to the product of force of gravity and the perpendicular distance between the line of action of force of gravity and the origin O.

$\vec{\tau} = rF\sin\theta\,\hat{k}$ Or $\tau = r_{\perp}F = x_0 mg$

$= r\,mg\dfrac{x_0}{r} = mgx_0\hat{k}$

6.2 Torque About An Axis

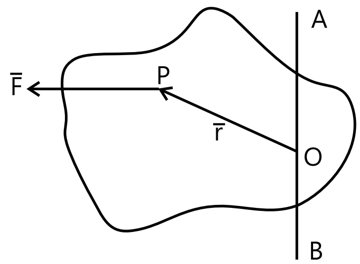


Figure 7.20: Torque about an axis

The torque of a force $\vec{F}$ about an axis AB is the component of the torque of $\vec{F}$ about point A along the axis AB.

Alternatively to find torque of force $\vec{F}$ about axis AB we choose any point O on the axis AB and find the torque of $\vec{F}$ about O as $\vec{\tau}_0 = \vec{r} \times \vec{F}$. Then we calculate the component of $\vec{\tau}_0$ along AB to get $\vec{\tau}_{AB}$ (see Fig. 7.20).

There are a few special cases of torque of a force about an axis:

Case I: Applied force is parallel to the axis of rotation, i.e. $\vec{F} \parallel \overrightarrow{AB}$

Therefore torque $\vec{r} \times \vec{F}$ about any point on the axis will be perpendicular to $\vec{F}$ and hence perpendicular to $\overrightarrow{AB}$. Therefore the component of $\vec{r} \times \vec{F}$ along $\overrightarrow{AB}$ will be zero.

Case II: The line of action of the applied force intersects the axis of rotation

($\vec{F}$ intersects $\overrightarrow{AB}$)

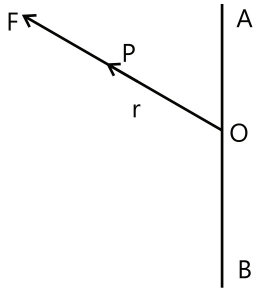


Figure 7.21: Force intersects axis

If we choose the point of intersection of line of action of $\vec{F}$ and $\overrightarrow{AB}$ as the origin O then the position vector $\vec{r}$ and applied force $\vec{F}$ will be collinear (see Fig. 7.21). Therefore the torque about O is $\vec{r} \times \vec{F} = 0$ and thus the component of this torque along line AB will also be zero.

Case III: Line of action of $\vec{F}$ and axis AB are skew and $\vec{F} \perp \overrightarrow{AB}$

Let O be the origin on the axis AB and P be the point of application of force $\vec{F}$ such that OP is perpendicular to the axis AB (see Fig. 7.22). Then torque $\vec{\tau} = \overrightarrow{OP} \times \vec{F}$ will be parallel to axis AB and the component of $\vec{\tau}$ along AB will be equal to its magnitude i.e.

$$\tau_{AB} = F \times (OP)\sin\theta = F \times l$$

where ℓ = (OP)sinθ is the length of the common perpendicular to the line of action of force and the axis called the lever arm or moment arm of this force.

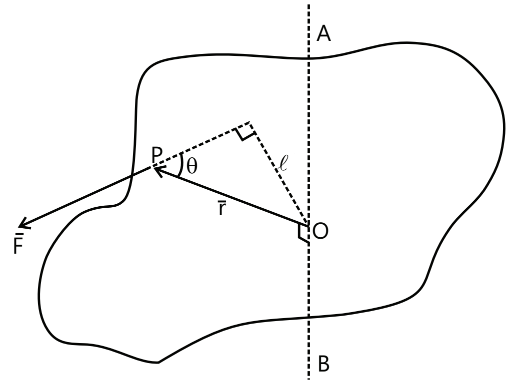


Figure 7.22: Force and axis are skew

Illustration 14: Find the torque of weight about the axis passing through point P. **(JEE MAIN)**

Sol: Required torque is equal to the product of force of gravity and the perpendicular distance between the line of action of force of gravity and the point P.

$\vec{F}$ = mg Downwards

$\vec{\tau} = \vec{F} \times \vec{r} \quad = F.r \sin\theta$

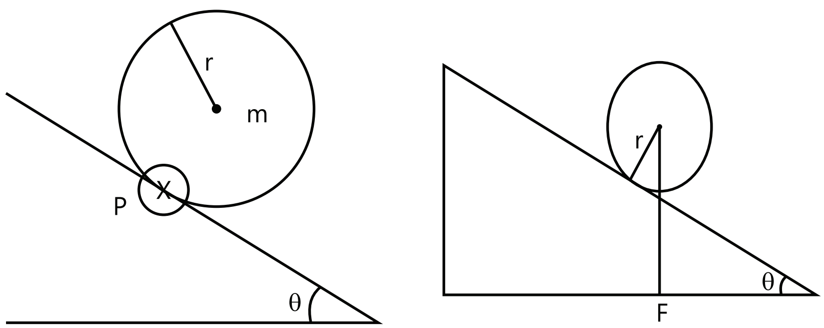


Figure 7.23

Illustration 15: A bob of mass m is suspended at point O by string of length ℓ. Bob is moving in a horizontal circle find out. (i) Torque of gravity and tension about point O and O′ (ii) Net torque about axis OO′. **(JEE ADVANCED)**

Sol: Torque of a force about an origin is equal to the product of force and the perpendicular distance between the line of action of force and the origin.

(i) Torque about point O

Torque of tension (T), τ_{net} = 0 (tension is passing through point O)

Torque of gravity $\tau_{mg} = mg\ell \sin\theta$ (along negative $\hat{j}$)

Torque about point O′

Torque of gravity $\tau_{mg} = mgr = mg\ell \sin\theta$ (along negative $\hat{j}$)

Torque of tension $\tau_T = Tr\sin(90+\theta)$ (T cos θ= mg)

$$\tau_T = Tr\cos\theta = \frac{mg}{\cos\theta}(\ell\sin\theta)\cos\theta = mg\ell\sin\theta \text{ (along positive } \hat{j})$$

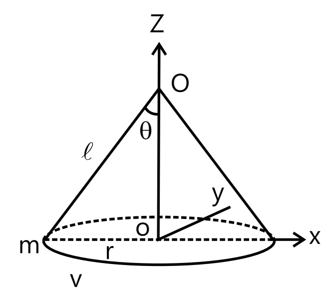


Figure 7.24

(ii) Torque about axis OO′

Torque of gravity about axis OO′ τ_{mg} = 0 (force mg is parallel to axis OO′)

Torque of tension about axis OO′ τ_T = 0 (force T intersects the axis OO′)

Net torque about axis OO′ τ_{net} = 0

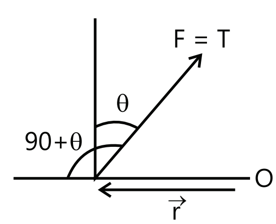


Figure 7.25

6.3 Force Couple

A pair of forces each of same magnitude and acting in opposite directions is called a force couple.

Torque due to couple = magnitude of one force x distance between their lines of action.

Magnitude of torque = τ = F (d)

A couple does not exert a net force on an object even though it exerts a torque.

Net torque due to a force couple is the same about any point (see Fig. 7.26).

Total torque about A = $x_1F + x_2F = F(x_1 + x_2) = Fd$

Total torque about B = $y_1F - y_2F = F(y_1 - y_2) = Fd$

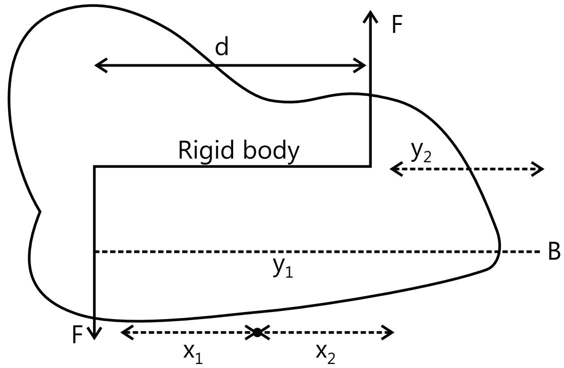


Figure 7.26: Force couple

6.4 Torque on a Rigid Body Executing Pure Rotation

Suppose I is the moment of inertia of a rigid body about the axis of rotation which is stationary in a given reference frame. The body is executing pure rotational motion about this fixed axis.

τ_{ext} = resultant torque about the axis of rotation due to all the external forces acting on the body

α = instantaneous angular acceleration of the body.

ω = instantaneous angular velocity of the body.

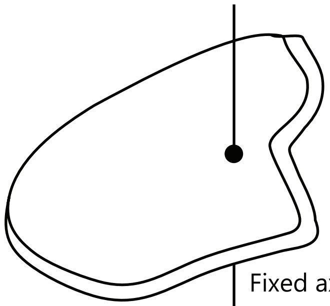


Figure 7.27: Rigid body executing pure rotation

Consider one particle of the body say i[th] particle of mass m_i at perpendicular distance r_i from the axis.

Radial force on the particle $F_r = m\omega^2 r$ towards the center of its circular path.

Tangential force on the particle $F_t = m_i a_t = m_i \alpha r_i$

Torque of the radial force about the axis of rotation is zero as it intersects the axis. Torque of tangential force about the axis will be,

$$\tau_i = r_i F_t = m_i r_i^2 \alpha$$

To find the total torque on the rigid body about the axis we take summation of torques acting on all the particles of the body. The total torque comes out to be equal to the resultant torque due to external forces only as the torques due to internal forces cancel each other in pairs when summation is taken on all the particles of the body (By Newton's third law of motion internal forces form pairs of equal and opposite collinear forces. So the lever arms of the forces of a pair with respect to the axis will be equal so their torques will have equal magnitude but opposite directions and cancel each other in the summation). So

$$\tau_{ext} = \sum_i \tau_i = (\sum_i m_i r_i^2)\, \alpha = I\,\alpha \qquad \text{...(i)}$$

Remember: This formula is applicable only for pure rotational motion of a rigid body about a fixed axis.

7. KINETIC ENERGY OF BODY IN PURE ROTATION

When a rigid body performs pure rotational motion about a given axis, all of its constituent particles move in circular paths with centers on the axis and radii r_1 , r_2 ...... and r_n (say), and with linear velocities $v_1 = \omega r_1$, $v_2 = \omega r_2$,........... and $v_n = \omega r_n$. If m_1, m_2,..... and m_n are the masses of the particles then the total kinetic energy of the rigid body is given by

$$K = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \text{.........} + \frac{1}{2} m_n v_n^2 \qquad \text{...(i)}$$

$$= \frac{1}{2} m_1\omega^2 r_1^2 + \frac{1}{2} m_2\omega^2 r_2^2 + \ldots\ldots\ldots + \frac{1}{2} m_n\omega^2 r_n^2$$

$$= \frac{1}{2}(m_1 r_1^2 + m_2 r_2^2 + \ldots\ldots\ldots + m_n r_n^2)\omega^2$$

Now as we have learnt the term $m_1 r_1^2 + m_2 r_2^2 + \ldots\ldots\ldots + m_n r_n^2$ is the moment of inertia of the rigid body.

Hence, the rotational kinetic energy of a body is given by

$$K = \frac{1}{2} I\omega^2 \quad \ldots(ii)$$

PLANCESS CONCEPTS

Most of the problems involving incline and a rigid body, can be solved by using the conservation of energy. Care has to be taken in writing down the total Kinetic energy. Rotational Kinetic Energy term has to be taken into consideration along with translational kinetic energy. And while writing the rotational energy, the axis about which the moment of inertia is taken should be carefully chosen.

The point about which the conservation is done should be inertial to avoid calculating the work done by pseudo forces or the point itself should be the COM so that the work done by the torque of pseudo forces would be 0.

Shrikant Nagori (JEE 2009 AIR 30)

Illustration 16: A uniform circular disc has radius R and mass m. A particle, also of mass m, is fixed at a point A on the edge of the disc as shown in Fig 7.28. The disc can freely rotate about a fixed horizontal chord PQ that is at a distance R/4 from the center C of the disc. The line CA is ⊥ to PQ. Initially the disc is held vertical with point A at its highest point. It is then allowed to fall so that it starts rotating about PQ. Find the linear speed of the particle as it reaches lowest point. **(JEE ADVANCED)**

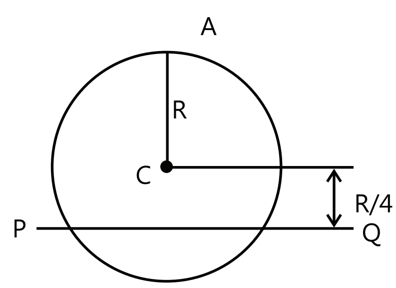


Figure 7.28

Sol: Find the moment of inertia of circular disc and the particle at point A about the chord PQ. The loss in potential energy of the system comprising the disc and the particle will be equal to the gain in its rotational kinetic energy.

$$I = \frac{1}{2} \times \frac{mR^2}{2} + m\left(\frac{R}{4}\right)^2 + m\left(\frac{5R}{4}\right)^2 = \frac{15mR^2}{8}$$

Energy equation

$$mg\frac{5R}{4} + \frac{mgR}{4} = \frac{1}{2}I\omega^2 - mg\frac{5R}{4} - \frac{mgR}{4}$$

$$\omega = 4\sqrt{\frac{g}{5R}}$$

$$V = \frac{5R\omega}{4} = \sqrt{5gR}$$

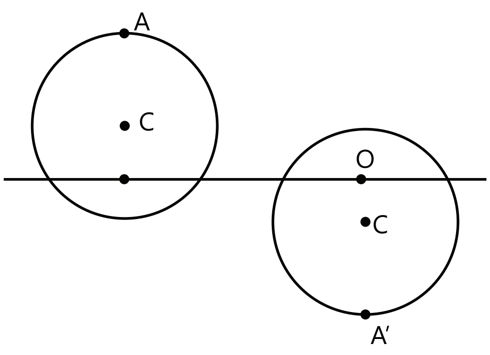


Figure 7.29

Illustrations 17: A pulley having radius r and moment of inertia I about its axis is fixed at the top of an inclined plane of inclination θ as shown in Fig 7.30. A string is wrapped round the pulley and its free end supports a block of mass m which can slide on the plane initially. The pulley is rotated at a speed ω_0 in a direction such that the block slides up the plane. Calculate the distance moved by the block before stopping? **(JEE ADVANCED)**

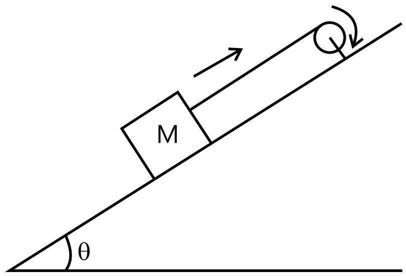


Figure 7.30

Sol: Apply Newton's second law of motion for block M along the inclined plane. Find the torque (about its axis) of force of tension acting on pulley. This will be equal to the product of moment of inertia I and the angular acceleration of pulley.

Suppose the deceleration of the block is a. The linear deceleration of the rim of the pulley is also a. The angular deceleration of the pulley is α = a/r. If the tension in the string is T, the equations of motion are as follows:

$$mg\sin\theta - T = ma \text{ and } Tr = I\alpha = Ia/r.$$

Eliminating T from these equations,

$$mg\sin\theta - I\frac{a}{r^2} = ma; \text{ Giving, } a = \frac{mgr^2\sin\theta}{I + mr^2}$$

The initial velocity of the block up the incline is v = $\omega_0 r$ Thus, the distance moved by the block before stopping is

$$x = \frac{v^2}{2a} = \frac{(I + mr^2)\omega_0^2}{2mg\sin\theta}$$

Illustration 18: A uniform rod of mass m and length ℓ can rotate in vertical plane about a smooth horizontal axis hinged at point H.

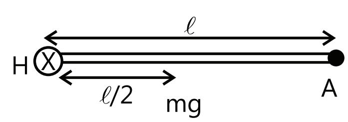


Figure 7.31

(i) Find angular acceleration α of the rod just after it is released from initial horizontal position from rest?

(ii) Calculate the acceleration (tangential and radial) of point A at this moment.

(iii) Calculate net hinge force acting at this moment.

(iv) Find α and ω when rod becomes vertical.

(v) Find hinge force when rod become vertical. **(JEE ADVANCED)**

Sol: The axis of rotation passing through H is fixed. So the torque of force of gravity about axis through H is equal to the product of moment of inertia about axis through H and angular acceleration of rod. Angular acceleration at an instant can be found if the torque of force of gravity at the instant is known.

(i) $\tau_H = I_H\ \alpha$

$$mg.\frac{\ell}{2} = \frac{m\ell^2}{3}\alpha \Rightarrow \alpha = \frac{3g}{2\ell}$$

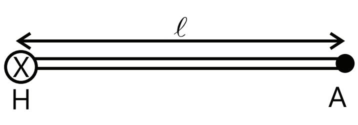


Figure 7.32

(ii) $a_{tA} = \alpha\ell = \frac{3g}{2\ell}.\ell = \frac{3g}{2}$

$a_{CA} = \omega^2 r = 0.\ell = 0$ ($\because \omega = 0$ just after release)

(iii) Suppose hinge exerts normal reaction in component form as shown

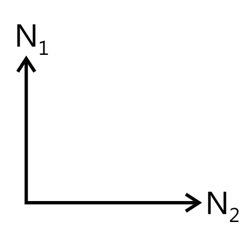


Figure 7.33

In vertical direction

$F_{ext} = ma_{CM}$

$$\Rightarrow mg - N_1 = m.\frac{3g}{4}$$

(We get the value of a_{CM} from previous example)

$$\Rightarrow N_1 = \frac{mg}{4}$$

In horizontal direction

$F_{ext} = ma_{CM} \Rightarrow N_2 = 0$ ($\because a_{CM}$ in horizontal = 0 as $\omega = 0$ just after release)

(iv) Torque = 0 when rod becomes vertical so $\alpha = 0$

Using energy conservation $\frac{mg\ell}{2} = \frac{1}{2}I\omega^2 \quad \left(I = \frac{m\ell^2}{3}\right)$

(Work done by gravity when COM moves down by (½) ℓ = change in K.E.)

$$\omega = \sqrt{\frac{3g}{\ell}}$$

(v) When rod becomes vertical

$\alpha = 0, \omega = \sqrt{\frac{3g}{\ell}}$ (Using $F_{net} = Ma_{CM}$)

$F_H - mg = \frac{m\omega^2\ell}{2}$ (a_{CM} = centripetal acceleration of COM)

Ans. $F_H = \frac{5mg}{2}$

Illustration 19: A bar of mass m is held as shown between 4 disks each of mass m′ and radius r = 75 mm. Determine the acceleration of the bar immediately after it has been released from rest, knowing that the normal forces exerted on the disks are sufficient to prevent any slipping and assuming that. (a) m = 5 kg and m′ = 2 kg.

(b) The mass m′ of the disks is negligible.

(c) The mass m of the bar is negligible **(JEE ADVANCED)**

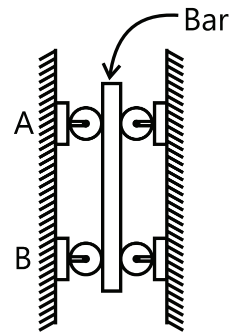


Figure 7.34

Sol: Apply Newton's second law of motion in vertical direction for the motion of center of mass of bar. Write the equation of torque due to force of friction acting on disc, for rotational motion about fixed axis through center of disk. Acceleration of rod will be equal to the tangential acceleration of the disc in the case of no slipping.

(a) Equation of center of mass of rod,

$mg - 4f = ma$ ....(i)

(where f is frictional force from one disk)

Torque acting on each disk due to frictional force is

$fr = \frac{m'r^2}{2}\frac{a}{r}$ ....(ii)

From (i) and (ii) we get

$mg - 2m'a = ma$ ....(iii)

$5g = (5 + 2 \times 2)a; \quad a = \frac{5g}{9}$

(b) Putting m′=0 in eqn. (iii) we get a = g

(c) Putting m = 0 in eqn. (iii) we get a = 0

(a) $\frac{5g}{9}\downarrow$

(b) $g\downarrow$ c) 0

7.1 Work Done and Power Delivered by Torque

If a torque τ rotates a body through an angle $d\theta$, the work, dW done by it is given by

$$dW = \tau . d\theta$$

The total work done W in rotating a body from the initial angle θ_1 to the final angle θ_2, is

$$W = \int_{\theta_1}^{\theta_2} \tau . d\theta = \int_{\omega_1}^{\omega_2} I \frac{d\omega}{dt} \omega dt = \int_{\omega_1}^{\omega_2} I \omega \, d\omega = \frac{1}{2} I\omega_2^2 - \frac{1}{2} I\omega_1^2$$

So the work done by torque is equal to the change in the rotational kinetic energy.

$$W = \Delta K_{rot} = \Delta\left(\frac{1}{2} I\omega^2\right) \quad ...(i)$$

This is called the **Work-Energy Theorem for rotation of rigid body**.

The rate at which work is done is called power P, given by

$$P = \frac{dW}{dt} = \tau \frac{d\theta}{dt} = \tau\omega \quad ...(ii)$$

Also, the power P delivered by the torque on the rigid body is equal to the rate of change of kinetic energy

$$K = \frac{1}{2} I\omega^2 \qquad \therefore P = \frac{dK}{dt} = \frac{d}{dt}\left(\frac{1}{2} I\omega^2\right)$$

$$\therefore P = \frac{1}{2} \times I \times 2\omega \frac{d\omega}{dt} = I \frac{d\omega}{dt} \omega = \tau \omega$$

8. EQUILIBRIUM OF RIGID BODIES

A rigid body can be in linear equilibrium as well as in rotational equilibrium. If a rigid body is in linear equilibrium, then the vector sum of all the forces acting on it should be zero.

i.e. $\Sigma \vec{F}_{ext} = 0$

Taking scalar components along the three axes x, y and z we get $\Sigma F_x = 0, \Sigma F_y = 0, \Sigma F_z = 0$

If a rigid body is in rotational equilibrium then the vector sum of all the external torques acting on it with respect to an axis in a given reference frame must be zero.

$$\Sigma \tau_{ext} = 0 \Rightarrow \Sigma \tau_x = 0, \Sigma \tau_y = 0, \Sigma \tau_z = 0$$

Illustrations 20: Two boys weighing 20 kg and 25 kg are trying to balance a seesaw of total length 4 m, with the fulcrum at the center. If one of the boys is sitting at an end, where should the other sit? **(JEE MAIN)**

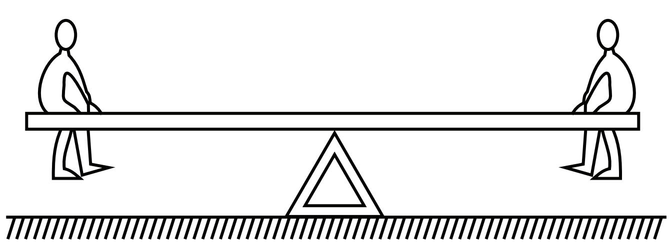


Figure 7.35

Sol: For rotational equilibrium, the net torque about the fulcrum of all the forces acting on the boys and the seesaw should be zero.

It is clear that the 20 kg kid should sit at the end and the 25 kg kid should sit closer to the center. Suppose his distance from the center is x. As the boys are in equilibrium, the normal force between a boy and the seesaw equals the weight of that boy. Considering the rotational equilibrium of the seesaw, the torque of the forces acting on it should add to zero. The forces are

(a) (25kg) g downward by the 25 kg boy

(b) (20kg) g downward by the 20 kg boy

(c) Weight of the seesaw and

(d) The normal force by the fulcrum.

Taking torques about the fulcrum.

(25 kg) g x = (20 kg) g (2 m) or x = 1.6 m

9. ANGULAR MOMENTUM

9.1 Angular Momentum of a Particle About a Point

If $\vec{p}$ is the linear momentum of a particle in a given reference frame, then angular momentum of the particle about an origin O in this reference frame is defined as

$$\vec{L} = \vec{r} \times \vec{p} \quad ...(i)$$

where $\vec{r}$ is the position vector of the particle with respect to origin O (see Fig. 7.36).

Magnitude of angular momentum is $L = rp\sin\theta$

or $L = r_\perp p$ or $L = p_\perp r$

θ = angle between vectors $\vec{r}$ and $\vec{p}$

$r_\perp$ = component of position vector $\vec{r}$ perpendicular to vector $\vec{p}$.

$p_\perp$ = component of vector $\vec{p}$ perpendicular to position vector $\vec{r}$.

SI unit angular momentum is $kg\,m^2\,s^{-1}$.

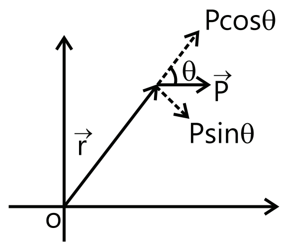


Figure 7.36: Angular momentum about a point

Relation between Torque and Angular Momentum

$\because \quad \vec{L} = \vec{r} \times \vec{p}$

Differentiating with respect to time we get

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{v} \times (m\vec{v}) + \vec{r} \times \vec{F} = 0 + \vec{r} \times \vec{F} = \vec{\tau}$$

$$\Rightarrow \frac{d\vec{L}}{dt} = \vec{\tau} \quad ...(ii)$$

For a single particle moving in a circle of radius r with angular velocity ω we have

$v = \omega r$ and $p = m\omega r$

So angular momentum comes out to be $L = r\,p = mr^2\omega$

Illustration 21: A particle of mass m is projected at time t = 0 from a point O with a speed u at an angle of 45° to the horizontal. Calculate the magnitude and direction of the angular momentum of the particle about the point O at time t = u/g. **(JEE ADVANCED)**

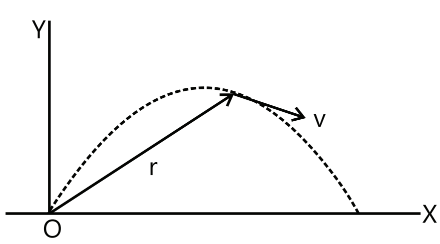


Figure 7.37

Sol: Express the position and velocity of particle in Cartesian coordinates in terms of unit vectors $\hat{i}$ and $\hat{j}$ and then calculate the cross product in

Cartesian coordinates.

Let us take the origin at O, X –horizontal axis and

Y – Axis along the vertical upward direction as shown in Fig 7.37 for horizontal during the time 0 to t.

$v_x = u\cos 45° = u/\sqrt{2}$ and $x = v_x t = \frac{u}{\sqrt{2}} \cdot \frac{u}{g} = \frac{u^2}{\sqrt{2}g}$

For vertical motion,

$v_y = u\sin 45° - gt = \frac{u}{\sqrt{2}} - u = \frac{(1-\sqrt{2})}{\sqrt{2}}u$

and $y = (u\sin 45°)\,t - \frac{1}{2}gt^2 = \frac{u^2}{\sqrt{2}g} - \frac{u^2}{2g} = \frac{u^2}{2g}(\sqrt{2}-1)$

The angular momentum of the particle at time t about the origin is

$$\vec{L} = \vec{r} \times \vec{P} = m\vec{r} \times \vec{v} = m(\hat{i}x + \hat{j}y) \times (\hat{i}v_x + \hat{j}v_y) = m(\hat{k}xv_y - \hat{k}yv_x)$$

$$= m\hat{k}\left[\left(\frac{u^2}{\sqrt{2}g}\right)\frac{u}{\sqrt{2}}(1-\sqrt{2}) - \frac{u^2}{2g}(\sqrt{2}-1)\frac{u^2}{\sqrt{2}}\right] = -\hat{k}\frac{mu^3}{2\sqrt{2}g}$$

Thus, the angular momentum of the particle is $\frac{mu^3}{2\sqrt{2}g}$ in the negative z – direction i.e., perpendicular to the plane of motion, going into the plane.

Illustration 22: A cylinder is given angular velocity ω_0 and kept on a horizontal rough surface the initial velocity is zero. Find out distance travelled by the cylinder before it performs pure rolling and work by frictional force.

(JEE ADVANCED)

Sol: Due to backward slipping force of friction will act forwards. The cylinder is accelerated forwards. The torque due to friction and hence the angular acceleration is opposite to the initial angular velocity. So the angular velocity will decrease and the linear velocity of center of mass of cylinder will increase in the forward direction, till the slipping stops and pure rolling starts. The work done by frictional force is equal to change in the kinetic energy of the cylinder. The kinetic energy includes both rotational kinetic energy and translational kinetic energy.

$\mu Mg\,R = \frac{MR^2\alpha}{2}$

$\alpha = \frac{2\mu g}{R}$ ... (i)

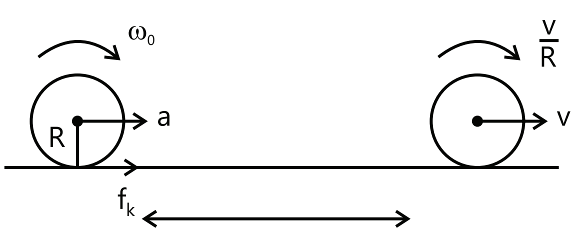


Figure 7.38

Initial velocity u = 0

$v^2 = u^2 + 2as$

$v^2 = 2as$ ... (ii)

$f_k = ma$; $\mu Mg = Ma$; $a = \mu g$ ... (iii)

$\omega = \omega_0 - \alpha t$

From equation (i) $\omega = \omega_0 - \frac{2\mu g}{R}t$; $V = u + at$

From equation (iii) $v = \mu g t$

$\omega = \omega_0 - \frac{2v}{R}$; $\omega = \omega_0 - 2\omega$; $\omega = \frac{\omega_0}{3}$

From equation (ii)

$\left(\frac{\omega_0 R}{3}\right)^2 = (2as) = 2\mu gs$; $S = \left(\frac{\omega_0^2 R^2}{18\mu g}\right)$

Work done by the frictional force

$$W = (-f_k R d\theta + f_k \Delta s) = -\mu mgR\Delta\theta + \frac{\mu mg \times \omega_0^2 R^2}{18\mu g};$$

$$\Delta\theta = \omega_0 \times t - \frac{1}{2}\alpha t^2 = \left\{\omega_0 \times \left(\frac{\omega_0 R}{3\mu g}\right)\frac{1}{2} \times \frac{2\mu g}{R}\left(\frac{\omega_0 R}{3\mu g}\right)^2\right\} = \left(\frac{\omega_0^2 R}{3\mu g} - \frac{\omega_0^2 R}{9\mu g}\right) = \frac{2\omega_0^2 R}{9\mu g}$$

$$W = \left\{\left(-\mu mg \times R\,\frac{2\omega_0^2 R}{9\mu g}\right) + \left(\mu mg \times \frac{\omega_0^2 R^2}{18\mu g}\right)\right\} = -\frac{m\omega_0^2 R^2}{6}$$

Illustration 23: A hollow sphere is projected horizontally along a rough surface with speed v and angular velocity ω_0. Find out the ratio $\frac{v}{\omega_0}$, so that the sphere stops moving after some time. **(JEE ADVANCED)**

Sol: For the sphere to stop after sometime, the acceleration should be opposite to velocity, i.e. the force of friction should be backwards (forward slipping). Also, the torque due to friction should be opposite to angular velocity, i.e. if the torque due to friction is clockwise (see Fig. 7.39), then the initial angular velocity should be anti-clockwise.

Torque about lowest point of sphere

$f_K \times R = I\alpha$; $\mu mg \times R = \frac{2}{3}mR^2\alpha$; $\alpha = \frac{3\omega g}{2R}$ (Angular acceleration in opposite direction of angular velocity)

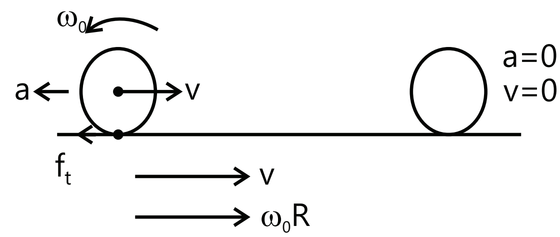


Figure 7.39

$\omega = \omega_0 - \alpha t$ (Final angular velocity $\omega = 0$)

$\omega_0 = \frac{3\omega g}{2R} \times t$; $t = \frac{\omega_0 \times 2R}{3ì g}$

Acceleration a = μg

$v_t = v - at$ (Final velocity $v_t = 0$);

$v = \mu g \times t$; $t = \frac{v}{\mu g}$

To stop the sphere, time at which v and ω are zero, should be same.

$$\frac{v}{\mu g} = \frac{2\omega_0 R}{3ì g}; \Rightarrow \frac{v}{\omega_0} = \frac{2R}{3}$$

Illustration 24: A rod AB of mass 2m and length ℓ is lying on a horizontal frictionless surface. A particle of mass m travelling along the surface hits the end of the rod with a velocity v_0 in a direction perpendicular to AB. The collision is elastic. After the collision the particle comes to rest. Find out after collision

(a) Velocity of center of mass of rod

(b) Angular velocity. **(JEE ADVANCED)**

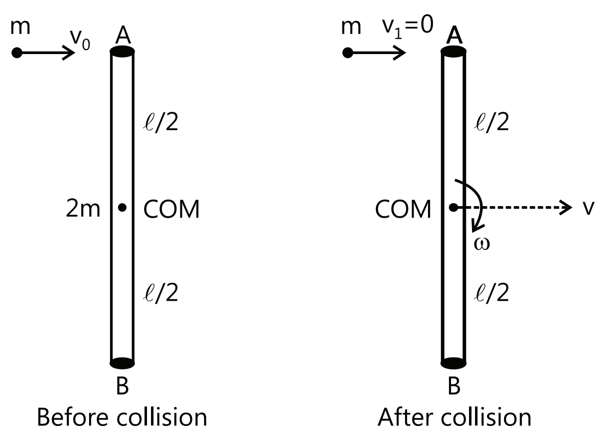


Figure 7.40

Sol: Conserve linear momentum and angular momentum of the system constituting "the rod and the particle" before and after collision. Here the linear and angular momentum of the rod before collision is zero. Angular momenta of the rod and particle are calculated about the center of the rod.

(a) Let just after collision the speed of COM of rod is v and angular velocity about COM is ω.

External force on the system (rod + mass) in horizontal plane is zero.

Apply conservation of linear momentum in x direction;

$$mv_0 = 2mv \qquad \text{..... (i)}$$

(b) Net torque on the system about any point is zero

Apply conservation of angular momentum about COM of rod.

$$mv_0\frac{\ell}{2} = I\omega \Rightarrow mv_0\frac{\ell}{2} = \frac{2m\ell^2}{12}\omega \qquad \text{...... (ii)}$$

From equation (i) velocity of center of mass $v = \frac{v_0}{2}$

From equation (ii) angular velocity $\omega = \frac{3v_0}{\ell}$.

9.2 Angular Momentum of a Rigid Body Rotating About Fixed Axis

For a system of particles the total angular momentum about an origin is the sum of the angular momenta of all the particles calculated about the same origin.

$$\vec{L} = \sum_i \vec{L}_i$$

Differentiating with respect to time we get,

$$\frac{d\vec{L}}{dt} = \sum_i \frac{d\vec{L}_i}{dt} = \sum_i\left(\sum_k \vec{\tau}_{ik} + \vec{\tau}_i^{\,ext}\right) = \sum_i\sum_k \vec{\tau}_{ik} + \sum_i \vec{\tau}_i^{\,ext} = 0 + \vec{\tau}^{\,ext}$$

The double summation term corresponds to the sum of torques due to internal forces and as explained earlier, according to Newton's third law of motion these internal torques cancel out in pairs.

So for a system of particles

$$\frac{d\vec{L}}{dt} = \vec{\tau}^{\,ext} \qquad \text{...(xvii)}$$

Impulse of a torque is defined as $J = \int d\vec{L} = \int \vec{\tau}^{\,ext} dt$

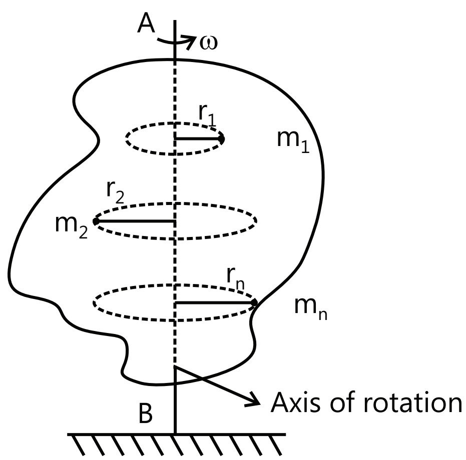


Figure 7.41: Angular momentum of rigid body

Angular momentum of a rigid body rotating about a fixed axis can be calculated as below:

Angular momenta of its individual particles about the axis are

$L_1 = m_1r_1^2\omega$, $L_2 = m_2r_2^2\omega$, $L_3 = m_3r_3^2\omega$, ... $L_n = m_nr_n^2\omega$ where ω is the instantaneous angular velocity of the rigid body

Total angular momentum of the body

$$L = m_1r_1^2\omega + m_2r_2^2\omega + m_3r_3^2\omega \text{ ....} + m_nr_n^2\omega$$

$$L = \sum_i m_i(r_i)^2\omega = I\omega$$

So $L = I\omega$

Remember: This formula is applicable only for rotation of the rigid body about a fixed axis.

Again differentiating this relation with respect to time we get,

$$\frac{dL}{dt} = I\frac{d\omega}{dt} = I\alpha = \tau_{ext}$$

Illustration 25: Two small balls of mass m each are attached to a light rod of length ℓ, one at its center and the other at its free end. The rod is fixed at the other end and is rotated in horizontal plane at an angular speed ω. Calculate the angular momentum of the ball at the end with respect to the ball at the center. **(JEE MAIN)**

Sol: Both the balls A and B have same angular velocity but different linear velocities.

The situation is shown in Fig 7.42. The velocity of the ball A with respect to the fixed end O is $v_A = \omega(\ell/2)$ and that of B with respect to O is $v_B = \omega\ell$. Hence the velocity of B with respect to A is $v_B - v_A = \omega(\ell/2)$. The angular momentum of B with respect to A is, therefore,

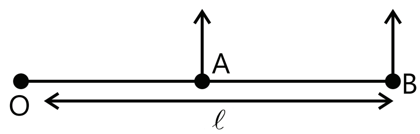


Figure 7.42

$$L = mvr = m\omega\left(\frac{\ell}{2}\right)\frac{\ell}{2} = \frac{1}{4}m\omega\ell^2$$

along the direction perpendicular to the plane of rotation.

9.3 Conservation of Angular Momentum

In the previous article we have proved the relation

$$\frac{d\vec{L}}{dt} = \vec{\tau}^{ext} \text{ where } \vec{L} \text{ and } \vec{\tau}^{ext} \text{ are evaluated about the same origin.}$$

From the above equation we see that if $\vec{\tau}^{ext} = 0$ then $\vec{L}$ of the system of particles remains constant.

In some situations the component of external torque about an axis is zero even if the net external torque is not zero. So in these cases the component of the total angular momentum, about the particular axis, remains constant.

Illustration 26: A uniform rod of mass m and length ℓ can rotate freely on a smooth horizontal plane about a vertical axis hinged at point H. A point mass having same mass m coming with an initial speed u perpendicular to the rod strikes the rod in-elastically at its free end. Find out the angular velocity of the rod just after collision?

(JEE MAIN)

Sol: After collision the rod and the particle execute pure rotational motion about vertical axis through fixed point H.

Angular momentum is conserved about H because no external force is present in horizontal plane which is producing torque about H.

$$mul = \left(\frac{m\ell^2}{3} + m\ell^2\right)\omega \quad \Rightarrow \quad \omega = \frac{3u}{4\ell}$$

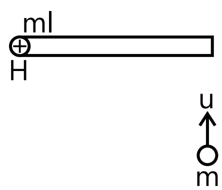


Figure 7.43

Illustration 27: A uniform rod of mass m_1 and length ℓ lies on a frictionless horizontal plane. A particle of mass m_2 moving at a speed v_0 perpendicular to the length of the rod strikes it at a distance $\ell/3$ from the center and stops after the collision. Calculate (a) the velocity of the center of the rod and (b) the angular velocity of the rod about its center just after the collision. **(JEE ADVANCED)**

Sol: Conserve the linear momentum of the system comprising "the rod and the particle" before and after the collision. Conserve the angular momentum, about the center of the rod, of the system comprising "the rod and the particle" before and after the collision.

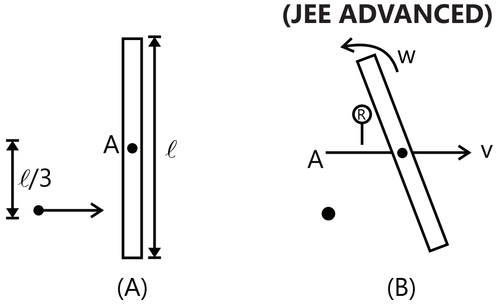


Figure 7.44

The situation is shown in the Fig 7.44. Consider the rod and the particle together as the system. As there is no external resultant force, the linear momentum of the system will remain constant. Also there is no resultant external torque on the system and so the resultant external torque on the system and the angular momentum of the system about the line will remains constant. Suppose the velocity of the center of the rod is V and the angular velocity about the center is ω .

(a) The linear momentum before the collision is m_2v_0 and that after the collision is M_1V.

Thus $m_2v_0 = m_1V$, or $V = \left(\frac{m_2}{m_1}\right)v_0$

(b) Let A be the center of the rod when at rest. Let AB be the line perpendicular to the plane of the Fig 7.44. Consider the angular momentum of N "the rod plus the particle" system about AB.

Initially the rod is at rest. The angular momentum of the particle about AB is

$$L = m_2v_0(\ell/3)$$

After collision the particle comes to rest. The angular momentum of the rod about a is

$$\vec{L} = \vec{L}_{CM} + m_1\vec{r_0} \times \vec{V}$$

As $\vec{r_0} \parallel \vec{V}$, $\vec{r_0} \times \vec{V} = 0$ thus, $\vec{L} = \vec{L}_{CM}$

Hence the angular momentum of the rod about AB is

$$L = I\omega = \frac{m_1\ell^2}{12}\omega \quad \text{Thus,} \quad \frac{m_2v\ell}{3} = \frac{m_1\ell^2}{12}\omega \quad \text{Or} \quad \omega = \frac{4m_2v_0}{m_1\ell}$$

10. RIGID BODY IN COMBINED TRANSLATIONAL AND ROTATIONAL MOTION

As discussed earlier, in this type of motion the rigid body is performing pure rotational motion about an axis and the axis itself is performing pure translational motion relative to a given reference frame.

Consider a car moving over a straight horizontal road with some instantaneous velocity v with respect to a reference frame K fixed to the road. Now let us observe the motion of a wheel of the car from the K frame. This motion of the wheel in K frame is an example of combined translational and rotational motion. Let us suppose a reference frame K' which is translating with respect to frame K with same instantaneous velocity v. In other words frame K' is rigidly fixed to the body of the car. In this frame the wheel of the car performs pure rotational motion. The body of the car itself is performing pure translational motion.

Take another example of motion of a fan fixed inside the car.

If the fan is switched off while the car is moving on the road, the motion of fan is **pure translational** with respect to K frame.

If the fan is switched on while the car is at rest, the motion of fan is **pure rotational** about its axis, as the axis is at rest in the K frame.

If the fan is switched on while the car is moving on the road, the motion of the fan with respect to K frame is neither pure translational nor pure rotational but a combination of both. Now if an observer A is sitting inside the car, as the car moves, the motion of fan will appear to him as pure rotational while the motion of the observer A with respect to K frame is pure translational. Hence in this case we can see that the motion of the fan can be resolved into two components, pure rotational motion relative to observer A and pure translational motion of observer A relative to K frame.

Such a resolution of motion of a rigid body into components of pure rotational and pure translational motion is an important tool used in the study of their dynamics.

10.1 Kinematics of a General Rigid Body Motion

For a rigid body the value of angular displacement θ, angular velocity ω, and angular acceleration α is same for all points on the rigid body. Also, if we choose any point of the rigid body as origin O and any other point, say A, of the body has a position vector $\vec{r}$ relative to O, and during any time interval the vector $\vec{r}$ rotates by an angle θ relative to its initial direction, then position vector of any other point, say B, relative to any other origin, say O′, inside the rigid body will also rotate by the same angle θ. This means the angular variables θ, ω, and α do not depend on the choice of origin in the rigid body.

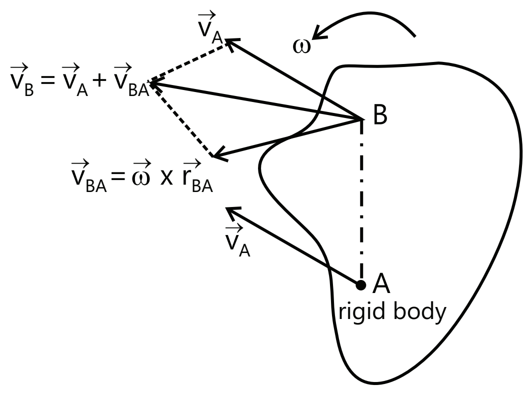


Figure 7.45: Kinematics of rigid body

The above concept is very important as it enables us to calculate the velocity of each point of the rigid body if we know the velocity of any one point (say A) in the rigid body with respect to a reference frame K and angular velocity of any point in the rigid body relative to any other point in the rigid body.

Suppose we want to calculate the velocity of a point B in the rigid body which has a position vector $\vec{r}_{BA}$ relative to A (see Fig. 7.45).

The velocity of point A is $\vec{v}_A$, so we have velocity of B as

$$\vec{v}_B = \vec{v}_A + \vec{v}_{BA} = \vec{v}_A + \vec{\omega} \times \vec{r}_{BA}$$

Direction of $\vec{\omega}$ is given by right hand thumb rule. If we curl the fingers of the right hand in the direction of rotation of the body, thumb gives the direction of $\vec{\omega}$.

Similarly the acceleration of point B is: $\vec{a}_B = \vec{a}_A + \vec{\alpha} \times \vec{r}_{BA}$

Illustration 28: Consider the general motion of a wheel (radius r) which can be viewed as pure translation of its center O (with the velocity v) and pure rotation about O (with angular velocity $\vec{\omega}$)

Find out $\vec{v}_{AO}, \vec{v}_{BO}, \vec{v}_{CO}, \vec{v}_{DO}$ and $\vec{v}_A, \vec{v}_B, \vec{v}_C, \vec{v}_D$ **(JEE MAIN)**

Sol: Express the angular velocity, linear velocity of point O and position vectors of points A, B and C relative to O in Cartesian coordinates.

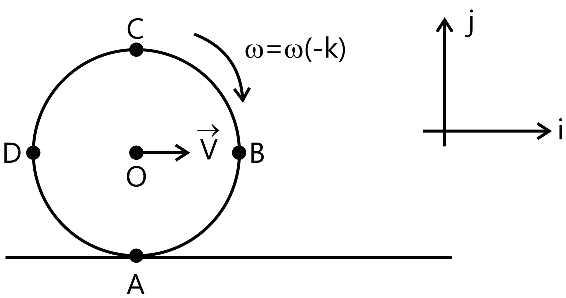


Figure 7.46

$$\vec{v}_{AO} = (\vec{\omega} \times \vec{r}_{AO}) = \left(\omega(-\hat{k}) \times \vec{OA}\right)$$
$$= \left(\omega(-\hat{k}) \times r(-\hat{j})\right) = -\omega r\hat{i}$$

Similarly $\vec{v}_{BO} = \omega r\,(-\hat{j})$; $\vec{v}_{CO} = \omega r(\hat{i})$; $\vec{v}_{DO} = \omega r(\hat{j})$

$$\vec{v}_A = \vec{v}_O + \vec{v}_{AO} = v\hat{i} - \omega r\hat{i} \ ;$$
$$\vec{v}_B = \vec{v}_O + \vec{v}_{BO} = v\hat{i} + \omega r\hat{j}$$
$$\vec{v}_C = \vec{v}_O + \vec{v}_{CO} = v\hat{i} + \omega r\hat{i} \ ; \qquad \vec{v}_D = \vec{v}_O + \vec{v}_{DO} = v\hat{i} + \omega r\hat{j}$$

10.2 Dynamics of a General Rigid Body Motion

Combined rotation and translation of a rigid body is considered as combination of pure rotation in C frame about an axis passing through the center of mass and translation of center of mass in a reference frame K. Dynamics of combined rotational and translational motion of a rigid body in K frame is defined by two vector equations. One of them describes the dynamics of the center of mass of the rigid body in the K frame, and the other the equation of dynamics of pure rotation of the body about center of mass in the C frame.

So if the total mass of the rigid body is M and the net external force acting on it is $\vec{F}_{ext}$ then we have in the K frame,

$$M\frac{d\vec{V}_C}{dt} = \vec{F}_{ext} \qquad \text{...(i)}$$

If I_C is the moment of inertia of the rigid body about the axis passing through center of mass and $\vec{\tau}_C$ is the net torque of all external forces about the axis passing through the center of mass, then we have in the C frame,

$$\vec{\tau}_C = I_C\frac{d\vec{\omega}}{dt} = I_C\vec{\alpha} \qquad \text{...(ii)}$$

If $\vec{P}_{total}$ is the total linear momentum of the rigid body in the K frame, $\vec{L}_C$ is angular momentum of the body in C frame about center of mass and $\vec{r}_C$ is the position vector of center of mass relative to some origin in K frame, then we have,

$$\vec{P}_{total} = M\vec{V}_C$$

Total Kinetic energy

$$K = \frac{1}{2}MV_C^2 + \frac{1}{2}I_C\omega^2 \quad \text{...(iii)}$$

$$\vec{L}_C = I_C \vec{\omega} \quad \text{...(iv)}$$

Angular momentum in K frame = $\vec{L}_C$ about C.O.M + $\vec{L}$ of the C.O.M about some origin in K frame

$$\vec{L} = \vec{I}_C \vec{\omega} + \vec{r}_C \times M\vec{V}_C \quad \text{...(v)}$$

10.3 Pure Rolling (Rolling Without Slipping)

Pure rolling is a special case of combined translational and rotational motion of a rigid body with circular cross section (e.g. wheel, disc, ring, cylinder, sphere etc.) moving on a surface. Here, there is no slipping between the rolling body and the surface at the point of contact.

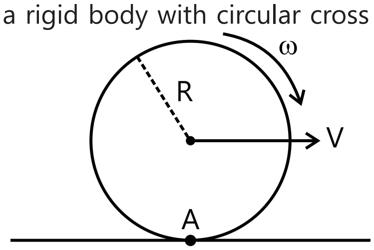


Figure 7.47: Sphere rolling on a stationary surface.

Suppose a sphere rolls on a stationary surface and the point of contact between the sphere and the surface is A (see Fig. 7.47). Let the velocity of the center of sphere be v, radius be R and its angular velocity be ω. For pure rolling the relative velocity between the point A of the sphere and the surface must be zero. As the surface is at rest, the velocity of point A is also zero.

$\therefore v_A = v - \omega R = 0$

$\therefore v = \omega R$

If sphere is rolling on a plank moving velocity v_0, then for pure rolling, $v_A = v - \omega R = v_0$ (see Fig. 7.48)

Same is true for the tangential acceleration of the point of contact in case of pure rolling.

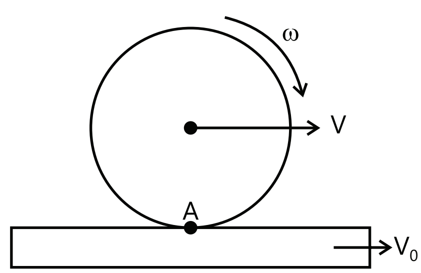


Figure 7.48: Sphere rolling on a moving surface.

Now let's discuss the case where a rolling cylinder of mass m moves forward on a rough plate of same mass with acceleration "a" and the rough plate moves forward with an acceleration "a_0" under action of force F on a smooth surface.

As the cylinder accelerates in the forward direction, so by Newton's second law, the friction on the cylinder at the point of contact will be in forward direction and on the plate in backward direction by Newton's third law (see Fig. 7.50).

Equation of torque about center of cylinder:

$$\tau_C = fR = \frac{mR^2}{2}\alpha$$

$$\Rightarrow \alpha = \frac{2f}{mR} \quad \text{....... (i)}$$

Equation of motion of center of cylinder:

$$f = ma \quad \text{...... (ii)}$$

From (i) and (ii) we get

$$a = \frac{\alpha R}{2}$$

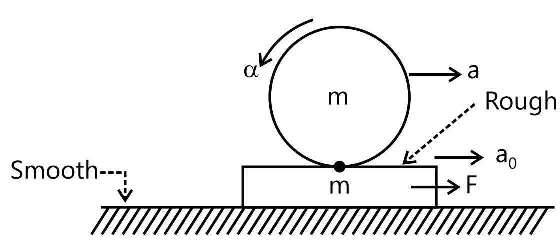


Figure 7.49: Cylinder rolling on an accelerating plate.

At contact point

$\alpha_0 = a + \alpha R = \frac{3\alpha R}{2} = 3a$ ...... (iii)

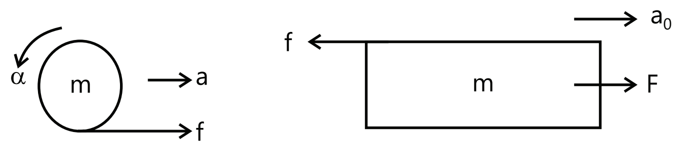


Figure 7.50: (a) FBD of Cylinder. (b) FBD of Plate.

Equation of motion of plate:

$F - f = ma_0$

$F = m(a + a_0)$

$F = 4ma$; $a = \frac{F}{4m}$; $a_0 = \frac{3F}{4m}$

Illustration 29: A wheel of radius r rolls (rolling without slipping) on a level road as shown in fig 7.51.

Find out velocity of point A and B. **(JEE MAIN)**

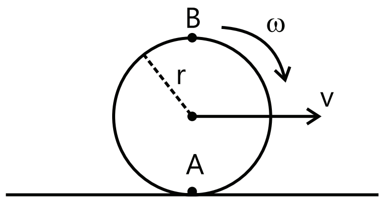


Figure 7.51

Sol: Linear velocity of any point on the rim of the wheel has magnitude ωr in the reference frame of center of wheel (C-frame). Velocity in ground frame is the vector sum of velocity in C-frame and the velocity of center of wheel.

Contact point at surface is in rest for pure rolling

Velocity of point is A zero.

So $v = \omega r$

Velocity of point B = $v + \omega r = 2v$

Illustration 30: A uniform sphere of mass 200 g rolls without slipping on a plane surface so that its center moves at a speed of 2.00 cm s^{-1}. Find its kinetic energy. **(JEE MAIN)**

Sol: The kinetic energy of sphere is the sum of the translational kinetic energy and the rotational kinetic energy.

As the sphere rolls without slipping on the plane surface its angular speed about center is

$\omega = \frac{v_{cm}}{r}$. The kinetic energy is $K = \frac{1}{2} I_{cm}\omega^2 + \frac{1}{2}Mv_{cm}^2 = \frac{1}{2}.\frac{2}{5}Mr^2\omega^2 + \frac{1}{2}Mv_{cm}^2$

$= \frac{1}{5}Mv_{cm}^2 + \frac{1}{2}Mv_{cm}^2 = \frac{7}{10}Mv_{cm}^2 = \frac{7}{10}(0.200\text{ kg})(0.02\text{ m s}^{-1})^2 = 5.6 \times 10^{-5}$ J

Illustration 31: A constant force F acts tangentially at the highest point of a uniform disc of mass m kept on a rough horizontal surface as shown in Fig 7.52. If the disc rolls without slipping, calculate the acceleration of the Center C and point A and B of the disc. **(JEE ADVANCED)**

Sol: Apply Newton's second law for the motion of center of mass of the disc. Find the torque of the force F and the force of friction acting on the disc at point A about the center of mass of the disc and thus obtain the equation relating the angular acceleration in the C-frame to the torques of all the external forces.

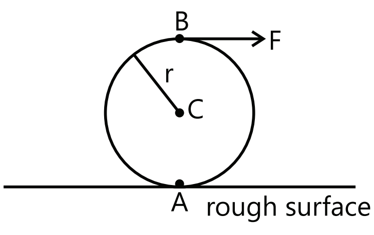


Figure 7.52

The situation is shown in Fig 7.52. As the force F rotates the disc, the point of contact has a tendency to slip towards left so that the static friction on the disc will act towards right. Let r be the radius of the disc and be the linear acceleration of the center of the disc. The angular acceleration about the center of the disc is

$\alpha = a/r$, as there is no slipping.

For the linear motion of the center,

$$F + f = ma \qquad \text{........ (i)}$$

And for the rotation motion about the center,

$$Fr - fr = I\alpha = Fr - fr = I = \left(\frac{1}{2}mr^2\right)\left(\frac{a}{r}\right) \text{ or } \quad F - f = \frac{1}{2}ma \qquad \text{........ (ii)}$$

From (i) and (ii),

$$2F = \frac{3}{2}ma \quad \text{or} \quad a = \frac{4F}{3m}$$

Acceleration of point A is zero

Acceleration of point B is $2a = 2\left(\frac{4F}{3m}\right) = \left(\frac{8F}{3m}\right)$.

Illustration 32: A circular rigid body of mass m, radius R and radius of gyration (k) rolls without slipping on an inclined plane of an inclination θ. Find the linear acceleration of the rigid body and force of friction on it. What must be the minimum value of coefficient of friction so that rigid body may roll without sliding?

(JEE ADVANCED)

Sol: Apply Newton's second law for the motion of center of mass of the rigid body. Find the torque of the force F and the force of friction acting on the rigid body about the center of mass of the disc and thus obtain the equation relating the angular acceleration in the C-frame to the torques of all the external forces.

If a is the acceleration of the center of mass of the rigid body and f the force of friction between sphere and the plane, the equation of translational and rotational motion of the rigid body will be

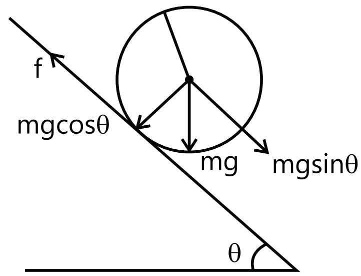


Figure 7.53

$$Mg\sin\theta - f = ma \text{ (Translational motion)}$$

$$fR = I\alpha \text{ (Rotational motion)}$$

$$f = \frac{I\alpha}{R} \quad I = mk^2, \text{ due to pure rolling } a = \alpha R$$

$$mg\sin\theta - \frac{I\alpha}{R} = m\alpha R = m\alpha R + \frac{I\alpha}{R} = ma + \frac{mk^2\alpha}{R} = a\left[\frac{R^2+k^2}{R^2}\right]$$

$$a = \frac{g\sin\theta}{\left(\frac{R^2+k^2}{R^2}\right)} = \frac{g\sin\theta}{\left(1+\frac{k^2}{R^2}\right)}; \qquad f = \frac{I\alpha}{R} = \frac{mk^2a}{R^2} \Rightarrow \frac{mg\,k^2\sin\theta}{R^2+k^2}$$

$$f \le \mu N; \qquad \frac{mk^2}{R^2}a \le \mu \le mg\cos\theta$$

$$R^2\frac{k^2}{R^2} \times \frac{g\sin\theta}{\left(k^2+R^2\right)} \le \mu g\cos\theta;\ \mu \ge \frac{\tan\theta}{\left[1+\frac{R^2}{k^2}\right]}; \qquad \mu_{min} \ge \frac{\tan\theta}{\left[1+\frac{R^2}{k^2}\right]}$$

PLANCESS CONCEPTS

- From above example if rigid bodies are solid cylinder, hollow cylinder, solid sphere and hollow sphere (having radius 'r' and mass 'm')
- Increasing order of acceleration

  $a_{\text{solid sphere}} > a_{\text{hollow sphere}} > a_{\text{solid cylinder}} > a_{\text{hollow cylinder}}$

- Increasing order of required friction force for pure rolling

  $f_{\text{hollow cylinder}} > f_{\text{hollow sphere}} > f_{\text{solid cylinder}} > f_{\text{solid sphere}}$

- Increasing order of required minimum friction coefficient for pure rolling

  $\mu_{\text{hollow cylinder}} > \mu_{\text{hollow sphere}} > \mu_{\text{solid cylinder}} > \mu_{\text{solid sphere}}$

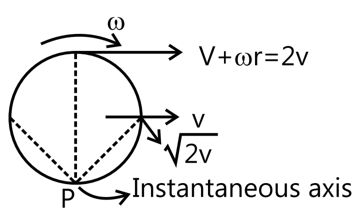
- I would advise you to derive these, verify and remember!

Anand K (JEE 2011 AIR 47)

10.4 Instantaneous Axis of Rotation

The combined translational and rotational motion of a rigid body can be reduced to a purely rotational motion. When we know the velocity V_C of the center of mass and the instantaneous angular velocity ω of the body then we can find a point whose velocity comes out to be zero at a given moment of time. The axis passing through this point at the given moment is called instantaneous axis of rotation and the rigid body performs pure rotation about this axis with same angular velocity at that moment.

The position of the instantaneous axis of rotation changes with time. E.g. in pure rolling the point of contact with the surface is the instantaneous axis of rotation (see Fig. 7.54).



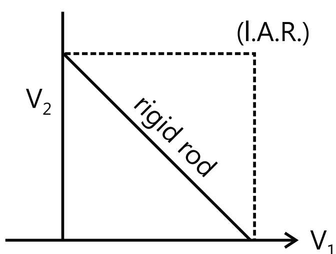


Figure 7.54: IAR (a) pure rolling; (b) Rod slipping down a wall

Geometrical construction of instantaneous axis of rotation (I.A.R). If we know the velocity vectors of any two points in the rigid body then the I.A.R. is the axis passing through the point of intersection of the perpendiculars drawn to the velocity vectors at those points.

Once location of I.A.R is known, we find the moment of inertia of the body about this axis, and then the equations of rotation about fixed axis can be used for this axis.

Illustration 33: Prove that kinetic energy = $1/2\ I_P\omega^2$ **(JEE MAIN)**

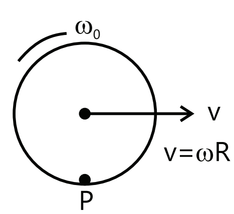


Figure 7.55

Sol: Kinetic energy is the sum of the translational kinetic energy and the rotational kinetic energy.

$$K.E. = \frac{1}{2} I_{cm}\omega^2 + \frac{1}{2} Mv_{cm}^2 = \frac{1}{2}I_{cm}\omega^2 + \frac{1}{2} M\omega^2R^2 = \frac{1}{2}(I_{cm} + MR^2)\omega^2 = \frac{1}{2}\left(I_{contact\ point}\right)\omega^2$$

Notice that in pure rolling of uniform object, equation of torque can also be applied about the contact point.

Illustration 34: A uniform bar of length ℓ and mass m stands vertically touching a vertical wall (y – axis). When slightly displaced, its lower end begins to slide along the floor (x – axis). Obtain an expression for the angular velocity (ω) of the bar as a function of θ. Neglect friction everywhere. **(JEE ADVANCED)**

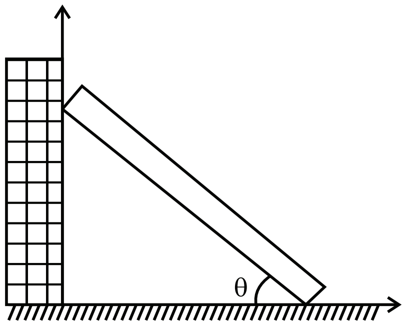


Figure 7.56

Sol: As the rod falls, it executes pure rotational motion about the instantaneous axis of rotation. The loss in gravitational potential energy is equal to the gain in the rotational kinetic energy.

The position of instantaneous axis of rotation (IAOR) is shown in Fig 7.57.

$C = \left(\frac{\ell}{2}\cos\theta, \frac{\ell}{2}\sin\theta\right)$; $r = \frac{\ell}{2}$ = half of the diagonal

All surfaces are smooth, therefore, mechanical energy will remain conserved.

∴ Decrease in gravitational potential energy of bar = increase in rotational kinetic energy of bar about IAOR.

$$mg\frac{\ell}{2}(1 - \sin\theta) = \frac{1}{2}I\omega^2 \quad \text{... (i)}$$

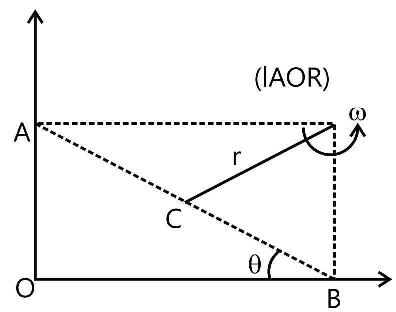


Figure 7.57

Here, $I = \frac{m\ell^2}{12} + mr^2$ (about IAOR) or I

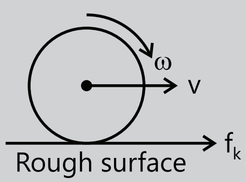
$= \frac{m\ell^2}{12} + \frac{m\ell^2}{4} = \frac{m\ell^2}{3}$ Substituting in Eq. (i)

We have $mg\frac{\ell}{2}(1 - \sin\theta) = \frac{1}{2}\left(\frac{m\ell^2}{3}\right)\omega^2$ or $\omega = \sqrt{\frac{3g(1-\sin\theta)}{\ell}}$

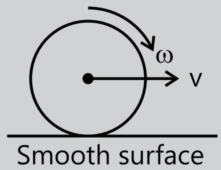
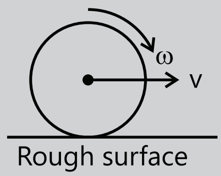
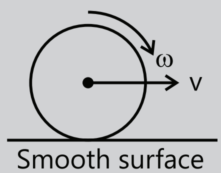
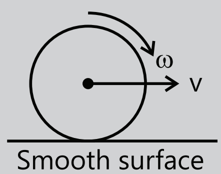
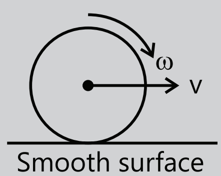
PLANCESS CONCEPTS

Nature of friction for rigid bodies:

- A rigid body rolling with a speed of v and angular velocity of ω at an instant. Then it falls under one of the following cases.

Cases	Rough/Smooth	Diagram	Inference
$V < r\omega$	Rough Surface		1. There is relative motion at point of contact. With respect to the body the surface moves slower than itself. So the surface tries to decrease its angular velocity by a frictional force in forward direction. And this friction is kinetic friction. 2. It increases v and decreases ω So, after sometime, $v = r\omega$ and pure rolling will resume.

PLANCESS CONCEPTS

Cases	Rough/Smooth	Diagram	Inference
	Smooth Surface		No friction is possible and it is not pure rolling.
$v > r\omega$	Rough surface		With respect to the COM of the cylinder, the surface moves at a higher speed than itself. So the surface tries to increase its angular velocity by exerting a frictional force in backward direction. And this friction would be kinetic friction. 2. The friction tries to reduce V and increase ω
$V > r\omega$	Smooth Surface		No friction and no pure rolling.
$V = r\omega$	Rough Surface		This is pure rolling. However there might be static friction acting on the body.
	Smooth Surface		No friction is possible and it is pure rolling

Rohit Kumar (JEE 2012 AIR 79)

Illustration 35: A rigid body of mass m and radius r rolls without slipping on a surface. A force is acting on the rigid body at x distance from the center as shown in Fig 7.58. Find the value of x so that static friction is zero.

(JEE MAIN)

Sol: For static friction to be zero, the linear and angular accelerations a and α caused by the force F should be related as a = αR, for rolling without slipping.

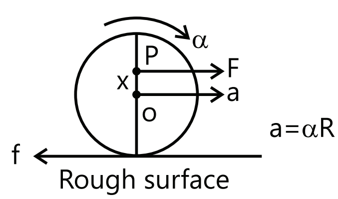


Figure 7.58

Torque about center of mass $Fx = I_{cm}\alpha$ ... (i)

$F = ma$ ... (ii)

From equation (i) & (ii) $max = I_{cm}\alpha$ $(a = \alpha R)$;

$x = \frac{I_{cm}}{mR}$

Illustration 36: There are two cylinders of radii R_1 and R_2 having moments of inertia I_1 and I_2 about their respective axes as shown in Fig 7.59. Initially, the cylinders rotate about their axes with angular speed ω_1 and ω_2 as shown in the Fig 7.59. The cylinders are moved close to touch each other keeping the axes parallel. The cylinders first slip over each other at the contact but the slipping finally ceases due to the friction between them. Calculate the angular speeds of the cylinders after the slipping ceases. **(JEE ADVANCED)**

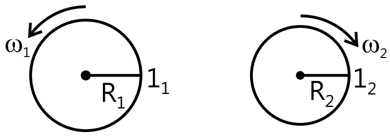


Figure 7.59

Sol: The force of friction acting on the cylinder moving faster will be such that its angular velocity decreases. The force of friction acting on the cylinder moving slower will be such that its angular velocity increases. When slipping ceases, the linear speeds of the points of contact of the two cylinders will be equal.

If ω'_1 and ω'_2 be the respective angular speeds at the instant slipping ceases, we have

$$\omega'_1 R_1 = \omega'_2 R_2 \quad \text{...(i)}$$

The change in the angular speed is brought about by the frictional force which acts as long as the slipping exists. If this force f acts for a time t. the torque on the first cylinder is fR_1 and that on the second is fR_2. Assuming $\omega_1 > \omega_2$. The corresponding angular impulses are $-fR_1 t$ and $fR_2 t$,

We therefore, have

$$-fR_1 t = I_1(\omega'_1-\omega_1) \text{ and } fR_2 t = I_2(\omega'_2-\omega_2)$$

$$\text{or } -\frac{I_1}{R_1}(\omega'_1-\omega_1) = \frac{I_2}{R_2}(\omega'_2-\omega_2) \quad \text{...(ii)}$$

Solving (i) and (ii) $\omega'_1 = \dfrac{I_1\omega_1R_2 + I_2\omega_2R_1}{I_2R_1^2 + I_1R_2^2}R_2$ and $\omega'_2 = \dfrac{I_1\omega_1R_2 + I_2\omega_2R_1}{I_2R_1^2 + I_1R_2^2}R_1$.

10.5 Energy Method in Solving Problems of Rolling Body

We can conserve energy in case of pure rolling of a rigid body because the point of contact between the surfaces remains at rest and so the frictional forces acting at the point of contact do not do any work. Thus only conservative force do work on the body.

Thus Potential energy + total K.E. = constant

As shown in the Fig. 7.60, a disc is rolling down on an inclined plane. Then we can conserve total mechanical energy. If the disc falls a height h then loss in potential energy is equal to gain in kinetic energy.

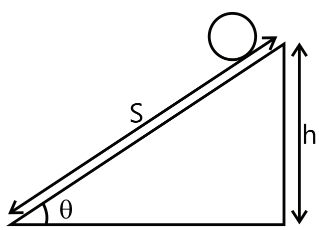


Figure 7.60

$$Mgh = \frac{1}{2}I\omega^2 + \frac{1}{2}MV_C^2 \quad \text{... (i)}$$

$$\text{Its total kinetic energy} = \frac{1}{2}MV_C^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}MV_C^2\left(1+\frac{K^2}{R^2}\right) \quad \text{...(ii)}$$

where K is the radius of gyration of the disc and V_C the velocity of center of mass.

$$\text{So } \frac{1}{2}MV_C^2\left(1+\frac{K^2}{R^2}\right) = Mgh; \quad V_C^2 = \frac{2gh}{\left(1+\frac{K^2}{R^2}\right)}$$

Thus the velocity of center of mass of a body rolling down an inclined plane is given by

$$V_C = \frac{\sqrt{2gh}}{\left(1 + \frac{K^2}{R^2}\right)^{1/2}}$$

If a_C is linear acceleration of center of mass down this plane, and distance covered on the plane is s, then if the body starts from rest we have

$$V_C^2 = 2a_C s \therefore a_C = \frac{V_C^2}{2\,s} = \frac{2\,gh}{\left(1 + \frac{K^2}{R^2}\right) \times 2 \times \frac{h}{\sin\theta}} \text{ or } a_C = \frac{g\sin\theta}{1 + \frac{K^2}{R^2}}$$

PLANCESS CONCEPTS

Rather than going in a conventional way, using this method greatly simplifies our effort. But take care while writing the kinetic energy!

Nitin Chandrol (JEE 2012 AIR 134)

Illustration 37: A solid sphere is released from rest from the top of an incline of inclination θ and length ℓ. If the sphere rolls without slipping. What will be its speed when it reaches the bottom? **(JEE MAIN)**

Sol: The loss in the gravitational potential energy of the solid sphere is equal to the gain in the kinetic energy. The kinetic energy of the sphere comprises the rotational kinetic energy as well as the translational kinetic energy.

Let the mass of the sphere be m and its radius be r. Suppose the linear speed of the sphere when it reaches the bottom is v. As the sphere rolls without slipping, its angular speed ω about its axis is v/r. The kinetic energy at the bottom will be

$$K = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2 = \frac{1}{2}\left(\frac{2}{5}mr^2\right)\omega^2 + \frac{1}{2}mv^2 = \frac{1}{5}mv^2 + \frac{1}{2}mv^2 = \frac{7}{10}mv^2$$

This should be equal to the loss of potential energy mg ℓ sinθ. Thus

$$\frac{7}{10}mv^2 = mg\ell\sin\theta \qquad \text{Or} \qquad v = \sqrt{\frac{10}{7}g\ell\sin\theta}.$$

11. TOPPLING

When an external force is applied to the upper edge of a body with a flat base to cause it to slide along a surface, the body may topple before sliding starts. Toppling is more likely to happen when the width of the base of the body is small.

Toppling occurs due to the turning effect of torques of applied force at the upper edge and frictional force at the base.

Let the surface be quite rough and the force F is applied at height h above the base of the block as shown in Fig. 7.61. Width of the base is b. The static friction at the base is f = F. The normal reaction is N = mg. The couple of forces F and f try to topple the block about point S. To cancel the effect of this unbalanced torque the normal reaction N shifts towards S by a distance x so that torque of N counter balances torques of F and f.

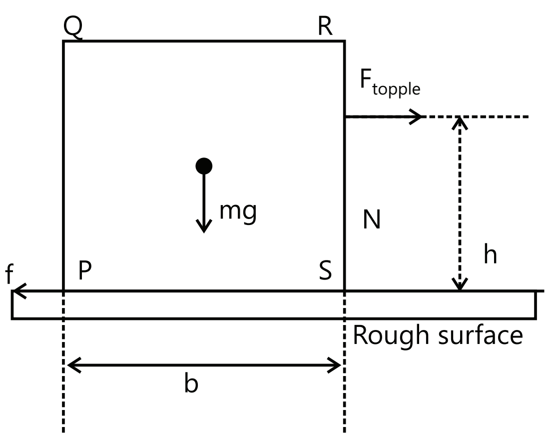


Figure 7.61: Block toppling on rough surface

$Fh = (mg)x \quad \text{or} \quad x = \frac{Fh}{mg}$

If F or h or both increase, distance x also increases, but it cannot go beyond the maximum value of $x_{max} = b/2$ i.e in extreme case N passes through edge S. If F is further increased block will topple.

So, $F_{topple} = \frac{mgb}{2h}$

Here we assumed that the surface is sufficiently rough so that sliding starts only when

$F = f_{max} = \mu mg > F_{topple}$ or $\mu > \frac{b}{2h}$ (toppling before sliding)

If surface is not sufficiently rough, the body slides before F is increased to F_{topple} i.e. the body will slide before toppling. This is the case when

$F = f_{max} = \mu mg < F_{topple}$ or $\mu < \frac{b}{2h}$

Illustration 38: A uniform cube of side 'a' and mass m rests on a rough horizontal table. A horizontal force F is applied normal to one of the faces at a point directly below the center of the face, at a height $\frac{a}{4}$ above the base.

(i) What is the minimum value of F for which the cube begins to tip about an edge?

(ii) What is the minimum value of μ_s so that toppling occurs?

(iii) If $f_1 = \mu_{min}$, find minimum force for toppling.

(iv) Minimum μ_s so that F_{min} can cause toppling. **(JEE ADVANCED)**

Sol: For part (i) we consider toppling before sliding. The normal reaction will pass through the edge. In part (ii) it is not mentioned whether the toppling occurs before sliding or sliding occurs before toppling. So the toppling will occur for any value of μ_s, sliding or no sliding. Part (iii) is same as part (i). Part (iv) is the case of toppling before sliding.

(i) In the limiting case normal reaction will pass through O. The cube will tip about O if torque of F about O exceeds the torque of mg.

Hence, $F\left(\frac{a}{4}\right) > mg\left(\frac{a}{2}\right)$ or $F > 2\ mg$

Therefore, minimum value of F is 2 mg.

(ii) In this case since it is not acting at COM, toppling can occur even after body started sliding even if there is no friction by increasing the torque of F about COM. Hence $\mu_{min} = 0$.

(iii) Now body is sliding before toppling. O is not I.A.R., torque equation cannot be applied across it. It can be applied about COM.

$F \times \frac{a}{4} = N \times \frac{a}{2}$ ... (i)

$N = mg$ ... (ii)

From (i) and (ii) -> F = 2 mg

(iv) F > 2 mg .... (i) (From sol. (i))

$N = mg$ ... (ii)

$F = \mu_s N = \mu_s mg$ ... (iii)

From (i) and (iii) $\mu_s = 2$

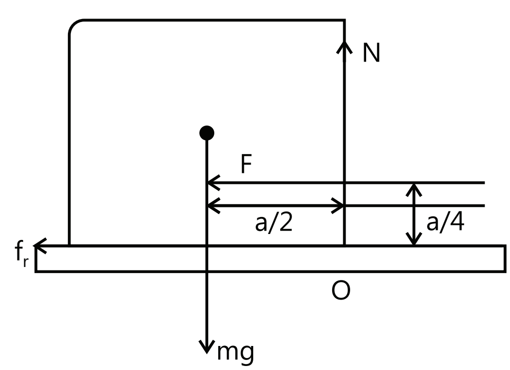


Figure 7.62

Illustration 39: Find minimum value of ℓ so that truck can avoid the dead end, without toppling the block kept on it. **(JEE ADVANCED)**

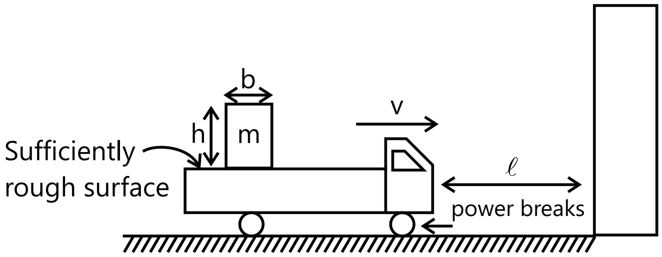


Figure 7.63

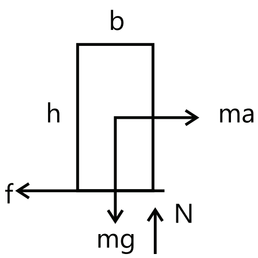


Figure 7.64

Sol: The block kept on truck will experience pseudo force in forward direction and friction force due to the floor of the truck in backward direction. We assume the case of toppling before sliding. In extreme case the normal reaction N = mg will pass through the edge.

$$ma\frac{h}{2} \le mg\frac{b}{2} \Rightarrow a \le \frac{b}{h}g$$

Final velocity of truck is zero. So that $0 = v^2 - 2(\frac{b}{h}g)\ell$

$$\ell = \frac{h}{2b}\frac{v^2}{g}$$

PROBLEM-SOLVING TACTICS

- Most of the problems involving incline and a rigid body can be solved by using conservation of energy during pure rolling. In case of non-conservative forces, work done by them also has to be taken into consideration in the equation. Care has to be taken in writing down the Kinetic energy. Rotational Kinetic Energy term has to be taken into consideration. And while writing the rotational energy, the axis about which the moment of inertia is taken should pass through the COM.
- The motion of a body in pure rolling can be viewed as pure rotation about the bottommost point of the body or the point of contact with the ground. Hence an axis passing through the point of contact and tangential to the point would be the Instantaneous axis of rotation. So problems on pure rolling can be solved easily by using the concept of instantaneous axis of rotation.
- Problems on toppling can be easily solved by writing the moments on the body and visualizing them as forces acting on the body. If the net moment is tending to stabilize the body, then the body doesn't topple. For any condition else it may get toppled.
- Problems which include the concept of sliding and rolling can be solved easily by using the concept of conservation of angular momentum. But care has to be taken in selecting the proper axis so that net moment about that axis vanishes.

FORMULAE SHEET

S. No	Term	Description	Linear Motion	Rotational motion & relation
1	Displacement	Displacement (linear or angular) is the physical change in the position of the body when a body moves linearly or angular in position. **(a)** The linear displacement Δs is difference between final and initial position measured in linear direction. **S.I. unit**: meter **m** **(b)** The angular displacement of the body while rotating about a fixed axis is the displacement $\Delta\theta$ it swept out with respect to its initial position in sense of rotation. It can be positive (anti clockwise) or negative (clockwise) **S.I. unit:** radians **rad**,	s	θ $(s = r\theta)$
2	Velocity	Velocity of any moving object is the time rate of change of position. The velocity is the vector quantity. Linear velocity is in the plane of motion. Angular velocity can be positive or negative & its direction is perpendicular to the plane of rotation Linear velocity is categorized as - Average velocity= $\Delta s / \Delta t$ - Instantaneous velocity= ds/dt. **S.I. unit: m/s** Angular velocity is categorized as - Average angular velocity $\Delta\theta / \Delta t$ - Instantaneous angular velocity $\omega = d\theta / dt$ **S.I. unit: rad/s**	$v = \frac{ds}{dt}$	$\omega = \frac{d\theta}{dt}$ $(v = r\omega)$
3	Acceleration	Acceleration is the time rate change of velocity of a body. It's a vector quantity. Linear acceleration can be positive or negative and related to direction of motion. Linear acceleration is categorized as - Average acceleration= $\Delta v / \Delta t$ - Instantaneous acceleration = dv/dt.	$a = \frac{dv}{dt}$	$\alpha = \frac{d\omega}{dt}$ $(a = r\alpha)$

S. No	Term	Description	Linear Motion	Rotational motion & relation
		S.I. unit: m/s^{-2} Angular acceleration is categorized as - Average angular acceleration $\Delta\omega / \Delta t$ - Instantaneous angular acceleration $\alpha = d\omega / dt$ **S.I. unit: rad/s^{-2}**		
4	Mass	Mass is the basic entity of any body by virtue of which the body gains weight. In linear kinematics the mass of whole body is constant. **S.I. unit:** kilogram **kg** In angular kinematics mass of body is distributed among various tiny rigid points so mass is measured about inertia of rotating body- moment of inertia **I**	M	I ($I = \sum mr^2$)
5	Momentum	Momentum of body is product of mass and its velocity of motion. It's a vector quantity. Linear momentum= mv **S.I. unit: kg m/s** Angular momentum of body is a vector in direction perpendicular to plane of rotation given by $\vec{L}$ **S.I. unit: kg m^2/s**	p = mv	$\vec{L}$ = I $\vec{L} = \vec{r} \times \vec{p}$
6	Impulse	Impulse is the product of force and time period And it is categorized as -Linear impulse -Angular impulse	$\int F\, dt$	$\int \tau dt$
7	Force (Newton's second law of motion)	From the newton second law of motion, force is time rate of change of momentum. It's a vector quantity. Linear force $F = \frac{dp}{dt} = ma$ **S.I. unit:** Newton **N** Angular force $\vec{\tau} = I \times \vec{\alpha}$ **Laws of conservation of momentum** - Linear momentum is said to be conserved if $\frac{dp}{dt}$ = 0, than P remains constant - Angular momentum is said to be conserved if $\frac{d\vec{L}}{dt}$ = 0 than L remains constant	F = ma If = 0 the body is in equilibrium with its surrounding	$\vec{\tau} = r \times \vec{F} = I \times \vec{\alpha}$ $= \frac{d\vec{L}}{dt}$ If = 0 the body is in equilibrium with its surrounding

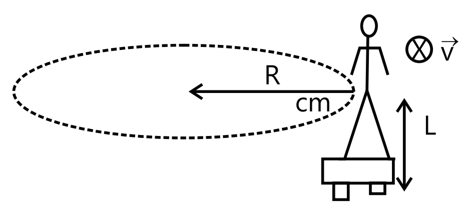
S. No	Term	Description	Linear Motion	Rotational motion & relation
8	Work	Work is the product of displacement of body under action of external applied force.	$W = \int F\, ds$	$W = \int \tau d\theta$
9	Power	Power is the time rate change of work done	P =F	$P = \tau\, \omega$
10	Kinetic energy	The phenomenon associated with the moving bodies	$K.E._{tran} = \frac{1}{2}mv^2$	$K.E._{rot} = \frac{1}{2}I\omega^2$
11	Kinematics of Motion	Kinematical equation are the interrelation of displacement, velocity, acceleration and time and are categorized as follows: -Linear kinematical equation -Angular kinematical equation	$v = u + at$ $s = ut + \frac{1}{2}at^2$ $v^2 = u^2 + 2as$	$\omega = \omega_0 + \alpha t$ $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ $\omega^2 = \omega_0^2 + 2\alpha\theta$
12	Parallel Axis Theorem	$I_{xx} = I_{cc} + Md^2$ where I_{cc} is the moment of inertia about the center of mass		
13	Perpendicular Axis Theorem	$I_{xx} + I_{yy} = I_{zz}$ It is valid for plane laminas only.		
14	Work energy principle	Work energy principle is used to determine the change in the kinetic energy of moving body	$W = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$	$W = \frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_0^2$

Solved Examples

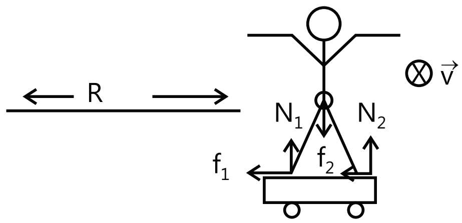
JEE Main/Boards

The first five Examples discussed below show us the strategy to tackle down any problem in the rigid body motion. Hence follow them up properly! They may be lengthy but are very learner friendly!!

Example 1: A person of mass M is standing on a railroad car, which is rounding an unbanked turn of radius at speed v. His center of mass is at a height of L above the car midway between his feet, which are separated by a distance of d. The man is facing the direction of motion. What is the magnitude of the normal force on each foot?



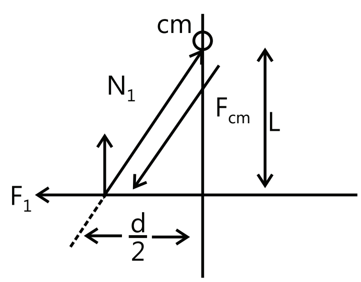
Sol: The frictional forces acting on the feet of man will provide the necessary centripetal acceleration to move in a circular path. Apply the Newton's second law of motion at the center of mass of the man to get the equation of motion along the circular path. In the vertical plane the man is in rotational and translational equilibrium under the action of its weight acting vertically downwards and the normal reactions at its feet acting vertically upwards. Get one equation each for rotational and translational equilibrium in vertical plane.



We draw the free body diagram of the man, as shown in figure.

Static friction $\vec{f}_1$ and a normal reaction $\vec{N}_1$ is acting on the inner foot. Static friction $\vec{f}_2$ and normal reaction $\vec{N}_2$ is acting on the outer foot. We do not assume the limiting value of frictional forces. The weight of the man acts at its center of mass.

As the man is moving in a circular path with speed, by Newton's Second Law the forces of friction should act towards the center of the circular path.



$$f_1 + f_2 = m\frac{v^2}{R} \qquad \text{....(i)}$$

For vertical equilibrium we should have

$$N_1 + N_2 - mg = 0$$

or $\quad N_1 + N_2 = mg \qquad \text{....(ii)}$

For rotational equilibrium of the man about its center of mass we have $\quad \vec{\tau}_{cm}^{\,total} = 0$

The gravitational force does not contribute to the torque about center of mass because it is acting at the center of mass itself. We draw a torque diagram in the figure showing the line of action of the forces at the inner foot.

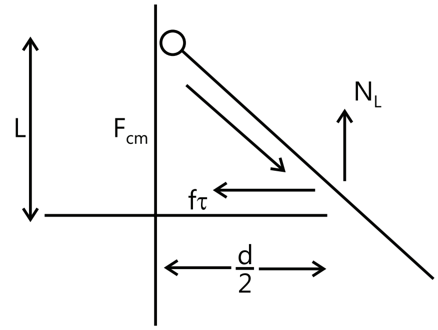
The torque on the inner foot about COM is given by

$$\tau_{cm1} = \frac{d}{2}N_1 + Lf_1 \quad \text{(clockwise)}$$

We draw a similar torque diagram for the forces at the outer foot.

The torque on the outer foot about COM is given by

$$\tau_{cm2} = -\frac{d}{2}N_2 + Lf_2 \quad \text{(clockwise)}$$



Both these torques about the center of mass must add up to zero.

Therefore

$$\left(\frac{d}{2}N_1 + Lf_1\right) + \left(-\frac{d}{2}N_2 + Lf_2\right) = 0$$

$$\frac{d}{2}(N_1 - N_2) + L(f_1 + f_2) = 0 \qquad \text{....(iii)}$$

Putting (i) in (iii) we get,

$$\frac{d}{2}(N_1 - N_2) + Lm\frac{v^2}{R} = 0$$

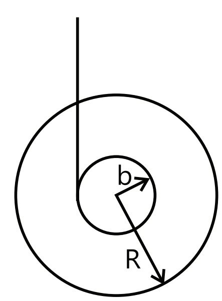
or $\quad N_2 - N_1 = \dfrac{2Lmv^2}{Rd} \qquad \text{....(iv)}$

Solving (ii) and (iv) we get

$$N_1 = \frac{1}{2}\left(mg - \frac{2Lmv^2}{Rd}\right) \qquad \text{....(v)}$$

$$N_2 = \frac{1}{2}\left(\frac{2Lmv^2}{Rd} + mg\right) \qquad \text{.... (vi)}$$

Example 2: A Yo-Yo of mass m has an axle of radius b and a spool of radius R. Moment of inertia about the center can be taken to be I_{cm} = (1/2) MR^2 and the thickness of the string can be neglected. The Yo-Yo is released from rest. You will need to assume that the center of mass of the Yo-Yo descends vertically, and that the string is vertical as it unwinds.

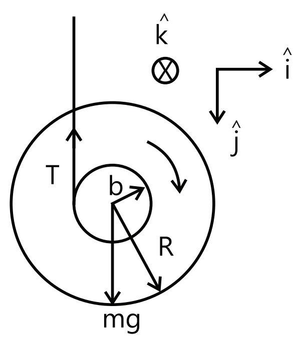


(a) What is the tension in the cord as the Yo-Yo descends?

(b) What is the magnitude of the angular acceleration as the Yo-Yo descends and the magnitude of the linear acceleration?

(c) Find the angular velocity of the Yo-Yo when it reaches the bottom of the string when a length L of the string has unwound.

Sol: Apply the Newton's second law of motion at the center of mass of Yo-Yo to get the equation of motion along the vertical direction. Get the relation between net torque, of all the external forces acting on Yo-Yo, and its moment of inertia, both these quantities calculated about the axis passing through the center of mass of Yo-Yo. As the Yo-Yo descends, the loss in the gravitational potential energy is equal to the gain in the translational and rotational kinetic energy.



(a) The torque of tension in the cord about the center of mass of the Yo-Yo is in the clockwise direction. So as the Yo-Yo descends with linear acceleration a_{cm}, it rotates in the clockwise direction with angular acceleration α.

$$\tau_{cm} = bT = I_{cm}\alpha \text{ (clockwise)} \quad ....(i)$$

Applying Newton's Second Law for the motion of COM in the vertical direction,

$$Mg - T = Ma_{cm} \quad ....(ii)$$

As the string is stationary, and the Yo-Yo does not slip on the string, the angular acceleration and the linear acceleration of COM are related by the constraint condition,

$$a_{cm} - b\alpha = 0 \Rightarrow a_{cm} = b\alpha \quad ....(iii)$$

From (ii) and (iii) we get,

$$Mg - T = Mb\alpha \quad ....(iv)$$

Eliminating α from (i) and (iv) we get

$$Mg - T = \frac{Mb^2T}{I_{cm}}$$

or

$$T = \frac{Mg}{\left(1+\frac{Mb^2}{I_{cm}}\right)} = \frac{Mg}{\left(1+\frac{Mb^2}{(1/2)MR^2}\right)} = \frac{Mg}{\left(1+\frac{2b^2}{R^2}\right)} \quad ....(v)$$

(b) Substitute Eq. (v) into Eq. (i) to determine the angular acceleration

$$\alpha = \frac{bT}{I_{cm}} = \frac{2bg}{(R^2+2b^2)} \quad ....(vi)$$

From (iii) and (vi) we get

$$a_{cm} = b\alpha = \frac{2b^2g}{(R^2+2b^2)} = \frac{g}{1+(R^2/2b^2)} \quad ....(vii)$$

For a typical Yo-Yo, the acceleration is much less than that of an object in free fall.

(c) Use conservation of energy to determine the angular velocity of the Yo-Yo when it reaches the bottom of the string (Tension force does not perform any work because point of contact between string and Yo-Yo is always at rest).

Loss in gravitational potential energy = Gain in kinetic energy MgL

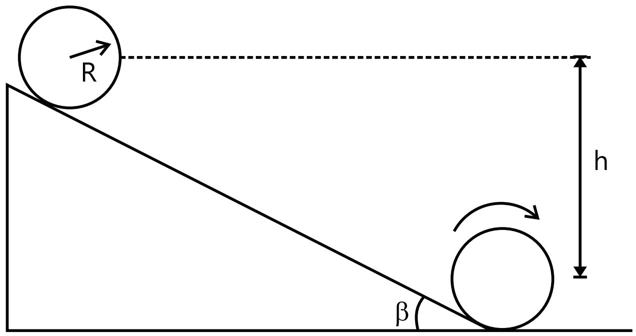
$$= MgL = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2 = \frac{1}{2}M(v_{cm}^2 + \frac{1}{2}R^2\omega^2) \quad ....(viii)$$

Linear velocity of COM and angular velocity are related by the constraint condition,

$$v_{cm} - b\omega = 0 \Rightarrow v_{cm} = b\omega \quad ....(ix)$$

Solving (viii) and (ix) for ω, we get $\omega = \sqrt{\frac{4gL}{(2b^2+R^2)}}$

Example 3: A uniform cylinder of radius R and mass M with moment of inertia about the center of mass I_{cm} = (1/2) MR^2 starts rolling due to the mass of the cylinder, and has dropped a vertical distance h when it reaches the bottom of the incline. Let g denote the gravitational constant. The coefficient of static friction between the cylinder and the surface is μ. The cylinder rolls without slipping down the incline. The goal of this problem is to find the magnitude of the velocity of the center of mass of the cylinder when it reaches the bottom of the incline.



Sol: This problem can be solved either by applying law of conservation of mechanical energy, or by applying Newton's laws of motion.

We shall solve this problem in three different ways.

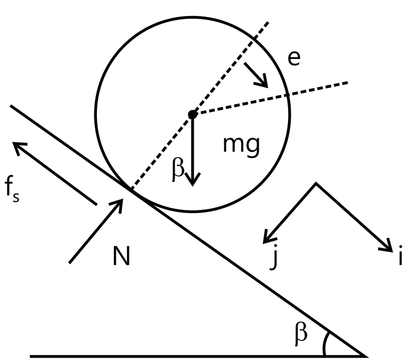
1. Applying the torque equation about the center of mass and the force equation for the center of mass motion.

2. Applying the energy equation.

3. Using torque about a fixed point that lies along the line of contact between the cylinder and the surface.

Applying the torque equation about the center of mass and the force equation for the center of mass motion.

We will find the acceleration and hence the speed at the bottom of the incline using kinematics. A figure showing the force is shown below.

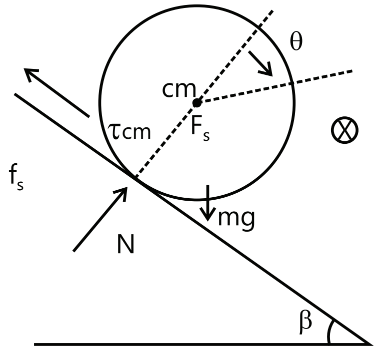


Choose x = 0 as the point where the cylinder just starts to roll. With the unit vectors shown in the figure above, Newton's second Law, applied in the x – and y – directions in turn, yields

$Mg\sin\beta - f_s = Ma_x$ ...(i)

$-N + Mg\cos\beta = 0$ ...(ii)

Choose the center of the cylinder to compute the torque about (see figure below).



Then the only force exerting a torque about the center of mass is the friction force, and so we have $f_s R = I_{cm}\alpha_z$ ...(iii)

Use $I_{cm} = (1/2)\ M\ R^2$ and the kinematic constraint for the no-slipping condition $\alpha_z = a_x/R$ in Eq. (xxxiv) to solve for the magnitude of the static friction force yielding

$f_x = (1/2)Ma_x$ ...(iv)

Substituting Eq. (iv) into Eq. (v)

$Mg\sin\theta = (1/2)\ M a_x = M a_x$ ...(v)

Which we can solve for the acceleration

$a_x = \frac{2}{3}\sin\beta$ ...(vi)

The displacement of the cylinder is $x_f = h/\sin\beta$ in the time it takes the bottom, t_f. The x – component of the velocity v_x at the bottom is $v_{x.f} = a_{x.f}$. The displacement in the time interval t_f satisfies $x_f = (1/2)\ a_x t_f^2$. After eliminating t_f, we have $x_f = v_{x\cdot f}^2/2a_x$, so the magnitude of the velocity when the cylinder reaches the bottom of the inclined plane is

$$v_{x\cdot f} = \sqrt{1a_x x_f} = \sqrt{2((2/3)g\ \sin\ \beta)(h/\sin\beta)} = \sqrt{(4/3)gh}$$ ...(viii)

Note that if we substitute Eq. (vi) into Eq. (iv) the magnitude of the friction force is

$fs = (1/3)\ Mg\sin\beta$ ...(ix)

In order for the cylinder to roll without slipping.

$f_s \le \mu_s Mg\cos\beta$ ...(x)

So combining Eq. (ix) and Eq. (x) we have the condition that

$(1/3)\ Mg\sin\beta \le \mu_s\ Mg\cos\beta$ ...(xi)

Thus in order to roll without slipping the coefficient of static friction must satisfy

$$\mu_s \geq \frac{1}{3}\tan\beta \qquad \text{...(xii)}$$

Applying the energy equation

We shall use the fact that the energy of the cylinder-earth system is constant since the static frictional force does no work. Choose a zero reference point for potential energy at the center of mass when the cylinder reaches the bottom of the incline plane.

Then the initial potential energy is $U_t = Mgh$ ...(xiii)

$Mg - N = 0$

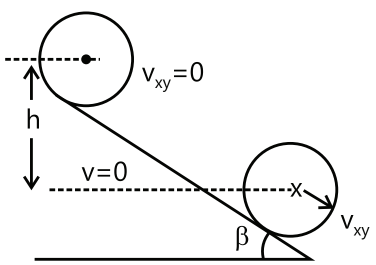
For the given moment of inertia, the final kinetic energy is

$$K_f = \frac{1}{2}Mv_{x.f}^{2} + \frac{1}{2}I_{cm}\omega_{z.f}^{2}$$

$$= \frac{1}{2}Mv_{x.f}^{2} + \frac{1}{2}(1/2)MR^2(v_{x.f}/R)^2 \qquad \text{... (xiv)}$$

$$= \frac{3}{4}Mv_{x.f}^{2}$$

Setting the final kinetic energy equal to the initial gravitational potential energy leads to



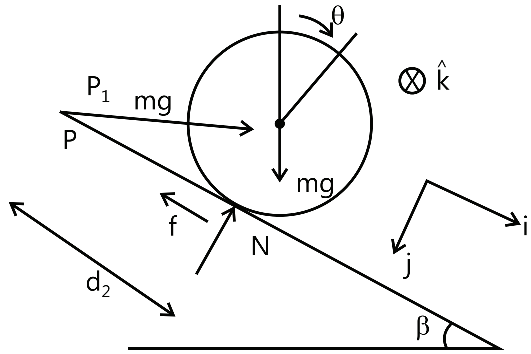
$$Mgh = \frac{3}{4}Mv_{x.f}^{2} \qquad \text{...(xv)}$$

The magnitude of the velocity of the center of mass of the cylinder when it reaches the bottom of the incline is $v_{x.f} = \sqrt{(4/3)gh}$ ...(xvi)

In agreement with Eq. (viii)

Using torque about a fixed point that lies along the line of contact between the cylinder and the surface

Choose a fixed point that lies along the line of contact between the cylinder and the surface. Then the torque diagram, is shown below.



The gravitational force $M\vec{g} = Mg\sin\beta\,\hat{j}$ acts at the center of mass. The vector from the point P to the center of mass is given by $\vec{r}_{p.mg} = d_p\,\hat{i} - R\hat{j}$, so the torque due to the gravitational force about the point P is given by

$$\vec{\tau}_{p.mg} = \vec{r}_{p.mg} \times M\vec{g} = (d_P\hat{i} - R\hat{j}) \times (Mg\sin\beta\,\hat{i} + Mg\cos\beta\hat{j})$$

$$= (d_p Mg\cos\beta + RMg\sin\beta)\hat{k} \qquad \text{...(xvii)}$$

The normal force acts at the point of contact between the cylinder and the surface and is given by $\vec{N} = -N\hat{j}$. The vector from the point P to the point of contact between the cylinder and the surface is $\vec{\tau}_{P.N} = d_p\hat{i}$. So the torque due to the normal force about the point P is given by

$$\vec{\tau}_{p.N} = \vec{\tau}_{p.N} \times \vec{N} = (d_p\hat{i}) \times (-N\hat{j}) = -d_p\hat{k} \qquad \text{...(xviii)}$$

Substituting Eq. (xxxiii) for the normal force into Eq. (xviii) yields

$$\vec{\tau}_{p.N} = -d_P Mg\cos\beta\hat{k} \qquad \text{...(xix)}$$

Therefore the sum of the torques about the point P is

$$\vec{\tau}_p = \vec{\tau}_{p.Mg} + \vec{\tau}_{p.N} = (Mg\cos\beta + RMg\sin\beta)\,\hat{k} - d_P\,Mg\cos\beta\,\hat{k} = Rmg\sin\beta\,\hat{k} \qquad \text{...(xx)}$$

The angular momentum about the point P is given by

$$\vec{L}_P = \vec{L}_{cm} + \vec{r}_{p.cm} \times M\vec{V}_{cm}$$

$$= I_{cm}\omega_z\hat{k} + (d_p\hat{i} - R\hat{j}) \times (Mv_x)\hat{i}$$

$$= (I_{cm}\omega_z + RMv_x)\hat{k} \qquad \text{...(xxi)}$$

The time derivative of the angular momentum about the point P is then

$$\frac{d\vec{L}_P}{dt} = I_{cm}\alpha_z + RMa_x\hat{k} \qquad \text{...(xxii)}$$

Therefore the torque equation

$$\vec{\tau} = \frac{d\vec{L}_P}{dt} \qquad \text{...(xxiii)}$$

Becomes $RMg\sin\beta\hat{k} = (I_{cm}\alpha_z + RMa_x)\hat{k}$ ...(xxiv)

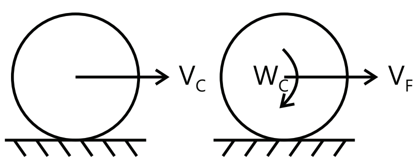
Using the fact that $I_{cm} = (1/2)MR^2$ and $\alpha_x = a_x/R$, we can conclude that $RM\ a_x = (3/2)\ MR\ a_x$ ...(xxv)

We can now solve Eq. (xxv) for the x – component of the acceleration

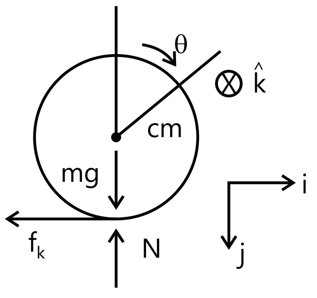
$$a_x = (2/3)\ g\sin\ \beta \quad ...(xxvi)$$

In agreement with Eq. (vi).

Example 4: A bowling ball of mass m and radius R is initially thrown down an alley with an initial speed v_0 and it slides without rolling but due to friction it begins to roll. The moment of inertia of the ball about its center of mass is I_{cm} = (2/5) mR^2. Using conservation of angular momentum about a point (you need to find that point), find the speed v_f of the bowling ball when it just start to roll without slipping?



Sol: The angular momentum of any rigid body about a fixed point in ground reference frame is the sum of the angular momentum in the C-frame and the angular momentum corresponding to the translation of the center of mass relative to the fixed point in ground frame.



At t = 0, when the ball is released $v_{cm,i} = v_0$ towards right and $\omega_i = 0$, so the ball slips towards right on the surface and hence the frictional force on the ball, will be towards left.

The frictional force will pass through the point of contact S with the surface.

The weight of the ball as well as the normal reaction from the surface are equal in magnitude and opposite in direction and have same line of action and will also pass through the point of contact S. (Point of contact is vertically below the COM of the ball).

Thus we choose the initial point of contact S as the origin and the net torque of all the forces about the origin S comes out to be zero at all times. So we can conserve the angular momentum of the ball about the initial point of contact (origin S).

The initial angular momentum about the origin S is only due to the translation of the center of mass.

$$L_i = mv_0R$$

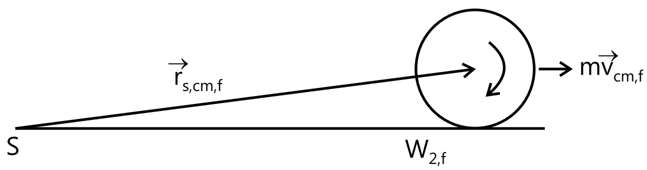
The final angular momentum about the origin S has both translational and rotational contribution.

$$L_f = m\ v_{cm,f}\ R + I_{cm}\omega$$

Finally the ball rolls without slipping, so we have $v_{cm,f} = R\omega$

Now I_{cm} = (2/5)mR^2

Therefore the final angular momentum about the origin S is



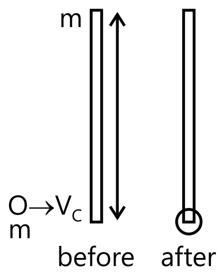
$$L_f = (mR + (2/5)mR)v_{cm,f} = (7/5)mRv_{cm,f}$$

Now equating $L_i = L_f$ we get

$$mR\ v_0 = (7/5)\ m\ R\ v_{cm,f}$$

or $v_{cm,f} = (5/7)\ v_0$

Example 5: A long narrow uniform stick of length ℓ and mass m lies motionless on ice (assume the ice provides a frictionless surface). The center of mass of the stick is the same as the geometric center (at the midpoint of the stick). The moment of inertia of the stick about its center of mass is I_{cm}. A puck (with putty on one side) has same mass m as the stick. The puck slides without spinning on the ice with a speed of v_0 towards the stick, hits one end of the stick, and attaches to it. You may assume that the radius of the puck is much less than the length of the stick so that moment of inertia of the puck about its center of mass is negligible compared to I_{cm}.



(a) How far from the midpoint of the stick is the center of mass of the stick–puck combination after the collision?

(b) What is the linear velocity of the stick plus puck after the collision?

(c) Is mechanical energy conserved during the collision? Explain your reasoning.

(d) What is the angular velocity of the stick plus puck after the collision?

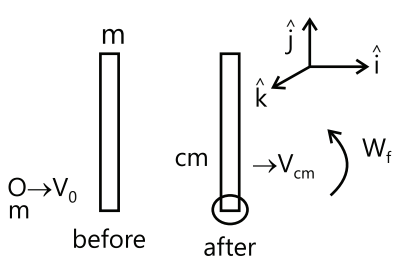
(e) How far does the stick's center of mass move during one rotation of the stick?

Sol: Apply the law of conservation of linear momentum and law of conservation of angular momentum before and after collision. The angular momentum is to be calculated about the center of mass of "stick-puck system".

(a) From the midpoint of the stick the center of mass of the stick–puck combination after the collision is at a distance d_{cm}.

(b) $d_{cm} = \frac{m_{stick}d_{stick} + m_{puck}d_{puck}}{m_{stick} + m_{puck}} = \frac{m \times 0 + m(\ell/2)}{m+m} = \frac{\ell}{4}$

There are no external forces acting on this system comprising "stick and puck" so the momentum of the system before and after the collision is conserved.



After the collision, suppose the center of mass of the system is moving with speed v_f

Equating initial and final linear momentum we get,

$$mv_0 = (2m)v_f \Rightarrow v_f = \frac{v_0}{2}$$

The direction of the velocity is the same as the initial direction of the puck's velocity.

(c) As the collision is perfectly inelastic the mechanical energy of the system is not conserved.

(d) Choose the center of mass of the stick-puck combination, as found in part (a) as the point about which we find the angular momentum before and after the collision. This choice is advantageous as there will be no angular momentum due to the translation of the center of mass just after the collision about the center of mass itself. Before the collision, the angular momentum was entirely due to the motion of the puck,

$$L_0 = (\ell/4)(mv_0)$$

After the collision, the angular momentum is $L_f = I_{cm}\omega_f$ where I_{cm} is the moment of inertia about the center of mass of the stick-puck combination.

This moment of inertia of the stick about the new center of mass is found from the parallel axis theorem, and the moment of inertia of the puck is $m(\ell/4)^2$, and so

$$I_{cm} = m(\ell^2/12) + m(\ell/4)^2 + m(\ell/4)^2 = 5m\ell^2/24$$

Equating $L_0 = L_f$ we get $(\ell/4)(mv_0) = 5m\ell^2\omega_f/24$

This gives $\omega_f = \frac{6v_0}{5\ell}$

(e) Time taken by stick-puck system for one rotation is $T = 2\pi/\omega_f$

Distance travelled by COM during this time is

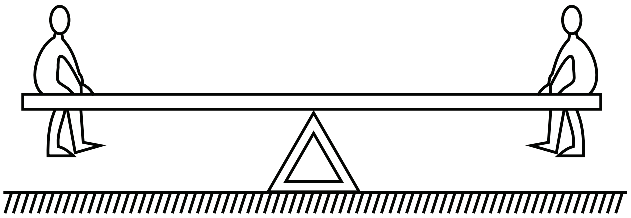
$$x_{cm} = v_{cm}T = \frac{2\pi}{\omega_f}v_{cm} = \frac{2\pi}{6v_0/5\ell}(v_0/2)$$

This gives $x_{cm} = \frac{5\pi\ell}{6}$

Example 6: Two small kids of masses 10 kg and 15 kg are trying to balance a seesaw of total length 5.0 m with the fulcrum at the center. If one of the kids is sitting at ends, where should the other sit?

Sol: For rotational equilibrium, the net torque about the fulcrum of all the forces acting on the boys and the seesaw should be zero.

It is clear that the 10 kg kid should sit at the end and the 15 kg kid should sit closer to the center. Suppose this distance from the center is X. As the kids are in equilibrium, the normal force between a kid and the see-saw equals the weight of that kid. Considering the rotational equilibrium of the seesaw, the torques of the forces acting on it should add to zero. The forces are.



(a) (15kg) g downward by the 15kg kid,

(b) (10 kg) g downward by the 10kg kid,

(c) Weight of the seesaw and

(d) The normal force by the fulcrum.

Taking torques about by the fulcrum

(15kg) g x = (10 kg) g (2.5m) or x = 1.7 m.

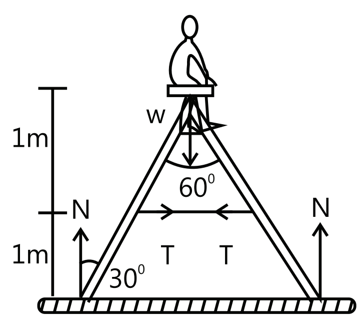
Example 7: The ladder shown in figure has negligible mass and rests on a frictionless floor. The crossbar connects the two legs of the ladder at the middle. The angle between the two legs is 60°.

The fat person sitting on the ladder has a mass of 80 kg. Find the contact force exerted by the floor on each leg and the tension in the crossbar.

Sol: The forces of normal reaction at the feet of the ladders balance the weight of the person. For rotational equilibrium of ladder the toque due to normal reaction at the foot is balanced by the torque due to tension in the crossbar. Both the torques are calculated about the upper end of the ladder.

The forces acting on the different parts are shown in figure. Consider the vertical equilibrium of "the ladder plus the person" system. The forces acting on this system are its weight (80kg)g and the contact force N + N = 2N due to the floor. Thus

2N = (80kg) g or N = (40 kg) (9.8m/s²) = 392 N

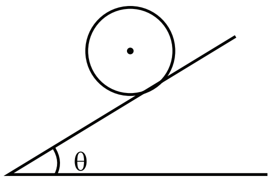


Next consider the equilibrium of the left leg of the ladder. Taking torque of the forces acting on it about the upper end.

N (2 m) tan 30°= T (1 m) or

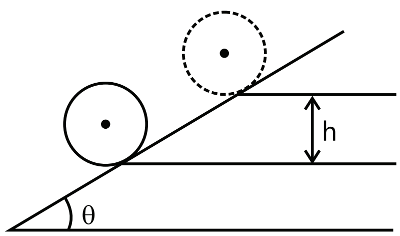
$$T = N\frac{2}{\sqrt{3}} = (392N) \times \frac{2}{\sqrt{3}} = 450 \text{ N.}$$

Example 8: A solid cylinder of mass m and radius r starts rolling down an inclined plane of inclination θ. Friction is enough to prevent slipping. Find the speed of its center of mass when, its center of mass has fallen a height h.



Sol: Loss in the gravitational potential energy of the cylinder is equal to the gain in rotational plus translational kinetic energy.

Consider the two shown positions of the cylinder. As it does not slip, total mechanical energy will be conserved.



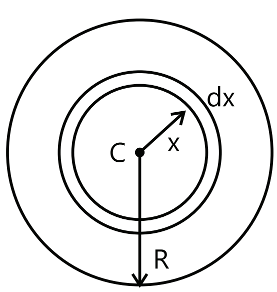
Energy at position 1 is $E_1 = mgh$

Energy at position 2 is $E_2 = \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2$

$\because \frac{v_{cm}}{r} = \omega$ and $I_{c.m} = \frac{mr^2}{2} \Rightarrow E_2 = \frac{3}{4}mv^2_{c.m}$

From COE, $E_1 = E_2$; $v_{c.m} = \sqrt{\frac{4}{3}gh}$

Example 9: A uniform disc of radius R and mass M is rotated to an angular speed ω_0 in its own plane about its center and then placed on a rough horizontal surface such that plane of the disc is parallel to the horizontal surface. If co-efficient of friction between the disc and the surface is μ then how long will it take for the disc to come to stop.



Sol: The disc can be thought of made-up of elementary rings of infinitesimal thickness. The torque about the center of disk due to friction force on each ring will be different from the other rings in the disc as the radii of rings are different, varying from 0 to R. So use the method of integration to find the torque on the entire disc.

Consider a differential circular strip of the disc of radius x and thickness dx. Mass of this strip is $dm = 2\rho\,\pi x\,dx$,

where $\rho = \frac{M}{\pi R^2}$. Frictional force on this strip is along the tangent and is equal to $dF = 2\mu\rho\pi g\,x\,dx$

Torque on the strip due to frictional force is equal to $d\tau = \mu\rho g 2\pi x^2 dx$

The disc is supposed to be the combination of Number of such strips hence torque on the disc is given by

$$\tau = \int d\tau = \mu\rho\, g 2\pi x \int_0^R x^2 dx = \mu\rho g 2\pi \frac{R^3}{3}$$

$$\Rightarrow \tau = \mu Mg(2/3)R$$

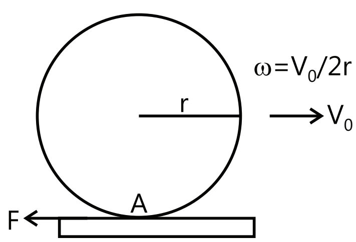
$$\Rightarrow \alpha = \frac{2\mu MgR}{3\left(\frac{MR^2}{2}\right)} = \frac{4}{3}\frac{\mu g}{R}$$

The α is opposite to the ù

$$\because \omega(t) - \omega_0 + \alpha t \qquad \Rightarrow 0 = \omega_0 \frac{4\mu g}{3R} t$$

$$\Rightarrow t = \frac{3\omega_0 R}{4\mu g}.$$

Example 10: A sphere of mass M and radius r shown in figure slips on a rough horizontal plane. At some instant it has translational velocity v_0 and rotational velocity about the center is $\frac{v_0}{2r}$. Find the translational velocity after the sphere starts pure rolling.



Sol: Due to forward slipping the friction will act backwards. So the sphere will decelerate. The torque due to friction will be in the direction of initial angular velocity. So the angular velocity will increase.

The slipping will stop when the condition of pure rolling is satisfied.

Velocity about the center $= \frac{v_0}{2r}$. Thus $v_0 > \omega_0 r$. The sphere slips forward and thus the friction by the plane on the sphere will act backward. As the friction is kinetic its value of N is given by $\mu N = \mu Mg$ and sphere will be decelerated by $a_{cm} = f/M$. Hence.

This friction will also have a torque $T = fr$ about the center. This torque is clockwise and in the direction of ω_0. Hence the angular acceleration about the center will be

$$\alpha = f\frac{r}{(2/5)Mr^2} = \frac{5f}{2Mr}$$

and the clockwise angular velocity at time t will be

$$\omega(t) = \omega_0 + \frac{5f}{2Mr}t = \frac{v_0}{2r} + \frac{5f}{2Mr}t$$

Pure rolling starts when

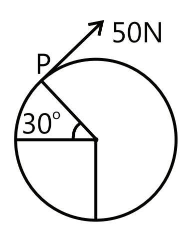
Eliminating t from (i) and (ii)

$$\frac{5}{2}v(t) + v(t) = \frac{5}{2}v_0 + \frac{v_0}{2} \quad \text{Or}$$

Thus the sphere rolls with translational velocity $6v_0/7$ in the forward direction.

JEE Advanced/Boards

Example 1: A carpet of mass M made of inextensible material is rolled along its length in the form of a cylinder of radius R and is kept on a rough floor. The carpet starts unrolling without sliding on the floor when a negligibly small push is given to it. Calculate the horizontal velocity of the axis of the cylindrical part of the carpet when its radius decreases to (R/2).



Sol: As the carpet unrolls, the radius and mass of cylindrical part decreases and center of mass descends. Thus loss in the gravitational potential energy is equal to gain in rotational plus translational kinetic energy.

If ρ is the density of material of the carpet, initial mass of the carpet (cylinder) M will be $\pi R^2 L\rho$. When its radius becomes half, the mass of cylindrical part will be

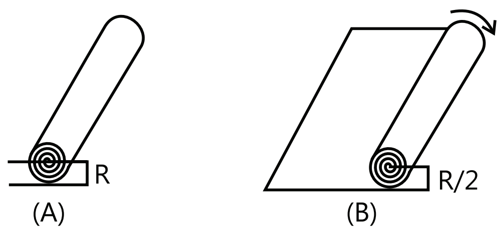
$$M_F = \pi(R/2)^2 L\rho = M/4$$

So initial PE of the carpet is MgR while final

(M/4) g(R/2) = MgR/8

So loss in potential energy when due to unrolling radius changes from R to R/2

MgR (1–(1/8)) = (7/8)MgR ... (i)



This loss in potential energy is equal to increase in rotation KE which is

$$K = K_T + K_R = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

If v is the velocity when half the carpet has unrolled, then as

$$v = \frac{R}{2}\omega,\ M \to \frac{M}{4} \text{ and } I = \frac{1}{2}\left[\frac{M}{4}\right]\left[\frac{R}{2}\right]^2$$

$$K = \frac{1}{2}\left[\frac{M}{4}\right]v^2 + \frac{1}{2}\left[\frac{MR^2}{32}\right]\left[\frac{2v}{R}\right]^2$$

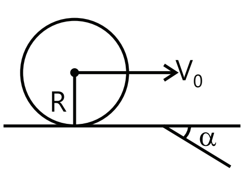
i. e., $K = \frac{1}{8}Mv^2 + \frac{1}{16}Mv^2 = \frac{3}{16}Mv^2$ ... (i)

So from equation (i) and (ii)

$$\left(\frac{3}{16}\right)Mv^2 = \left(\frac{7}{8}\right)MvR$$

i.e., $v = \sqrt{(14gR)/3}$

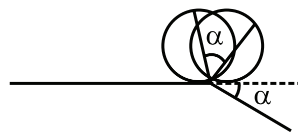
Example 2: A uniform solid cylinder of radius R = 15 cm rolls over a horizontal plane passing into an inclined plane forming an angle α = 30° with the horizontal. Find the maximum value of the velocity v_0 which still permits the cylinder to roll onto the inclined plane section without a jump. The sliding is assumed to be absent.



Sol: As the cylinder rolls into the inclined plane section, its center of mass descends, thus loss in gravitational potential energy will be equal to increase in kinetic energy. At the edge the COM moves in circular arc during the time interval when the vertical radius through the point of contact turns by angle α to become normal to the inclined plane. During this interval normal reaction from edge should always be greater than zero.

Initial energy $E_1 = \frac{1}{2}mv_0^2 + \frac{1}{2}I_{c.m}\omega^2 + mgR$

For rolling $\frac{v_0}{R} = \omega$



$$\Rightarrow E_1 = \frac{1}{2}mv_0^2 + \frac{1}{2}.\frac{1}{2}mR^2\frac{v_0}{R^2} + mgR$$

$$= \frac{3}{4}mv_0^2 + mgR$$

$$E_2 = \frac{1}{2}mv^2 + \frac{1}{2}I_{c.m}\omega'^2 + mgR\cos\alpha$$

$$= \frac{3}{4}mv^2 + mgR\cos\alpha$$

From COE

$$\frac{3}{4}mv^2 + mgR\cos\alpha = \frac{3}{4}mv_0^2 + mgR$$

$mv^2 = mv_0^2 + \frac{4}{3}mgR(1-\cos\alpha)$ .... (i)

F.B.D. of the cylinder when it is at the edge.

Center of mass of the cylinder describes circular motion about P.

Hence $mg\cos\alpha - N = mv^2/R$

$$\Rightarrow N = mg\cos\alpha - mv^2/R$$

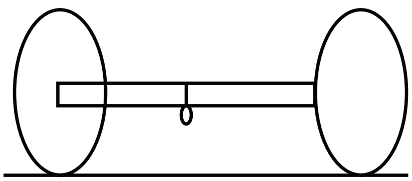
$$= mg\cos\alpha - \frac{mv_0^2}{R} - \frac{4}{3}mg + \frac{4}{3}mg\cos\alpha$$

For no jumping, $N \geq 0$

$$\Rightarrow \frac{7}{3}mg\cos\alpha - \frac{4}{3}mg - \frac{mv_0^2}{R} \geq 0$$

$$\Rightarrow v_0 \leq \sqrt{\frac{7gR}{3}\cos\alpha - \frac{4}{3}g}$$

Example 3: Two thin circular disc of the mass 2 kg each and radius 10 cm each are joined by a rigid massless rod of length 20 cm. The axis of the rod is perpendicular to the plane of the disc through their centers as shown in the figure. The object is kept at the center as shown in the figure. The object is kept on a truck in such a way that the axis of the object is horizontal and perpendicular to the direction of motion of the truck. Its friction with the floor of the truck is large enough to prevent slipping. If the truck has an acceleration of 9 m/s^2 calculate.



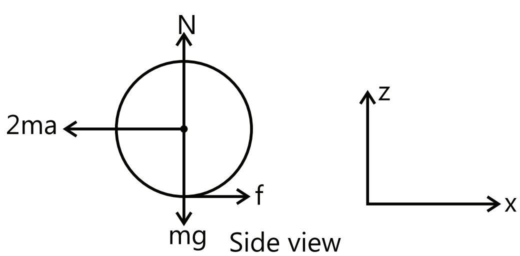
(a) The force of friction on each disc.

(b) The magnitude and direction of the frictional torque acting on each disc about the center of mass 'O' of the object. Take x-axis along the direction of the motion of the truck, and z-axis along vertically upwards direction. Express the torque in the vector form in terms of unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ in the x, y and z directions.

(c) Find the minimum value of the co- efficient of friction μ between the object and the floor of the truck which makes rolling of the object possible.

Sol: This problem is best solved in the reference frame of truck. Each disc will experience pseudo force as well as frictional force. Get two equations, one by applying Newton's second law at the center of mass of the object, and the other relating the torques of forces about the center of mass and the moment of inertia about axis passing through the center of mass.

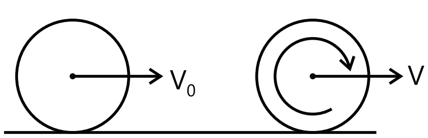
F.B.D. of the object with respect to truck.



Side view

In the reference frame of truck it experiences a pseudo force $F = -2ma\,\hat{i}$

where a = acceleration of the truck.



Pseudo force does not provide torque about the center of the disc. Because of this force object has tendency to slide along – Ve x –axis, hence frictional force will act along + Ve x – axis. For translational motion.

$2ma - f = ma'$ .... (i)

Here a′ = acceleration of the center of mass of the object.

For rotational motion

$fR = I\alpha$

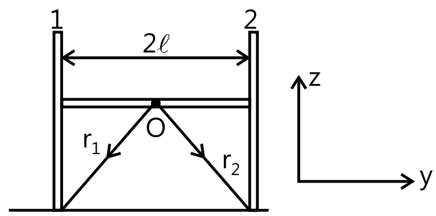
$= 2.\dfrac{mR^2}{2}.\dfrac{a'}{R}$ for no slipping $\alpha = a/R$

$\Rightarrow a' = \dfrac{f}{m}$ ..... (ii)

From (i) and (ii) we get

$F = \dfrac{2}{3}ma\,\hat{i} \Rightarrow$ Force of friction on each disc is

$\dfrac{f}{2} = \dfrac{ma}{3}\hat{i} = 6\hat{i}$ N



$\vec{f}_1 = \dfrac{ma}{3}\hat{i}$

$\vec{r}_1 = \ell\hat{j} - R\hat{k}$

$\vec{\tau}_{f_1} = \vec{r}_1 \times \vec{f}_1 = -(\ell\hat{j} + R\hat{k}) \times \dfrac{ma}{3}\hat{i}$

$= -\dfrac{maR}{3}\hat{j} + \dfrac{ma\ell}{3}\hat{k}$

$= 6 \times 0.1\,\hat{i} + 6 \times 0.1\,\hat{k}$

$= -0.6\,\hat{j} + 0.6\,\hat{k}$

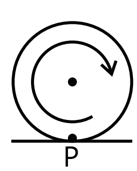
$\vec{\tau}_{(f_2)} = -0.6\,\hat{j} - 0.6\,\hat{k}$

(c) Maximum value of frictional force is 2μ mg

$\Rightarrow \dfrac{2}{3}ma \le 2\mu mg \qquad \Rightarrow \mu > \dfrac{a}{3g}$.

Example 4: A uniform disc of mass m and radius r is projected horizontal with velocity v_0 on a rough horizontal floor so that it starts of with a purely sliding motion at t = 0. At t = t_0 second it acquires a purely rolling motion.

(a) Calculate the velocity of the center of mass of the disc at t = t_0.



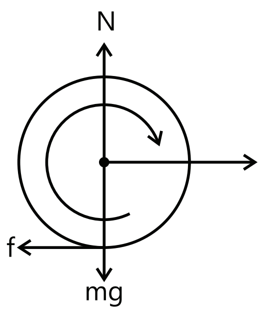
(b) Assuming coefficient of friction to be μ calculate t_0.

(c) The work done by the frictional force as a function of time.

(d) Total work done by the friction over a time t much longer then t_0.

Sol: This problem can be solved either by applying Newton's laws of motion or by law of conservation of angular momentum about the point of contact of the disc with the floor. The force of friction will act opposite to the direction of motion, and the work done by friction will be equal to loss in kinetic energy. The friction will stop doing any work once pure rolling starts.

F.B.D. of the disc.



When the disc is projected it starts sliding and hence there is a relative motion between the points of contact. Therefore frictional force acts on the disc in the direction opposite to the motion.

(a) Now for translational motion

$$a_{c.m} = \frac{f}{m}$$

$f = \mu N$ (as it slides) $= \mu mg$

$\Rightarrow a_{c.m} = -\mu g$, negative sign indicates that

$a_{c.m}$ is opposite to $v_{c.m}$

$$\Rightarrow v_{c.m(t)} = v_0 - \mu g t_0$$

$$\Rightarrow t_0 = \frac{(v_0 - v)}{\mu g} \quad \text{...(i)}$$

where $v_{c.m(t_0)} = v$

For rotational motion about center

$$\tau_f + \tau_{mg} = I_{c.m}\alpha \Rightarrow \mu mgr = \frac{mr^2}{2}\alpha$$

$$\Rightarrow \alpha = \frac{2\mu g}{r} \quad \text{...(ii)}$$

Therefore $\omega_{(t_0)} = 0 + \frac{2\mu g}{r} t_0$

Using $\omega_t = \omega_0 + \alpha t$

$$\Rightarrow \omega = \frac{2(v_0 - v)}{r} \quad \text{... (iii)}$$

(Using (i)) $V_{c.m} = \omega r$

$\Rightarrow v = 2(v_0 - v)$ (using (iii))

$$\Rightarrow v = \frac{2}{3}v_0$$

Alternative method: Since frictional force passes through the point of contact, hence about this point no external torque is acting.

Therefore angular momentum of the disc about point of contact does not change.

Initial angular momentum about p is given by

$L_1 = 0 + mv_0 r$ (Using $L_P = L_{c.m} + \vec{r} \times P_{c.m}$)

When it starts pure rolling its angular momentum about P is given by

$L_2 = L_{c.m} + \omega + mvr$

For rolling $v = \omega r$

$$\Rightarrow L_2 = \frac{mr^2}{2}\frac{v}{r} + mvr = \frac{3}{2}mvr$$

From COAM

$$L_1 = L_2 \Rightarrow v = \frac{2}{3}v_0$$

(b) Putting the value of v in equation (i)

We get $t_0 = \frac{v_0}{3\mu g}$

(c) Work done by the frictional force is equal to change in K.E.

$\Rightarrow W_{friction}$

$$= \frac{1}{2}m(v_0 - \mu g t)^2 + \frac{1}{2}\left(\frac{mr^2}{2}\right)\left(\frac{2\mu g t}{r}\right)^2 - \frac{1}{2}mv_0^2$$

$$= m\left(\frac{3}{2}\mu^2 g^2 t^2 - v_0 \mu g t\right), \text{ For } t \le t_0$$

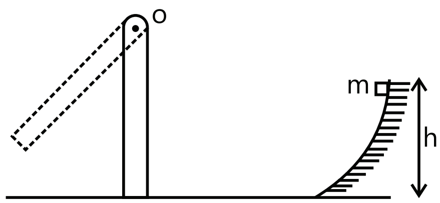
(d) For time $t \ge t_0$ work done by the friction is zero.

For longer time total work done is same as that in part

$$\text{(c)} \Rightarrow W = m\left(\frac{3}{2}\mu^2 g^2 t^2 \left(\frac{v_0}{3\mu g}\right)^2 - v_0 \mu g t \frac{v_0}{3\mu g}\right).$$

$$= -\frac{mv_0^2}{6}$$

Example 5: In the shown figure a mass m slides down a frictionless surface from height h and collides with a uniform vertical rod of length L and mass M. After collision the mass m sticks to the rod. The rod is free to rotate in a vertical plane about fixed axis through O. find the maximum angular deflection of the rod from its initial position.



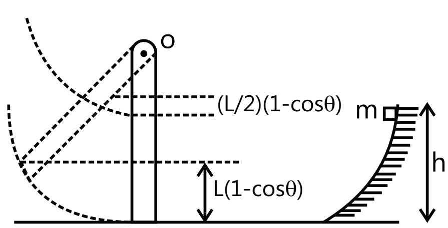
Sol: During the collision between rod and the mass, the linear momentum is not conserved because of the reaction force acting on the rod due to the hinge at the fixed point O. The torque of reaction force at the hinge will be zero about the hinge itself, i.e. about point O. So we can conserve the angular momentum of the "rod and mass system" before and after collision. As the rod rotates, the gain in gravitational potential energy is equal to the loss in the kinetic energy.

Just before collision, velocity of the mass m is along the horizontal and is equal to $v_0 = \sqrt{2gh}$. In the process of collision only angular momentum of the system will be conserved about the point O.

If L_1 and L_2 are the angular momentum of the system just before and just after the collision then $L_1 = mv_0L$

And $L_2 = I\omega = \left(\frac{ML^2}{3} + mL^2\right)\omega$

From Conservation of Angular Momentum



$$\left(\frac{M}{3} + m\right)L^2\omega = mv_0L$$

$$\Rightarrow \omega = \frac{mv_0}{\left(\frac{M}{3} + m\right)L}$$

Let the rod deflect through an angle θ.

Initial energy of rod and mass system = $\frac{1}{2}I\omega^2$

Where $I = \left(\frac{ML^2}{3} + mL^2\right)$

Gain in potential energy of the system

∵ From conservation of energy

$$\frac{1}{2}I\omega^2 = \left(m + \frac{M}{2}\right)gL(1 - \cos\theta)$$

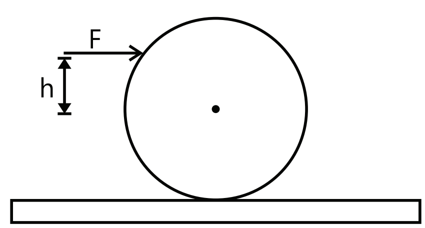
$$\Rightarrow \frac{1}{2}\left(\frac{ML^2}{3} + mL^2\right) \times \frac{m^2v_0^2}{\left(\frac{M}{3} + m\right)^2 L^2}$$

$$= \left(m + \frac{M}{2}\right)gL(1 - \cos\theta)$$

$$\frac{1}{2}\frac{m^2v_0^2}{\left(\frac{M}{3} + m\right)} = \left(m + \frac{M}{2}\right)gL(1 - \cos\theta)$$

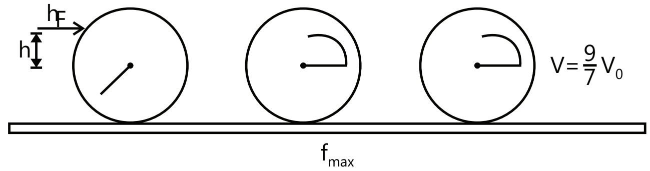
$$\cos\theta = \frac{1}{2}\frac{m^2v_0^2}{\left[\frac{M}{3} + m\right]\left[\frac{M}{2} + m\right]gL}.$$

Example 6: A billiard ball, initially at rest, is given a sharp impulse by a cue. The cue is held horizontally a distance h above the center line as shown in figure. The ball leaves the cue with a speed v_0 and because of its forward rotation (backward slipping) eventually acquires a final speed $\frac{9}{7}v_0$ show that.



$h = \frac{4}{5}R$ where R is the radius of the ball.

Sol: Initial linear and angular velocity of ball is found by calculating the linear and angular impulse delivered by the cue. The angular momentum of the ball about the point of contact with ground surface, during its combined translational and rotational motion remains conserved.



Let ω_0 be the angular speed of the ball just after it leaves the cue. The maximum friction acts in forward direction till the slipping continues. Let v be linear speed and ω the speed when slipping is ceased.

$$\therefore v = R\omega \text{ or } \omega = \frac{v}{R}$$

Given, $v = \frac{9}{7} v_0$ ..... (i)

$$\therefore \omega = \frac{9}{7}\frac{v_0}{R} \quad \text{..... (ii)}$$

Applying Linear impulse = change in linear momentum

$F\,dt = V_0$ .... (iii)

Applying Angular impulse = change in angular momentum

or $Fh\,dt = \frac{2}{5} mR^2 \omega_0$ .... (iv)

Angular momentum about bottommost point will remain conserved.

i.e., $L_i = L_f$

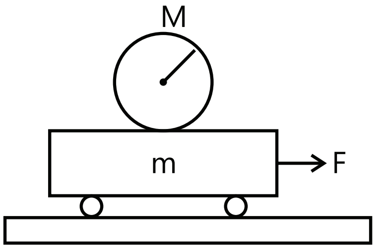
or $I\omega_0 + mRv_0 = I\omega + mRv$

$$\therefore \frac{2}{5}mR^2\omega_0 + mRv_0$$

$$= \frac{2}{5}mR^2\left(\frac{9}{7}\frac{v_0}{R}\right) + \frac{9}{7}mRv_0 \quad \text{...... (v)}$$

Solving Eqs. (iii), (iv) and (v), we get $h = \frac{4}{5}R$

Example 7: Determine the maximum horizontal force F that may be applied to the plank of mass m for which the solid does not slip as it begins to roll on the plank. The sphere has a mass M and radius R (see figure). The coefficient of static and kinetic friction between the sphere and the plank are μ_s and μ_k respectively.



Sol: As the plank moves forward, the sphere, due to its inertia, has a tendency to slip backwards relative to the plank. So the force of friction acts on the sphere in the forward direction. For maximum force F, the friction will be limiting. Write the equations of Newton's second law and torque about center of mass for the sphere and the equation of Newton's second law for the plank.

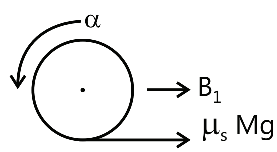
The free body diagram of the sphere and the plank are as shown below:

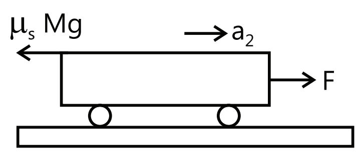
Writing equation of motion:

For sphere: Linear acceleration $a_1 = \frac{\mu_S Mg}{M} = \mu_S g$ ....(i)

Angular acceleration: $\alpha = \frac{(\mu_S Mg)R}{\frac{2}{5}MR^2} = \frac{5}{2}\frac{\mu_S g}{R}$ ....(ii)

For plank: Linear acceleration





$$\alpha_2 = \frac{F - \mu_S Mg}{m}$$

For no slipping $\alpha_2 = a_1 + R\alpha$

Solving the above four equations, we get

$$F = \mu_S g\left(M + \frac{7}{2}m\right)$$

Thus, maximum value of F can be $\mu_S g\left(M + \frac{7}{2}m\right)$

Example 8: A uniform disc of radius r_0 lies on a smooth horizontal plane. A similar disc spinning with the angular velocity ω_0 is carefully lowered onto the first disc. How soon do both discs spin with the same angular velocity if the friction coefficient between them is equal to μ ?

Sol: The initial angular momentum about its center of the disc being lowered will be equal to the combined angular momentum of both the discs about their centers once they start rotating together. Each disc can be thought of made-up of elementary rings of infinitesimal thickness. The torque about the center of disk due to friction force on each ring will be different from that on the other rings in the disc as the radii of rings are different, varying from 0 to r_0. So use the method of integration to find the torque on the entire disc.

From the law of conservation of angular momentum. $I\omega_0 = 2I\omega$

Here, I = moment of inertia of each disc relative to common rotation axis

$\therefore\ \omega = \frac{\omega_0}{2}$ = steady state angular velocity

The angular velocity of each disc varies due to the torque τ of the frictional forces. To calculate τ, let us take an elementary ring with inner and outer radii r and r + dr. The torque of the friction acting on the given is equal to.

$$d\tau = \mu r\left(\frac{mg}{\pi r_0^2}\right)2\pi r\, dr = \left(\frac{2\mu mg}{r_0^2}\right)r^2\, dr$$

where m is the mass each disc. Integrating this respect to r between 0 and r_0, we get

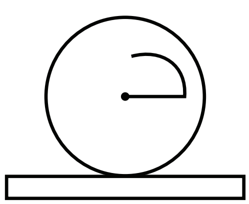
$$\tau = \frac{2}{3}\mu mgr_0$$

The angular velocity of the lower disc increases by $d\omega$ over the time interval

$$= \left(\frac{3r_0}{4\mu g}\right)d\omega$$

Integrating this equation with respect to ω between 0 and $\frac{\omega_0}{2}$, we find the desired time $t = \frac{3r_0\omega_0}{8\mu g}$

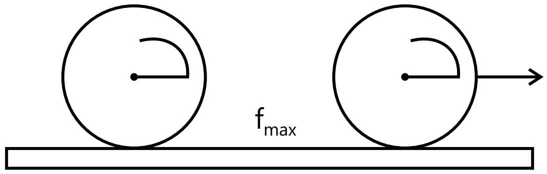
Example 9: A solid sphere of radius r is gently placed on a rough horizontal ground with an initial angular speed ω_0 and no linear velocity. If the coefficient of friction is μ, find the time when the slipping stops. In addition, state the linear velocity v and angular velocity ω at the end of slipping.



Sol: Due to backward slipping the force of friction will act forwards and torque due to friction will be anti-clockwise. This problem can be solved either by Newton's second law and torque about center of mass method or by applying the law of conservation of angular momentum about the point of contact of the sphere with the ground.

Let m be the mass of the sphere.

Since, it is a case of backward slipping, force of friction is in forward direction. Limiting friction will act in this case.



Linear acceleration $a = \frac{f}{m} = \frac{\mu mg}{m} = \mu g$

Angular retardation $\alpha = \frac{\tau}{I} = \frac{f.r}{\frac{2}{5}mr^2} = \frac{5}{2}\frac{\mu g}{r}$

Slipping ceases when $v = r\omega$

Or $(at) = r(\omega_0 - \alpha t)$

Or $\mu gt = r\left(\omega_0 - \frac{5}{2}\frac{\mu g}{r}\right)$

$\frac{7}{2}\mu gt = r\omega_0$; $t = \frac{2}{7}\frac{r\omega_0}{\mu g}$

$v = at = \mu gt = \frac{2}{7}r\omega_0$

And $\omega = \frac{v}{r} = \frac{2}{7}r\omega_0$

Alternative solution: Net torque on the sphere about the bottommost point is zero. Therefore, angular momentum of the sphere will remain conserved about the bottommost point.

$$L_t = L_f$$

$$\therefore\ I\omega_0 = I\omega + mrv$$

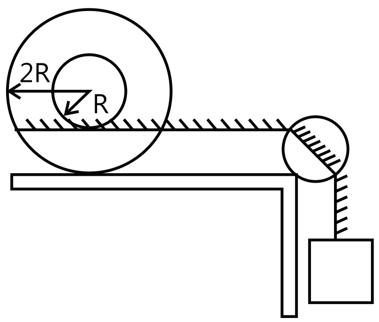
Or $\frac{2}{5}mr^2\omega_0 = \frac{2}{5}mr^2\omega + mr(\omega r)$

$\therefore\ \omega = \frac{2}{7}\omega_0$

And $v = r\omega = \frac{2}{7}r\omega_0$

Example 10: A thin massless thread is wound on reel of mass 3 kg and moment of inertia 0.6 kg-m². The hub radius is R = 10 cm and peripheral radius is 2R = 20 cm. The reel is placed on a rough table and the friction is enough to prevent slipping. Find the acceleration of the center of reel and of hanging mass of 1 kg (see figure).

Sol: Apply Newton's second law at the center of mass of reel in horizontal direction. Find relation between net torque about center of mass of reel and moment of inertia about axis passing through the center of mass. Apply Newton's second law for hanging mass in vertical direction.



Let, a_1 = acceleration of center of mass of reel

a_2 = acceleration of 1 kg block

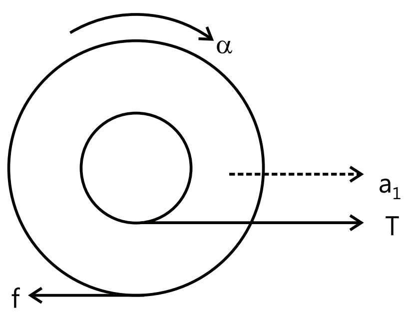
α = angular acceleration of reel (clockwise)

T = tension in the string

and f = force of friction

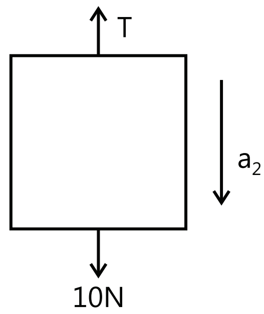
Free body diagram of reel is as shown below; (only horizontal forces are shown.).

Equations of motion are: $T - f = 3a_1$ ... (i)



$$\alpha = \frac{\tau}{I} = \frac{f(2R) - T.R}{I} = \frac{0.2f - 0.1T}{0.6}$$

$$= \frac{f}{3} - \frac{T}{6} \quad \text{...(ii)}$$



Free body diagram of mass is

Equation of motion is,

$10 - T = a_2$ ...(iii)

For no slipping condition,

$a_1 = 2R\alpha$ or $a_1 = 0.2\alpha$ ...(iv)

And $a_2 = a_1 - R\alpha$ or $a_2 = a_1 - 0.1\alpha$ .....(v)

Solving the above five equations, we get

$a_1 = 0.27\ m/s^2; a_2 = 0.135\ m/s^2$

JEE Main/Boards

Exercise 1

Q.1 What are the units and dimensions of moment of inertia? Is it a vector?

Q.2 What is rotational analogue of force?

Q.3 What is rotational analogue of mass of a body?

Q.4 What are two theorems of moment of inertia?

Q.5 What is moment of inertia of a solid sphere about its diameter?

Q.6 What is moment of inertia of a hollow sphere about an axis passing through its center.

Q.7 What are the factors on which moment of inertia of a body depends?

Q.8 Is radius of gyration of a body a constant quantity?

Q.9 There are two spheres of same mass and same radius, one is solid and other is hollow. Which of them has a larger moment of inertia about its diameter?

Q.10 Two circular discs A and B of the same mass and same thickness are made of two different metals whose densities are d_A and d_B ($d_A > d_B$). Their moments of inertia about the axes passing through their centers of gravity and perpendicular to their planes are I_A and I_B. Which is greater, I_A or I_B?

Q.11 The moments of inertia of two rotating bodies A and B are I_A and I_B ($I_A > I_B$) and their angular moments are equal. Which one has a greater kinetic energy?

Q.12 Explain the physical significance of moment of inertia and radius of gyration.

Q.13 Obtain expression of K.E. for rolling motion.

Q.14 State the laws of rotational motion.

Q.15 Establish a relation between torque and moment of inertia of a rigid body.

Q.16 State and explain the principle of conservation of angular momentum. Give at least two examples.

Q.17 Derive an expression for moment of inertia of a thin circular ring about an axis passing through its center and perpendicular to the plane of the ring.

Q.18 The moment of inertia of a circular ring about an axis passing through the center and perpendicular to its plane is 200 g cm^2. If radius of ring is 5 cm, calculate the mass of the ring.

Q.19 Calculate moment of inertia of a circular disc about a transverse axis through the center of the disc. Given, diameter of disc is 40 cm, thickness = 7 cm and density of material of disc = 9 g cm^{-3}

Q.20 A uniform circular disc and a uniform circular ring each has mass 10kg and diameter 1m. Calculate their moment of inertia about a transverse axis through their center.

Q.21 Calculate moment of inertia of earth about its diameter, taking it to be a sphere of radius 6400 km and mass 6×10^{24} kg.

Q.22 Calculate moment of inertia of a uniform circular disc of mass 700 g and diameter 20 cm about

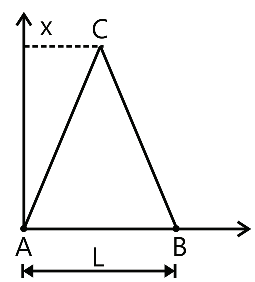
(i) An axis through the center of disc and perpendicular to its plane,.

(ii) A diameter of disc

(iii) A tangent in the plane of the disc,

(iv) A tangent perpendicular to the plane of the disc.

Q.23 Three particles, each of mass m, are situated at the vertices of an equilateral triangle ABC of side L. Find the moment of inertia of the system about the line AX perpendicular to AB in the plane of ABC, in the given figure.



Q.24 Calculate K.E. of rotation of a circular disc of mass 1 kg and radius 0.2 m rotating about an axis passing through its center and perpendicular to its plane. The disc is making 30/π rpm.

Q.25 A circular disc of mass M and radius r is set into pure rolling on a table. If ω be its angular velocity, show that it's total K.E. is given by (3/4) Mv^2, where v is its linear velocity. M.I. of circular disc = (1/2) mass × $(radius)^2$.

Q.26 The sun rotates around itself once in 27 days. If it were to expand to twice its present diameter, what would be its new period of revolution?

Q.27 A 40 kg flywheel in the form of a uniform circular disc 1 meter in radius is making 120 r.p..m. Calculate its angular momentum about transverse axis passing through center of fly wheel.

Q.28 A body is seated in a revolving chair revolving at an angular speed of 120 r.p.m. By some arrangement, the body decreases the moment of inertia of the system from 6 kg m^2 to 2 kg m^2. What will be the new angular speed?

Exercise 2

Single Correct Choice Type

Q.1 Thee bodies have equal masses m. Body A is solid cylinder of radius R, body B is square lamina of side R, and body C is a solid sphere of radius R. Which body has the smallest moment of inertia about an axis passing through their center of mass and perpendicular to the plane (in case of lamina)

(A) A (B) B (C) C (D) A and C both

Q.2 For the same total mass which of the following will have the largest moment of inertia about an axis passing through its center of mass and perpendicular to the plane of the body.

(A) A disc of radius a

(B) A ring of radius a

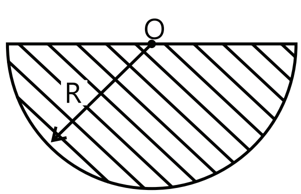
(C) A square lamina of side 2a

(D) Four rods forming a square of side 2a

Q.3 A thin uniform rod of mass M and length L has its moment of inertia I_1 about its perpendicular bisector. The rod is bend in the form of semicircular arc. Now its moment of inertia perpendicular to its plane is I_2. The ratio of $I_1 : I_2$ will be

(A) < 1 (B) >1 (C) =1 (D) Can't be said

Q.4 Moment of inertia of a thin semicircular disc (mass = M & radius = R) about an axis through point O and perpendicular to plane of disc, is given by:



(A) $\frac{1}{4}MR^2$ (B) $\frac{1}{2}MR^2$ (C) $\frac{1}{8}MR^2$ (D) MR^2

Q.5 A rigid body can be hinged about any point on the x-axis. When it is hinged such that the hinge is at x, the moment of inertia is given by $I = 2x^2 - 12x + 27$

The x-coordinate of center of mass is

(A) x = 2 (B) x = 0 (C) x = 1 (D) x = 3

Q6. A weightless rod is acted upon by upward parallel forces of 2N and 4N at ends A and B respectively. The total length of the rod AB=3m. To keep the rod in equilibrium, a force of 6N should act in the following manner:

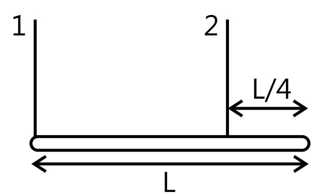
(A) Downwards at any point between A and B.

(B) Downwards at mid-point of AB

(C) Downwards at a point C such that AC=1m

(D) Downwards at a point D such that BD= 1m.

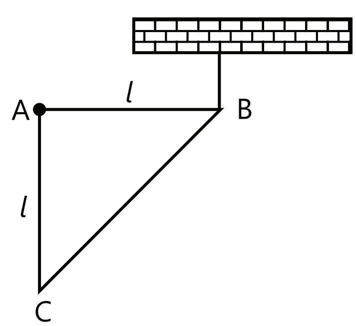
Q.7 A heavy rod of length L and weight W is suspended horizontally by two vertical ropes as shown. The first rope is attached to the left end of rod while the second rope is attached a distance L/4 from right end. The tension in the second rope is:



(A) (1/2) W (B) (1/4) W

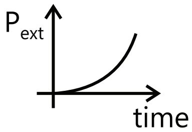
(C) (1/3) W (D) (2/3) W (E) W

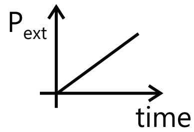
Q.8 A right triangular plate ABC of mass m is free to rotate in the vertical plane about a fixed horizontal axis though A. It is supported by a string such that the side AB is horizontal. The reaction at the support A in equilibrium is:

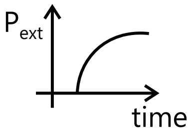


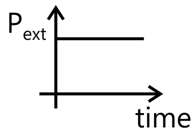
(A) $\frac{mg}{3}$ (B) $\frac{2mg}{3}$ (C) $\frac{mg}{2}$ (D) mg

Q.9 A rod is hinged at its center and rotated by applying a constant torque from rest. The power developed by the external torque as a function of time is:

(A)


(B)


(C)


(D)


Q.10 Two uniform spheres of mass M have radii R and 2R. Each sphere is rotating about a fixed axis through its diameter. The rotational kinetic energies of the spheres are identical. What is the ratio of the angular moments of these sphere? That is, $\frac{L^{2}2R}{L_R}$ =

(A) 4 (B) $2\sqrt{2}$ (C) 2 (D) $\sqrt{2}$ (E) 1

Q.11 A spinning ice skater can increase his rate of rotation by bringing his arms and free leg closer to his body. How does this procedure affect the skater's momentum and kinetic energy?

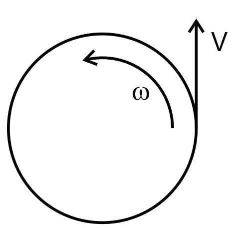
(A) Angular momentum remains the same while kinetic energy increases.

(B) Angular momentum remains the same while kinetic energy decreases

(C) Both angular momentum and kinetic energy remains the same.

(D) Both angular momentum and kinetic energy increase.

Q.12 A child with mass m is standing at the edge of a disc with moment of inertia I, radius R, and initial angular velocity ω. See figure given below. The child jumps off the edge of the disc with tangential velocity v with respect to the ground. The new angular velocity of the disc is



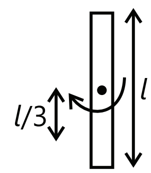
(A) $\sqrt{\frac{I\omega^2 - mv^2}{I}}$

B) $\sqrt{\frac{(I+mR^2)\omega^2 - mv^2}{I}}$

(C) $\frac{I\omega - mvR}{I}$

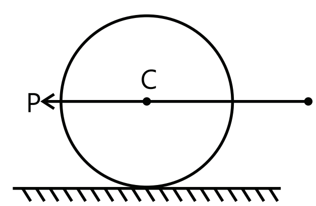
(D) $\frac{(I+mR^2)\omega - mvR}{I}$

Q.13 A uniform rod of length l and mass M is rotating about a fixed vertical axis on a smooth horizontal table. It elastically strikes a particle placed at a distance $l/3$ from its axis and stops. Mass of the particle is



(A) 3M (B) $\frac{3M}{4}$ (C) $\frac{3M}{2}$ (D) $\frac{4M}{3}$

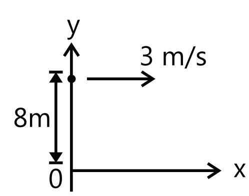
Q.14 a disc of radius R is rolling purely on a flat horizontal surface, with a constant angular velocity. The angle between the velocity and acceleration vectors of point P is



(A) Zero (B) 45°

(C) 135° (D) $\tan^{-1}(1/2)$

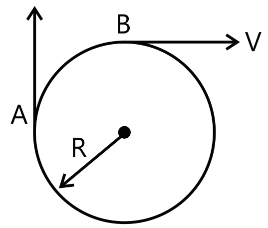
Q.15 A particle starts from the point (0m, 8m) and moves with uniform velocity of 3m/s. After 5 second, the angular velocity of the particle about the origin will be:



(A) $\frac{8}{289}$rad/s (B) $\frac{3}{8}$rad/s

(C) $\frac{24}{289}$rad/s (D) $\frac{8}{17}$rad/s

Q.16 Two points of a rigid body are moving as shown. The angular velocity of the body is:



(A) $\frac{v}{2R}$ (B) $\frac{v}{R}$ (C) $\frac{2v}{R}$ (D) $\frac{2v}{3R}$

Q.17 A yo-yo is released from hand with the string wrapped around your finger. If you hold your hand still, the acceleration of the yo-yo is

(A) Downward, much greater than g

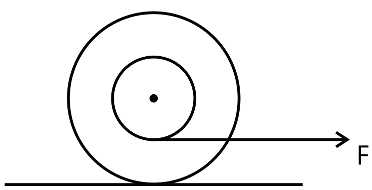
(B) Downward much greater than g

(C) Upward, much less than g

(D) Upward, much greater than g

(E) Downward, at g

Q.18 Inner and outer radii of N a spool are r and R respectively. A thread is wound over its inner surface and placed over a rough horizontal surface. Thread is pulled by a force F as shown in figure. Then in case of pure rolling.



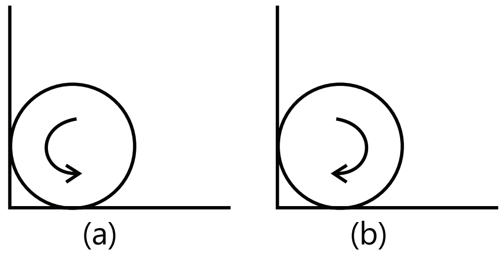
(A) Thread unwinds, spool rotates anticlockwise and friction acts leftwards

(B) Thread unwinds, spool rotates clockwise and friction acts leftwards

(C) Thread wind, spool moves to the right and friction acts rightwards.

(D) Thread winds, spool moves to the right and friction does not come into existence.

Q.19 A sphere is placed rotating with its center initially at rest in a corner as shown in figure (a) & figure (b). Coefficient of friction between all surfaces and the sphere is $\frac{1}{3}$. Find the ratio of the frictional force $\frac{f_a}{f_b}$ by ground in situations (a) & (b).



(A) 1 (B) $\frac{9}{10}$ (C) $\frac{10}{9}$ (D) None

Q.20 A body kept on a smooth horizontal surface is pulled by a constant horizontal force applied at the top point of the body. If the body rolls purely on the surface, its shape can be:

(A) Thin pipe (B) Uniform cylinder

(C) Uniform sphere (D) Thin spherical shell

Q.21 A uniform rod AB of mass m and length l is at rest on a smooth horizontal surface. An impulse j is applied to the end B, perpendicular to the rod in the horizontal direction. Speed of point P at A distance $\frac{l}{6}$ from the center towards a of the rod after time $t = \frac{\pi ml}{12J}$ is

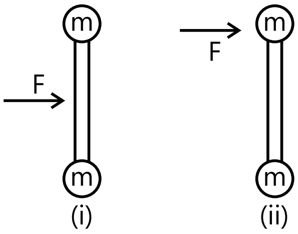
(A) $2\frac{J}{m}$ (B) $\frac{J}{\sqrt{2}m}$

(C) $\frac{J}{m}$ (D) $\sqrt{2}\frac{J}{m}$

Q.22 The moment of inertia of a solid cylinder about its axis is given by (1/2) MR^2. If this cylinder rolls without slipping, the ratio of its rotational kinetic energy to its translational kinetic energy is

(A) 1: 1 (B) 2: 2 (C) 1: 2 (D) 1: 3

Q.23 A force F is applied to a dumbbell for a time interval t, first as in (i) and then as in (ii). In which case does the dumbbell acquire the greater center-of-mass speed?



(A) (i)

(B) (ii)

(C) There is no difference

(D) The answer depends on the rotation inertia of the dumbbell

Q.24 A hoop and a solid cylinder have the same mass and radius. They both roll, without slipping on a horizontal surface. If their kinetic energies are equal

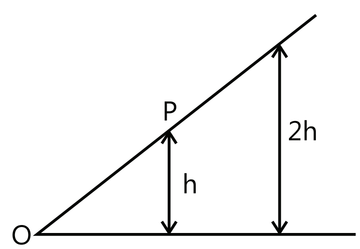
(A) The hoop has a greater translational speed then the cylinder

(B) The cylinder has a greater translational speed then the hoop

(C) The hoop and the cylinder have the same translational speed

(D) The hoop has a greater rotational speed then the cylinder.

Q.25 A ball rolls down an inclined plane, as shown in figure. The ball is first released from rest from P and then later from Q. Which of the following statement is /are correct?



(i) The ball takes twice as much time to roll from Q to O as it does to roll from P to O.

(ii) The acceleration of the ball at Q is twice as large as the acceleration at P.

(iii) The ball has twice as much K.E.at O when rolling from Q as it does when rolling from P.

(A) i, ii only (B) ii, iii only

(C) i only (D) iii only

Q.26 If a person is sitting on a rotating stool with his hands outstretched, suddenly lowers his hands, then his

(A) Kinetic energy will decrease

(B) Moment of inertia will decrease

(C) Angular momentum will increase

(D) Angular velocity will remain constant

Q.27 Choose the correct statement(s)

(A) The momentum of the ring is conserved

(B) The angular momentum of the ring is conserved about its center of mass

(C) The angular momentum of the ring is conserved about a point on the horizontal surface.

(D) The mechanical energy of the ring is conserved.

Previous Years' Questions

Q.1 Let I be moment of inertia of a uniform square plate about an axis AB that passes through its center and is parallel to two of its sides. CD is a line in the plane of the plate that passes through the center of the plate and makes an angle è with AB. The moment of inertia of the plate about the axis CD is then equal to. ***(1998)***

(A) I (B) $I\sin^2\theta$

(C) $I\cos^2\theta$ (D) $I\cos^2(\theta/2)$

Q.2 A smooth sphere is moving on a frictionless horizontal plane with angular velocity ω and center of mass velocity v. It collides elastically and head on with an identical sphere B at rest, Neglect friction everywhere. After the collision their angular speeds are ω_A and ω_B respectively. Then, ***(1999)***

(A) $\omega_A < \omega_B$ (B) $\omega_A = \omega_B$

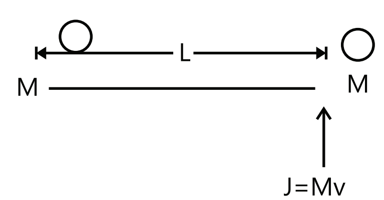
(C) $\omega_A = \omega$ (D) $\omega_B = \omega$

Q.3 A particle of mass m is projected with a velocity v making an angle of 45° with the horizontal. The magnitude of the angular momentum of the projectile about the point of projection when the particle is at its maximum height h is ***(1990)***

(A) Zero (B) $mv^3/(4\sqrt{2}g)$

(C) $mv^3/(\sqrt{2}g)$ (D) $m\sqrt{2gh^3}$

Q.4 Consider a body, shown in figure, consisting of two identical balls, each of mass M connected by a light rigid rod. If an impulse J = Mv is imparted to the body at one of its end, what would be its angular velocity? ***(2003)***



(A) v/L (B) 2v/L (C) v/3L (D) v/4L

Q.5 A tube of length L is filled completely with an incompressible liquid of mass M and closed at both the ends. The tube is then rotated in a horizontal plane about one of its ends with a uniform angular velocity ω. The force exerted by the liquid at the other end is ***(1992)***

(A) $\frac{M\omega^2 L}{2}$ (B) $M\omega^2 L$ (C) $\frac{M\omega^2 L}{4}$ (D) $\frac{M\omega^2 L^2}{2}$

Q.6 A cylinder rolls up an inclined plane, reaches some height and then rolls down (without slipping throughout these motions.) The directions of the frictional force acting on the cylinder are ***(2002)***

(A) Up the incline while ascending and down the incline while descending.

(B) Up the incline while ascending as well as descending

(C) Down the incline while ascending and up the inline while descending.

(D) Down the incline while ascending as well as descending.

Q.7 Two point masses of 0.3 kg and 0.7 kg fixed at the ends of a rod of length 1.4 m and of negligible mass. The rod is set rotating about an axis perpendicular to its length with a uniform angular speed. The point on the rod through which rotation of the rod is minimum, is located at a distance of ***(1995)***

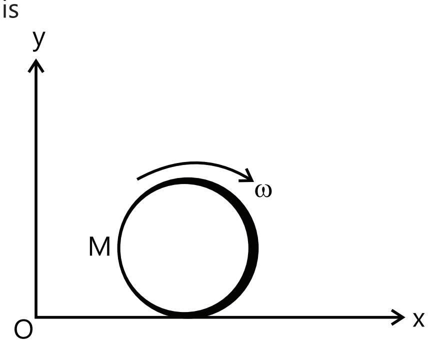
(A) 0.42 m from mass of 0.3kg

(B) 0.70 m from mass of 0.7 kg

(C) 0.98 m form mass of 0.3 kg

(D) 0.98 m form mass of 0.7 kg

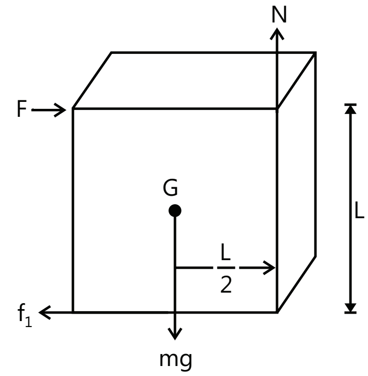
Q.8 A disc of mass M and radius R is rolling with angular speed ω on a horizontal plane as shown. The magnitude of angular momentum of the about the origin O is ***(1999)***



(A) $\left(\frac{1}{2}\right)MR^2\omega$ (B) $MR^2\omega$

(C) $\left(\frac{3}{2}\right)MR^2\omega$ (D) $2MR^2\omega$

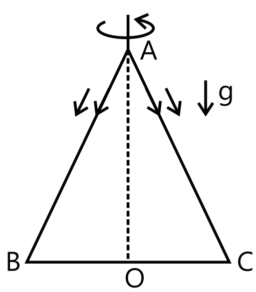
Q.9 A cubical block of side L rests on a rough horizontal surface with coefficient of friction μ. A horizontal force F is applied sufficient high, so that the block does not slide before toppling, the minimum force required to topple the block is ***(2000)***



(A) Infinitesimal (B) mg/4

(C) mg/2 (D) mg $(1-\mu)$

Q.10 An equilateral triangle ABC formed from a uniform wire has two small identical beads initially located at A. The triangle is set rotating about the vertical axis AO. Then the beads are released from rest simultaneously and allowed to slide down, one along AB and other along AC as shown. Neglecting frictional effects, the quantities that are conserved as beads slide down are ***(2000)***



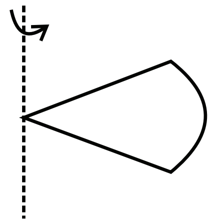
(A) Angular velocity and total energy (kinetic and potential)

(B) Total angular momentum and total energy

(C) Angular velocity and moment of inertia about the axis of rotation

(D) Total angular momentum and moment of inertia about the axis of rotation.

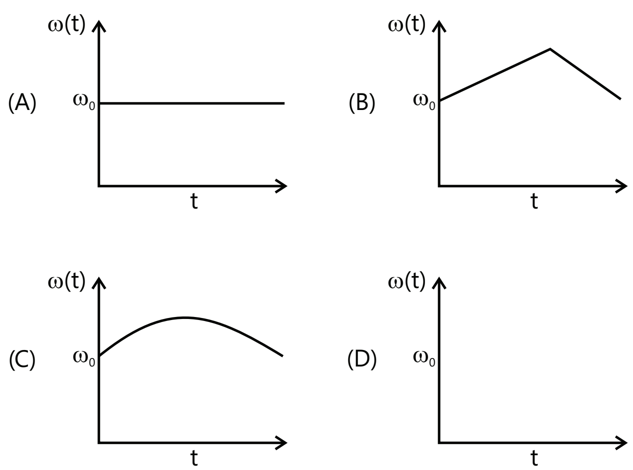
Q.11 One quarter section is cut from a O uniform circular disc of radius R. This section has a mass M. It is made to rotate about a line perpendicular to its plane and passing through the center of the original disc. Its moment of inertia about the axis of rotation is ***(2001)***



(A) $\frac{1}{2}MR^2$ (B) $\frac{1}{4}MR^2$ (C) $\frac{1}{8}MR^2$ (D) $\sqrt{2}MR^2$

Q.12 A circular platform is free to rotate in a horizontal plane about a vertical axis passing through its center. A tortoise is sitting at the edge of the platform. Now the platform is given an angular velocity ω_0

When the tortoise moves along a chord of the platform with a constant velocity (with respect to the platform). The angular velocity of the platform $\omega(t)$ will vary with time t as ***(2002)***



(A)

Q.13 A thin circular ring of mass M and radius r is rotating about its axis with a constant angular velocity ω. Two objects, each of mass m_2 are attached gently to the opposite ends of a diameter of the ring. The wheel now rotates with an angular velocity ***(2006)***

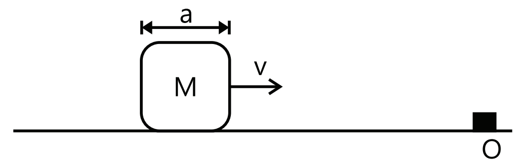
(A) $\omega M/(M + m)$

(B) $\omega(M - 2m)/(M + 2m)$

(C) $\omega M/(M + 2m)$

(D) $\omega(M + 2m)/M$

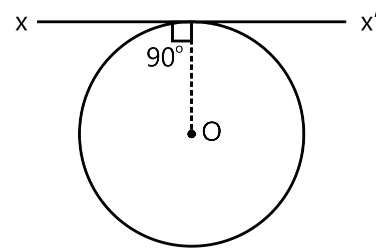
Q.14 A cubical block of side a moving with velocity v on a horizontal smooth plane as shown. It hits at point O. The angular speed of the block after it hits O is ***(1999)***



(A) $3v / 4a$ (B) $3v / 2a$

(C) $\sqrt{3} / \sqrt{2}a$ (D) Zero

Q.15 A thin wire of length L and uniform linear mass density ρ is bent into a circular loop with center at O as shown. The moment of inertia of the loop about the axis XX′ is ***(2000)***



(A) $\frac{\rho L^3}{8\pi^2}$ (B) $\frac{\rho L^3}{16\pi^2}$ (C) $\frac{5\rho L^3}{16\pi^2}$ (D) $\frac{3\rho L^3}{8\pi^2}$

Q.16 A diatomic molecule is made of two masses m_1 and m_2 which are separated by a distance r. If we calculate its rotational energy by applying Bohr's rule of angular momentum quantization, its energy will be given by (n is an integer) ***(2012)***

(A) $\frac{(m_1+m_2)^2 n^2 \hbar^2}{2m_1^2 m_2^2 r^2}$ (B) $\frac{n^2 \hbar^2}{2(m_1+m_2)r^2}$

(C) $\frac{2n^2 \hbar^2}{(m_1+m_2)r^2}$ (D) $\frac{(m_1+m_2)n^2 \hbar^2}{2m_1 m_2 r^2}$

Q.17 A hoop of radius r and mass m rotating with an angular velocity ω_0 is placed on a rough horizontal surface. The initial velocity of the centre of the hoop is zero. What will be the velocity of the centre of thehoop when it ceases to slip? ***(2013)***

(A) $\frac{r\omega_0}{3}$ (B) $\frac{r\omega_0}{2}$ (C) $r\omega_0$ (D) $\frac{r\omega_0}{4}$

Q.18 A bob of mass m attached to an inextensible string of length ℓ is suspended from a vertical support. The bob rotates in a horizontal circle with an angular speed ω rad/s about the vertical. About the point of suspension: ***(2014)***

(A) Angular momentum changes in direction but not in magnitude.

(B) Angular momentum changes both in direction and magnitude.

(C) Angular momentum is conserved.

(D) Angular momentum changes in magnitude but not in direction.

Q.19 The current voltage relation of diode is given by $I = \left(e^{1000V/T} - 1\right)mA$, where the applied voltage V is in volts and the temperature T is in degree Kelvin. If a student makes an error measuring $\pm 0.01\,V$ while measuring the current of 5 mA at 300 K, what will be the error in the value of current in mA? ***(2014)***

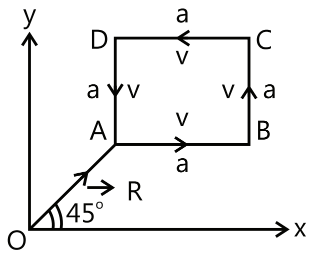
(A) 0.5 mA (B) 0.05 mA

(C) 0.2 mA (D) 0.02 mA

Q.20 From a solid sphere of mass M and radius R a cube of maximum possible volume is cut. Moment of inertia of cube about an axis passing through its center and perpendicular to one of its faces is: ***(2015)***

(A) $\frac{MR^2}{16\sqrt{2}\pi}$ (B) $\frac{4MR^2}{9\sqrt{3}\pi}$ (C) $\frac{4MR^2}{3\sqrt{3}\pi}$ (D) $\frac{MR^2}{32\sqrt{2}\pi}$

Q.21 A particle of mass m is moving along the side of a square of side 'a', with a uniform speed v in the x-y plane as shown in the figure:



Which of the following statements is false for the angular momentum $\vec{L}$ about the origin? ***(2016)***

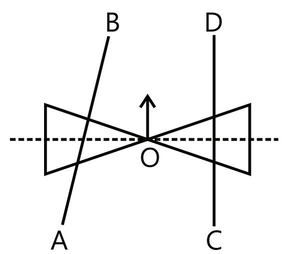
(A) $\vec{L} = mv\left[\frac{R}{\sqrt{2}} - a\right]\hat{k}$ when the particle is moving from C to D.

(B) $\vec{L} = mv\left[\frac{R}{\sqrt{2}} + a\right]\hat{k}$ when the particle is moving from B to C.

(C) $\vec{L} = \frac{mv}{\sqrt{2}}R\hat{k}$ when the particle is moving from D to A.

(D) $\vec{L} = -\frac{mv}{\sqrt{2}}R\hat{k}$ when the particle is moving from A to B.

Q.22 A roller is made by joining together two cones at their vertices O. It is kept on two rails AB and CD which are placed asymmetrically (see figure), with its axis perpendicular to CD and its centre O at the centre of line joining AB and CD (see figure). It is given a light push so that it starts rolling with its centre O moving parallel to CD in the direction shown. As it moves, the roller will tend to: ***(2016)***



(A) Turn right

(B) Go straight

(C) Turn left and right alternately

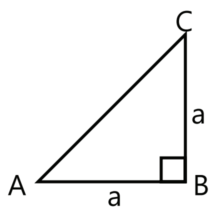
(D) Turn left

JEE Advanced/Boards

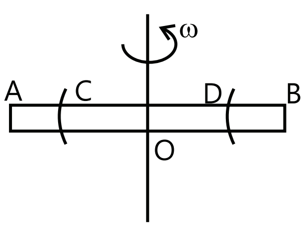
Exercise 1

Q.1 A thin uniform rod of mass M and length L is hinged at its upper end, and released from rest in a horizontal position. Find the tension at a point located at a distance L/3 from the hinge point, when the rod becomes vertical.

Q.2 A rigid body in shape of a triangle has V_A = 5 m/s downwords, V_B = 10 m/s downwords. Find velocity of point C.

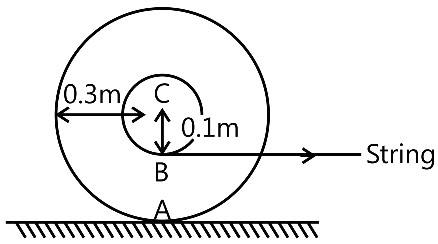


Q.3 A rigid horizontal smooth rod AB of mass 0.75 kg and length 40 cm can rotate freely about a fixed vertical axis through its mid-point O. Two rings each of mass 1 kg are initially at rest at a distance of 10 cm from O on either side of the rod. The rod is set in rotation with an angular velocity of 30 rad per second. Find the velocity of each ring along the length of the rod in m/s when they reach the ends of the rod.

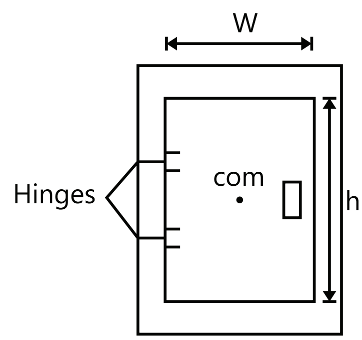


Q.4 A straight rod AB of mass M and length L is placed on a frictionless horizontal surface. A horizontal force having constant magnitude F and a fixed direction starts acting at the end A. The rod is initially perpendicular to the force. Find the initial acceleration of end B.

Q.5 A wheel is made to roll without slipping, towards right, by pulling a string wrapped around a coaxial spool as shown in figure. With what velocity the string should be pulled so that the center of wheel moves with a velocity of 3 m/s?



Q.6 A uniform wood door has mass m, height h, and width w. It is hanging from two hinges attached to one side; the hinges are located h/3 and 2h/3 from the bottom of the door.



Suppose that m = 20.0 kg, h = 2.20m, and W = 1.00 m and the bottom smooth hinge is not screwed into the door frame, find the forces acting on the door.

Q.7 A thin rod AB of length a has variable mass per unit length $\rho_0\left(1+\frac{x}{a}\right)$ where x is the distance measured from a and ρ_0 is a constant

(a) Find the mass M of the rod.

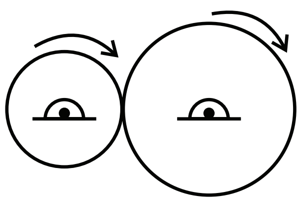
(b) Find the position of center of mass of the rod.

(c) Find moment of inertia of the rod about an axis passing through A and perpendicular to AB. Rod is freely pivoted at A and is hanging in equilibrium when it is struck by a horizontal impulse of magnitude P at the point B.

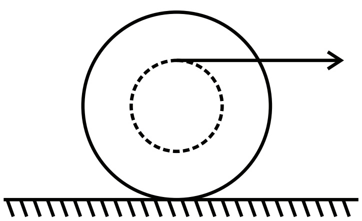
(d) Find the angular velocity with which the rod begins to rotate.

(e) Find minimum value of impulse P if B passes through a point vertically above A.

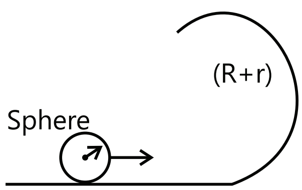
Q.8 Two separate cylinders of masses m (=1 kg) and 4m and radii R (=10cm) and 2R are rotating in clockwise direction with ω_1 = 100rad/sec and ω_2 = 200 rad/sec. Now they are held in contact with each other as in figure Determine their angular velocity after the slipping between the cylinders stops.



Q.9 A spool of inner radius R and outer radius 3R has a moment of inertia = MR^2 about an axis passing through its geometric center, where M is the mass of the spool. A thread wound on the inner surface of the spool is pulled horizontally with a constant force = Mg. Find the acceleration of the point on the thread which is being pulled assuming that the spool rolls purely on the floor.



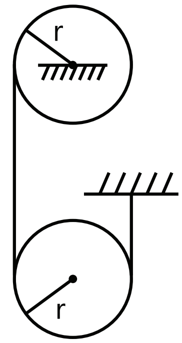
Q.10 A sphere of mass m and radius r is pushed onto a fixed horizontal surface such that it rolls without slipping from the beginning. Determine the minimum speed v of its mass center at the bottom so that it rolls completely around the loop of radius (R + r) without leaving the track in between.



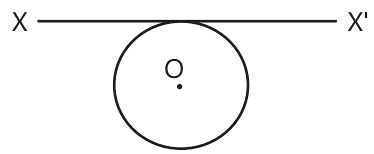
Q.11 Two uniform cylinders each of mass m = 10 kg and radius r = 150 mm, are connected by a rough belt as shown. If the system is released from rest, determine

(a) The tension in the portion of the belt connecting the two cylinder.

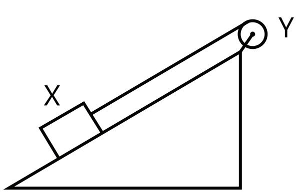
(b) The velocity of the center of cylinder a after it has moved through 1.2 m.



Q.12 A thin wire of length L and uniform linear mass density ρ is bent into a circular loop with center at O as shown in the figure. The moment of inertia of the loop about the axis XX` is



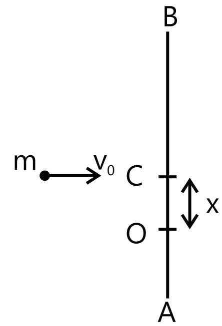
Q.13 A block X of mass 0.5 kg is held by a long massless string on a frictionless inclined plane of inclination 30° to the horizontal. The string is wound on a uniform solid cylindrical drum Y of mass 2 kg and of radius 0.2 m as shown in the figure. The drum is given an initial angular velocity such that the block X stands moving up the plane. (g = 9.8 m/s²)



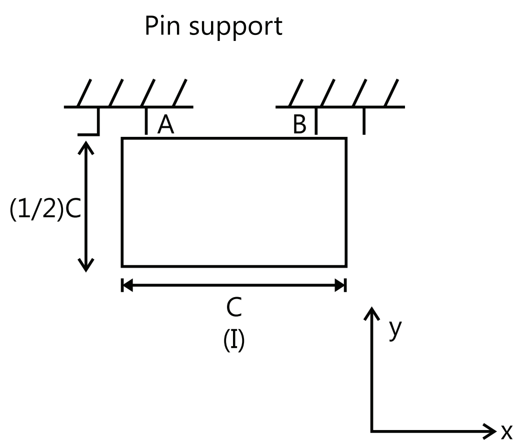
(i) Find the tension in the string during the motion.

(ii) At a certain instant of time the magnitude of the angular velocity of Y is 10rad/sec. Calculate the distance travelled by X form that instant of time unit it comes to rest.

Q.14 A uniform rod AB of length L and mass M is lying on a smooth table. A small particle of mass m strikes the rod with a velocity v_0 at point at distance from the center O. The particle comes to rest after collision. Find the value of x, so that of the rod remains stationary just after collision.



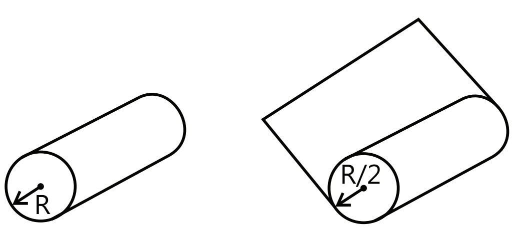
Q.15 A uniform plate of mass m is suspended in each of the ways shown. For each case determine immediately after the connection at B has been released:



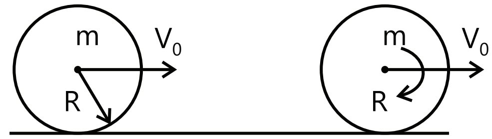
(a) The angular acceleration of the plate.

(b) The acceleration of its mass center.

Q.16 A carpet of mass 'M' made of inextensible material is rolled along its length in the form of cylinder of radius 'R' and is kept on a rough floor. The carpet starts unrolling without standing on the floor when a negligibly small push is given to it. The horizontal velocity of the axis of the cylindrical parts of the carpet when its radius decreases to R/2 will be:



Q.17 A uniform disk of mass m and radius R is projected horizontally with velocity v_0 on a rough horizontal floor so that it starts off with a purely sliding motion at t = 0. After t_0 seconds it acquires a purely rolling motion as shown in figure.



(i) Calculate the velocity of the center of mass of the disk at t_0

(ii) Assuming the coefficient of friction to be μ calculate t_0. Also calculate the work done by the frictional force as a function of time and the total work done by it over a time t much longer then t_0.

Q.18 A circular disc of mass 300 gm and radius 20 cm can rotate freely about a vertical axis passing through its center of mass o. A small insect of mass 100 gm is initially at a point A on the disc (which is initially stationary). The insect starts walking from rest along the rim of the disc with such a time varying relative velocity that the disc rotates in the opposite direction with a constant angular acceleration = 2π rad/s^2. After some time T, the insect is back at the point A. By what angle has the disc rotated till now, as seen by a stationary earth observer? Also find the time T.

Q.19 A uniform disc of mass m and radius R rotates about a fixed vertical axis passing through its center with angular velocity ω. A particle of same mass m and having velocity $2\omega R$ towards center of the disc collides with the disc moving horizontally and stick to its rim. Find

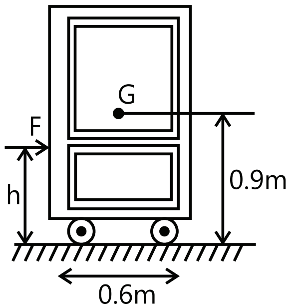
(a) The angular velocity of the disc

(b) The impulse on the particle due to disc.

(c) The impulse on the disc due to hinge.

Q.20 The door of an automobile is open and perpendicular to the body. The automobile starts with an acceleration of 2 ft/sec^2, and the width of the door is 30 inches. Treat the door as a uniform rectangle, and neglect friction to find the speed of its outside edge as seen by the driver when the door closes.

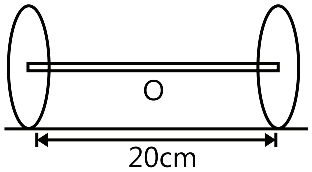
Q.21 A 20 kg cabinet is mounted on small casters that allow it to move freely (μ = 0) on the floor. If a 100 N force is applied as shown, determine.



(a) The acceleration of the cabinet,

(b) The range of values of h for which the cabinet will not tip.

Q.22 Two thin circular disks of mass 2 kg and radius 10 cm each are joined by a rigid massless rod of length 20 cm. The axis of the rod is along the perpendicular to the planes of the disk through their center. The object is kept on a truck in such a way that the axis of the object is horizontal and perpendicular to the direction of motion of the truck. Its friction with the floor of the truck is large enough so that the object can roll on the truck without slipping. Take x-axis as the vertically upwards direction. If the truck has an acceleration of $9 m/s^2$ calculate.



(a) The force of friction on each disk.

(b) The magnitude and the direction of the frictional torque acting on each disk about the center of mass O of the object. Express the torque in the vector form of unit vectors in the x-y and z direction.

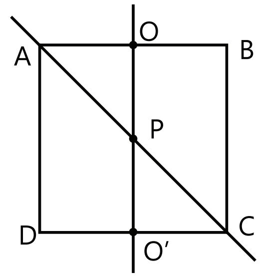
Q.23 Three particles A, B, C of mass m each are joined to each other by mass less rigid rods to form an equilateral triangle of side a. Another particle of mass m hits B with a velocity v_0 directed along BC as shown. The colliding particle stops immediately after impact.

(i) Calculate the time required by the triangle ABC to complete half-revolution in its subsequent motion. (ii) What is the net displacement of point B during this interval?

Exercise 2

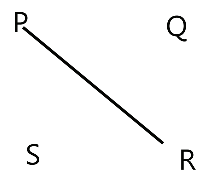
Single Correct Choice Type

Q.1 Let I_1 and I_2 be the moment of inertia of a uniform square plate about axes APC and OPO` respectively as shown in the figure. P is center of square. The ratio $\frac{I_1}{I_2}$ of moment of inertia is



(A) $\frac{1}{\sqrt{2}}$ (B) 2 (C) $\frac{1}{2}$ (D) 1

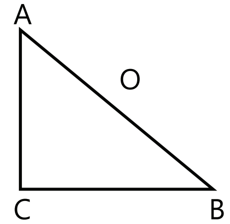
Q.2 Moment of inertia of a rectangular plate about an axis passing through P and perpendicular to the plate is I. Then moment of PQR about an axis perpendicular to the plane.



(A) About P = I/2 (B) About R = I/2

(C) About P > I/2 (D) About R > I/2

Q.3 Find the moment of inertia of a plate cut in shape of a right angled triangle of mass M, AC=BC=a about an axis perpendicular to plane, side the plane of the plate and passing through the mid-point of side AB.



(A) $\frac{Ma^2}{12}$ (B) $\frac{Ma^2}{6}$ (C) $\frac{Ma^2}{3}$ (D) $\frac{2Ma^2}{3}$

Q.4 Let I be the moment of inertia of a uniform square plate about an axis AB that passes through its center and is parallel to two of its sides. CD is a line in the plane of the plate that passes through the center of the plate and makes an angle θ with AB. The moment of inertia of the plate about the axis CD is then equal to

(A) I (B) $I \sin^2 \theta$

(C) $I \cos^2 \theta$ (D) $I \cos^2 (\theta/2)$

Q.5 A heavy seesaw (i.e., not mass less) is out of balance. A light girl sits on the end that is tilted downward, and a heavy body sits on the other side so that the seesaw now balances. If they both move forward so that they are one-half of their original distance from the pivot point (the fulcrum) what will happen to the seesaw?

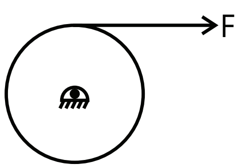
(A) The side the body is sitting on will tilt downward

(B) The side the girl is sitting on will once again tilt downward

(C) Nothing; the seesaw will still be balanced

(D) It is impossible to say without knowing the masses and the distances.

Q.6 A pulley is hinged at the center and a mass less thread is wrapped around it. The thread is pulled with a constant force F starting from rest,. As the time increases,



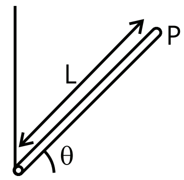
(A) Its angular velocity increases, but force on hinge remains constant

(B) Its angular velocity remains same, but force on hinge increases

(C) Its angular velocity increases and force of hinge increases

(D) Its angular velocity remains same and force on hinge is constant

Q.7 A uniform flag pole of length L and mass M is pivoted on the ground with a frictionless hinge. The flag pole makes an angle θ with the horizontal. The moment of inertia of the flag pole about one end is $(1/3)\ ML^2$. If it starts falling from the position shown in the accompanying figure, the linear acceleration of the free end of the flag pole – labeled P – would be:



(A) $(2/3)\ g\cos\theta$ (B) $(2/3)g$

(C) g (D) $\left(\frac{3}{2}\right)g\cos\theta$

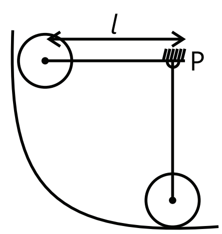
(E) $(3/2)g$

Q.8 A mass m is moving at speed v perpendicular to a rod of length d and mass M = 6m which pivots around a frictionless axle running through its center. It strikes and sticks to the end of the rod. The moment of inertia of the rod about its center is $Md^2/12$. Then the angular speed of the system right after the collision is.

(A) $2v/d$ (B) $2v/(3d)$

(C) v/d (D) $3v/(2d)$

Q.9 A sphere of mass M and radius R is attached by a light rod of length l to a point P. The sphere rolls without slipping on a circular track as shown. It is released from the horizontal position. The angular momentum of the system about P when the rod becomes vertical is:



(A) $M\sqrt{\frac{10}{7}gl}[l+R]$ (B) $M\sqrt{\frac{10}{7}gl}[l+\frac{2}{5}R]$

(C) $M\sqrt{\frac{10}{7}gl}[l+\frac{7}{5}R]$ (D) None of the above

Q.10 A ladder of length L is slipping with its ends against a vertical wall and a horizontal floor. At a certain moment, the speed of the end in contact with the horizontal floor is v and the ladder makes an angle $\alpha = 30°$ with the horizontal. Then the speed of ladder's center must be

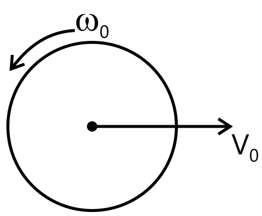
(A) $2v\sqrt{3}$ (B) v/2 (C) v (D) None

Q.11 In the previous question, if dv/dt = 0, then the angular acceleration of the ladder when $\alpha = 45°$ is

(A) $2v^2/L^2$ (B) $v^2/2L^2$

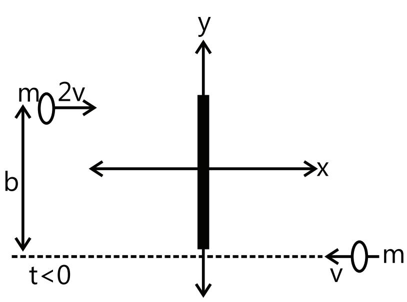
(C) $\sqrt{2}[v^2\ L^2]$ (D) None

Q.12 A uniform circular disc placed on a rough horizontal surface has initially a velocity v_0 and an angular velocity ω_0 as shown in the figure. The disc comes to rest after moving some distance in the direction of motion. Then $\frac{v_0}{r\omega_0}$ is

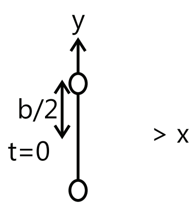


(A) 1/2 (B) 1 (C) 3/2 (D) 2

Q.13 An ice skater of mass m moves with speed 2v to the right, while another of the same mass m moves with speed v toward the left, as shown in figure I. Their paths are separated by a distance b. At t = 0, when they are both at x = 0, they grasp a pole of length



b and negligible mass. For r > 0 consider the system as a rigid body of two masses m separated by distance b, as shown in figure II. Which of the following is the correct formula for motion after t = 0 of the skater initially at y = b/2?



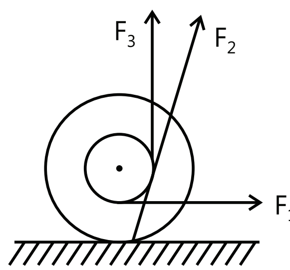
(A) $x = 2vt,\ y = b/2$

(B) $x = vt + 0.5\ b\sin(3vt/b),\ y = 0.5b\cos(3vt/b)$

(C) $x = 0.5c = vt + 0.5b\sin(3vt/b),\ y = 0.5b\cos(3vt/b)$

(D) $x = 0.5vt + 0.5b\sin(3vt/b),\ y = 0.5b\cos(3vt/b)$

Q.14 A yo-yo is resting on a perfectly rough horizontal table. Forces F_1, F_2 and F_3 are applied separately as shown. The correct statement is



(A) When F_3 is applied the center of mass will move to the right.

(B) When F_2 is applied the center of mass will move to the right.

(C) When F_1 is applied the center of mass will move to the right.

(D) When F_2 is applied the center of mass will move to the right.

Multiple Correct Choice Type

Q.15 A rod of weight w is supported by two parallel knife edges and B and is in equilibrium in a horizontal position. The knives are at a distance d from each other. The center of mass of the rod is at a distance x from A.

(A) The normal reaction at a is $\frac{wx}{d}$

(B) The normal reaction at a is $\frac{w(d-x)}{d}$

(C) The normal reaction at B is $\frac{wx}{d}$

(D) The normal reaction at B is $\frac{w(d-x)}{d}$

Q.16 A block with a square base measuring and height h, is placed on an inclined plane. The coefficient of friction is μ. The angle of inclination (α) of the plane is gradually increased. The block will

(A) Topple before sliding if $\mu > \frac{a}{h}$

(B) Topple before sliding if $\mu < \frac{a}{h}$

(C) Slide before toppling if $\mu > \frac{a}{h}$

(D) Slide before toppling if $\mu < \frac{a}{h}$

Q.17 A particle falls freely near the surface of the earth. Consider a fixed point O (not vertically below the particle) on the ground.

(A) Angular momentum of the particle about O is increasing.

(B) Torque of the gravitational force on the particle about O is decreasing.

(C) The moment of inertia of the particle about O is decreasing.

(D) The angular velocity of the particle about O is increasing.

Q.18 The torque τ on a body about a given point is found to be equal to a × L where a constant vector is and L is the angular momentum of the body about that point. From this it follows that

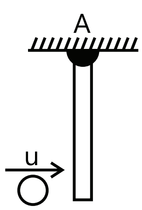
(A) dL/dt is perpendicular to L at all instants of time

(B) The components of l in the direction of a does not change with time.

(C) The magnitude of l does not change with time.

(D) L does not change with time.

Q.19 In the given figure. a ball strikes a uniform rod of same mass elastically and rod is hinged at point A. Then which of the statement (S) is /are correct?



(A) Linear momentum of system (ball + rod) is conserved.

(B) Angular momentum of system (ball + rod) about the hinged point A is conserved.

(C) Kinetic energy of system (ball + rod) before the collision is equal to kinetic energy of system just after the collision

(D) Linear momentum of ball is conserved.

Q.20 A hollow sphere of radius R and mass m is fully filled with non-viscous liquid of mass m. It is rolled down a horizontal plane such that its center of mass moves with a velocity v. If it purely rolls

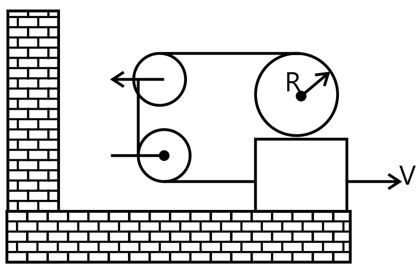
(A) Kinetic energy of the sphere is $\frac{5}{6}mv^2$

(B) Kinetic energy of the sphere is $\frac{4}{3}mv^2$

(C) Angular momentum of the sphere about a fixed point on ground is $\frac{8}{3}mvR$

(D) Angular momentum of the sphere about a fixed point on ground is $\frac{14}{5}mvR$

Q.21 In the figure shown, the plank is being pulled to the right with a constant speed v. If the cylinder does not slip then:



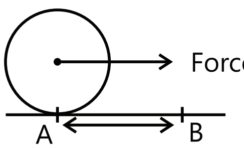
(A) The speed of the center of mass of the cylinder is 2v.

(B) The speed of the center of mass of the cylinder is zero.

(C) The angular velocity of the cylinder is v/R.

(D) The angular velocity of the cylinder is zero.

Q.22 A disc of circumference s is at rest at a point A on a horizontal surface when a constant horizontal force begins to act on its center. Between A and B there is sufficient friction to prevent slipping and the surface is smooth to the right of B. AB = s. The disc moves from A to B in time T. To the right of B,



(A) The angular acceleration of disc will disappear, linear acceleration will remain unchanged

(B) Linear acceleration of the disc will increase

(C) The disc will make one rotation in time T/2.

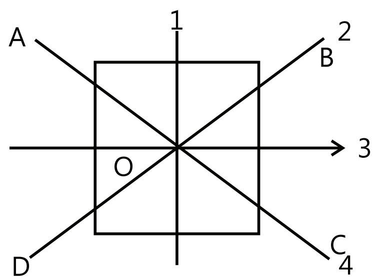
(D) The disc will cover a distance greater then s in further time T.

Q.23 A rigid object is rotating in a counterclockwise sense around a fixed axis. If the rigid object rotates though more than180° but less than 360°, which of the following pairs of quantities can represent an initial angular position and a final angular position of the rigid object.

(A) 3 rad, 6 rad (B) –1 rad, 1 rad

(C) 1 rad, 5 rad (D –1rad, 2.5 rad

Q.24 ABCD is a square plate with center O. The moments of inertia of the plate about the perpendicular axis through O is I and about the axes 1, 2, 3 & 4 are I_1, I_2, I_3, & I_4 respectively. If follows that:



(A) $I_2 = I_3$ (B) $I = I_1 + I_4$

(C) $I = I_2 + I_4$ (D) $I_1 = I_3$

Q.25 A body is in equilibrium under the influence of a number of forces. Each force has a different line of action. The minimum number of forces required is

(A) 2, if their lines of action pass through the center of mass of the body.

(B) 3, if their lines of action are not parallel.

(C) 3, if their lines of action are parallel.

(D) 4, if their lines of action are parallel and all the forces have the same magnitude.

Q.26 A block of mass m moves on a horizontal rough surface with initial velocity v. The height of the center of mass of the block is h from the surface. Consider a point a on the surface in line with the center of mass.

(A) Angular momentum about a is mvh initially

(B) The velocity of the block decreases as time passes.

(C) Torque of the forces acting on block is zero about a.

(D) Angular momentum is not conserved about A.

Q.27 A man spinning in free space changes the shape of his body, eg. By spreading his arms or curling up. By doing this, he can change his

(A) Moment of inertia

(B) Angular momentum

(C) Angular velocity

(D) Rotational kinetic energy

Q.28 A ring rolls without slipping on the ground. Its center C moves with a constant speed u. P is any point on the ring. The speed of P with respect to the ground is v.

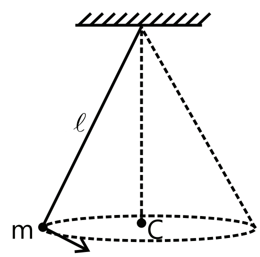
(A) $0 \leq v \leq 2u$

(B) v = u, if CP is horizontal

(C) v = u is CP makes an angle of 30° with the horizontal and P is below the horizontal level of c.

(D) v = $\sqrt{2}$u, if CP is horizontal

Q.29 A small ball of mass m suspended from the ceiling at a point O by a thread of length ℓ moves along a horizontal circle with a constant angular velocity ω.



(A) Angular momentum is constant about O

(B) Angular momentum is constant about C

(C) Vertical component of angular momentum about O is constant

(D) Magnitude of angular momentum about O is constant.

Q. 30 If a cylinder is rolling down the incline with sliding.

(A) After some time it may start pure rolling

(B) After sometime it will start pure rolling

(C) It may be possible that it will never start pure rolling

(D) None of these.

Q.31 Which of the following statements are correct.

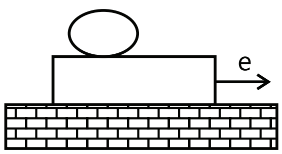
(A) Friction acting on a cylinder without sliding on an inclined surface is always upward along the incline irrespective of any external force acting on it

(B) Friction acting on a cylinder without sliding on an inclined surface may be upward may be downwards depending on the external force acting on it

(C) Friction acting on a cylinder rolling without sliding may be zero depending on the external force acting on it

(D) Nothing can be said exactly about it as it depends on the frictional coefficient on inclined plane.

Q. 32 A plank with a uniform sphere placed on it rests on a smooth horizontal plane. Plank is pulled to right by a constant force F. If sphere does not slip over the plank. Which of the following is correct?



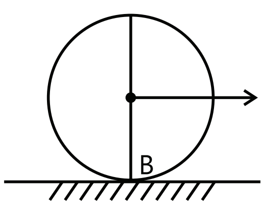
(A) Acceleration of the center of sphere is less than that of the plank

(B) Work done by friction acting on the sphere is equal to its total kinetic energy

(C) Total kinetic energy of the system is equal to work done by the force F

(D) None of the above.

Q. 33 a uniform disc is rolling on a horizontal surface. At a certain instant B is the point of contact and A is at height 2R from ground, where R is radius of disc.



(A) The magnitude of the angular momentum of the disc about B is thrice that about A

(B) The angular momentum of the disc about A is anticlockwise

(C) The angular momentum of the disc about B is clockwise

(D) The angular momentum of the disc about A is equal to that about B.

Q. 34 A wheel of radius r is rolling on a straight line, the velocity of its center being v. At a certain instant the point of contact of the wheel with the grounds is M and N is the highest point on the wheel (diametrically opposite to M). The incorrect statement is:

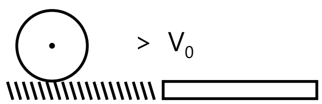
(A) The velocity of any point P of the wheel is proportional to MP

(B) Points of the wheel moving with velocity greater than v form a larger area of the wheel than points moving with velocity less than v

(C) The point of contact M is instantaneously at rest

(D) The velocities of any two parts of the wheel which are equidistant from center are equal.

Q.35 A ring of mass M and radius R sliding with a velocity v_0 suddenly enters into a rough surface where the coefficient of friction is μ, as shown in figure.



Choose the correct statement(s)

(A) As the ring enters on the rough surface, the limiting frictional force acts on it

(B) The direction of friction is opposite to the direction of motion.

(C) The frictional force accelerates the ring in the clockwise sense about its center of mass

(D) As the ring enters on the rough surface it starts rolling.

Q.36 Choose the correct statement (s)

(A) The ring starts its rolling motion when the center of mass is stationary

(B) The ring starts rolling motion when the point of contact becomes stationary

(C) The time after which the ring starts rolling is $\frac{v_0}{2\mu g}$

(D) The rolling velocity is $\frac{v_0}{2}$

Q.37 Choose the correct alternative (s)

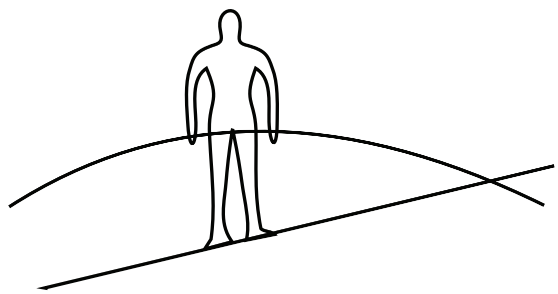
(A) The linear distance moved by the center of mass before the ring starts rolling is $\frac{3v_0^2}{8\mu g}$

(B) The net work done by friction force is $-\frac{3}{8}\ mv_0^2$

(C) The loss is kinetic energy of the ring is $\frac{mv_0^2}{4}$

(D) The gain in rotational kinetic energy is $+\frac{mv_0^2}{8}$

Q.38 A tightrope walker in a circus holds a long flexible pole to help stay balanced on the rope. Holding the pole horizontally and perpendicular to the rope helps the performer.



(A) By lowering the overall center-of- gravity

(B) By increasing the rotation inertia

(C) In the ability to adjust the center- of -gravity to be over the rope.

(D) In achieving the center of gravity to be under the rope.

Assertion Reasoning Type

(A) Statement-I is true, statement-II is true and statement-II is correct explanation for statement-I.

(B) Statement-I is true, statement-II is true and statement-II is NOT the correct explanation for statement-I.

(C) Statement-I is true, statement-II is false.

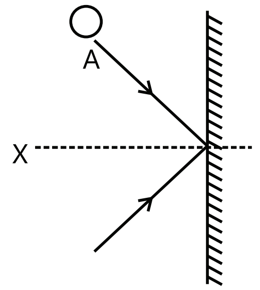
(D) Statement-I is false, statement-II is true.

Q.39 Consider the following statements

Statement-I: a cyclist always bends inwards while negotiating a curve

Statement-II: By bending he lowers his center of gravity of these statements.

Q.40 Statement-I A disc A moves on a smooth horizontal plane and rebounds elastically from a smooth vertical wall (Top view is shown in Fig 7.166), in this case about any point on line XY the angular momentum of the disc remains conserved.



Statement-II: About any point in the plane, the torque of gravity force and normal contact force by ground balance each other

Q.41 Statement-I: The angular velocity of all the points on the laminar rigid body lying in the plane of a body as seen from any other point on it is the same.

Statement-II: The distance between any 2 points on the rigid body remains constant.

Q.42 Consider the following statements:-

Statement-I: The moment of inertia of a rigid body reduces to its minimum value as compared to any other parallel axis when the axis of rotation passes through its center of mass.

Statement-II: The weight of a rigid body always acts through its center of mass in uniform gravitational field.

Q.43 Statement-I: The moment of inertia of any rigid body is minimum about axis which passes through its center of mass as compared to any other parallel axis.

Statement-II: The entire mass of a body can be assumed to be concentrated at its center of mass for applying Newton's force Law.

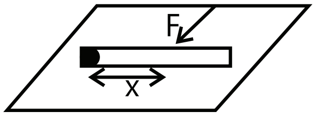
Q.44 A uniform thin rod of length L is hinged about one of its ends and is free to rotate about the hinge without friction, Neglect the effect of gravity. A force F is applied at a distance x from the hinge on the rod such that force is always perpendicular to rod. As the value of x is increased from zero to L,

Statement-I: The component of reaction force by hinge on the rod perpendicular to length of rod increases.

Statement-II: The angular acceleration of rod increases.

Q.45 Statement-I: For a round shape body of radius R rolling on a fixed ground, the magnitude of velocity of its center is given by ωR, where ω is its angular speed.

Statement-II: When distribution of mass is symmetrical then center of round shape body is its center of mass.



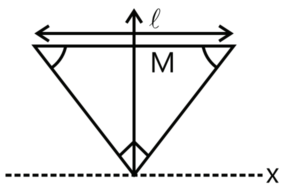
Q.46 Statement-I: a body cannot roll on a smooth horizontal Surface.

Statement-II: when a body rolls purely, the point of contact should be at rest with respect to surface.

Comprehension Type

Paragraph 1:

The figure shows an isosceles triangular plate of mass M and base L. The angle at the apex is 90°. The apex lies at the origin and base is parallel to X – axis



Q.47 The moment of inertia of the plate about the z – axis is

(A) $\frac{ML^2}{12}$ (B) $\frac{ML^2}{24}$ (C) $\frac{ML^2}{6}$ (D) None of these

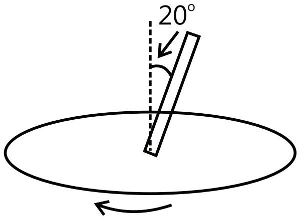
Q.48 The moment of inertia of the plate about the x axis is

(A) $\frac{ML^2}{8}$ (B) $\frac{ML^2}{32}$ (C) $\frac{ML^2}{24}$ (D) $\frac{ML^2}{6}$

Q.49 The moment of inertia of the plate about its base parallel to the x – axis is

(A) $\frac{ML^2}{18}$ (B) $\frac{ML^2}{36}$ (C) $\frac{ML^2}{24}$ (D) None of these

Q.50 The moment of inertia of the plate about the y – axis is



(A) $\frac{ML^2}{6}$ (B) $\frac{ML^2}{8}$ (C) $\frac{ML^2}{24}$ (D) None of these

Paragraph 2:

A uniform rod is fixed to a rotating turntable so that its lower end is on the axis of the turntable and it makes an angle of N20°to the vertical. (The rod is thus rotating with uniform angular velocity about a vertical axis passing through one end.) If the turntable is rotating clockwise as seen from above.

Q.51 What is the direction of the rod's angular momentum vector (calculated about its lower end)?

(A) Vertically downwards

(B) Down at 20°to the horizontal

(C) Up at 20° to the horizontal

(D) Vertically upwards

Q.52 Is there torque acting on it, and if so in what direction?

(A) Yes, vertically

(B) Yes, horizontal

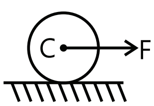
(C) Yes at 20°to the horizontal

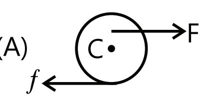
(D) No

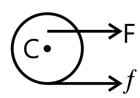
Paragraph 3:

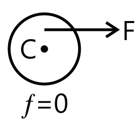
In the following problems, indicate the correct direction of friction force acting on the cylinder, which is pulled on a rough surface by a constant force F.

Q.53 A cylinder of mass M and radius R is pulled horizontal by a force F. The frictional force can be given by which of the following diagrams



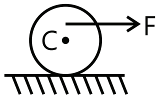
(A) 

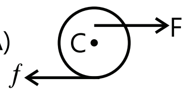
(B) 

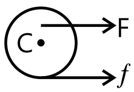
(C) 

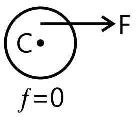
(D) Cannot be interpret

Q.54 A cylinder is pulled horizontally by a force F acting at a point below the center of mass of the cylinder, as shown in figure. The frictional force can be given by which of the following diagrams?



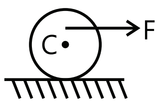
(A) 

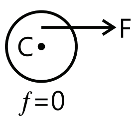
(B) 

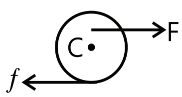
(C) 

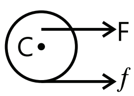
(D) Cannot be interpret

Q.55 A cylinder is pulled horizontally by a force F acting at a point above the center of mass of the cylinder, as shown in figure. The frictional force can be given by which of the following diagrams



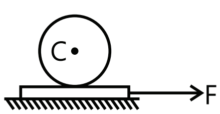
(A) 

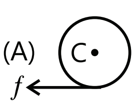
(B) 

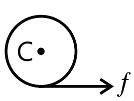
(C) 

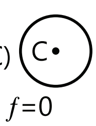
(D) Cannot be interpret

Q.56 A cylinder is placed on a rough plank which in turn is placed on a smooth surface. The plank is pulled with a constant force F. The frictional force can be given by which of the following diagrams.



(A) 

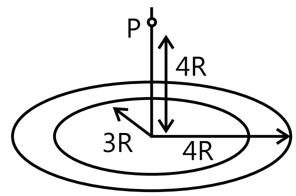
(B) 

(C) 

(D) Cannot be interpreted.

Previous Years' Questions

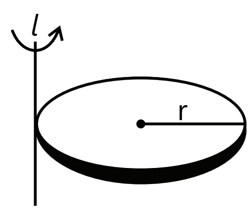
Q.1 A thin uniform angular disc (See figure) of mass M has outer radius 4R and inner radius 3R. The work required to take a unit mass from point P on its axis to infinity is ***(2010)***



(A) $\frac{2GM}{7R}(4\sqrt{2}-5)$ (B) $-\frac{2GM}{7R}(4\sqrt{2}-5)$

(C) $\frac{GM}{4R}$ (D) $\frac{2GM}{5R}(\sqrt{2}-1)$

Q.2 A solid sphere of radius R has moment of inertia I about its geometrical axis. If it's moment of inertia about the tangential axis (which is perpendicular to plane of the disc), is also equal to I, then the value of r is equal to ***(2006)***



(A) $\frac{2}{\sqrt{15}}R$ (B) $\frac{2}{\sqrt{5}}R$ (C) $\frac{3}{\sqrt{15}}R$ (D) $\frac{\sqrt{3}}{\sqrt{15}}R$

Q.3 A block of base 10 cm × 10 cm and height 15 cm is kept on an inclined plane. The coefficient of friction between them is $\sqrt{3}$. The inclination θ of this inclined plane from the horizontal plane is gradually increased from 0°. Then, ***(2009)***

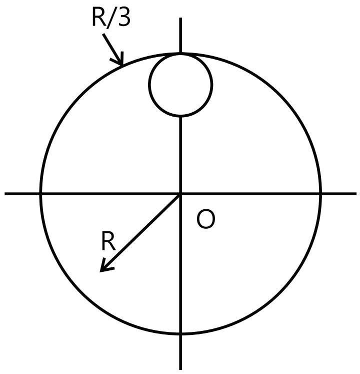
(A) At θ = 30°, the block will start sliding down the plane

(B) The block will remains at rest on the plane up to certain θ and then it will topple

(C) At θ = 60°, the block will start sliding down the plane and continue to do so at higher angles

(D) At θ = 60°, the block will start sliding down the plane and on further increasing θ, it will topple at certain θ.

Q.4 From a circular disc of radius R and mass 9M, a small disc of radius R/3 is removed from the disc. The moment of inertia of the remaining disc about axis perpendicular to the plane of the disc and passing through O is ***(2010)***



(A) 4 MR^2 (B) $\frac{40}{9}MR^2$ (C) $10MR^2$ (D) $\frac{37}{9}MR^2$

Q.5 Let I be moment of inertia of a uniform square plate about an axis AB that passes through its center and is parallel to two of its sides. CD is a line in the plane and makes an angle θ with AB. The moment of inertia of the plate about the axis CD is then equal to ***(1998)***

(A) I (B) $I \sin^2 \theta$

(C) $I \cos^2 \theta$ (D) $I \cos^2 (\theta/2)$

Q.6 A solid sphere is in pure rolling motion on an inclined surface having inclination θ ***(2006)***

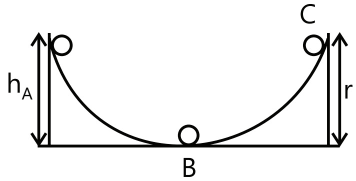
(A) Frictional force acting on sphere is $f = \mu$

(B) f Is dissipative force

(C) Friction will increase its angular velocity and decrease its linear velocity

(D) If θ decreases, friction will decrease

Q.7 A ball moves over a fixed track as shown in figure. From A to B the ball rolls without slipping. If surface BC is frictionless and K_A, K_B and K_C are kinetic energies of the ball at A, B and C respectively, then ***(2006)***



(A) $h_A > h_C$; $K_B > K_C$

(B) $h_A > h_C$; $K_C > K_A$

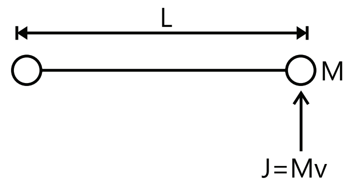
(C) $h_A = h_C$; $K_B = K_C$

(D) $h_A < h_C$; $K_B > K_C$

Q.8 A child is standing with folded hands at the center of platform rotating about its central axis. The kinetic energy of the system is K. The child now stretches his arms so that the moment of inertia of the system doubles. The kinetic energy of system now is ***(2004)***

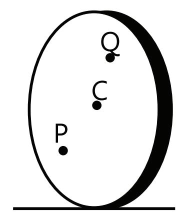
(A) 2K (B) $\frac{K}{2}$ (C) $\frac{K}{4}$ (D) 4 K

Q.9 Consider a body, shown in figure, consisting of two identical balls, each of mass M connected by a light rigid rod. If an impulse J = Mv is imparted to the body at one of its end, what would be its angular velocity? ***(2003)***



(A) v / L (B) $2v / L$ (C) $v / 3L$ (D) $v / 4L$

Q.10 A disc is rolling (without slipping) on a horizontal surface. C is its center and Q and P are two points equidistant from C. Let v_P, v_Q and v_C be the magnitude of velocity of points P, Q, and C respectively, then ***(2004)***



(A) $v_Q > v_C > v_P$

(B) $v_Q < v_C < v_P$

(C) $v_Q = v_P, v_C = \frac{1}{2}v_P$

(D) $v_Q < v_C > v_P$

Paragraph 1: Two discs A and B are mounted coaxially on a vertical axle. The discs have moments of inertia I and 2I respectively about the common axis. Disc A is imparted an initial angular velocity 2ω using the entire potential energy of a spring compressed by a distance x_1. Disc B is imparted an angular velocity ω by a spring having the same spring constant and compressed by a distance x_2. Both the discs rotate in the clockwise direction. ***(2007)***

Q.11 The ratio $\frac{x_1}{x_2}$ is

(A) 2 (B) $\frac{1}{2}$ (C) $\sqrt{2}$ (D) $\frac{1}{\sqrt{2}}$

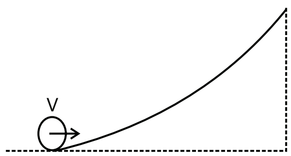
Q.12 When disc B is brought in contact with disc A, they acquire a common angular velocity in time t. The average frictional torque on one disc by the other during this period is.

(A) $\frac{2I\omega}{3t}$ (B) $\frac{9I\omega}{2t}$ (C) $\frac{9I\omega}{4t}$ (D) $\frac{3I\omega}{2t}$

Q.13 The loss of kinetic energy during the above process is

(A) $\frac{I\omega^2}{2}$ (B) $\frac{I\omega^2}{3}$ (C) $\frac{I\omega^2}{4}$ (D) $\frac{I\omega^2}{6}$

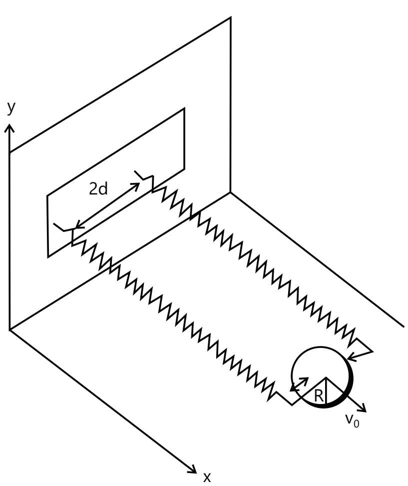
Q.14 A small object of uniform density rolls up a curved surface with an initial velocity v. If reaches up to a maximum height of $3v^2/4g$ with respect to the initial position. The object is



(A) Ring (B) Solid sphere

(C) Hollow sphere (D) Disc

Paragraph 2: A uniform thin cylindrical disk of mass M and radius R is attached to two identical mass less springs of spring constant k which are fixed to the wall as shown in the figure. The springs are attached to the disk diametrically on either side at a distance d from its center. The axle is mass less and both the springs and the axle are in a horizontal plane. The un-stretched length of each spring is L. The disk is initially at its equilibrium position with its center of mass (CM) at a distance L from the wall. The disk rolls without slipping with velocity $\vec{v}_0 = v_0\,\hat{i}$. The coefficient of friction is μ. ***(2008)***



Q.15 The net external force acting on the disk when its center of mass is at displacement x with respect to its equilibrium position is

(A) –kx (B) –2kx (C) $-\frac{2kx}{3}$ (D) $-\frac{4kx}{3}$

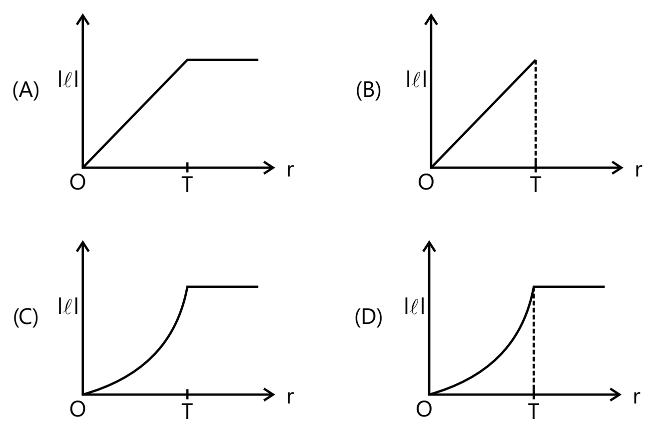
Q.16 The center of mass of the disk undergoes simple harmonic motion with angular frequency ω equal to

(A) $\sqrt{\frac{k}{M}}$ (B) $\sqrt{\frac{2k}{M}}$ (C) $\sqrt{\frac{2k}{3M}}$ (D) $\sqrt{\frac{4k}{3M}}$

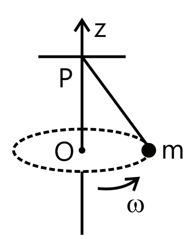
Q.17 The maximum value of v_0 for which the disk will roll without slipping is

(A) $\mu g\sqrt{\frac{M}{k}}$ (B) $\mu g\sqrt{\frac{M}{2k}}$ (C) $\mu g\sqrt{\frac{3M}{k}}$ (D) $\mu g\sqrt{\frac{5M}{2k}}$

Q.18 A thin uniform rod, pivoted at O, is rotating in the horizontal plane with constant angular speed ω, as shown in the figure. At time t = 0, a small insect starts from O and moves with constant speed v, with respect to the rod towards the other end. It reaches the end of the rod at t = T and stops. The angular speed of the system remains ω throughout. The magnitude of the torque $\left(|\vec{\tau}|\right)$ about O, as a function of time is best represented by which plot? ***(2012)***



Q.19 A small mass m is attached to a massless string whose other end is fixed at P as shown in the figure. The mass is undergoing circular motion in the x-y planewith centre at O and constant angular speed ω. If the angular momentum of the system, calculated about O and P are denoted by $\vec{L}_O$ and $\vec{L}_P$ respectively, then ***(2012)***



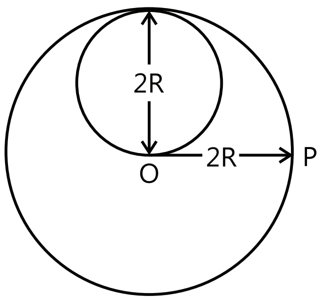
(A) $\vec{L}_O$ and $\vec{L}_P$ do not vary with time.

(B) $\vec{L}_O$ varies with time while $\vec{L}_P$ remains constant.

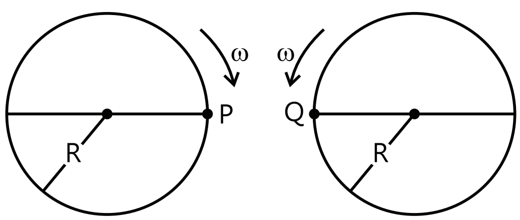
(C) $\vec{L}_O$ remains constant while $\vec{L}_P$ varies with time.

(D) $\vec{L}_O$ and $\vec{L}_P$ both vary with time.

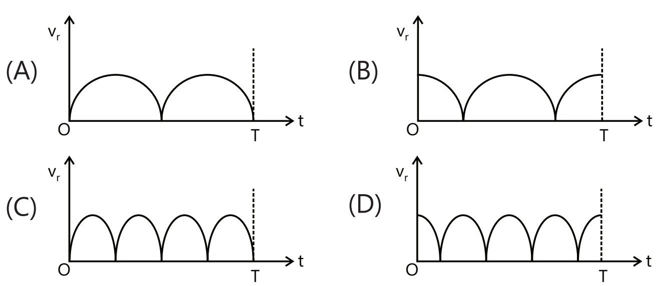
Q.20 A lamina is made by removing a small disc of diameter 2R from a bigger disc of uniform mass density and radius 2R, as shown in the figure. The moment of inertia of this lamina about axes passing though O and P is I_0 and I_P respectively. Both these axes are perpendicular to the plane of the lamina. The ratio I_P / I_O to the the nearest integer is ***(2012)***



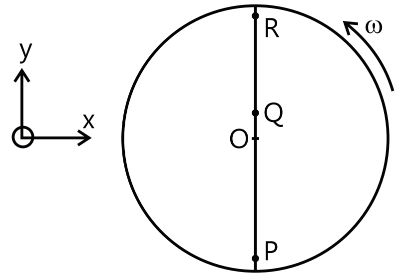
Q.21 Two identical discs of same radius R are rotating abouttheir axes in opposite directions with the same constant angular speed ω. The discs are in the same horizontal plane. At time t = 0, the points P and



Q are facing each other as shown in the figure. The relative speed between the two points P and Q is v_r In one time period (T) of rotation of the discs, v_r as a function of time is best represented by - ***(2012)***



Q.22 Consider a disc rotating in the horizontal plane with a constant angular speed ω about its centre O. The disc has a shaded region on one side of the diameter and an unshaded region on the other side as shown in the figure. When the disc is in the orientation as shown, two pebbles P and Q are simultaneously projected at an angle towards R. The velocity of projection is in the y-z plane and is same for both pebbles with respect to the disc. Assume that (i) they land back on the disc before the disc has completed $\frac{1}{8}$ rotation (ii) their range is less than half the disc radius and (iii) ω remains constant throughout. Then ***(2012)***



(A) P lands in the shaded region and Q in the unshaded region.

(B) P lands in the unshaded region and Q in the shaded region.

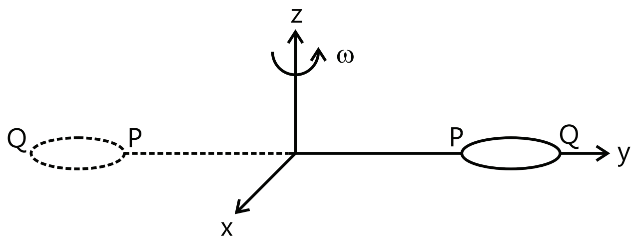
(C) Both P and Q land in the unshaded region.

(D) Both P and Q land in the shaded region.

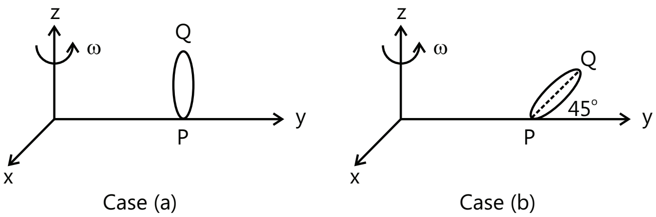
Paragraph for Questions 23 and 24

The general motion of a rigid body can be considered to be a combination of (i) a motion of its centre of mass about an axis, and (ii) its motion about an instantaneous axis passing through the centre of mass. These axes need not be stationary. Consider, for example, a thin uniform disc welded (rigidly fixed) horizontally at its rim to a massless stick, as shown in the figure. When the disc-stick system is rotated about the origin on a horizontal frictionless plane with angular speed ω, the motion at any instant can be taken as a combination of

(i) a rotation of the centre of mass of the disc about the z-axis, and (ii) a rotation of the disc through an instantaneous vertical axis passing through its centre of mass (as is seen from the changed orientation of points P and Q). Both these motions have the same angular speed ω in this chase.



Now consider two similar systems as shown in the figure: Case (a) the disc with its face vertical and parallel to x-y plane; Case (b) the disc with its face making an angle of $45°$ with x-y plane and its horizontal diameter parallel to x-axis. In both the cases, the disc is welded at point P, and the systems are rotated with constant angular speed ω about the z-axis.



Q.23 Which of the following statements about the instantaneous axis (passing through the centre of mass) is correct? ***(2012)***

(A) It is vertical for both the cases (a) and (b) (B) It is vertical for case (a); and is at $45°$ to the x-z plane and lies in the plane of the disc for case (b)

(C) It is horizontal for case (a); and is at $45°$ to the x-z plane and is normal to the plane of the disc for case (b)

(D) It is vertical for case (a); and is at $45°$ to the x-z plane and is normal to the plane of the disc for case (b)

Q.24 Which of the following statements regarding the angular speed about the instantaneous axis (passing through the centre of mass) is correct? ***(2012)***

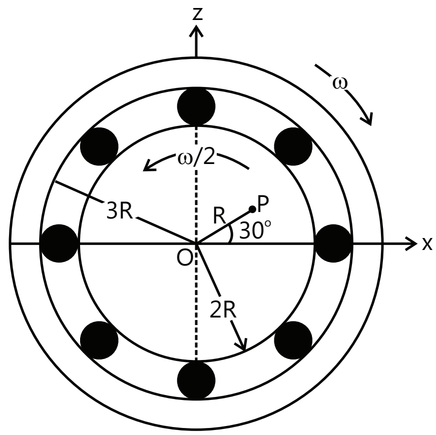
(A) It is $\sqrt{2}\omega$ for both the cases

(B) It is ω for case (a); and $\frac{\omega}{\sqrt{2}}$ for case (b)

(C) It is ω for case (a); and $\sqrt{2}\omega$ for case (b)

(D) It is ω for both the cases

Q.25 The figure shows a system consisting of (i) a ring of outer radius 3R rolling clockwise without slipping on a horizontal surface with angular speed ω and (ii) an inner disc of radius 2R rotating anti-clockwise with angular speed $\omega/2$. The ring and disc are separated by frictionless ball bearings. The system is in the x-z plane. The point P on the inner disc is at a distance R from the origin, where OP makes an angle of $30°$ with the horizontal. Then with respect to the horizontal surface, ***(2012)***



(A) The point O has a linear velocity $3R\omega\hat{i}$

(B) The point P has a linear velocity $\frac{11}{4}R\omega\hat{i}+\frac{\sqrt{3}}{4}R\omega\hat{k}$

(C) The point P has a linear velocity $\frac{13}{4}R\omega\hat{i}-\frac{\sqrt{3}}{4}R\omega\hat{k}$

(D) The point P has a linear velocity $\left(3-\frac{\sqrt{3}}{4}\right)R\omega\hat{i}+\frac{1}{4}R\omega\hat{k}$

Q.26 Two solid cylinders P and Q of same mass and same radius start rolling down a flixed inclined plane from the same height at the same time. Cylinder P has most of its mass concentrated near its surface, while Q has most of its mass concentrated near the axis. Which statement(s) is (are) correct? ***(2012)***

(A) Both cylinders P and Q reach the ground at the same time.

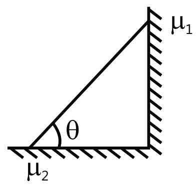
(B) Cylinder P has larger linear acceleration than cylinder Q.

(C) Both cylinders reach the ground with same translational kinetic energy

(D) Cylinder Q reaches the ground with larger angular speed

Q.27 A uniform circular disc of mass 50 kg and radius 0.4 m is rotating with an angular velocity of 10 rad s^{-1} about its own axis, which is vertical. Two uniform circular rings, each of mass 6.25 kg and radius 0.2 m, are gently placed symmetrically on the disc in such a manner that they are touching each other along the axis of the disc and are horizontal. Assume that the friction is large enough such that the rings are at rest relative to the disc and the system rotates about the original axis. The new angular velocity (in rad s^{-1}) of the system is ***(2013)***

Q.28 In the figure, a ladder of mass m is shown leaning against a wall. It is in static equilibrium making an angle θ with the horizontal floor. The coefficient of friction between the wall and the ladder is μ_1 and that between the floor and the ladder is μ_2. The normal reaction of the wall on the ladder is N_1 and that of the floor is N_2. If the ladder is about to slip, then ***(2014)***



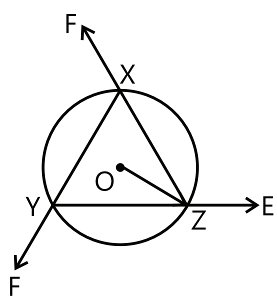
(A) $\mu_1 = 0\ \mu_2 \neq 0$ and $N_2 \tan\theta = \dfrac{mg}{2}$

(B) $\mu_1 \neq 0\ \mu_2 = 0$ and $N_1 \tan\theta = \dfrac{mg}{2}$

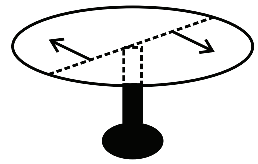
(C) $\mu_1 \neq 0\ \mu_2 \neq 0$ and $N_2 = \dfrac{mg}{1+\mu_1\mu_2}$

(D) $\mu_1 = 0\ \mu_2 \neq 0$ and $N_1 \tan\theta = \dfrac{mg}{2}$

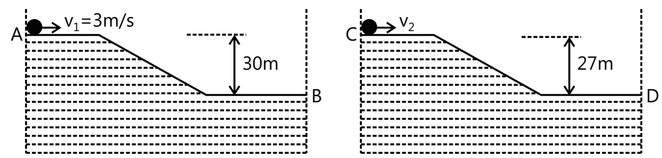
Q.29 A uniform circular disc of mass 1.5 kg and radius 0.5 m is initially at rest on a horizontal frictionless surface. Three forces of equal magnitude F = 0.5 N are applied simultaneously along the three sides of an equilateral triangle XYZ with its vertices on the perimeter of the disc (see figure). One second after applying the forces, the angular speed of the disc in rad s^{-1} is ***(2014)***



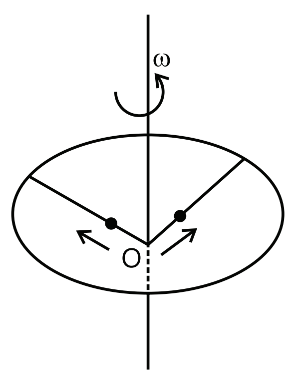
Q.30 A horizontal circular platform of radius 0.5 m and mass 0.45 kg is free to rotate about its axis. Two massless spring toy -guns, each carrying a steel ball of mass 0.05 kg are attached to the platform at a distance 0.25 m from the centre on its either sides along its diameter (see figure).Each gun simultaneously fires the balls horizontally and perpendicular to the diameter in opposite directions. After leaving the platform, the balls have horizontal speed of $9ms^{-1}$ with with respect to the ground. The rotational speed of the platform in rad s^{-1} after the balls leave the platform is ***(2014)***



Q.31 Two identical uniform discs roll without slipping on two different surfaces AB and CD (see figure) starting at A and C with linear speeds v_1 and v_2, respectively ,and always remain in contact with the surfaces. If they reach B and D with the same linear speed and $v_1 = 3\ m/s$, then v_2 in m/s is (g = 10 m/s^2) ***(2015)***



Q.32 A ring of mass M and radius R is rotating with angular speed ω about a fixed vertical axis passing through its centre O with two point masses each of mass $\dfrac{M}{8}$ at rest at O. These masses can move radially outwards along two massless rods fixed on the ring as shown in the figure. At some instant the angular speed of the system is $\dfrac{8}{9}\omega$ and one of the masses is at a distance of $\dfrac{3}{5}R$ from O. At this instant the distance of the other mass from O is ***(2015)***



(A) $\dfrac{2}{3}R$ (B) $\dfrac{1}{3}R$ (C) $\dfrac{3}{5}R$ (D) $\dfrac{4}{5}R$

Q.33 The densities of two solid spheres A and B of the same radii R vary with radial distance r as $\rho_A(r) = k\left(\dfrac{r}{R}\right)$ and $\rho_B(r) = k\left(\dfrac{r}{R}\right)^5$, respectively, where k is a constant. The moments of inertia of the individual spheres about axes passing through their centres are I_A and I_B, respectively. If $\frac{I_B}{I_A} = \frac{n}{10}$, the value of n is ***(2015)***

Q.34 A uniform wooden stick of mass 1.6 kg and length ℓ rests in an inclined manner on a smooth, vertical wall of height $h\left(<\ell\right)$ such that a small portion of the stick extends beyond the wall. The reaction force of the wall on the stick is perpendicular to the stick. The stick makes an angle of $30°$ with the wall and the bottom of the stick is on a rough floor. The reaction of the wall on the stick is equal in magnitude to the reaction of the floor on the stick. The ratio h/ℓ and the frictional force fat the bottom of the stick are $\left(g = 10\ ms^{-2}\right)$ ***(2016)***

(A) $\frac{h}{\ell} = \frac{\sqrt{3}}{16}, f = \frac{16\sqrt{3}}{3}$ N

(B) $\frac{h}{\ell} = \frac{3}{16}, f = \frac{16\sqrt{3}}{3}$ N

(C) $2\frac{h}{\ell} = \frac{3\sqrt{3}}{16}, f = \frac{8\sqrt{3}}{3}$ N

(D) $\frac{h}{\ell} = \frac{3\sqrt{3}}{16}, f = \frac{16\sqrt{3}}{3}$ N

Q.35 The position vector $\vec{r}$ of a particle of mass m is given by the following equation $\vec{r}\left(t\right) = \alpha t^3\,\hat{i} + \beta t^2\,\hat{j}$, where $\alpha = 10/3\ ms^{-3}$, $\beta = 5\ ms^{-2}$ and $m = 0.1$ kg. At t = 1 s, which of the following statement(s) is(are) true about the particle? ***(2016)***

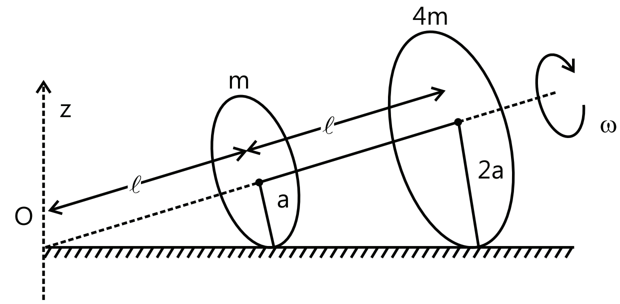
(A) The velocity $\vec{v}$ is given $\vec{v} = \left(10\,\hat{i} + 10\hat{j}\right)\ ms^{-1}$

(B) The angular momentum $\vec{L}$ with respect to the origin is given by $\vec{L} = -\left(5/3\right)\hat{k}$ Nms

(C) The force $\vec{F}$ is given by $\vec{F} = \left(\hat{i} + 2\,\hat{j}\right)$ N

(D) The torque $\vec{\tau} = -\left(20/3\right)\hat{k}$ N m

Q.36 Two thin circular discs of mass m and 4m, having radii of a and 2a, respectively, are rigidly fixed by a massless, rigid rod of length $\ell = \sqrt{24}\,a$ through their centers. This assembly is laid on a firm and flat surface, and set rolling without slipping on the surface so that the angular speed about the axis of the rod is ω. The angular momentum of the entire assembly about the point 'O' is $\vec{L}$ (see the figure). Which of the following statement(s) is(are) true? ***(2016)***



(A) The magnitude of angular momentum of the assembly about its center of mass is $17\ ma^2\ \omega/2$

(B) The magnitude of the z-component of $\vec{L}$ is $55\ ma^2\ \omega$

(C) The magnitude of angular momentum of center of mass of the assembly about the point O is $81\ ma^2\ \omega$

(D) The center of mass of the assembly rotates about the z-axis with an angular speed of $\omega/5$

PlancEssential Questions

JEE Main/Boards

Exercise 1

Q.19 Q.22 Q.23
Q.27 Q.28

Exercise 2

Q.12 Q.25

Previous Years' Questions

Q.4 Q.7 Q.9
Q.12

JEE Advanced/Boards

Exercise 1

Q.5 Q.7 Q.8
Q.10 Q.11 Q.21
Q.24

Exercise 2

Q.7 Q.9 Q.21
Q.22 Q.28 Q.41
Q.54

Previous Years' Questions

Q.1 Q.3

Answer Key

JEE Main/Boards

Exercise 1

Q.1 kg m^2, $[M^1L^2T^0]$, No

Q.2 Torque

Q.3 Inertia

Q.4 Theorem of parallel axes and theorem of perpendicular axes

Q.5 $I = \frac{2}{5}MR^2$, M = mass & R = radius

Q.6 $I = \frac{2}{3}MR^2$, M = mass & R = radius

Q.8 No

Q.9 Hollow sphere

Q.10 $I_B > I_A$

Q.11 $K_B > K_A$

Q.18 8 g

Q.19 1.584×10^7 g cm^2

Q.20 1.25 kg m^2;

Q.21 9.83×10^{37} kg m^2

Q.22 3.5×10^4 g cm^2 ; 1.75×10^4 g cm^2 ; 8.75×10^4 g cm^2 ; 10.5×10^4 g cm^2

Q.23 $\frac{5}{4}mL^2$

Q.24 0.01J

Q.26 108 days

Q.27 80π kg m^2 s^{-1}

Q.28 360 r.p.m

Exercise 2

Single Correct Choice Type

Q.1 B	**Q.2** B	**Q.3** A	**Q.4** B	**Q.5** D	**Q.6** D
Q.7 D	**Q.8** B	**Q.9** B	**Q.10** C	**Q.11** A	**Q.12** D
Q.13 B	**Q.14** B	**Q.15** C	**Q.16** B	**Q.17** B	**Q.18** B
Q.19 B	**Q.20** A	**Q.21** D	**Q.22** C	**Q.23** C	**Q.24** B
Q.25 D	**Q.26** B	**Q.27** C			

Previous Years' Questions

Q.1. A	**Q.2** C	**Q.3** B	**Q.4** A	**Q.5** A	**Q.6** B
Q.7 C	**Q.8** C	**Q.9** C	**Q.10** B	**Q.11** A	**Q.12** C
Q.13 C	**Q.14** A	**Q.15** D	**Q.16** D	**Q.17** B	**Q.18** A
Q.19 C	**Q.20** B	**Q.21** A, C	**Q.22** D		

JEE Advanced/Boards

Exercise 1

Q.1 2Mg

Q.2 $5\sqrt{5}$ m/s

Q.3 3

Q.4 2F/M

Q.5 2m/s

Q.6 $\vec{F}_A = (-133.64\,\hat{i} + 196\,\hat{j})$N and $\vec{F}_B = 133.64\,\hat{i}$

Q.7 (a) $\frac{3a\rho_0}{2}$; (b) $\frac{5a}{9}$; (c) $\frac{7\rho_0 a^3}{12}$; (d) $\frac{12}{7\rho_0 a^2}$;

(e) $\sqrt{\frac{7}{4}\rho_0^2 g a^3}$

Q.8 300 rad/sec, 150 rad/sec

Q.9 16 m/s^2

Q.10 $v = \sqrt{\frac{27}{7}gR}$

Q.11 (a) $\frac{200}{7}$ N; (b) $4\sqrt{\frac{3}{7}}$ m/s

Q.12 $\frac{3L^2\rho}{8\pi^2}$

Q.13 1.65 N, 1.224 m

Q.14 L/6

Q.15 (i) (a) $\frac{1.2g}{c}$ (clockwise) (b) $-0.3g(\hat{i} + 2\hat{j})$

(ii) (a) $\frac{2.4g}{c}$(clockwise) (b) 0.5g

Q.16 $V = \frac{\sqrt{14gR}}{3}$

Q.17 (i) $v = \frac{2v_0}{3}$ (ii) $t = \frac{v_0}{3\mu g}$

$$w = \frac{1}{2}[3\mu^2 mg^2 t^2 - 2\mu\, mg\, t\, v_0](t < t_0),$$

$$w = -\frac{1}{6}mv_0^2 (t > t_0)$$

Q.18 $t = 2\sqrt{5}$ sec, q = 4π/5 rad

Q.19 (a) $\omega/3$, (b) $\frac{\sqrt{37}}{3}\, m\omega R$, (c) $\frac{\sqrt{37}}{3}\, m\omega R$

Q.20 $\sqrt{15}$ ft/sec

Q.21 (a) 5 m/s^2, (b) $0.3 < h < 1.5$ m

Q.22 6N, $-0.6\hat{j} \pm 0.6\hat{k}$

Q.23 (i) $t = \frac{6a\pi}{\sqrt{3}v_0}$; (ii) $s = \frac{a}{\sqrt{3}}\sqrt{1+(2\pi+\sqrt{3})^2}$

Exercise 2

Single Correct Choice Type

Q.1 D **Q.2** C **Q.3** B **Q.4** A **Q.5** B **Q.6** A
Q.7 D **Q.8** B **Q.9** D **Q.10** C **Q.11** A **Q.12** A
Q.13 C **Q.14** C

Multiple Correct Choice Type

Q.15. B, C **Q.16** A, D **Q.17** A, C, D **Q.18** A, B, C **Q.19** B, C **Q.20** B, C
Q.21 B, C **Q.22** B, C, D **Q.23** C, D **Q.24** A, B, C, D **Q.25** B, C **Q.26** A, B, D
Q.27 A, C, D **Q.28** A, C, D **Q.29** B, C, D **Q.30** A, C **Q.31** B, C **Q.32** A, C
Q.33 A, B, C **Q.34** A, B, C **Q.35** A, B, C **Q.36** B, C, D **Q.37** A, B, C, D **Q.38** B, C

Assertion Reasoning Type

Q.39 B **Q.40** B **Q.41** A **Q.42** B **Q.43** B **Q.44** D
Q.45 B **Q.46** D

Comprehension Type

Q.47 C **Q.48** A **Q.49** C **Q.50** C **Q.51** B **Q.52** B
Q.53 A **Q.54** A **Q.55** D **Q.56** B

Previous Years' Questions

Q.1 A **Q.2** A **Q.3** B **Q.4** A **Q.5** A **Q.6** D
Q.7 A **Q.8** B **Q.9** A **Q.10** A **Q.11** C **Q.12** A
Q.13 B **Q.14** D **Q.15** D **Q.16** D **Q.17** C **Q.18** B
Q.19 C **Q.20** C **Q.21** A **Q.22** C, D **Q.23** A **Q.24** D
Q.25 A, B **Q.26** D **Q.27** 8 **Q.28** C, D **Q.29** 2 **Q.30** 4
Q.31 7 **Q.32** D **Q.33** 6 **Q.34** D **Q.35** A, B, D **Q.36** A, D

Solutions

JEE Main/Boards

Exercise 1

Sol 1: Moment of inertia of any body is given by $I = \int r^2 dm$ where 'dm' is mass of a small element under consideration and 'r' is the distance between the axis of rotation and the element. Then,

S.I. units of moment of inertia will be m^2 kg or kg.m^2

DIMENSIONS –

$[M^0LT^0]^2[M^1L^0T^0] = [M^1L^2T^0]$

It is not a vector quantity since direction is nowhere considered.

Sol 2: Torque is the rotational analogue of force.

Sol 3: Moment of inertia is the rotational analogue of mass of a body.

Sol 4: The theorem of parallel axes

$I = I_{cm} + m.d^2$

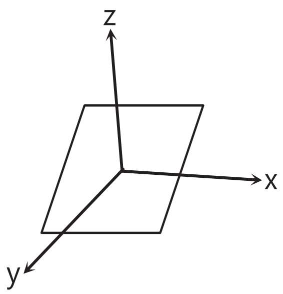
where I_{cm} = Moment of Inertia about an axis passing through center of mass and parallel to the considered axis

m = mass of the body

d = distance between the axis (about which the value of I is required) and the axis passing through center of mass and parallel to the considered axis.

The theorem of perpendicular axes

$I_z = I_x + I_y$



This theorem is applicable for planar bodies only and I_x and I_y should be about two perpendicular axes lying in plane.

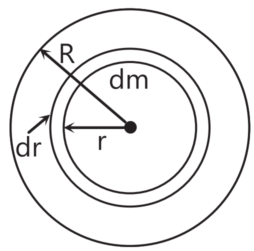
Sol 5: Moment of inertia of a solid sphere about its diameter is

$I = \frac{2}{5}MR^2$

where M = mass of the sphere

R = Radius of the sphere

Derivation:



Considering, the solid sphere as a large group of hollow spheres whose radii range from 0 to R with a thickness of dr. Integrating moment of inertia of these dements gives the required value.

$dm = \rho.\,4\pi r^2.dr$ ($dm = \rho.\,dv$) and $I = \int dI$

$I = \int_0^R \frac{2}{3}.r^2 \;\; \rho.\,4\pi r^2 dr$ ($\because I = \int r^2 dm$) and

$(I_{\text{hollow sphere}} = \frac{2}{3}MR^2)$

$\Rightarrow dI = \frac{2}{3}(dm)r^2 = \frac{2}{3}.\,4\pi\rho\left[\frac{r^5}{5}\right]_0^R$

$= 4\pi\rho\,.\,\frac{R^4}{5} = \frac{2}{5}MR^2 \left[\because \rho = \frac{M}{\frac{4\pi}{3}R^3}\right]$

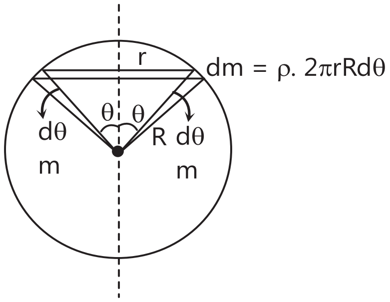
Sol 6: The moment of inertia of a hollow sphere about an axis passing through its center is $I = \frac{2}{3}MR^2$

where M = mass of sphere

R = Radius of sphere

Derivation:

Considering the hollow sphere, as a large group of rings with thickness dr and radii ranging from 0 to R. Integrating the moment of inertia of these rings we will get the required moment of inertia.



$I = \int dI = \int r^2 dm = \int r.\rho 2\pi rRd\theta$

[$\because I_{ring} = MR^2$]

$\Rightarrow dI = dmr^2 \quad \Rightarrow I = \int 2\pi\rho R\,(R^3 \sin^3\theta)d\theta$

$I = 2\pi\rho R^4 \int_0^{\pi} \sin^3\theta d\theta$

$\Rightarrow I = \frac{MR^2}{2}\int_0^{\pi}\frac{(3\sin\theta - \sin 3\theta)}{4}.\,d\theta$

$\Rightarrow I = \frac{MR^2}{2}\left[-\frac{3\cos\theta}{4} + \frac{\cos 3\theta}{12}\right]_0^{\pi}$

$\Rightarrow I = \frac{MR^2}{2}\left[\frac{3}{4} - \frac{1}{12} - \left[-\frac{3}{4} + \frac{1}{12}\right]\right] \Rightarrow I = \frac{2}{3}MR^2$

Sol 7: Factors on which moment of inertia depend

→ Mass of the body

→ Mass distribution of the body

→ Size of the body

→ Axis about which moment of inertia is required

Sol 8: No, it is not a constant

It depends on the axis about which moment of inertia is calculated

Since, $K = \sqrt{\frac{I_{axis}}{M}}$

I_{axis} = moment of inertia about a given axis

M = mass of the body (constant)

Sol 9: Given

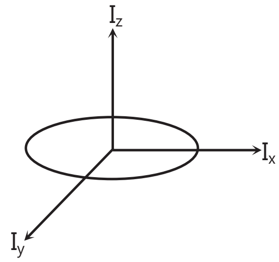
A solid sphere and hollow sphere have same mass and same radius

$I_{solid\ sphere} = \frac{2}{5}MR^2$

$I_{hollow\ sphere} = \frac{2}{3}MR^2$

$\therefore I_{hollow\ sphere} > I_{solid\ sphere}$

Sol 10: Given,



Circular discs A and B of same mass and same thickness but different densities d_A and d_B ($d_A > d_B$)

$M_A = M_B$

$\Rightarrow \pi R_A^2 .\, d_A\, t_A = \pi R_B^2 .\, d_B\, .t_B$

$\Rightarrow \frac{R_A}{R_B} = \left(\frac{d_B}{d_A}\right)^{1/2}$

$(I_A)_x = \frac{M_A R_A^2}{4} = (I_A)_y$

$\Rightarrow (I_A)_z = (I_A)_x + (I_A)_y = \frac{M_A R_A^2}{2}$

[By perpendicular axes theorem]

also

$(I_B)_z = \frac{M_B R_B^2}{2}$

$\frac{I_A}{I_B} = \left(\frac{R_A}{R_B}\right)^2 = \frac{d_B}{d_A} \Rightarrow I_A < I_B$ since $d_B < d_A$

Sol 11: Given.

Moment of inertia of two rotating bodies A and B as

I_A and I_B ($I_A > I_B$)

and

Angular moment (L_A and L_B) are equal

$\Rightarrow L_A = L_B$

then kinetic energies

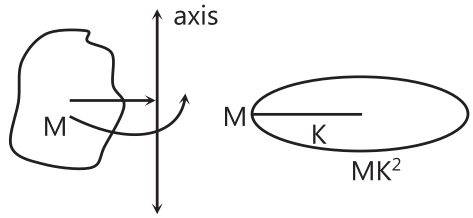
$(K.E.)_A$ and $(K.E.)_B$ will be

$\frac{1}{2}L_A^2 / I_A$ and $\frac{1}{2}L_B^2 / I_B$ $\left[\because K.E. = \frac{1}{2}I\omega^2 = \frac{1}{2}L\omega = \frac{1}{2}\frac{L^2}{I}\right]$

$\Rightarrow (K.E.)_A < (K.E.)_B$ since $I_B < I_A$

Sol 12: Moment of inertia is the measure of tendency of a body to resist rotational motion if it was at rest or resist being stopped it was rotating.

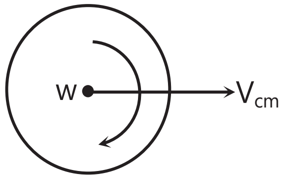
Radius of Gyration is the radius of the circle is which a zero sized particle of mass M (which is equal to the mass of a body considered) such that the moment of inertia of both the particle and the body about any axis are equal



I $\qquad$ $I = MK^2$

$\Rightarrow K = \sqrt{\frac{I}{M}}$

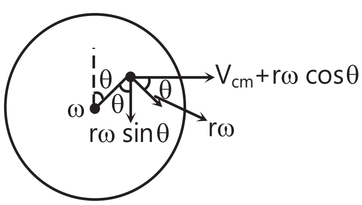
Sol 13: Kinetic energy of rolling motion



Consider a body rolling with angular velocity ω and linear center of mass velocity v_{cm}

Velocity of a particle at any point is given by

$(V_{cm} + r\omega\cos\theta)\,\hat{i} + (r\omega\sin\theta)\,\hat{j}$



$$K.E. = \int d(K.E.)$$

$$= \int \frac{1}{2} dm\left((V_{cm} + r\omega\cos\theta)^2 + (r\omega\sin\theta)^2\right)$$

$$= \frac{1}{2}\int_0^M r^2\omega^2 dm + \frac{1}{2}\int_0^M V_{cm}^2\, dm +$$

$$\frac{1}{2}\int_0^M 2V_{cm}\, r\omega\cos\theta\, dm$$

$$= \frac{1}{2} I_{cm}\,\omega^2 + \frac{1}{2} V_{cm}^2 + V_{cm}\,\omega \int_0^m r\cos\theta\, dm$$

$$= \frac{1}{2} I_{cm}\,\omega^2 + \frac{1}{2} MV_{cm}^2 + V_{cm}\,\omega \int_0^R\int_0^{2\pi} r\cos\theta . d\theta . dr$$

$$= \frac{1}{2} I_{cm}\omega^2 + \frac{1}{2} MV_{cm}^2 \quad \left[\because \int_0^{2\pi} \cos\theta\, d\theta = 0\right]$$

$$\therefore K.E. = \frac{1}{2} I_{cm}\,\omega^2 + \frac{1}{2} MV_{cm}^2$$

Sol 14: Laws of rotational motion

First law: Every body has tendency to be in rest or is state of rotation unless acted upon by a torque

Second law: The torque applied on a body is moment of inertia times the angular acceleration of the body J – Iα

Third law: Every action (torque) has an equal and opposite reaction (torque) on the body which gave the action.

Sol 15: Newton's second law of linear motion is

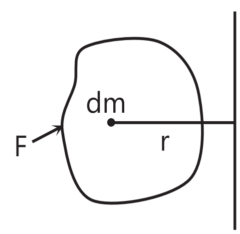
F = ma

where F = force acting on the body

m = mass of body

a = acceleration of body

Now, consider 'dm' part of as body acted upon by a force F and the body is rotating with an angular acceleration of 'α'



$dT = r \times F$

$= r.\, dm\, (r\alpha)$

$= r^2 .\, dm .\, \alpha$

$\int dT = \int r^2 dm.\alpha$

$\Rightarrow T_{axis} = I_{aixs}\,\alpha_{axis}$

Sol 16: Principle of conservation of angular momentum.

In absence of a torque, the angular momentum of a body is always conserved (constant)

$T = \frac{dL}{dt} = 0$

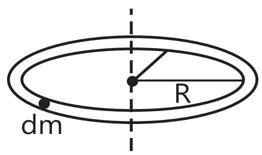
$\Rightarrow L = \text{constant} \Rightarrow I\omega = \text{constant}$

or $I_1\omega_1 = I_2\omega_2$

Example: By stretching hands a ballet decreases the angular speed of the body by increasing the moment of inertia

Example: A person sitting is a chair which can rotate holds a rotating wheel in hands and when he flips the wheel, the person along with chair rotates conserving angular momentum.

Sol 17: Consider a circular ring of radius R and mass M



$I = \int dI = \int R^2.dm = MR^2$

Sol 18: Given,

$I = 200\ g\ cm^2$

$r = 5\ cm$

We know that for a thin ring the moment of inertia about an axis passing through center is Mr^2

$\Rightarrow M = \frac{T}{r^2} = 8$ grams

Sol 19: Given,

diameter of disc = 40 cm

thickness of disc = 7 cm

density of disc = $9\ gm\ cm^{-3}$

mass of the disc = $\rho.V = \rho.\frac{\pi D^2}{4}.t$

$= 9 \times \pi \times 400 \times 7 = 79168.13$ grams

moment of inertia of disc about a transverse axis through the center of the disc is

$I = \frac{MR^2}{2}$

$\Rightarrow I = \frac{79168.13 \times 20 \times 20}{2} \Rightarrow I = 1.584 \times 10^7\ g\ cm^2$

Sol 20: Given,

A uniform circular disc and A Uniform circular ring of same mass of 10 kg and diameter of 1 m

Moment of inertia of disc about a transverse axis through center of the disc is $I = \frac{1}{2}MR^2$

$\Rightarrow I = \frac{1}{2} \times 10 \times \left(\frac{1}{2}\right)^2 = 1.25\ kg\ m^2$

Sol 21: Given,

Radius of earth assuming it as a sphere as 6400 km mass of earth is 6×10^{24} kg

Moment of inertia of the earth is

$I = \frac{2}{5}MR^2$ ($\because I = \frac{2}{5}MR^2$ for a sphere (solid))

$\Rightarrow I = \frac{2}{5} \times 6 \times 10^{24} \times (6400 \times 10^3)^2$

$\Rightarrow I = 9.8304 \times 10^{37}\ kgm^2$

Sol 22: Given,

A Uniform circular disc of mass 700 gms and diameter 20 cm

(i) Moment of inertia about the transverse axis through center of disc is

$I = \frac{MR^2}{2}$

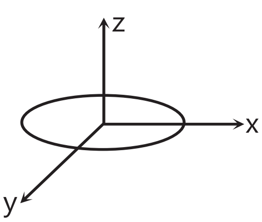
$\Rightarrow I = \frac{700 \times 10 \times 10}{2} = 3.5 \times 10^4\ gcm^2$

(ii) Moment of inertia about the diameter of disc is

$I = \frac{MR^2}{4}$

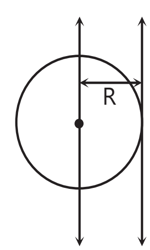
[$\because I_x = I_y$ and $I_x + I_y = I_z$

(perpendicular axis theorem)]



$\Rightarrow I = \frac{700 \times (10 \times 10)}{4} = 1.75 \times 10^4 gcm^2$

(iii)



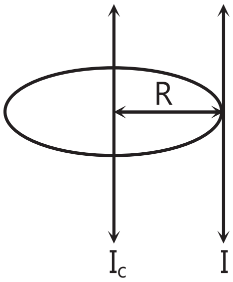
Moment of inertia about a tangent is plane is

$I = I_d + MR^2$ (parllel axis theorem)

$\Rightarrow I = \frac{5MR^2}{4}$

$\Rightarrow I = \frac{5}{4} \times (700) \times (10 \times 10) = 8.75 \times 10^4\ gcm^2$

(iv)



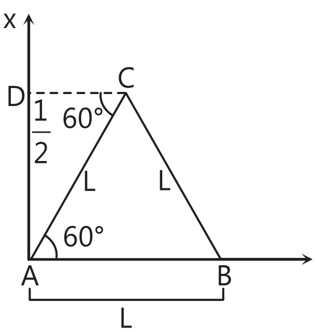
Moment of inertia about a tangent perpendicular to the plane is

$I = I_C + MR^2$

$\Rightarrow I = \frac{3}{2}MR^2$

$\Rightarrow I = \frac{3}{2} \times 700 \times 10 \times 10 = 10.5 \times 10^4 \text{ gcm}^2$

Sol 23:



Given, particles of masses m are placed at A, B and C and side of triangle ABC is L

Then,

$CD = AC \cos 60° = \frac{AC}{2} = \frac{AB}{2} = \frac{L}{2}$

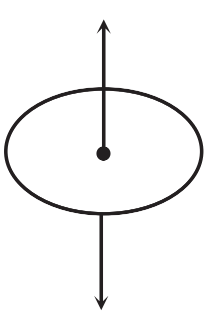
moment of inertia of system is

$I = I_A + I_B + I_C$

$\Rightarrow I = m(0)^2 + m(L)^2 + m\left(\frac{L}{2}\right)^2$

$\Rightarrow I = \frac{5mL^2}{4}$

Sol 24: Given,



A circular disc of mass 1 kg and radius 0.2 m rotating about transverse axis passing through its center.

Moment of inertia about the given axis as

$I = \frac{MR^2}{2} \Rightarrow I = \frac{1}{2} \times (0.2)^2 = 0.02 \text{ kg–m}^2$

Given, disc makes $\frac{30}{\pi}$ rotations per minute then,

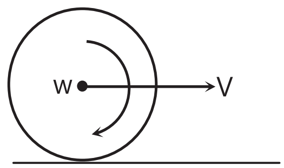
Angular velocity $= 2\pi \times \frac{30}{11} \times \frac{1}{60}$ rad/s

= 1 rad/s

kinetic energy $= \frac{1}{2} I\omega^2$

$= \frac{1}{2} \times 0.02 \times (1)^2 = 0.01$ joules

Sol 25: Given,



A circular dice of mass M and radius r is set rolling on table the kinetic energy of the disc is given by

$K.E = \frac{1}{2} I\omega^2 + \frac{1}{2}mv^2$(refer Q. 14)

$\Rightarrow K.E. = \frac{1}{2}\frac{mr^2}{2}\omega^2 + \frac{1}{2}mv^2$

$\Rightarrow K.T. = \frac{mv^2}{4} + \frac{mv^2}{2}$

[For pure rolling $v = r\omega$]

$\Rightarrow K.E. = \frac{3}{4}mv^2$

Sol 26: Given,

Sun rotates around itself once in 27 days angular velocity of sum $= \frac{2\pi}{27}$ rad/days

If it expands to truce its present diameter, then moment of inertia becomes 4

$\therefore I = \frac{2}{5}MR^2$ (for sphere)

$\frac{I_2}{I_1} = \frac{R_2^2}{R_1^2}$

$\Rightarrow I_2 = 4I_1$

$I_1\omega_1 = I_2\omega_2$

(By principle of conservation of angular momentum)

$\Rightarrow \omega_2 = \frac{\omega_1}{4} = \frac{\omega_1}{108}$ rad/days

Then, the new period of revolution

$= \frac{2\pi}{(2\pi/108)} = 108$ days

$\because \left(T = \frac{\pi}{\omega}\right)$

Sol 27: Given,

A 40 kg of flywheel is form of a uniform circular disc 1 meter in radius is making 120 r.p.m.

Angular velocity = $120 \times 2\pi \times \frac{1}{60}$ rad/s

= 4π rad/s

Moment of inertia about transverse axis passing through center is

$I = \frac{MR^2}{2}$

$\Rightarrow I = \frac{40 \times (1)^2}{2} = 20$ kg–m^2

angular momentum = I.ω

= $80\,\pi$ kgm^2/s = 251.33 kgm^2/s

Sol 28: Given,

Angular velocity of the system = 120 r.p.m

Moment of inertia of the system initially = 6 kgm^2

Moment of inertia of the system finally = 2 kgm^2

$I_1\omega_1 = I_2\omega_2$

(By principle of conservation of angular momentum)

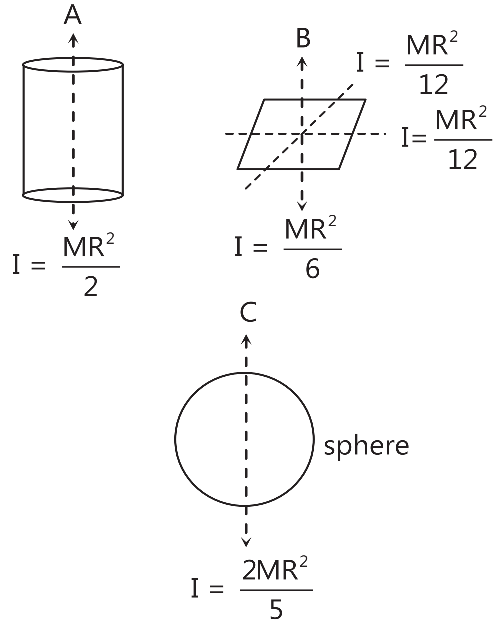
$\Rightarrow \omega_2 = \frac{I_1}{I_2} \times \omega_1 = 360$ r.p.m.

$\therefore$ Final angular velocity = 360 r.p.m

Exercise 2

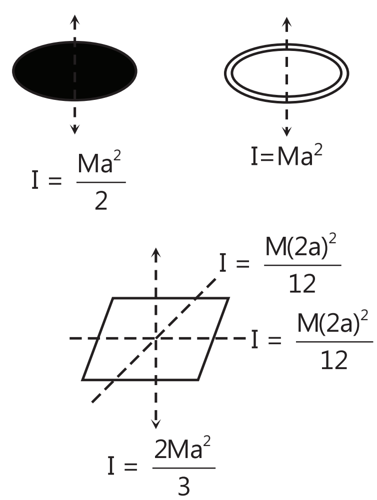
Single Correct Choice Type

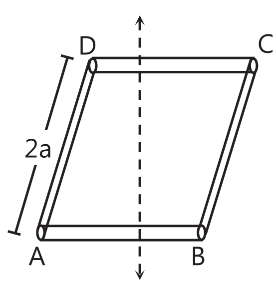
Sol 1: (B)



B< C < A

Sol 2: (B)





Mass of each rod = $\frac{M}{4}$

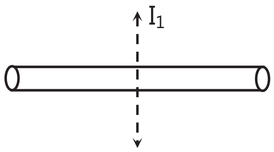
Moment of inertia of one rod about the given axis is

$$I = \left(\frac{M(2a)^2}{4\times 12}\right) + \frac{Ma^2}{4} = \frac{4Ma^2}{4\times 3} = \frac{Ma^2}{3}$$

Total moment of inertia = $4I = \frac{Ma^2}{3}$

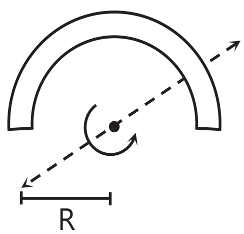
Sol 3: (A) Given,

I_1 is the moment of inertia about perpendicular bisector of rod



$I_1 = \frac{ML^2}{12}$

Now rod is bent into semi-circular arc



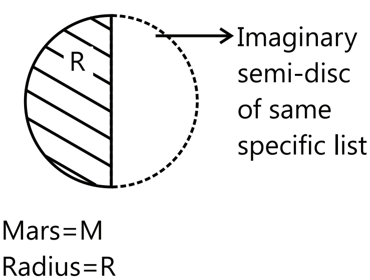
Length of arc = L = πR

Moment of inertia of the arc = MR^2

$= \frac{ML^2}{\pi^2} = I_{2}$

$I_2 > I_1$ (since $\pi^2 < 12$)

Sol 4: (B) So, $2I = \frac{1}{2}(M+M)R^2 \Rightarrow 2I = \frac{1}{2}(2M)R^2$



$\Rightarrow \boxed{I = \frac{1}{2}MR^2}$

Sol 5: (D) Given,

I' as a function of x of a rigid body

$I = 2x^2 - 12x + 27$

The value of moment of inertia is minimum at center of mass point

To calculate min value of I, differentiate w.r.t. x and equate it to 0

$\frac{dI}{dx} = 0$

$\Rightarrow 4x - 12 = 0$

$\Rightarrow x = 3$

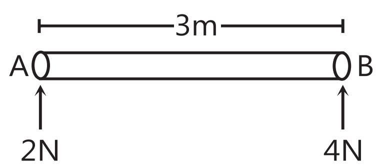
To check whether it is minimum,

$\frac{d^2I}{dx^2} > 0$

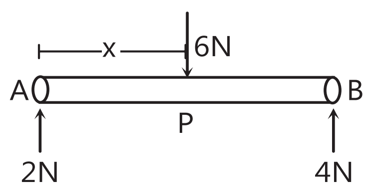
$\Rightarrow 4 > 0$

$\therefore$ x = 3 is the x-coordinate of center of mass.

Sol 6: (D)



To keep the body in equilibrium a 6 N acts at point x from A



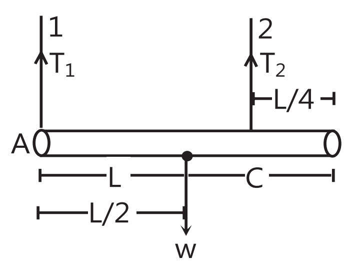
Taking moment of torques about A

$\sum M_A = 0$ (In equilibrium)

$\Rightarrow 6 \times x - 4 \times 3 = 0$

$\Rightarrow x = 2m$

Sol 7: (D)



Let T_1 and T_2 be tensions is the strings considering force equilibrium

$T_1 + T_2 = W$

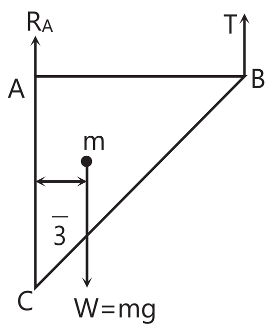
Also Considering moment equilibrium about A

$\sum M_A = 0$

$\Rightarrow T_2 \times \frac{3L}{4} - W \times \frac{L}{2} = 0$

$\Rightarrow T_2 = \frac{2W}{3} \Rightarrow T_1 = \frac{W}{3}$

Sol 8: (B)



Center of mass of the triangular plate is at a distance of $\frac{\ell}{3}$ from AC.

Considering force equilibrium

$R_A + T = W = mg$

Considering torque or moment equilibrium about B

$\Rightarrow R_A(\ell) = W\left(\frac{2\ell}{3}\right)$

$\Rightarrow R_A = \frac{2mg}{3}$

Sol 9: (B) Given, A constant torque is applied on a rod which hinged the angular acceleration in rod will be

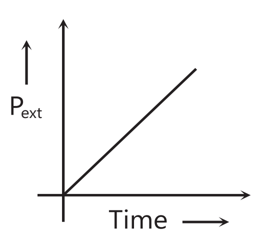
$\propto = \frac{T}{I}$

Angular velocity $\omega = \int_0^t \alpha\, dt = \frac{Tt}{I} + \omega_0$

Power developed $= T.\omega = T.\left(\frac{Tt}{I} + \omega_0\right)$

If $\omega_0 = 0$

$P = \frac{T^2}{I}(t)$



Sol 10: (C) Given,

Two spheres of same mass M and radii R and 2R and also have equal rotational kinetic energies

$\Rightarrow \frac{1}{2}I_1\omega_1^2 = \frac{1}{2}I_2\omega_2^2 \Rightarrow \frac{L_1^2}{I_1} = \frac{L_2^2}{I_2}$

[$\because$ $I\omega = L$ (Angular momentum)]

$\Rightarrow \frac{L_2}{L_1} = \sqrt{\frac{I_2}{I_1}} = \sqrt{\frac{R_2^2}{R_1^2}} = \frac{R_2}{R_1} = 2$

Sol 11: (A) Angular momentum remains constant since, no torque is acting on the skater.

While kinetic energy increases, since,

K.E. $= \frac{1}{2}L.\omega.$

and as L = constant and ω increase K.E. increases

Sol 12: (D) Energy is not conserved in this case because the disc is fixed at its center and a force is acting on it when the child jumps.

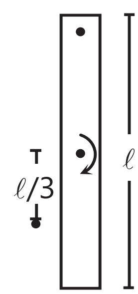
But Angular momentum can be conserved since No torque is present in the boy-disc system

$\therefore$ Initial angular momentum = final angular momentum

$(I + mR^2)\omega = I\omega' + MRv$

$\Rightarrow \omega' = \frac{(I+mR^2)\omega - mRv}{I}$

Sol 13: (B)



Moment of inertia of the rod a bout pivot $= \frac{M\ell^2}{12}$

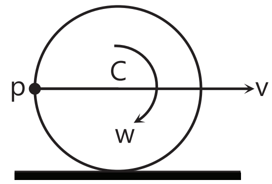
Since, the collision is elastic and the rod stops, velocity of the particle is $v = \frac{\ell}{3}.\omega$

By principle of Angular Momentum,

$I\omega = m.v.\frac{\ell}{3}$

$\Rightarrow \frac{M\ell^2}{12}.\omega = m.\frac{\ell}{3}\omega.\frac{\ell}{3} \Rightarrow m = \frac{3M}{4}$

Sol 14: (B)



The point P experiences centripetal acceleration towards centre

$\therefore \vec{a} = a\hat{i}$

the velocity of point P is

$\vec{v} = v\hat{i} + R\omega\hat{j}$

$\Rightarrow \vec{v} = v\hat{i} + v\hat{j}$

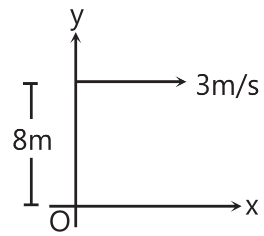
[$\because$ v = Rω pure rolling]

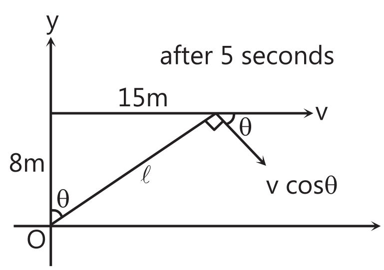
$\Rightarrow \vec{v} = v(\hat{i} + \hat{j})$

$$\cos\theta = \frac{\vec{a}.\vec{v}}{|\vec{a}||\vec{v}|} = \frac{1}{\sqrt{2}}$$

$\Rightarrow \theta = 45°$

Sol 15: (C)

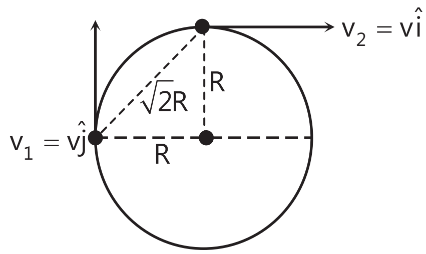




$$\text{Angular velocity} = \frac{v\cos\theta}{\ell} = \frac{v8}{\ell^2}$$

$$= \frac{24}{8^2+15^2} = \frac{24}{289}\ \text{rad/s}$$

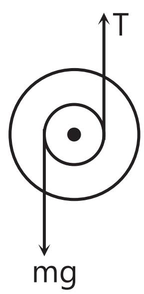
Sol 16: (B)



$$\text{Angular velocity} = \frac{\text{relative velocity}}{AB}$$

$$= \frac{|v\hat{i} - v\hat{j}|}{\sqrt{2}R} = \frac{v}{R}$$

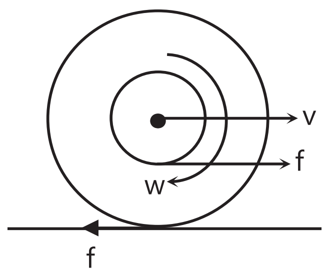
Sol 17: (B)



$$\text{Acceleration of yo-yo} = \frac{mg - T}{m} = g - \frac{T}{m}$$

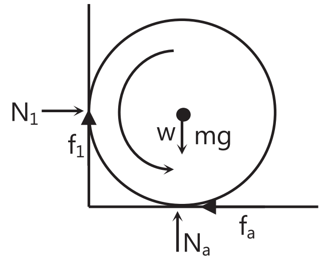
Sol 18: (B) In case of pure rolling v = Rω

(Bottom–most point has zero velocity)

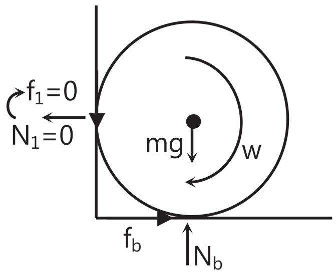


as ω is in clockwise direction, thread winds also friction acts leftwards to increase w.

Sol 19: (B)



(a)



$f_1 + N_a = mg \quad N_b = mg$

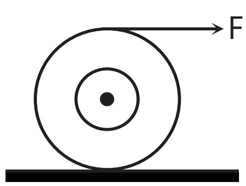
$\frac{N_1}{3} + N_a = mg \quad f_b = \frac{mg}{3}$

$\frac{f_a}{3} + Na = mg \; [\because N_1 = f_a]$

$N_a = \frac{9mg}{10} \left[\because f_a = \frac{N_a}{3}\right] \Rightarrow f_a = \frac{3mg}{10}$

$\frac{f_a}{f_b} = \frac{9}{10}$

Sol 20: (A)



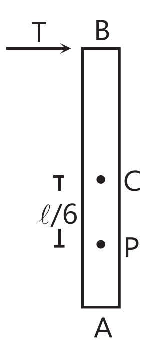
F = Ma (By Newton's second law), also T = Ta

$\Rightarrow FR = \frac{Ia}{R} \; (\because \alpha = \frac{a}{R}$ for pvre rolling)

$\Rightarrow I = MR^2$

This is satisfied for thin pipe

Sol 21: (D)

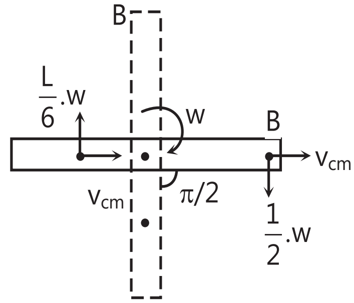


By conservation of linear momentum,

$mv_{cm} = J \Rightarrow v_{cm} = \frac{\hat{j}}{m}$

By conservation of angular momentum, $I\omega = J.\frac{\ell}{2}$

$\Rightarrow \omega = \frac{6J}{m\ell}$



After time $t = \frac{\pi m\ell}{12J}$

The angle rotated by rod $\theta = \omega t = \frac{\pi}{12}\frac{m\ell}{J}.\frac{6J}{m\ell} = \frac{\pi}{2}$

Velcoity of point P $= \sqrt{\left(\frac{\ell\omega}{6}\right)^2 + (v_{cm})^2}$

$= \sqrt{\left(\frac{J}{m}\right)^2 + \left(\frac{J}{m}\right)^2} = \sqrt{2}\frac{J}{m}$

Sol 22: (C) Solid cylinder rolls without slipping

$\Rightarrow v_{cm} = R\omega$

$\frac{(K.E.)_{rotational}}{(K.E.)_{translational}} = \frac{\frac{1}{2}I\omega^2}{\frac{1}{2}mv^2}$

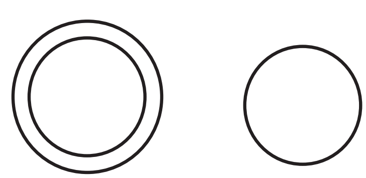
$\Rightarrow \frac{(K.E.)_r}{(K.E.)_t} = \frac{mR^2\omega^2}{2mv^2} = \frac{1}{2}$

Sol 23: (C) There will be no diff in velocity of centre of mass

$\Rightarrow F = ma_{cm}$

$\therefore a_{cm}$ = same in both cases

Sol 24: (B)



A hoop & solid cylinder of same mass

$\therefore I_{hoop} > I_{cylinder}$ as mass is distributed away from center

$\therefore$ Since gain in potential energy in both case is same

Let P for hoop

$$P = \frac{1}{2}mv^2 + \frac{1}{2}I_{hoop}\left(\frac{V}{R}\right)^2$$

and $P = \frac{1}{2}mv^2 + \frac{1}{2}I_{cy}\left(\frac{V}{R}\right)^2$

$\therefore V_{hoop} < V_{cylinder}$

Sol 25: (D) (i) Upward acceleration = a = same for both case

$$\Rightarrow 2h = 0 \times t_Q + \frac{1}{2}a \times t_Q^2 \Rightarrow t_Q = \sqrt{\frac{4h}{g}}$$

$$h = 0 \times t_p + \frac{1}{2}a \times t_p^2$$

$$\Rightarrow t_p = \sqrt{\frac{2h}{g}}$$

$\therefore t_Q \neq 2t_p$

(ii) The acceleration of both the balls is same = g sin q

(iii) $\Delta KE = \Delta PE$

$\therefore \Delta KE_Q = 2mgH$

$\Delta KE_P = mgH$

$\therefore \Delta KE_Q = 2\Delta KE_P$

Sol 26: (B) Moment of inertia decreases since mass is closer to axis while.

Angular momentum remains constant which implies angular velocity increases and which intern implies increase is kinetic energy.

($\because L = I\omega$ and K.E $= \frac{1}{2}L\omega$)

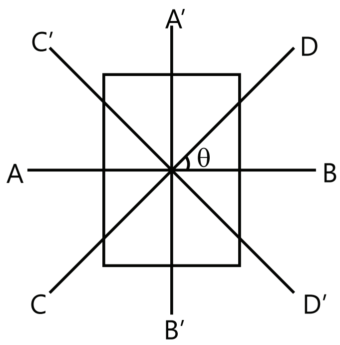
Sol 27: (C) Angular momentum is conserved on any point on the ground since the only force present passes through that point making torque zero.

Previous Years' Questions

Sol 1: (A) A\`B\` $\perp$ AB and C\`D\` $\perp$ CD

From symmetry $I_{AB} = I_{A`B`}$ and $I_{CD} = I_{C`D`}$

From theorem of perpendicular axes,



$$I_{ZZ} = I_{AB} + I_{A'B'} = I_{CD} + I_{C'D'} = 2I_{AB} = 2I_{CD}$$

Alternate:

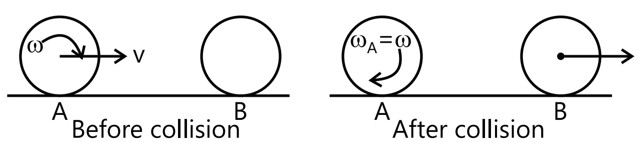
The relation between I_{AB} and I_{CD} should be true for all value of θ

At $\theta = 0$, $I_{CD} = I_{AB}$

Similarly, at è $= \pi/2$, $I_{CD} = I_{AB}$ (by symmetry)

Keeping these things in mind, only option (A) is correct.

Sol 2: (C) Since, it is head on elastic collision between two identical spheres, they will exchange their linear velocities i.e., A comes to rest and B starts moving with linear velocity v, As there is no friction anywhere, torque on both the spheres about their center of mass is zero and their angular velocity remains unchanged. Therefore,



Sol 3: (B) $L = m\frac{v}{\sqrt{2}}r_\perp$

Here, $r_\perp = h = \frac{v^2 \sin^2 45°}{2g} = \frac{v^2}{4g}$

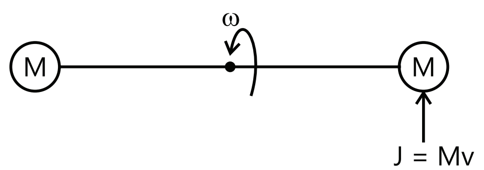
$$\therefore L = m\left(\frac{v}{\sqrt{2}}\right)\left(\frac{v^2}{4g}\right) = \frac{mv^3}{4\sqrt{2}g}$$

Sol 4: (A) Let ω be the angular velocity of the rod. Applying, angular impulse = change in angular momentum about center of mass of the system

$$J.\frac{L}{2} = I_c\omega$$

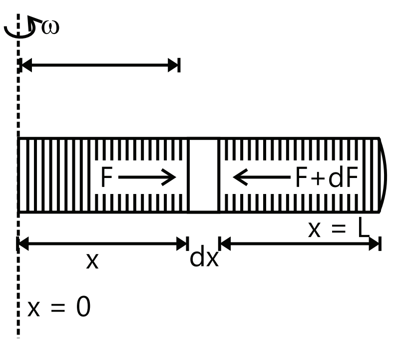
$$\therefore (Mv)\left(\frac{L}{2}\right) = (2)\left(\frac{ML^2}{4}\right)\omega$$

$$\therefore \ \omega = \frac{v}{L}$$



Sol 5: (A) Mass of the element dx is m = $\frac{M}{L}$ dx.

This element needs centripetal force for rotation.



$$\therefore \ dF = mx\omega^2 = \left(\frac{M}{L}x\omega^2 dx\right)$$

$$\therefore \ F = \int_0^L dF = \frac{m}{L}.\omega^2 \int_0^L x dx = \frac{M\omega^2 L}{2}$$

This is the force exerted by the liquid at the other end.

Sol 6: (B) mg sin θ component is always down the plane whether it is rolling up or rolling down. Therefore, for no slipping, sense of angular acceleration should also be same in both the cases.

Therefore, force of friction f always act upwards.

Sol 7: (C) Work done $W = \frac{1}{2}I\omega^2$

If x is the distance of mass 0.3 kg from the center of mass, we will have

$$I = (0.3)\ x^2 + (0.7)(1.4 - x^2)$$

For work to be minimum, the moment of inertia (I) should be minimum or

$$\frac{dI}{dx} = 0$$

or 2(0.3) – 2 (0.7)(1.4 – x) = 0 or (0.3)x = (0.7)(1.4 – x)

$$\Rightarrow \ x = \frac{(0.7)(1.4)}{0.3 + 0.7} = 0.98\text{m}$$

Sol 8: (C) From the theorem

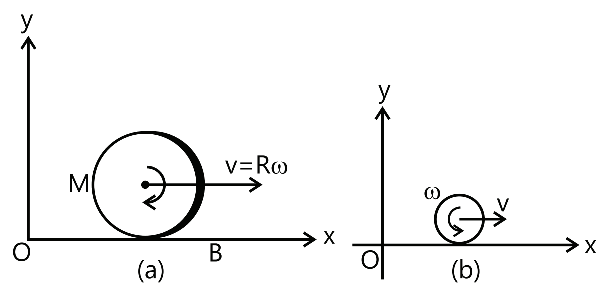
$$\vec{L}_0 = \vec{L}_{CM} + M(\vec{r} \times \vec{v}) \qquad \text{... (i)}$$

We may write

Angular momentum about O = Angular momentum about CM + Angular momentum of CM about origin

$$\therefore \ L_0 = I\omega + MRv$$

$$= \frac{1}{2}MR^2\omega + MR(R\omega) = \frac{3}{2}MR^2\omega$$



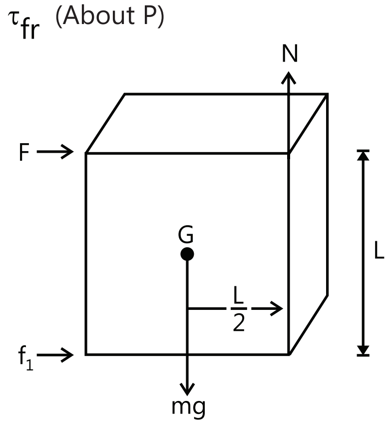
Note that is this case both the terms in

Eq. (i) i.e., $\vec{L}_{CM}$ and M $(\vec{r} \times \vec{v})$

Have the same direction. That is why we have used $L_0 = I\omega \sim MRv$ if they are in opposite direction as shown in figure (b).

Sol 9: (C) At the critical condition, normal reaction N will pass through point P. In this condition.

$\tau_N = 0 = \tau_{fr}$ (About P)



The block will topple when

$$\tau_F > \tau_{mg} \text{ or } FL > (mg)\frac{L}{2}$$

$$\therefore\ F > \frac{mg}{2}$$

Therefore, the minimum force required to topple the block is

$$F = \frac{mg}{2}$$

Sol 10: (B) Net external torque on the system is zero. Therefore, angular momentum is conserved. Force acting on the system are only conservative. Therefore, total mechanical energy of the system is also conserved.

Sol 11: (A) Mass of the whole disc = 4M

Moment of inertia of the disc about the given axis

$$= \frac{1}{2}(4M)R^2 = 2MR^2$$

$\therefore$ Moment of inertia of quarter section of the disc

$$= \frac{1}{4}(2MR^2) = \frac{1}{2}MR^2$$

Note: These type of questions are often asked in objective. Students generally error in taking mass of the whole disc. They take if M instead of 4 M.

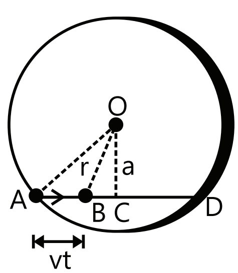
Sol 12: (C) Since, there is no external torque, angular momentum will remains conserved. The moment of inertia will first decrease till the tortoise moves from A to Can then increase as it moves from C and D. Therefore, ω will initially increase and then decrease.

Let R be the radius of platform, m the mass of disc and M is the mass of platform.

Moment of inertia when the tortoise is at a

$$I_1 = mR^2 + \frac{MR^2}{2}$$

And moment of inertia when the tortoise is at B.



$$I_2 = mr^2 + \frac{MR^2}{2}$$

Here, $r^2 = a^2 + [\sqrt{R^2 - a^2} - vt]^2$

Form conservation of angular momentum

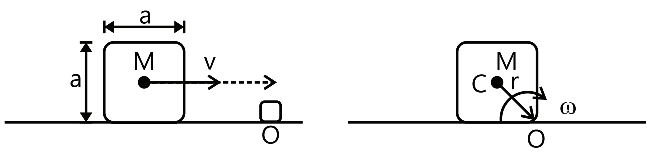
$$\omega_0 I_1 = \omega(t) I_2$$

Substituting the values we can see that variation of ω (t) is non-linear.

Sol 13: (C) $\therefore\ I_1\omega_1 = I_2\omega_2$

$$\omega_2 = \frac{I_1}{I_2}\omega = \left(\frac{Mr^2}{Mr^2 + 2mr^2}\right)\omega = \left(\frac{M}{M + 2m}\right)\omega$$

Sol 14: (A) $r = \sqrt{2}\frac{a}{2}$ or $r^2 = \frac{a^2}{2}$



Net torque about O is zero. Therefore, angular momentum (L) about O will be conserved,

Or $L_i = L_f$

$$Mv\left(\frac{a}{2}\right) = I_0\omega = (I_{CM} + Mr^2)\omega$$

$$= \left\{\left(\frac{Ma^2}{6}\right) + M\left(\frac{a^2}{2}\right)\right\}\omega = \frac{2}{3}Ma^2\omega$$

$$\omega = \frac{3v}{4a}$$

Sol 15: (D) Mass of the ring $M = \rho L$

Let R be the radius of the ring, then

$$L = 2\pi R \text{ or } R = \frac{1}{2\pi}$$

Moment of inertia about an axis passing through O and parallel to XX` will be

$$I_0 = \frac{1}{2}MR^2$$

Therefore, moment of inertia about XX` (from parallel axis theorem) will be given by

$$I_{XX'} = \frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2$$

Substituting values of m and R

$$I_{XX'} = \frac{3}{2}(\rho L)\left(\frac{L^2}{4\pi^2}\right) = \frac{3\rho L^3}{8\pi^2}$$

Sol 16: (D) $r_1 = \dfrac{m_2 r}{m_1 + m_2}$; $r_2 = \dfrac{m_1 r}{m_1 + m_2}$

$$(I_1 + I)\omega = \frac{nh}{2\pi} = n\hbar$$

$$K.E. = \frac{1}{2}(I_1 + I_2)\,\omega^2 = \frac{n^2\hbar^2(m_1 + m_2)}{2m_1 m_2 r^2}$$

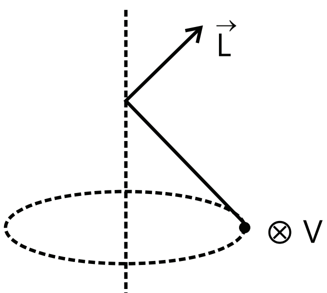
Sol 17: (B) From conservation of angular momentum about any fix point on the surface

$$mr^2\,\omega_0 = 2mr^2\,\omega$$

$$\therefore \omega = \frac{\omega_0}{2}$$

$$\therefore V_{CM} = \frac{\omega_0 r}{2}$$

Sol 18: (A)



$\vec{L}$ changes in direction not in magnitude

Sol 19: (C) $5 = e^{1000\frac{V}{T}} - 1$

$$\Rightarrow e^{1000\frac{V}{T}} = 6 \qquad \text{....(i)}$$

Again, $I = e^{1000\frac{V}{T}} - 1$

$$\frac{dI}{dV} = e^{\frac{1000V}{T}}\frac{1000}{T}$$

$$dI = \frac{1000}{T}e^{\frac{1000}{T}V}dV$$

Using (i)

$$\Delta I = \frac{1000}{T} \times 6 \times 0.01 = \frac{60}{T} = \frac{60}{300} = 0.2\text{mA}$$

Sol 20: (B) For maximum possible volume of cube

$2R = \sqrt{3}\,a$, a is side of the cube.

Moment of inertia about the required axis $= I = \rho a^3 \dfrac{a^2}{6}$,

where $\rho = \dfrac{M}{\frac{4}{3}\pi R^3}$

$$I = \frac{3M}{4\pi R^3}\frac{1}{6}\left(\frac{2R}{\sqrt{3}}\right)^5 = \frac{3M}{4\pi R^3}\frac{1}{6}\frac{32R^5}{9\sqrt{3}} = \frac{4MR^2}{9\sqrt{3}\,\pi} = \frac{4MR^2}{9\sqrt{3}\,\pi}$$

Sol 21: (A, C) $\vec{L}_0 = mv\dfrac{R}{\sqrt{2}}(-\hat{k})$ [D to A]

$\vec{L}_0 = mv\left[\dfrac{R}{\sqrt{2}} + a\right]\hat{k}$ [C to D]

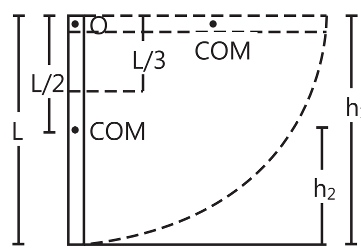
Sol 22: (D) From normal reactions of roller, we can conclude it moves towards left.

JEE Advanced/Boards

Exercise 1

Sol 1: Given,

A thin uniform rod of mass M and length L is hinged at its upper end, and is released from rest in a horizontal position.



Let angular velocity of the rod about hinge 'O' when it is vertical be 'ω'

Moment of inertia of rod about o is

$I = I_{com} + M\left(\dfrac{L}{2}\right)^2$ (parallel axis theorem)

$$\Rightarrow I = \frac{ML^2}{12} + \frac{ML^2}{4} = \frac{ML^2}{3}$$

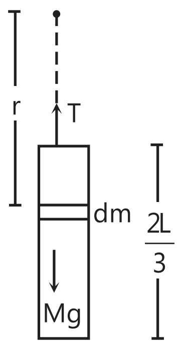
By using principle of conservation of energy

$\Delta K.E = -\Delta P.E$

$$\Rightarrow \frac{1}{2}I\omega^2 - 0 = -Mg(h_2 - h_1)_{com}$$

$$\Rightarrow \frac{1}{2}\frac{ML^2}{3}.\omega^2 = Mg\frac{L}{2} \Rightarrow \omega = \sqrt{\frac{3g}{L}}$$

The tension in the rod at a point $\frac{L}{3}$ from hinge would be due to weight below that point and centrifugal force of that part.



$$m = \rho.A.\frac{2L}{3} = \frac{2M}{3}\left[\because \rho = \frac{M}{A.L}\right]$$

$$\text{Centrifugal force} = \int_{L/3}^{L} r\omega^2.dm$$

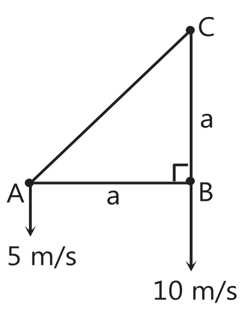
$$= \omega^2\int_{L/3}^{L} r.\rho.Adr = \omega^2\rho.A\left[\frac{r^2}{2}\right]_{L/3}^{L}$$

$$= \frac{3g}{L}.\frac{M}{L}\left[\frac{1}{2}\right][L^2]\left[\frac{8}{9}\right] = \frac{4Mg}{3}$$

Tension at the point is $T = mg + F_C$

$$= \frac{2Mg}{3} + \frac{4Mg}{3} = 2Mg$$

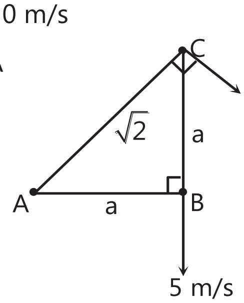
Sol 2:



Given,

$V_A = 5m/s$ $V_B = 10$ m/s

In the frame of A



A is stationary

Since, angular velocity of system would be same through

$$\omega = \frac{V_B}{a} = \frac{5}{a}\text{rad/s}$$

and

$V_C = \sqrt{2}\,a.\omega = 5\sqrt{2}$ m/s perpendicular to AC in vector form.

$V_C = +5\hat{i} + (-5)\hat{j}$ if co-ordinate system is along AB and BC

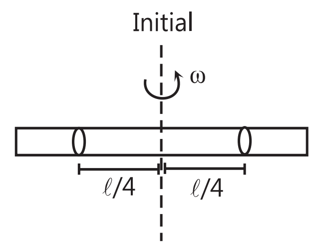
Velocity of 'C' in original frame

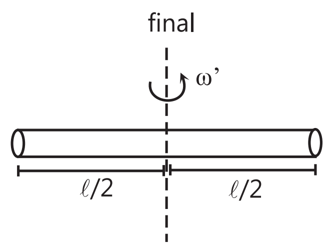
$$\vec{V}_c = \vec{V}_c + \vec{V}_a = +5\hat{i} - 5\hat{j} - 5\hat{j} \;\; \therefore (V_A = -5\hat{j})$$

$$\Rightarrow \vec{V}_c = 5\hat{i} - 10\hat{j}$$

$$|\vec{V}_c| = \sqrt{5^2 + 10^2} = 5\sqrt{5}\text{ m/s}$$

Sol 3:





Angular momentum of the system is conserved, since no torque is applied

$L_i = L_f$

$\Rightarrow I_i\omega = I_f\omega^1$

$$\Rightarrow \left(\frac{m_r\ell^2}{12} + m\left(\frac{\ell}{4}\right)^2 \times 2\right)\omega$$

$$= \left(\frac{m_r\ell^2}{12} + m\left(\frac{\ell}{2}\right)^2 \times 2\right)\omega^1$$

$$\Rightarrow \frac{0.03}{0.09} \times 30 = \omega^1$$

$\Rightarrow \omega_1 = 10$ rad/s

By energy conservation,

$\frac{1}{2}I_1\omega^2 = \frac{1}{2}I_f(\omega_1)^2 + \left(\frac{1}{2}mv^2\right) \times 2$

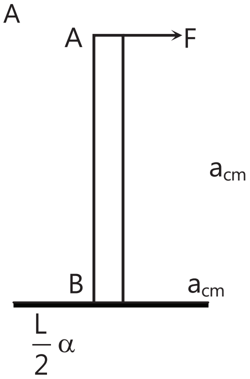
$\Rightarrow \frac{L}{2m}(\omega - \omega^1) = V^2 \Rightarrow \frac{0.03 \times 30 \times (20)}{0.2} = V^2$

$\Rightarrow v = 3$ m/s

$\therefore$ Velocity of ring along rod = 3m/s

Sol 4: Given,

A straight rod AB of mass M and length L, a horizontal force F starts on A



$F = Ma_{cm}$ (by newton's second law)

$T = I\alpha$

$\Rightarrow F.\frac{L}{2} = \frac{ML^2}{12} \times a$

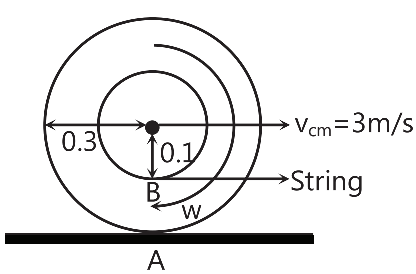
$\Rightarrow \alpha = \frac{6F}{ML}$

Acceleration of end B= $a_{cm}\hat{i} - \frac{1}{2}x\hat{i}$

$= -\frac{2F}{M}\hat{i}\left(\because a_{cm} = \frac{F}{M}\right)$

$\therefore$ Magnitude of acceleration of end B = $\frac{2F}{M}$

Sol 5:



Given, the wheel is rolling without slipping

$r_A\omega = V_{cm}(\because V_A = 0)$

(pure rolling)

The velocity of the string should be

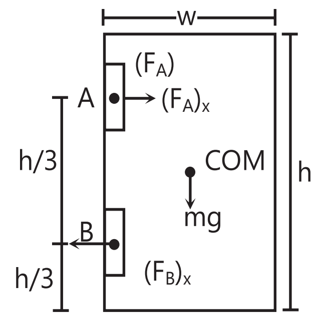
$V_B = V_{cm} - r_B\omega = V_{cm}\left(1 - \frac{r_B}{r_A}\right) = 2$m/s

Sol 6: Given,

A uniform wood door of mass m, height h and width w.

Location of hinges are $\frac{h}{3}$ and $\frac{2h}{3}$ from the bottom of the door.

Let the hinges be named A and B.



Given, hinge A is screwed while B is not, So, the upward component of force by hinge B is absent.

By equilibrium equations,

$\sum F_x = 0 \Rightarrow (F_A)_x = (F_B)_x$

$\sum F_y = 0 \Rightarrow (F_A)_y = mg$

$\sum M_{COM} = 0$

(Moment about center of mass)

$\Rightarrow (F_A)_y\left(\frac{w}{2}\right) + (F_A)_x\left(\frac{h}{6}\right) + (F_B)_x\left(\frac{h}{6}\right) = 0$

$\Rightarrow mg.\frac{\omega}{2} + (F_B)_x\left(\frac{h}{3}\right) = 0 \Rightarrow (F_B)_x = -\frac{3mg\omega}{2h}$

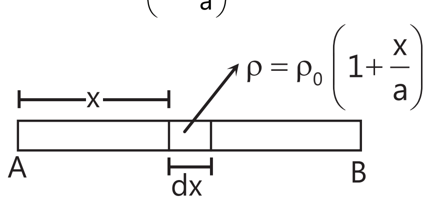
and $\vec{F_A} = (F_A)_x\hat{i} + (F_B)\hat{j}$, $\vec{F_B} = F_B(-\hat{i})$

$\vec{F_A} = -\frac{3mg\omega}{2h}\hat{i} + mg\hat{j}$, $\vec{F_B} = \frac{3mg\omega}{2h}\hat{i}$

Given m = 20 kg, h = 2.2 m, ω = 1m

$\Rightarrow \vec{F_A} = (-133.64\hat{i} + 196\hat{j})$N and $\vec{F_B} = 133.64\hat{i}$

Sol 7: Given, A thin rod of length 'a' with variable mass per unit length $\rho = \rho_0\left(1 + \frac{x}{a}\right)$ where x is distance from A.



(a) Mass of the elemental part is $dm = \rho.dx$

$$\Rightarrow dm = \rho_0\left(1+\frac{x}{a}\right).dx$$

Mass of the rod

$$m = \int_0^M dm = \int_0^a \rho_0\left(1+\frac{x}{a}\right).dx$$

$$\Rightarrow m = \rho_0\left[x+\frac{x^2}{2a}\right]_0^a \quad m = \rho_0\left(\frac{3a}{2}\right)$$

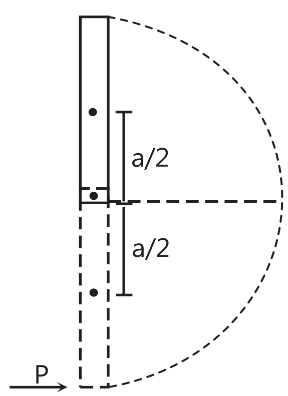
$$\therefore \text{Mass of rod} = \frac{3a\rho_0}{2}$$

(b) Center of mass is situated at distance of C from A where

$$C = \frac{\int_0^M x.dm}{\int_0^M dm}$$

$$\text{Value of } \int_0^M x.dm = \int_0^a x(\rho_0)\left(1+\frac{x}{a}\right)dx$$

$$= \rho_0\left[\frac{x^2}{2}+\frac{x^3}{3a}\right]_0^a \Rightarrow \rho_0\left(\frac{5a^2}{6}\right)$$



for minimum value of P, the angular velocity rod in the final position should be zero

by applying conservation of energy

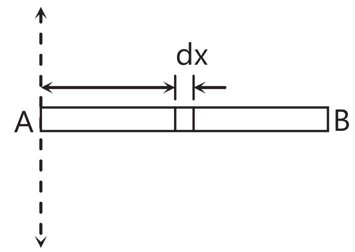
K.E = – ΔP.E.

$$\Rightarrow K.E_f - K.E_i = -mg(h_f - h_i)$$

$$0 - \frac{1}{2}I\omega^2 = -mg\ (a)$$

$$\Rightarrow C = \frac{\frac{5a^2}{6}}{\frac{3a^2}{2}} \Rightarrow C = \frac{5a}{9}$$

(c) Given, to find the moment of inertia about axis perpendicular to rod and passing through A.



$$I = \int_0^M x^2 dm \ (\because dI = x^2\, dm)$$

$$\text{but } dm = \rho_0\left(1+\frac{x}{a}\right).dx$$

$$\Rightarrow I = \int_0^a \rho_0\left(x^2+\frac{x^3}{a}\right).dx$$

$$\Rightarrow I = \rho_0\left[\frac{x^3}{3}+\frac{x^4}{4a}\right]_0^a \Rightarrow I = \frac{7\rho_0 a^3}{12}$$

(d) We know that,

Angular momentum $L = I.\omega$

$$P.a = \frac{7\rho_0 a^2}{12}.\omega$$

$$\omega = \frac{12}{7\rho_0 a^2}$$

(e) Given, an impulse of ‘P’ is applied at point B, then

Angular impulse about the axis will be

$L = P.a$

$\Rightarrow (I\omega).\omega = 2mga$

$$\Rightarrow \frac{Pa.12.P}{7\rho_0 a^2} = 2\ mga$$

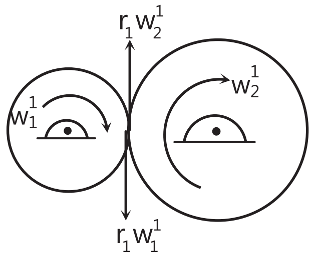
$$\Rightarrow P^2 = \frac{7}{6}\rho_0\, ga^2\left(\frac{3}{2}P_0 a\right) = \sqrt{\frac{7}{4}\rho_0^2 g a^3}$$

Sol 8: Given, two cylinders of mass 1 kg and 4 kg with radii 10 cm and 20 cm respectively.

also initial angular velocities as

ω_1 = 100 rad/s and ω_2 = 200 rad/s

final angular velocities will be such that there is no slip at point of contact



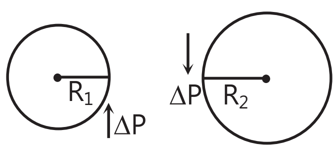
$\Rightarrow V_{contact} = 0 \Rightarrow r_1\omega_1^1 - r_2\omega_2^1 = 0$

$\Rightarrow \omega_2^1 = \dfrac{r_1\omega_1^1}{r_2}$

Angular impulse on one cylinder due to other is

$I_1(\omega_1^1 - \omega_1) = (\Delta P)(R_1)$

where ΔP = linear impulse while for the other



$I_2(\omega_2^1 - \omega_2) = (\Delta P)(R_2)$

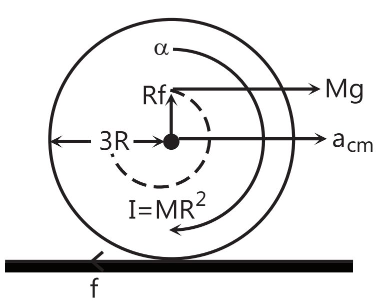
$\Rightarrow \dfrac{I_1}{I_2}\dfrac{(\omega_1^1 - \omega_1)}{(\omega_2^1 - \omega_2)} = \dfrac{R_1}{R_2}$

$\Rightarrow 8(\omega_2^1 - \omega_2) = \omega_1^1 - \omega_1$

$\Rightarrow \omega_1^1 = \dfrac{\omega_1 - 8\omega_2}{5} - 300$ rad/s

while $\omega_2^1 = -\dfrac{\omega_1}{2} = 150$ rad/s

Sol 9:



Let,

a_{cm} be acceleration of center of mass α be angular acceleration

by Newton's second law,

$Mg - f = Ma_{cm}$ .... (i)

and by considering torque

$Mg \times R + f \times 3R = I \,.\, \alpha$ .... (ii)

Since, the body is in pure rolling

$(V_{cm} = 3R\omega) \Rightarrow (a_{cm} = 3R\alpha)$

Solving (i) and (ii) we get,

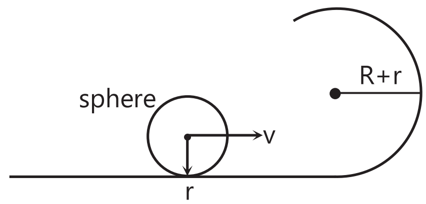
$a_{cm} = \dfrac{6}{5}.g$

Acceleration of the point where force is applied B

$\vec{a} = a_{cm}\hat{i} + R(\alpha)\hat{i} = \dfrac{3}{4}a_{cm}\hat{i} = \dfrac{8}{5}g\hat{i}$

$|\vec{a}| \cong 16$ m/s^2

Sol 10:



Given, A sphere of mass m and radius r and radius of loop as R + r the velocity of the sphere at the top most point should be such that the centrifugal force balances the weight of sphere

$\Rightarrow \dfrac{mv_f^2}{R} = mg$

($\because$ Center of mass makes circle of radius R)

$\Rightarrow v_f = \sqrt{Rg}$

Since, the sphere is in pure rolling at every point of time,

$v_{cm} = r\omega$

By principle of conservation of energy

$K.E_1 + P.E._i = K.E._f + P.E._f$

$\Rightarrow \dfrac{1}{2}mv^2 + \dfrac{1}{2}I\omega^2 + mg(r)$

$= \dfrac{1}{2}mv_f^2 + \dfrac{1}{2}I\omega_f^2 + mg(2R + r)$

$\Rightarrow \dfrac{1}{2}mv^2 + \dfrac{1}{2}\left(\dfrac{2}{5}\right)mv^2 = \dfrac{1}{2}mv_f^2 +$

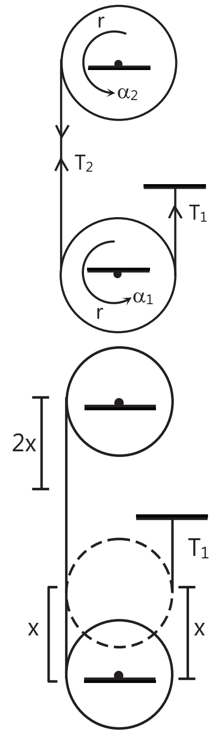
$\dfrac{1}{2}\left(\dfrac{2}{5}\right)\left(\dfrac{2}{5}\right)mv_f^2 + mg(2R)$

$\Rightarrow v^2 = v_f^2 + g(2R)\times\dfrac{10}{7}$

$\Rightarrow v = \sqrt{Rg + \dfrac{20Rg}{7}} = \sqrt{\dfrac{27Rg}{7}}$

$\therefore v = \sqrt{\dfrac{27Rg}{7}}$

Sol 11: $r.T_2 = I_2.\alpha_2;$



$\alpha_2 = \frac{a_2}{r_2}$

$mg - T_1 - T_2 = ma_1;$

$\alpha_1 = \frac{a_1}{r_1}$

(a) $(T_1 - T_2).r = I_1\alpha_1$ also,

$a_2 = 2a_1$ by constraint relations.

$\Rightarrow a_1 = \frac{2g}{7}$

$\Rightarrow T_2 = \frac{2mg}{7} = \frac{200}{7}$ N

(b) And also velocity 'v' of the body after traveling 1.2 m

$v = \sqrt{2 \times a \times s} = 4\sqrt{\frac{3}{7}}$ m/s

Sol 12: The moment of inertia of a thin hoop about it's diameter is

$\frac{1}{2} M R^2$

Here $M = L\rho$

Also we have $2\pi R = L$

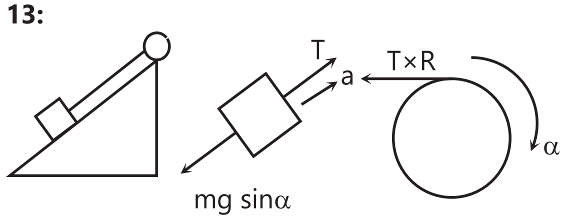
$\Rightarrow R = \frac{1}{2\pi}$

So we have, $I = \frac{1}{2} M R^2 = \frac{1}{2} L\rho\left(\frac{L}{2\pi}\right)^2 = \frac{L^3\rho}{8\pi^2}$

Now using parallel axis theorem we have

$I_{xx'} = I_{cm} + M R^2 = \frac{L^3\rho}{8\pi^2} + L\rho\left(\frac{L}{2\pi}\right)^2 = \frac{3L^2\rho}{8\pi^2}$

Sol 13:



(a) Now $T - mg\sin\alpha = ma$ ... (i)

$T \times R = -I\alpha = -\frac{MR^2}{2} \times \alpha = -\frac{MR\alpha}{2}$ ... (ii)

$R\alpha = a$ ... (iii)

$\Rightarrow T = -\frac{Ma}{2}$

$-\frac{Ma}{2} - mg\sin\alpha = ma$

$\Rightarrow -mg\sin\alpha = \left(m + \frac{M}{2}\right)a$

$\Rightarrow a = -\frac{mg\sin\alpha}{\left(m + \frac{M}{2}\right)}$

So, $T = -\frac{Ma}{2} = \frac{M}{2} \times \frac{(+mg\sin\alpha)}{\left(m + \frac{M}{2}\right)}$

$= \frac{M.m.g\sin\alpha}{(2m + M)} = \frac{2 \times (1/2) \times (9.8)(1/2)}{(2 \times 0.5 + 2)} = \frac{9.8}{6} = 1.65$ N

(b) $\omega = 10$ rad/s

$\Rightarrow \alpha = a/R = -\frac{mg\sin\alpha}{R\left(m + \frac{M}{2}\right)} = -\frac{mg}{R(2m + M)} = -\frac{(0.5)(9.8)}{0.2(1 + 2)}$

$= --\frac{5}{6} . 9.8$ rad/s^2

$= -8.166$ rad/s^2

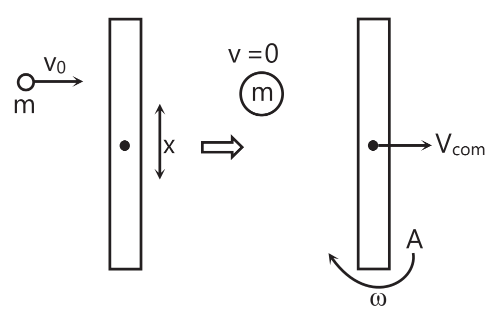
Now, $\omega^2 = \omega_0^2 + 2\alpha\theta$

$\Rightarrow 0 = (10)^2 + 2(-8.166) \times \theta$

$\Rightarrow \theta = \frac{100}{2 \times 8.166} = 6.123$ rad,

So distance $= \theta R = 6.123 \times 0.2 = 1.224$ m

Sol 14:



$v_A = V_{com} - \frac{\omega L}{2}$; so for $v_A = 0$

$\Rightarrow V_{com} = \frac{\omega L}{2}$

Now, $mv_0 = M.v_{com}$ (moment cons.)

$\Rightarrow V_{com} = \frac{mv_0}{M}$

and $mv_{0x} = I\omega$

[Angular momentum conservation about O]

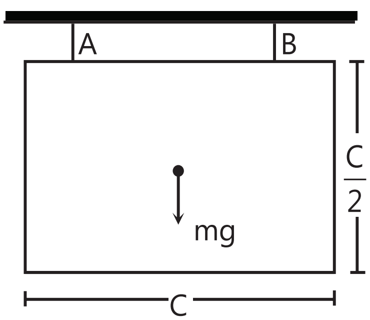
So $mv_0x = \frac{ML^2}{12} \times \omega$

$\Rightarrow \omega = \frac{12mv_0x}{ML^2}$

So $V_{com} = \omega L/2'$

$\Rightarrow \frac{mv_0}{M} = \frac{12mv_0x}{ML^2} \times \frac{L}{2} \Rightarrow x = L/6$

Sol 15:



Before connection B is released

$T_A + T_B = mg$ (By force equilibrium)

and

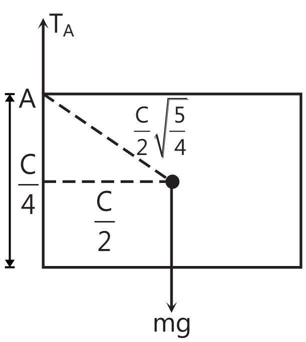
$T_A = T_B$ (for torque equilibrium)

$T_A = T_B = \frac{Mg}{2}$

$\Rightarrow$ Just after B is released

T_A is mg but $T_B = 0$

$\Rightarrow$ FBD



Moment of inertia about

$A = \frac{m}{12}\left(C^2 + \left(\frac{C}{2}\right)^2\right) + m\left(\frac{C}{2}.\sqrt{\frac{5}{4}}\right)^2$

$\Rightarrow I = \frac{mc^2}{12}\left(\frac{5}{4}\right) + \frac{5}{16}mc^2 \Rightarrow I = \frac{5}{12}mc^2$

Torque about A, $T = (mg)\left(\frac{c}{2}\right) = Ia$

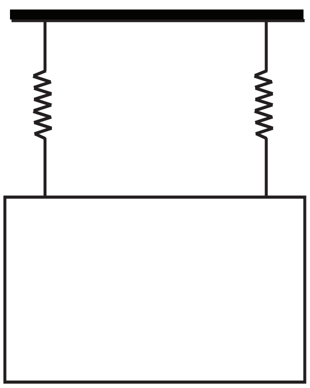
$\Rightarrow \alpha = \frac{mg\frac{c}{2}}{\frac{5mc^2}{12}} = \frac{6g}{5c} = \frac{1.2g}{c}$

Acceleration of the center is $a = \frac{\sqrt{5}c}{4} \times \left(\frac{1.2g}{c}\right)$

$= 0.3(\sqrt{5}g)$

or

$\vec{a} = -0.3(\hat{i} + 2\hat{j})$



After connection B is released

TA is still $\frac{mg}{2}$, while $T_B = 0$

linear acceleration $a = \frac{g}{2}\left(\frac{mg - \frac{mg}{2}}{m}\right)$

angular acceleration = 0.5 g

Sol 16: By energy conservation,

$K.E_i + P.E_i = K.E_f + P.E_f$

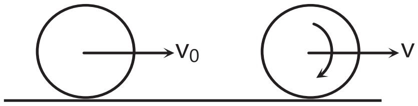
$\Rightarrow 0 + mgR$

$= \frac{1}{2}\left(\frac{m}{4}\right)(v^2)\left(1 + \frac{1}{2}\right) + \left(\frac{m}{4}\right)(g)\left(\frac{R}{2}\right)$

$\Rightarrow v = \sqrt{\frac{14gR}{3}}$

$\therefore$ Velocity of the axis of cylinder $= \sqrt{\frac{14gR}{3}}$

Sol 17:



Angular momentum about any point on ground is conserved

$\Rightarrow mv_0R = I\omega + mvR$

$mv_0R = \frac{mR^2\omega}{2} + mvR$

$\Rightarrow v = \frac{2}{3}v_0$ [$\because$ v = Rω]

Work done by frictional for time t_o

$$W = -\frac{1}{2}mv_0^2 + \left(\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2\right)$$

$$W = -\frac{mv_0^2}{2} + \left(\frac{3}{4}m.\frac{4}{9}v_0^2\right)$$

$$W = -\frac{1}{6}mv_0^2$$

Also for t > t_o No frictional force exists

$$\Rightarrow W = -\frac{1}{6}mv_0^2 \text{ for } t \geq t_o$$

Also ma = – μmg $\Rightarrow$ a = – μg

and v = v_0 – μgt

$$t_o = \frac{v_0}{3\mu g}$$

$$\alpha = \frac{T}{I} = \frac{f.R}{\frac{MR^2}{2}} = \frac{2\mu g}{R}$$

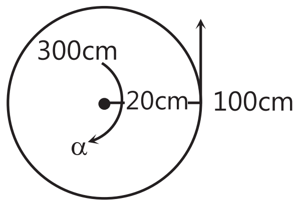
Work done by friction for t < t_o = $K.E_f - K.E_i$

$$\Rightarrow W = \left(\frac{1}{2}m(v_0 - \mu gt)^2 + \frac{1}{2}I\left(\frac{2\mu gt}{R}\right)^2\right) - \left(\frac{1}{2}mv_0^2\right)$$

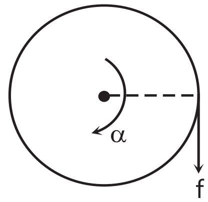
$$\Rightarrow W = \left(\frac{1}{2}m\mu^2g^2t^2 + \frac{1}{2}m(2\mu^2g^2t^2) - mv_0\mu gt\right)$$

$$\Rightarrow W = \frac{1}{2}(3m\mu^2g^2t^2 - 2mv_0\mu gt)$$

Sol 18:

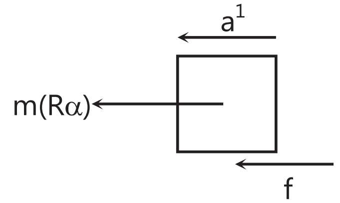


FBD of disc is



$$f.R = I\alpha \Rightarrow f.\ R = \frac{MR^2}{2}\alpha \Rightarrow \alpha = \frac{2f}{MR}$$

FBD of ant, in frame of disc



$$\Rightarrow mR\alpha + f = ma^1 \Rightarrow mR\alpha + \frac{mR\alpha}{2} = a^1$$

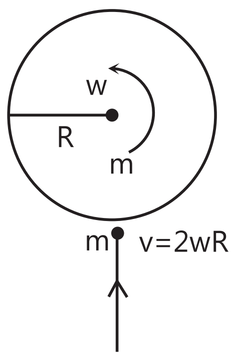
$$\Rightarrow a^1 = \frac{5R\alpha}{2} \therefore M = 3m$$

Given, after time T, the ant reaches same point

$$\Rightarrow \frac{1}{2}\left(\frac{5R\alpha}{2}\right)T^2 = 2\pi .R \Rightarrow T = \frac{2}{\sqrt{5}} \text{ seconds}$$

Also the angle moved by disc = $\frac{1}{2}\alpha T^2 = \frac{4\pi}{5}$ radians

Sol 19:



Angular momentum of the system is conserved

$L_i = L_f$

$$\Rightarrow \frac{mR^2\omega}{2} = \left(\frac{mR^2}{2} + mR^2\right)\omega^1$$

$$\Rightarrow \omega^1 = \frac{\omega}{3}$$

Impulse in the direction of velocity

= m ($v_f - v_i$) = – mv = – 2mωR

Impulse perpendicular to direction of velocity

$= m(v_f - m_i) = m(R\omega^1)$

$= \dfrac{mR\omega}{3}$

Net Impulse $= -2m\omega R\,\hat{j} + \dfrac{mR\omega}{3}\,\hat{i}$

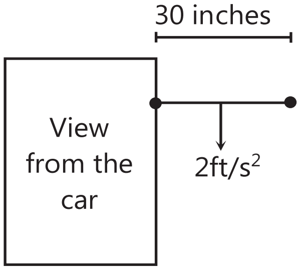
$|\vec{\Delta}p| = \dfrac{\sqrt{37}}{3} mR\omega$

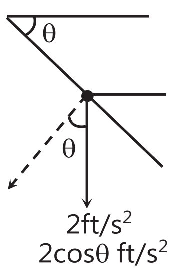
Impulse on particle due to disc =

Impulse on hinge due to disc

$\Rightarrow |\Delta\vec{P}|_{\text{disc due to hinge}} = \dfrac{\sqrt{37}}{3} mR\omega$

Sol 20:





Angular acceleration of door $= \dfrac{2\cos\theta}{\left(\dfrac{w}{2}\right)}$ ft/s^2

w is the width of the door

We know that,

$\dfrac{d\omega}{dt} = \alpha \Rightarrow \dfrac{d\omega}{d\theta}\cdot\dfrac{d\theta}{dt} = \alpha$

$\Rightarrow \omega.d\omega = \alpha\,.\,d\theta$

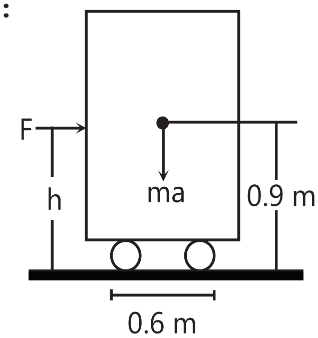
$\Rightarrow \dfrac{\omega^2}{2} - 0 = \dfrac{4}{w}\int_0^{\pi/2}\cos\theta.d\theta$

$\Rightarrow \dfrac{\omega^2}{2} = \dfrac{4}{w}[1]$

$\Rightarrow \omega = 2$

Velocity of the outer edge $= (\omega)(w) = \sqrt{8w}$

Sol 21:



F = 100 N

$F = ma \Rightarrow a = 5$ m/s^2

The cabinet will tip when

$F.\,h > mg(0.3) + ma(0.9)$

$h > \dfrac{20\times10\times0.3}{100} + \dfrac{20\times5\times0.9}{100}$

$\Rightarrow h > 1.5$ m

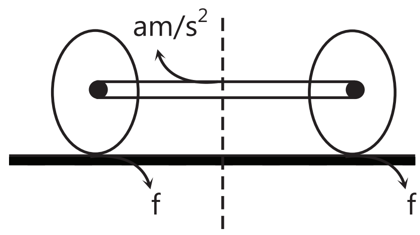
and also when

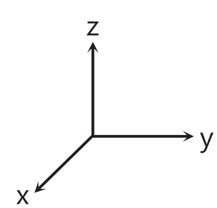
$F.h > m = (0.9) - mg\,(0.3)$

$\Rightarrow h > 0.3$ m

$\therefore\ 0.3 < h < 1.5$ m is the range of values of h for which cabinet will not tip.

Sol 22: In the frame of truck,





$2M(a) - 2f = 2Ma_1$ (force equation)

$(2f)R = (2T).\alpha$ (Moment equation)

$a^1 = R\alpha$ (pure rolling)

$f = \dfrac{I\alpha}{R} = \dfrac{Ma_1}{2}$

$\Rightarrow a = \dfrac{3a_1}{2} \Rightarrow a_1 = \dfrac{2a}{3} = 6$ m/s^2

$f = \frac{Ma_1}{2} = 6N$

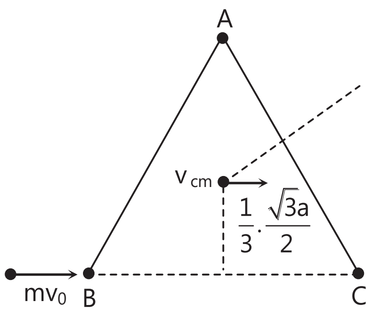
Frictional torque magnitude about rod is

f.R = 0.6 Nm

Friction torque about O is

$= \pm 0.6\left(\frac{0.2}{2}\right)(\hat{k}) - 0.6(0.1)\hat{j} = -0.6(\hat{j} \pm \hat{k})$

Sol 23:



Conserving linear momentum, we get

$(3m)(v_{cm}) = mv_0$

$\Rightarrow v_{cm} = \frac{v_0}{3}$

Conserving angular momentum about COM we get

$mv_0 = \frac{a}{2\sqrt{3}} = I.\omega$

$\Rightarrow \frac{mv_0 a}{2\sqrt{3}} = 3m\left(\frac{a}{\sqrt{3}}\right)^2.\omega$

$\Rightarrow \omega = \frac{v_0}{2\sqrt{3}a}$

Time taken to complete one revolution = $\frac{\pi}{\omega}$

$= \frac{\sqrt{3}\pi a}{v_0}$

Displacement of point B will be

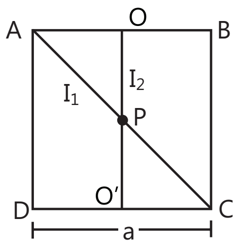
$v_{cm}\,t\,\hat{i} + \left(\frac{2a}{\sqrt{3}} \times \frac{\sqrt{3}}{2}\hat{i} + \frac{2a}{\sqrt{3}}.\frac{1}{2}\hat{j}\right)$

$\Rightarrow \left(\frac{2\pi}{\sqrt{3}} + 1\right)(a)\hat{i} + \frac{a}{\sqrt{3}}\hat{j}$

Exercise 2

Single Correct Choice Type

Sol 1: (D)



We know the moment of inertia of a square plate along OO' as $\frac{Ma^2}{12}$

$\therefore I_2 = \frac{Ma^2}{12}$

Also $I_z = I_x + I_y$

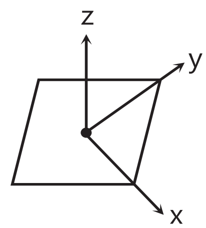
(Perpendicular axis theorem)

$\Rightarrow I_z = 2I_x$

($\because$ $I_x = I_y$ fn square)

$\Rightarrow I_z = \frac{Ma^2}{6}$

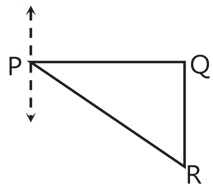
(I_z is about transverse axis through p) to find I_{APC} or I_1, we take x and y axis as diagonals of square and apply perpendicular axis theorem again



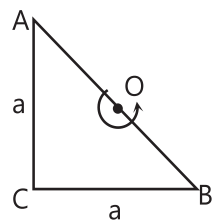
$\Rightarrow I_x = I_y = \frac{I_z}{2} \Rightarrow I_1 = I_x = \frac{Ma^2}{12}$

$\frac{I_1}{I_2} = 1$

Sol 2: (C) Given, moment of inertia of rectangular plate about transverse axis through P as I then the moment of inertia of PQR about P will be greater that $\frac{I}{2}$ since mass is distributed away from P unlike in PSR. Since, I depends on distance 'r', the farther the mass, the more the moment of inertia. The moment of inertia of PQR will be less than $\frac{I}{2}$ about R since mass is distributed closes to R

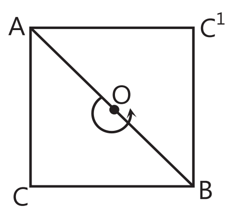


Sol 3: (B) Given, A triangle ABC such AB = BC = a and ∠ACB = 90° of mass M



O is the midpoint

Consider a counterpart with same mass such a square is formed



we know the moment of inertia of a square about transverse axis through center as $\frac{Ma^2}{6}$

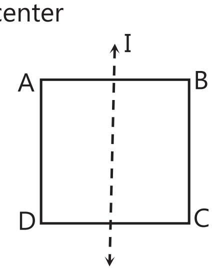
But, here m = 2M (total mass)

$$\Rightarrow I_{square} = \frac{Ma^2}{3}$$

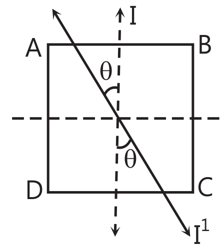
Since both triangles are symmetric about axis through O they have equal moment of inertia about axis through O.

$$\Rightarrow I_{required} = \frac{I_{square}}{2} = \frac{Ma^2}{6}$$

Sol 4: (A) Given, I is moment of inertia of a uniform square plate about axis parallel to two of its sides and passing through center

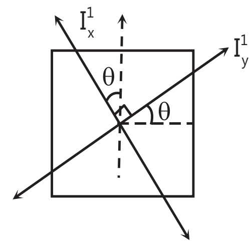


then about any axis as shown below will be



Consider, this axis as x-axis and y-axis perpendicular to a line to it

$(I^1)_{x\text{-axis}} = (I^1)_{y\text{-axis}}$ |By symmetry|



$$I_z = I_x^1 + I_y^1 = 2I_x^1$$

$$I_x^1 = \frac{I_z}{2}$$

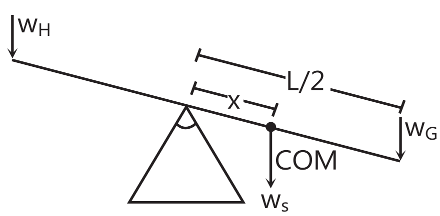
We know that I_z is independent θ since $I_z = \frac{Ma^2}{6}$

$\Rightarrow I_x^1$ is also independent of q

$\Rightarrow I_x^1(\theta) = I_x^1(0) = I$

Sol 5: (B) Given, see-saw is out of balance.

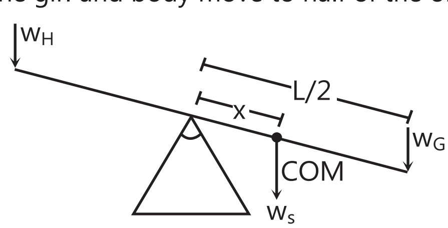
⇒ Centre of mass is not at the center of see-saw.



Let 'x' be the distance of COM from center. By moment equilibrium at center

$$[w_H - w_G]\frac{L}{2} = w_S \cdot x \Rightarrow x = \frac{(w_H - w_G)}{w_S}\frac{L}{2}$$

Now if the girl and body move to half of the original



moment due to heavy body and girl is $(w_H - w_G)\frac{L}{4}$

(opposite in direction to w_S.x)

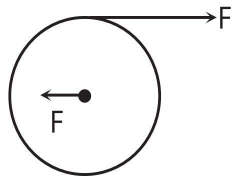
while

$w_S \,.\, x = (w_H - w_G)\frac{L}{2}$

$\therefore w_S \,.\, x > (w_H - w_G)\frac{L}{4}$

The side the girl is sitting on will once again tilt downward

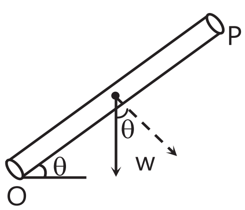
Sol 6: (A)



The force on the hinge is same as the force on the thread this can be found by using force equilibrium conditions.

Since, there is a torque always about hinge on pulley, angular velocity increases.

Sol 7: (D)



torque acting on the pole due to weight about point O is

$T = \vec{r} \times \vec{F} = \frac{L}{2} W \cos\theta$

T = Ia

$\Rightarrow \alpha = \frac{\frac{L}{2} W\cos\theta}{\frac{ML^2}{3}} = \frac{3mg\cos\theta}{2mL} = \frac{3g\cos\theta}{2L}$

Acceleration of point P=L.$\alpha = \frac{3g\cos\theta}{2}$

Sol 8: (B)

Moment of inertia of rod about point I_{COM}

$= \frac{Md^2}{12}$

Angular momentum is conserved as no torque is acting on the system. (while energy is not since a force acts on rod at point)

∴ Initial angular momentum = final angular momentum

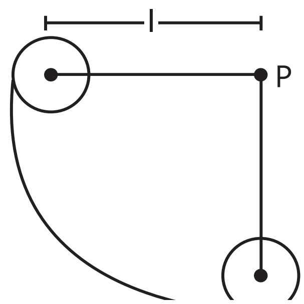
$I_1\omega_1 + m\frac{d}{2}v_1 = I_2\omega_2$

$\Rightarrow \frac{Md^2}{3}(0) + m\frac{dv}{2} = \left(\frac{Md^2}{12} + \frac{Md^2}{4}\right)\omega^1$

$\Rightarrow \frac{md}{2}v = \frac{3}{4}md^2\omega^1$

$\Rightarrow \omega^1 = \frac{2v}{3d}$

Sol 9: (D)



By the principle of conservation of energy

$K.E_i + P.E_i = K.E_f + P.E._f$

$\Rightarrow 0 + mg(l) = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + 0$

$\Rightarrow mgl = \frac{1}{2}mr^2w^2 + \frac{1}{2}.\frac{2}{5}mr^2\omega^2$

$\Rightarrow \omega = \sqrt{\frac{10}{7}\frac{gl}{r^2}}$

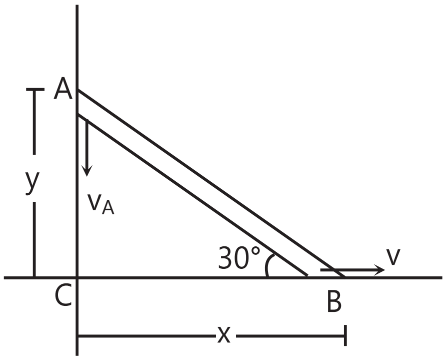
Angular momentum of the sphere about P is

L = Iω + m.l.v

$\Rightarrow L = \frac{2}{5}mr^2\,\omega + m.\,l.\,r\,\omega$

$\Rightarrow L = m.\sqrt{\frac{10}{7}gl}\,.\left[\frac{2}{5}r + l\right]$

Sol 10: (C)



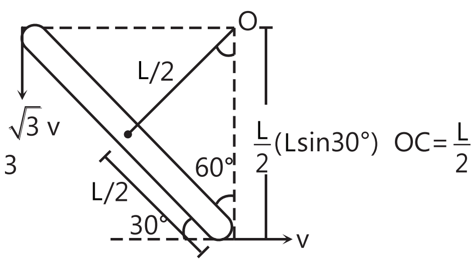
$x^2 + y^2 = l^2$

L = length of ladder = constant

$\Rightarrow x.\dfrac{dx}{dt} + y.\dfrac{dy}{dt} = 0$

$\Rightarrow x.\,v + y.v_A = 0$

$\Rightarrow v_A = -\dfrac{x}{y}.v = -\sqrt{3}\,v$



Angular velocity of rod

$= \dfrac{|-\sqrt{3}v\hat{j} + v\hat{i}\,|}{L} = \dfrac{2v}{L}$

Velocity of center $= \dfrac{L}{2}.\omega = v$

Sol 11: (A) If $\dfrac{dv}{dt} = 0$

$x^2 + y^2 = l^2$

$\Rightarrow x.\dfrac{dx}{dt} + y.\dfrac{dy}{dt} = 0$

$\Rightarrow x.\,v + y\,v_A = 0$

$\Rightarrow \dfrac{dx}{dt}.v + x.\dfrac{dv}{dt} + \dfrac{dy}{dt}.\,v_A + y.\dfrac{dv_A}{dt} = 0$

$\Rightarrow v^2 + v_A^2 + y.\dfrac{dv_A}{dt}$

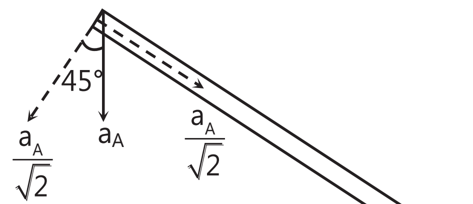
$\Rightarrow \dfrac{dv_A}{dt} = \dfrac{0-(v^2+v_A^2)}{y}$

when $\alpha = 45°$

$v_A = -v$ and $y = \dfrac{\ell}{\sqrt{2}}$

$\Rightarrow \dfrac{dv_A}{dt} = -\dfrac{2v^2.\sqrt{2}}{\ell}$

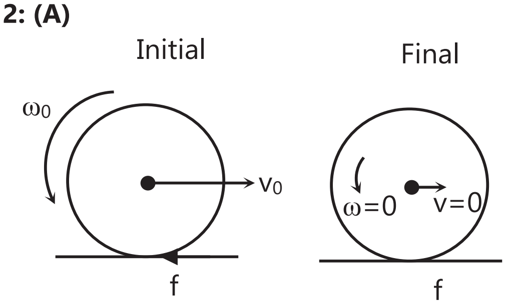
Angular acceleration =



$\alpha = \dfrac{\frac{a_A}{\sqrt{2}}}{(L)}$

$\alpha = \dfrac{2v^2}{L^2}$

Sol 12: (A)



Frictional forces acts to reduce the velocity of bottom most point.

$f = -ma_{cm}$ (By Newton's second law)

$T = I\alpha$

$\Rightarrow f.R = \dfrac{MR^2}{2}.\,a \Rightarrow f = \dfrac{MR\alpha}{2}$

After time t = t, angular velocity and lineal velocity becomes zero.

$\Rightarrow 0 - v_0 = -a_{cm}t$ and $0 - \omega_0 = -\alpha t$

$\Rightarrow \dfrac{v_0}{w_0} = \dfrac{a_{cm}}{\alpha} = \dfrac{\frac{f}{M}}{\frac{2f}{MR}}$

$\Rightarrow \dfrac{v_0}{\omega_0} = \dfrac{R}{2} \Rightarrow \dfrac{v_0}{R\omega_0} = \dfrac{1}{2}$

Sol 13: (C) Conserving linear momentum

$2mv - mv = 2m \times v_{cm}$

$v_{cm} = \frac{v}{2}$

Initial angular momentum

$= m \times 2v \times \frac{b}{2} + mv \times \frac{b}{2} = \frac{3vbm}{2}$

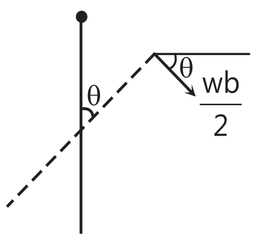
Final angular momentum

$= \left[m\left(\frac{b}{2}\right)^2 + m\left(\frac{b}{2}\right)^2\right] \times \omega = \frac{mb^2\omega}{2}$

$\Rightarrow \frac{3mvb}{2} = \frac{mb^2\omega}{2}$

$\therefore \omega^2 = \frac{3v}{b}$

For skater at x = b/2



$v_x = v + \frac{\omega b}{2}\cos\theta$

$v_y = -\frac{\omega b}{2}\sin\theta$

$\theta = \omega t$

$\therefore v_x$ at $t = \frac{v}{2}$

$v_x = v + \frac{3v}{2}\cos\left(\frac{3vt}{b}\right)$

$\therefore x = \int v_x\,dt = \int v + \frac{3V}{2}\cos\left(\frac{3vt}{b}\right)$

$= vt + \frac{3V}{2 \times 3V} \times b \times \sin\left(\frac{3vt}{b}\right)$

$= vt + \frac{b}{2}\sin\left(\frac{3vt}{b}\right)$

$y = \int v_y\,dt = \int -\frac{\omega b}{2}\sin(\omega t)$

$= \frac{+\omega b}{2 \times \omega}\cos(\omega t) = \frac{b}{2}\cos\left(\frac{3vt}{b}\right)$

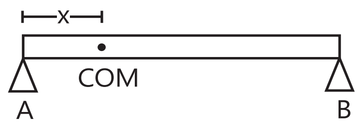
Sol 14: (C) When F_1 is applied, the body moves right and angular acceleration is developed accordingly by friction

when F_3 is applied, the angular acceleration developed it the body move left.

When F_2 is applied the body can move either left of right depending on angle of inclination.

Multiple Correct Choice Type

Sol 15: (B, C)



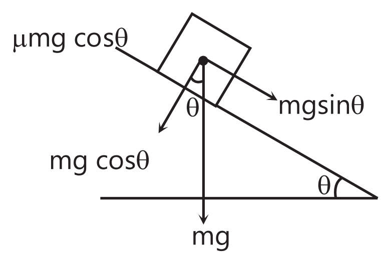
FBD of rod

$R_A + R_B = w$ (force equilibrium)

$R_B.d = w.x$ (torque equilibrium)

$\Rightarrow R_B = \frac{wx}{d}$ and $R_A = \frac{w(d-x)}{d}$

Sol 16: (A, D)



Sliding condition = mg sinθ > μmgcosθ

$\Rightarrow \tan\theta > m$

Toppling condition = $mg\sin\theta.\frac{h}{2} > mg\cos\theta.\frac{a}{2}$

$\Rightarrow \tan\theta > \frac{a}{h}$

If $\mu > \frac{a}{h}$

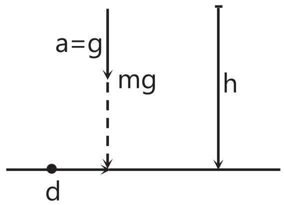
$\tan\theta > \frac{a}{h}$ is met earlier than $\tan\theta > m$

$\therefore$ Topples before sliding

If $\mu < \frac{a}{h}$

It will slide before toppling

Sol 17: (A, C, D)



Angular momentum = mvd = mgtd

Torque of gravitational force = $\frac{dL}{dt}$ = mgd

Moment of inertia = $m(d^2 + h^2)$

where $h = H_0 - \frac{1}{2}gt^2$

Angular velocity = $\frac{v}{d} = \frac{gt}{d}$

Sol 18: (A, B, C) $\vec{T} = \vec{A} \times \vec{L}$

$\Rightarrow \frac{d\vec{L}}{dt} = \vec{A} \times \vec{L}$

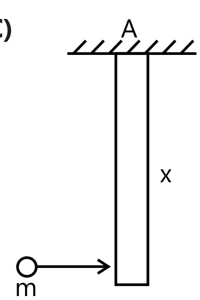
$\therefore \frac{d\vec{L}}{dt} \perp \vec{L}$

Components of $\vec{L}$ on $\vec{A}$ remain unchanged because if $\vec{L}$ component changes the L.H.S changes while R.H.S remains unchanged which is a contradiction. If magnitude of L changes with time, thin L.H.S and R.H.S vary differently with time which is a contradiction.

Suppose $\vec{L} = (x.t)\frac{\vec{L}}{|\vec{L}|}$

Then $\frac{d\vec{L}}{dt} = \frac{x\vec{L}}{|\vec{L}|}$ while $\vec{A} \times \vec{L} = xt\frac{\vec{A}\times\vec{L}}{|\vec{L}|}$

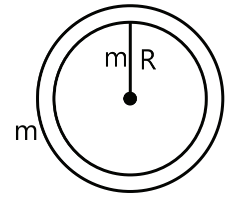
Sol 19: (B, C)



Linear momentum is not conserved because of hinge force angular momentum about A is conserved since torque at A is zero.

Kinetic energy of system before collision is equal to kinetic energy of system just after collision since, the collision is elastic

Sol 20: (B, C)



Kinetic energy of the body

$= \frac{1}{2}mv^2 + \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$

$= \left(1 + \frac{1}{2}.\frac{2}{3}\right)mv^2 = \frac{4}{3}mv^2$ ($\because$ v = Rω)

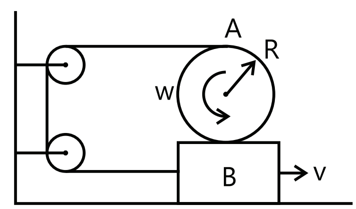
($\because I = \frac{2}{3}mR^2$ only hollow sphere

$\because$ Non-viscous liquid)

Angular momentum about any point on ground

$= 2mRv + \frac{2}{3}mR^2\omega = \frac{8}{3}mRv$

Sol 21: (B, C)



Since the cylinder does not slip

At point B velocity = 0

$\Rightarrow -\vec{V} + \vec{V}_{cm} + R\vec{\omega} = 0$

$\Rightarrow \vec{V}_{cm} = -(v - R\omega)\hat{i}$

At point A, velocity = 0

$\Rightarrow v = R\omega$

$\Rightarrow \vec{V}_{cm} = 0$

Sol 22: (B, C, D) To the right of B, angular acceleration will disappear but linear acceleration will increase since no friction is present angular velocity attained by disc after time T is

$\omega = \alpha T$

and $2\pi = \frac{1}{2}\alpha T^2$

Time to complete one rotation

$2\pi = \omega t$

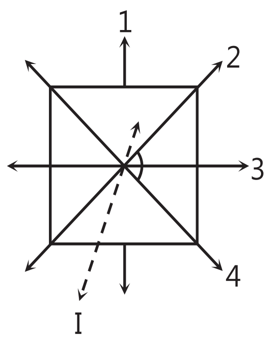
$\Rightarrow t = \dfrac{2\pi}{\alpha T} = \dfrac{2\pi.T}{2\pi.2} = \dfrac{T}{2}$

Sol 23: (C, D) Given,

$180° < Q_f - Q_i > 360°$

$\pi \text{ rad} < Q_f - Q_i < 2\pi \text{ rad}$

Sol 24: (A, B, C, D)



$I = I_1 + I_3$ (Perpendicular axes theorem)

Also $I_1 = I_3$ (by symmetry)

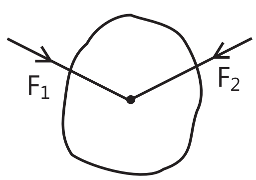
$\Rightarrow \dfrac{I}{2} = I_1 = I_3$

$I = I_2 + I_4$ (perpendicular axis theorem)

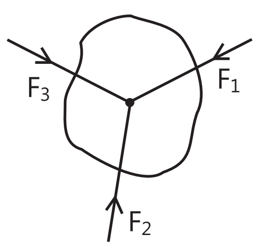
Also $I_2 = I_4$ (by symmetry)

$\Rightarrow \dfrac{I}{2} = I_2 = I_4$

Sol 25: (B, C) Option A is incorrect, since the statement indicates a force body system as below.

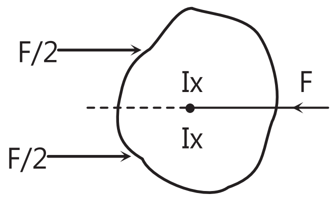


Which is not in equilibrium while B, C are possible

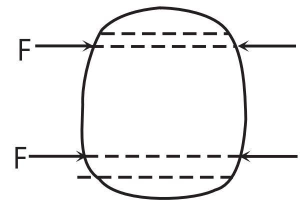


(b) $\vec{F_1} + \vec{F_2} + \vec{F_3} = 0$

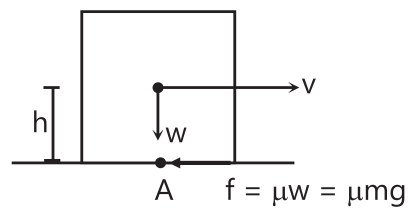
(c)



(d) While D is possible



Sol 26: (A, B, D)



L = mvh (Angular momentum)

$\text{acceleration} = -\dfrac{\mu mg}{m} = -\mu g$

$V_t = v - \mu gt$

$L_t = m(v - \mu gt)h$

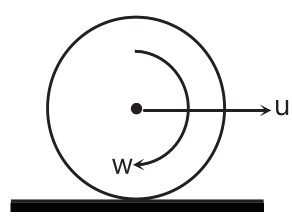
$\dfrac{dL_t}{dt} = T = -\mu mgh$

Sol 27: (A, C, D) If Re spreads or curls up his hands, moment of inertia changes, accordingly angular velocity changes too.

If $I\omega$ = Constant, it cant keep

$\dfrac{1}{2}I\omega^2$ the same, rotational kinetic energy would also change.

Sol 28: (A, C, D)

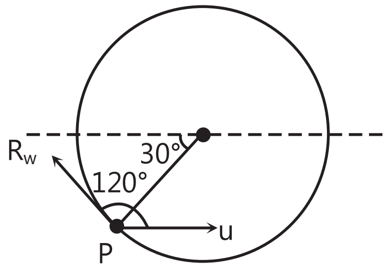


$u = R\omega$ (for pure rolling)

The velocity of bottom most point is $u - R\omega = 0$

the velocity of topmost point is u + Rω = 2u

$\therefore\ 0 \le v \le 2u$



Velocity of P is

$$\vec{V} = u\hat{i} + \left(-\frac{u}{2}\hat{i} + \frac{\sqrt{3}u}{2}\hat{j}\right)$$

$$\Rightarrow \vec{V} = \frac{u}{2}\hat{i} + \frac{\sqrt{3}u}{2}\hat{j}$$

$\Rightarrow V = u$

If CR is horizontal

$$\vec{V} = u\hat{i} + u\hat{j}$$

$V = \sqrt{2}\,u$

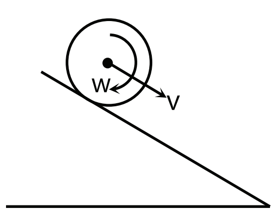
Sol 29: (B, C, D) Angular moment about O is not constant because a component of weight causes torque at point 'O'.

Angular moment about C is zero since weight is parallel axis

About O,

$\vec{L} = m(\vec{r} \times \vec{v})$ gives angular momentum in direction perpendicular to length of thread and velocity. The vertical component never changes direction.

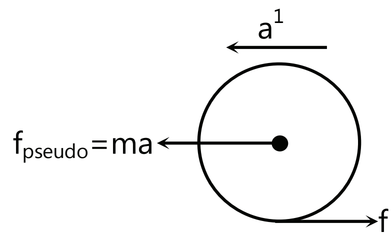
Sol 30: (A, C)



A cylinder rolling down with incline may or may not attain pure rolling. It depends on length of the incline

Sol 31: (B, C) Friction on cylinder under pure rolling depends on the external forces

Sol 32: (A, C)



In frame of plank,

$F_{pseudo} - f = ma_1$

$a_1 = a - \frac{f}{m}$ where a is acceleration of plank

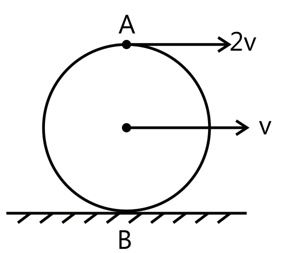
$a = \frac{F - f}{M}$

Total K.E. of system = work done by force F

($\because$ no other external forces is doing work)

Work done on sphere = work done by friction + work done by pseudo force = change in K.E.

Sol 33: (A, B, C)



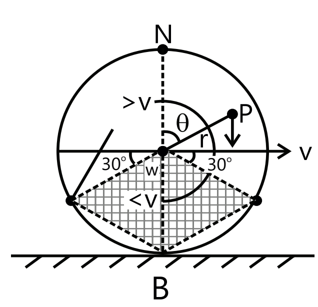
$V_A = V + R\omega = 2V$

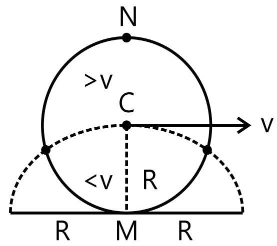
$V_B = V - R\omega = 0$

L about B = $mvR + \frac{1}{2}mR^2\omega = \frac{3mRV}{2}$ clockwise

L about A = $-mvR + \frac{1}{2}mR^2\omega = \frac{mRV}{2}$ anti-clockwise

Sol 34: (A, B, C)





$\vec{V}_P = (V + r\omega\cos\theta)\hat{i} + (r\omega\sin\theta)\hat{j}$

$|\vec{V}_P|^2 = V^2 + r^2\omega^2 + 2r\omega\cos\theta V$

$= (R^2 + r^2 + 2r R\cos\theta)\omega^2$

$|\vec{V}_P|^2 = M^2P^2\omega^2$

$\Rightarrow |\vec{V}_P| = (\overline{M}P).\omega$

Using the above relation we draw a circle with radius R to get point with velocities R.ω = v

Sol 35: (A, B, C) As the ring enters frictional force (limiting) acts on sphere increases angular acceleration in clockwise direction and slows down the linear motion.

Sol 36: (B, C, D) Rolling motion starts when the point of contact has zero velocity. Conserving angular momentum about point of ground gives

$mvR = I\omega + mv'R$

$\Rightarrow mvR = mR^2\omega^1 + mR^2\omega^1$ $(\because v^1 = R\omega^1)$

$\Rightarrow \omega^1 = \dfrac{v}{2R}$

$v^1 = \dfrac{v}{2}$

Time taken to achieve pure rolling is

$v - u = at \Rightarrow \dfrac{v}{2} - v = -\mu gt \Rightarrow t = \dfrac{v}{2\mu g}$

Sol 37: (A, B, C, D) Distance moved by ring

$= \dfrac{v^2 - u^2}{2a} = \dfrac{3v_0^2}{8\mu g}$

Work done by friction $= -\mu mg.\dfrac{3v_0^2}{8\mu g}$

$= -\dfrac{3mv_0^2}{8}$

Gain in rotational K.E. $= \dfrac{1}{2}mR^2.\dfrac{v_0^2}{4R^2} = \dfrac{mv_0^2}{8}$

Loss in K.E. $= -\left(-\dfrac{3mv_0^2}{8} + \dfrac{1}{8}mv_0^2\right) = \dfrac{mv_0^2}{4}$

Sol 38: (B, C) By holding a pole horizontally, the moment of inertia is increased leading to slower angular acceleration due to undesired torques.

Also, adjusts center of gravity to be vertically over rope to eliminate torque.

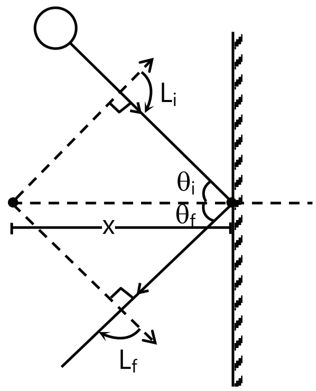
Assertion Reasoning Type

Sol 39: (B) A cyclist always bends inwards to reduce the centrifugal force.

Also, he lowers the center of gravity.

But the reason does not explain the assertion

Sol 40: (B)



Statement-I is true because the initial angular moment about any point on xy plane is $L_i = mv_i\sin\theta_i x$ and final angular momentum $L_f = mv_f x\sin\theta_f$

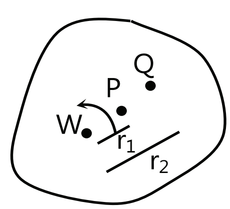
for elastic collision $v_f = v_i$, $\theta_i = \theta_f$

$\Rightarrow L_i = L_f$

Statement-II is also correct since the disc is in equilibrium

But II is explanation of I

Sol 41: (A)



In frame of O

$\omega_P = \omega_Q = \omega$

$v_P = r_P\omega_P$ $v_Q = r_Q\omega_Q$

In frame of P

$v_0 = -r_P\omega_P$ $v_Q = r_Q\omega_Q - r_P\omega_P$

$w_0 = -\dfrac{v_0}{r_P} = -\omega_P = \omega$

$$w_Q^1 = \frac{(r_Q - r_P)(\omega)}{(r_Q - w_P)} = \omega$$

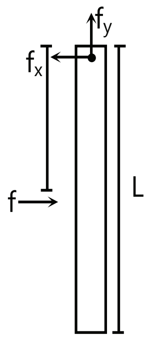
∴ Statement-I is true and statement-II is correct explanation of I

Sol 42: (B) Statement-I true by parallel axis theorem.

Statement-II is true but doesn't explain parallel axis theorem.

Sol 43: (B) See above

Sol 44: (D)



$F_x = F$ (force equilibrium in x-direction)

∴ Assertion is false

While reason is true since

$T = I\alpha = F.x \Rightarrow \alpha = \frac{F}{I}.x$

Sol 45: (B) Statement-I is true, which is the condition for pure rolling

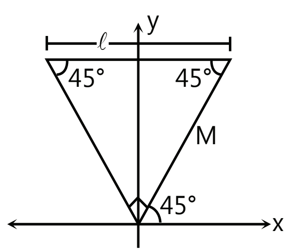
Statement-II is also correct by the definition of center of mass but II is not correct explanation of I.

Sol 46: (D) Statement-I is false, because a body can roll if we throw it with property determined linear and angular velocities.

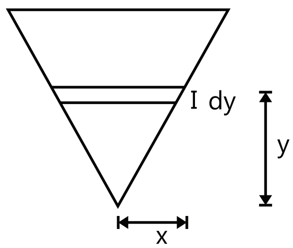
Statement-II is true by definition of pure rolling

Comprehension Type

Sol 47: (C)



Taking an elemental point dy



$dm = \rho.(2x)dy$

$= \rho.(2y)dy$ [∵ y = x]

$$\rho = \frac{M}{\frac{1}{4}\ell^2} = \frac{4M}{\ell^2}$$

$$dI_2 = \frac{dm(2x)^2}{12} + dm(y)^2$$

$$\left[\frac{m\ell^2}{12} \text{for rod, parallel axis theorem}\right]$$

$$\Rightarrow \int dI_2 = \int_0^{\ell/2}\left(\frac{y^2}{3} + y^2\right)(2y)\rho\, dy \ (\because y = x)$$

$$\Rightarrow \int dI_1 = I_2 = \frac{8\rho}{3}\int_0^{\ell/2} y^3.dy = \frac{8\rho}{3}.\left[\frac{y^4}{4}\right]_0^{\ell/2}$$

$$= \frac{2\rho}{3}\left[\frac{\ell^4}{16}\right] = \frac{ML^2}{6}$$

Sol 48: (A) $dI_x = (dm)(y^2)$

$$\Rightarrow \int dI_x = \int_0^{\ell/2} 2\rho y^3 dy$$

$$\Rightarrow I_x = 2\rho\left[\frac{y^4}{4}\right]_0^{\ell/2} = \frac{8M}{\ell^2}.\frac{\ell^4}{64} = \frac{M\ell^2}{8}$$

Sol 49: (C) Moment of inertia about base is

$$dI = dm\left(\frac{L}{2} - y\right)^2$$

$$\Rightarrow \int dI = \int_0^{L/2}\left(\frac{L}{2} - y\right)^2 (2\rho y)\, dy$$

$$\Rightarrow I = 2\rho\int_0^{L/2} y^2\left(\frac{L}{2} - y\right).\, dy$$

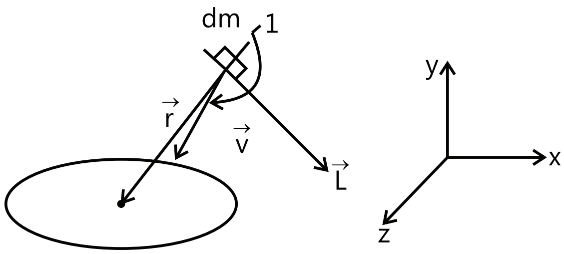
$$\Rightarrow I = 2\rho\left[\frac{L}{2}\cdot\frac{y^3}{3} - \frac{y^4}{4}\right]_0^{L/2}$$

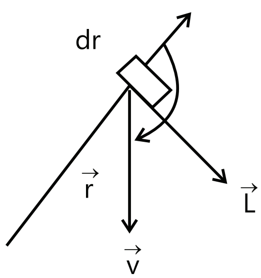
$$\Rightarrow I = 2\rho L^4\left[\frac{1}{48} - \frac{1}{64}\right] = 8ML^2\left(\frac{4-3}{64\times 3}\right) = \frac{ML^2}{24}$$

Sol 50: (C) $I_z = I_x + I_y$ (perpendicular axis theorem)

$$\Rightarrow I_y = \frac{ML^2}{6} - \frac{ML^2}{8} = \frac{ML^2}{24}$$

Sol 51: (B) $d\vec{L} = dm\,(\vec{r} \times \vec{v})$



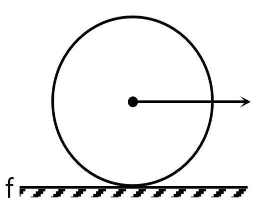


(z direction)

∴ Angular momentum is down at 20° to horizontal

Sol 52: (B) There is a torque since angular momentum is always changing direction.

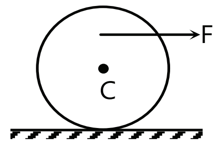
Sol 53: (A)



Friction reduces linear acceleration and increases angular velocity

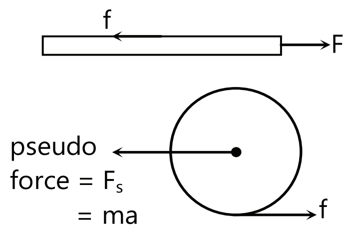
Sol 54: (A) Same as previous

So 55: (D)



Frictional force in this depends on distance between application of force and center

Sol 56: (B)



By basic FBD's we can understand that friction acts in forward direction

∴ Option B is correct

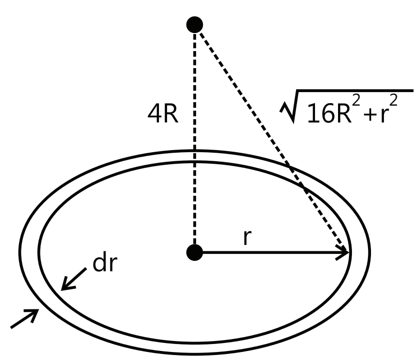
And the torque is acting horizontally, since the horizontal component of angular momentum is only changing.

Previous Years' Questions

Sol 1: (A) $W = \Delta U = U_f - U_i = U_\infty - U_P$

$= -U_P = -mV_P = -V_P$ (as m = 1)

Potential at point P will be obtained by integration as given below.



Let dM be the mass small ring as shown

$$dM = \frac{M}{\pi(4R)^2 - \pi(3R)^2}(2\pi r)dr = \frac{2Mrdr}{7R^2}$$

$$dV_P = -\frac{G\,dM}{\sqrt{16R^2 + r^2}}$$

$$= -\frac{2GM}{7R^2}\int_{3R}^{4R}\frac{r}{\sqrt{16R^2+r^2}}dr = -\frac{2GM}{7R}(4\sqrt{2}-5)$$

$$\therefore W = +\frac{2GM}{7R}(4\sqrt{2}-5)$$

Sol 2: (A) $\frac{2}{5}MR^2 = \frac{1}{2}Mr^2 + Mr^2$

Or $\frac{2}{5}MR^2 = \frac{3}{2}Mr^2$

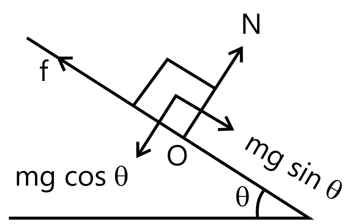
$\therefore r = \frac{2}{\sqrt{15}}R$

Sol 3: (B) Condition of sliding is

ms sin θ > μ mg cos θ or tan θ > μ

or tan $\theta > \sqrt{3}$ ... (i)

Condition of toppling is



Torque of mg sin θ about O>torque of mg about

$\therefore$ (mg sin θ)$\left(\frac{15}{2}\right)$ > (mg cos θ)$\left(\frac{10}{2}\right)$

or tan $\theta > \frac{2}{3}$ ... (ii)

With increase in value of θ, condition of sliding is satisfied first.

Sol 4: (A) $I_{remaining} = I_{whole} - I_{removed}$

or $I = \frac{1}{9}(9M)(R)^2 - \left[\frac{1}{2}m\left(\frac{R}{3}\right)^2 + \frac{1}{2}m\left(\frac{2R}{3}\right)^2\right]$ ....(i)

Here, $m = \frac{9M}{\pi R^2}\times\pi\left(\frac{R}{3}\right)^2 = M$

Substituting in Eq. (i), we have $I = 4MR^2$

Sol 5: (A) A'B' $\perp$ AB and C'D' $\perp$ CD

From symmetry $I_{AB} = I_{A`B`}$ and $I_{CD} = I_{C`D`}$ From theorem of perpendicular axes,

$I_{zz} = I_{AB} + I_{A`B`} = I_{CD} + I_{C`D`} = 2I_{AB} = 2I_{CD}$

$I_{AB} = I_{CD}$

Alternate The relation between I_{AB} and I_{CD} should be true for all values of Nθ

At θ = 0, $I_{CD} = I_{AB}$

Similarly, at $\theta = \pi/2$, $I_{CD} = I_{AB}$

(By symmetry)

Keeping these things in mind, only option (a) is correct.

Sol 6: (D) In case of pure rolling,

$f = \frac{mg\sin\theta}{1+\frac{mR^2}{I}}$ (Upwards)

$\therefore f \propto \sin\theta$

Therefore, as θ decreases force of friction will also decrease.

Sol 7: (A) On smooth part BC, due to zero torque, angular velocity and hence the rotational kinetic energy remains constant. While moving from B to C translational kinetic energy converts into gravitational potential energy.

Sol 8: (B) From conservation of angular momentum (I ω = constant), angular velocity will remains half. As

$K = \frac{1}{2}I\omega^2$

The rotational kinetic energy will become half. Hence, the correct option is (b).

Sol 9: (A) Let ω be the angular velocity of the rod. Applying angular impulse = change in angular momentum about center of mass of the system

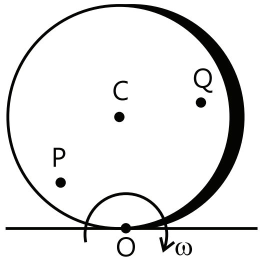
$J.\frac{L}{2} = I_C\omega$

$\therefore (Mv)\left(\frac{L}{2}\right) = (2)\left(\frac{ML^2}{4}\right)\omega \therefore \omega = \frac{v}{L}$

Sol 10: (A) In case of pure rolling bottom most point is the instantaneous center of zero velocity.

Velocity of any point on the disc,, where r is the distance of point from O.

$r_Q > r_C > r_P$



$\therefore\ v_Q > v_C > v_P$

Paragraph 1

Sol 11: (C) $\frac{1}{2}I(2\omega)^2 = \frac{1}{2}kx_1^2$ ...(i)

$\frac{1}{2}(2I)(\omega)^2 = \frac{1}{2}kx_2^2$ ..... (ii)

From Eqs. (i) and (ii), we have $\frac{x_1}{x_2} = \sqrt{2}$

Sol 12: (A) Let ω' be the common velocity. Then from conservation of angular momentum, we have

$$(I + 2I)\omega' = I(2\omega) + 2I(\omega)$$

$$\omega' = \frac{4}{3}\omega$$

From the equation,

Angular impulse = change in angular momentum, for any of the disc, we have

$$\tau.t = I(2\omega) - I\left(\frac{4}{3}\omega\right) = \frac{2I\omega}{3}$$

$$\therefore\ \tau = \frac{2I\omega}{3t}$$

Sol 13: (B) Loss of kinetic energy = $K_i - K_f$

$$= \left\{\frac{1}{2}I(2\omega)^2 + \frac{1}{2}(2I)(\omega)^2\right\} - \frac{1}{2}(3I)\left(\frac{4}{3}\omega\right)^2 = \frac{1}{3}I\omega^2$$

Sol 14: (D) $\frac{1}{2}mv^2 + \frac{1}{2}I\left(\frac{v}{R}\right)^2 = mg\left(\frac{3v^2}{4g}\right) \therefore\ I = \frac{1}{2}mR^2$

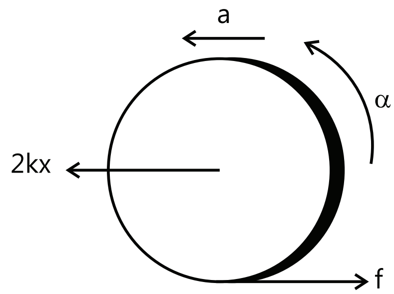
$\therefore$ Body is disc.

Paragraph 2

Sol 15: (D) $a = R\alpha$

$$\therefore\ \frac{2kx - f}{M} = R\left[\frac{fR}{\frac{1}{2}MR^2}\right]$$

Solving this equation, we get



$$f = \frac{2kx}{3}$$

$$\therefore\ |F_{net}| = 2kx - f = 2kx - \frac{2kx}{3} = \frac{4kx}{3}$$

This is opposite to displacement.

$$\therefore\ F_{net} = -\frac{4kx}{3}$$

Sol 16: (D) $F_{net} = -\left(\frac{4kx}{3}\right)x$

$$\therefore a = \frac{F_{net}}{M} = -\left(\frac{4k}{3M}\right)x = -\omega^2 x$$

$$\therefore\ \omega = \sqrt{\frac{4k}{3M}}$$

Sol 17: (C) In case of pure rolling mechanical energy will remains conserved.

$$\therefore\ \frac{1}{2}Mv_0^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(\frac{v_0}{R}\right)^2 = 2\left[\frac{1}{2}kx_{max}^2\right]$$

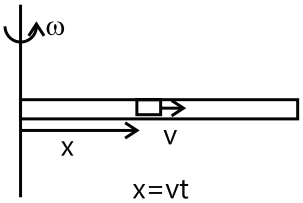
$$\therefore\ x_{max} = \sqrt{\frac{3M}{4k}}v_0$$

As $f = \frac{2kx}{3}$

$$\therefore\ F_{max} = \mu Mg = \frac{2kx_{max}}{3} = \frac{2k}{3}\sqrt{\frac{3M}{4k}}v_0$$

$$\therefore\ v_0 = \mu g\sqrt{\frac{3M}{k}}$$

Sol 18: (B) Angular momentum about rotational axis

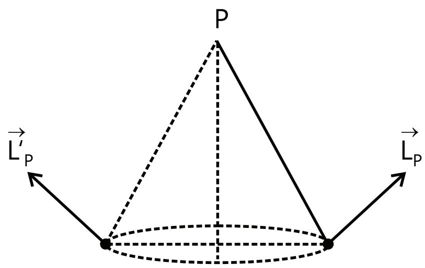


$$L_{(t)}=\left[I+m(vt)^2\right]\omega$$

$$\frac{dL_t}{dt}=2mv^2\,t\,\omega$$

Torque $\tau=\left(2mv^2\,\omega\right)t$

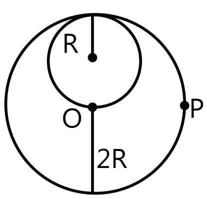
Sol 19: (C)



$$\vec{L}_0=\vec{r}_0\times\vec{p}$$

$\vec{L}_0$ is always directed along the axis & its magnitude is constant.

Sol 20: (C)



Let mass of original disc = m

The mass of disc removed = $\dfrac{m}{\pi\left(4R^2\right)}\times\pi R^2=\dfrac{m}{4}$

So M.O.I of remaining section about axis passing through "O" $I_O=\dfrac{m(2R)^2}{2}-\left[\dfrac{m}{4}\dfrac{R^2}{(2)}+\dfrac{m}{4}R^2\right]$

$$\Rightarrow 2mR^2-\left[\frac{mR^2+2mR^2}{8}\right]\Rightarrow\left[2-\frac{3}{8}\right]mR^2\Rightarrow\frac{13}{8}mR^2$$

MOI of remaining section about "P"

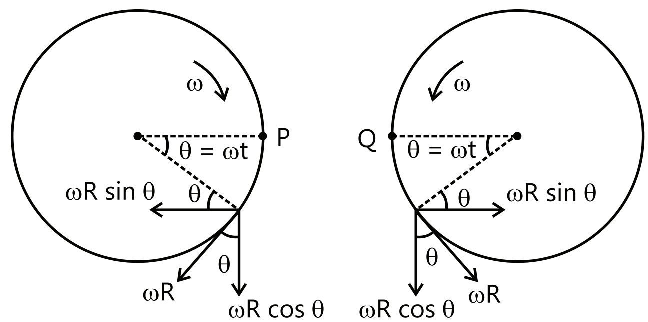
$$I_P=\left[\frac{m(2R)^2}{2}+m(2R)^2\right]-\left[\frac{m}{4}\frac{R^2}{(2)}+\frac{m}{4}5R^2\right]$$

$$\Rightarrow\left[2mR^2+4mR^2\right]-\left[\frac{mR^2}{8}+\frac{5mR^2}{4}\right]$$

$$\Rightarrow 6mR^2-\frac{11}{8}mR^2\ \Rightarrow\frac{37}{8}mR^2$$

$$\frac{I_P}{I_O}=\frac{37}{8}\times\frac{8}{13}\approx 3$$

Sol 21: (A)

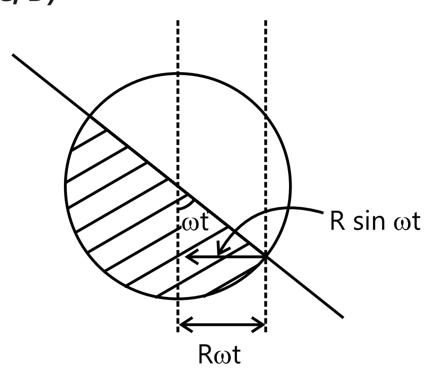


So, $v_r=2\omega R\ \sin(\omega t)$

At $t=T/2,\ v_r=0$

So two half cycles will take place.

Sol 22: (C, D)



According to problem particle is to land on disc.

If one consider a time 't' then x component of disc is Rωt
$R\sin\omega t<R\omega t$

This particle 'P' land on unshaded region. For "Q" x-component is very small and y-component equal to P it will also land in unshaded region.

Now repeat same thing when right part is shaded then correct answer is "C" or "D"

Sol 23: (A) In both the cases, the instantaneous axis will be along z-axis i.e. along vertical direction.

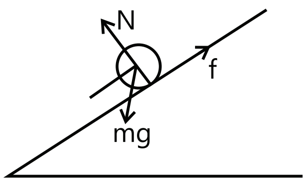
Sol 24: (D) w.r.t. centre of mass only pure rotation of disc will be seen. So in both the cases, angular speed about instantaneous axis will be "ω".

Sol 25: (A, B)

$$V_P = R\omega\hat{i} + \frac{\omega}{2}\left(-\hat{j}\right)\times\left(R\cos 30^\circ\hat{i} + R\sin 30^\circ\hat{k}\right)$$

$$= 3R\omega\hat{i} + \sqrt{3}\,\frac{\omega}{4}R\hat{k} - \frac{\omega}{4}R\hat{i} = \frac{11}{4}R\omega\hat{i} + \frac{\sqrt{3}}{4}R\omega\hat{k}$$

Sol 26: (D)



Translation motion:

$mg\ \sin\theta - f = ma_{cm}$ ...(i)

Rotational motion

$fR = I_{cm}\alpha$ ...(ii)

Rolling without slipping

$\alpha R = a_{cm}$ ...(iii)

From (ii) & (iii)

$$f = \frac{I_{cm}\,a_{cm}}{R^2}$$

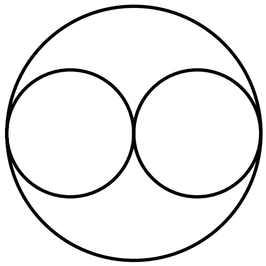
Put this in (i)

$$mg\sin\theta - \frac{I_{cm}\,a_{cm}}{R^2} = ma_{cm}$$

$$a_{cm} = \frac{mg\sin\theta}{\left(\frac{I_{cm}}{R^2} + m\right)}$$

As $I_P > I_Q$

Sol 27: (8) Conservation of angular momentum about vertical axis of disc



$$\frac{50(0.4)^2}{2}\times 10 = \left[\frac{50(0.4)^2}{2} + 4(6.25)(0.2)^2\right]\omega$$

$\omega = 8\,\text{rad/sec}$

Sol 28: (C, D) Condition of translational equilibrium

$N_1 = \mu_2 N_2$

$N_2 + \mu_1 N_1 = Mg$

Solving $N_2 = \dfrac{mg}{1+\mu_1\mu_2}$

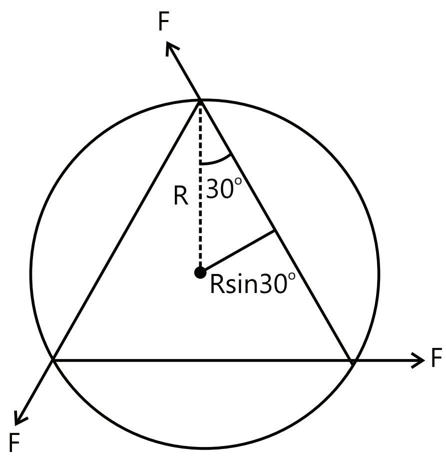
$N_1 = \dfrac{\mu_2\,mg}{1+\mu_1\mu_2}$

Applying torque equation about corner (left) point on the floor

$$mg\frac{\ell}{2}\cos\theta = N_1\,\ell\sin\theta + \mu_1 N_1\,\ell\cos\theta$$

Solving $\tan\theta = \dfrac{1-\mu_1\mu_2}{2\mu_2}$

Sol 29: (2)



$\tau = I\alpha$

$3FR\sin 30^\circ = I\alpha$

$I = \dfrac{MR^2}{2}$

$\alpha = 2$

$\omega = \omega_0 + \alpha t$

$\omega = 2\ \text{rad/s}$

Sol 30: (4) Since net torque about centre of rotation is zero, so we can apply conservation of angular momentum of the system about center of disc

$L_i = L_f$

$0 = I\omega + 2mv(r/2)$; comparing magnitude

$\therefore \left(\frac{0.45 \times 0.5 \times 0.5}{2}\right)\omega = 0.05 \times 9 \times \frac{0.5}{2} \times 2$

$\therefore \omega = 4$

Sol 31: (7) Kinetic energy of a pure rolling disc having velocity of centre of mass $v = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{mR^2}{2}\right)\frac{v^2}{R^2} = \frac{3}{4}mv^2$

So,

$\frac{3}{4}m(3)^2 + mg(30) = \frac{3}{4}m(v_2)^2 + mg(27) \therefore v_2 = 7\ m/s$

Sol 32: (D) Using conservation of angular momentum

$mR^2\omega$

$= \left(mR^2 \times \frac{8\omega}{9}\right) + \left(\frac{m}{8} \times \frac{9R^2}{25} \times \frac{8\omega}{9}\right) + \left(\frac{m}{8} \times x^2 \times \frac{8\omega}{9}\right)$

$\Rightarrow x = \frac{4R}{5}$

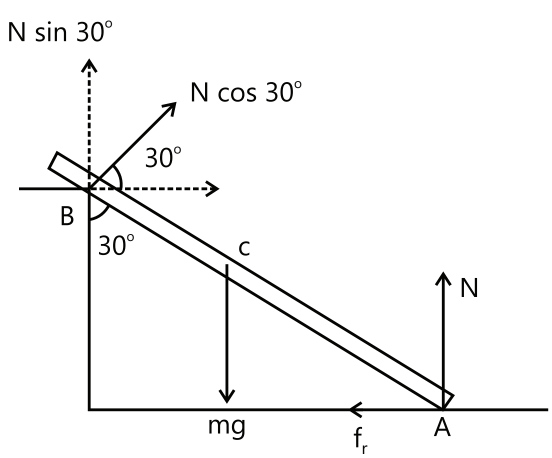
Sol 33: $I = \int \frac{2}{3}\rho 4\pi r^2 r^2\, dr$

$I_A \propto \int (r)(r^2)(r^2)dr$

$I_B \propto \int (r^5)(r^2)(r^2)dr$

$\therefore \frac{I_B}{I_A} = \frac{6}{10}$

Sol 34: (D) Force balance



$N + N \sin 30° = mg$

$\frac{3}{2}N = mg$

$\boxed{N = \frac{2}{3}mg}$

$f_r = \frac{mg}{\sqrt{3}} = \frac{16}{\sqrt{3}} = \frac{16\sqrt{3}}{3}$

Torque balance (about A)

$N \times \frac{h}{\cos 30°} = mg \times \frac{L}{2}\sin 30°$

$\frac{2}{3}mg \times \frac{2h}{\sqrt{3}} = mg \times \frac{L}{4}$

$\boxed{\frac{h}{L} = \frac{3\sqrt{3}}{16}}$

Sol 35: (A, B, D)

$\vec{v} = \frac{d\vec{r}(t)}{dt} = 3\alpha t^2\hat{i} + 2\beta t\hat{j}, \vec{a} = \frac{d\vec{v}}{dt} = 6\alpha t\hat{i} + 2\beta\hat{j}$

At t = 1, $\vec{v} = (10\hat{i} + 10\hat{j})\ ms^{-1}$

$\vec{a} = 20\hat{i} + 10\hat{j}\ ms^{-2}$

$\vec{r} = \frac{10}{3}\hat{i} + 5\hat{j}\ m$

$\vec{L}_0 = \vec{r} \times m\vec{v} = \left(-\frac{5}{3}\hat{k}\right) N\ m\ s$

$\vec{F} = m\frac{d\vec{v}}{dt} = (2\hat{i} + \hat{j})N$

$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times m\vec{a} = \left(-\frac{20}{3}\hat{k}\right) N\ m$

Sol 36: (A, D) $\omega_z = \frac{\omega a}{\ell}\cos\theta = \omega/5$