CHAPTER 3

THEORY OF EQUATION

Exercise 3.1

KEY POINTS

1. Polynomial equation $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0$
2. Quadratic equations $ax^2 + bx + c = 0$ where $a \neq 0$

- Two roots of quadratic equation $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

  where $b^2 - 4ac$ is called discriminant and denoted by Δ.

- If $\Delta = 0$ roots are equal

  If $\Delta > 0$ roots are real and distinct

  If $\Delta < 0$ roots are unreal (has no real roots)

3. Vieta's formula for quadratic equations

- Let α and β are the roots of the quadratic equation $ax^2 + bx + c = 0, a \neq 0$ then
- sum of roots $(\alpha + \beta) = \dfrac{-b}{a}$
- product of roots $(\alpha\beta) = \dfrac{c}{a}$
- quadratic equation $x^2 -$ (sum of roots) $x +$ product of roots $= 0$

4. Vieta's formula for polynomial equation degree 3.

   Let $ax^3 + bx^2 + cx + d = 0, a \neq 0$

   Let $\alpha, \beta,$ and γ be the roots, then

- $\alpha + \beta + \gamma = \dfrac{-b}{a}$
- $\alpha\beta + \beta\gamma + \gamma\alpha = \dfrac{c}{a}$

- $\alpha \beta \gamma = \frac{-d}{a}$
- equation: $x^3 - (\alpha + \beta + \gamma) x^2 + (\alpha \beta + \beta \gamma + \gamma \alpha) x - \alpha \beta \gamma = 0$

5. Vieta's formula for polynomial equation of degree $n > 3$.

- Let α, β, γ and δ be the roots of $ax^4 + bx^3 + cx^2 + dx + e = 0$

  Then,

$$\Sigma_1 = \alpha + \beta + \gamma + \delta = \frac{-b}{a}$$

$$\Sigma_2 = \alpha \beta + \alpha \gamma + \alpha \delta + \beta \gamma + \beta \delta + \gamma \delta = \frac{c}{a}$$

$$\Sigma_3 = \alpha \beta \gamma + \alpha \beta \delta + \alpha \gamma \delta + \beta \gamma \delta = \frac{-d}{a}$$

$$\Sigma_4 = \alpha \beta \gamma \delta = \frac{e}{a}$$

Note:

(i) $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0$ be a polynomial equation. Here a_n is called the **leading co-efficient** and the term $a_n x^n$ is called the **leading term** a_n may be any number, real or complex but $a_n \neq 0$

(ii) A polynomial with the leading co-efficient 1 is called a **monic polynomial**

(iii) **Remark**

- Polynomial functions are defined for all values of x.
- Every non-zero constant is a polynomial of degree 0.
- The constant 0 is also a polynomial called the **zero polynomial**. It s degree is not defined.
- The degree of a polynomial is a non-negative integer.
- The zero polynomial is the only polynomial with leading co-efficient 0.
- Polynomials of degree two are called **Quadratic polynomials.**

- Polynomials of degree three are called **cubic polynomials.**
- Polynomial of degree four are called **Quadratic polynomials**.

(iv) $\sin^2 x + \cos^2 x = 1$ is an identity on R while $\sin x + \cos x = 1$ and $\sin^3 x + \cos^3 x = 1$ are equations.

(v) The Fundamental theorem of Algebra

Every polynomial equation of degree $n \geq 1$ has atleast one root in C

(vi) $\Sigma\, \alpha_1 = \Sigma\, \alpha = \alpha + \beta + \gamma + \delta$

$\Sigma\, \alpha_1\, \alpha_2 = \Sigma\, \alpha\, \beta = \alpha\, \beta + \alpha\, \gamma + \alpha\, \delta + \beta\, \gamma + \beta\, \delta + \gamma\, \delta$

$\Sigma\, \alpha_1\, \alpha_2\, \alpha_3 = \Sigma\, \alpha\, \beta\, \gamma = \alpha\, \beta\, \gamma + \alpha\, \beta\, \delta + \alpha\, \gamma\, \delta + \beta\, \gamma\, \delta$

$\Sigma_n = \Sigma\, \beta\, \gamma\, \delta = \alpha\, \beta + \delta$

(vii) $(a + b + c + d)^2 = a^2 + b^2 + c^2 + d^2 + 2\,(ab + ac + ad + bc + bd + cd)$

Type I: [Quadratic equation (deg = 2) based sums]

Example 3.1, 3.2, 3.7, Q.No.9,10.

Example 3.1

If α and β are the roots of the quadratic equation $17x^2 + 43x - 73 = 0$, construct a quadratic equation whose roots are $\alpha + 2$ and $\beta + 2$

$$17x^2 + 43x - 73 = 0$$

$$a = 17 \; ; \; b = 43 \; ; \; c = -\,73$$

- sum of roots $= \dfrac{-b}{a}$

$$\boxed{\alpha + \beta = \frac{-43}{17}}$$

- product of roots $= \dfrac{c}{a}$

$$\boxed{\alpha\,\beta = \frac{-73}{17}}$$

Given roots: $\alpha + 2$ and $\beta + 2$

Sum	**Product**
$\alpha + 2 + \beta + 2$	$(\alpha + 2)(\beta + 2)$
$= \alpha + \beta + 4$	$\alpha\beta + 2\alpha + 2\beta + 4$
$= \frac{-43}{17} + 4$	$= \alpha\beta + 2(\alpha + \beta) + 4$
$= \frac{-43 + 68}{17}$	$= \frac{-73}{17} + 2\left(\frac{-43}{17}\right) + 4$
$= \frac{25}{17}$	$= \frac{-73 - 86}{17} + 4$
	$= \frac{-159 + 68}{17}$
	$= \frac{-91}{17}$

$\therefore$ **equation:**

$$x^2 - (\text{sum})\ x + \text{product} = 0$$

$$x^2 \frac{-15}{17} x \frac{-91}{17} = 0$$

i.e $\quad 17x^2 - 25x - 91 = 0$

Example 3.2

If α and β are the roots of the quadratic equation $2x^2 - 7X + 13 = 0$ construct a quadratic equation whose roots are α^2 and β^2

$$2x^2 - 7x + 13 = 0$$

$$a = 2\ ;\ b = -7\ ;\ c = 13$$

sum of roots $(\alpha + \beta) = \frac{-b}{a}$

$$\boxed{\alpha + \beta = \frac{7}{2}}$$

product of roots $= \frac{c}{a}$

$$\boxed{\alpha\beta = \frac{13}{2}}$$

Given roots: α^2 and β^2

Sum	**Product**
$\alpha^2 + \beta^2$	$\alpha^2 \beta^2$
$= (\alpha + \beta)^2 - 2\,\alpha\,\beta$	$= (\alpha\,\beta)^2$
$= \left(\frac{7}{2}\right)^2 - 2\left(\frac{13}{2}\right)$	$= \left(\frac{13}{2}\right)^2$
$= \frac{49}{4} - 13$	$= \frac{169}{4}$
$= \frac{49 - 52}{4}$	
$= \frac{-3}{4}$	

$\therefore$ **equation**

$x^2 -$ (sum) $x +$ product $= 0$

$$x^2 + \frac{3x}{4} + \frac{169}{4} = 0$$

$$4x^2 + 3x + 169 = 0$$

Example 3.7

If p is real, discuss the nature of the roots of the equation $\boldsymbol{4x^2 + 4px + p + 2 = 0}$ ***in terms of p.***

$$4x^2 + 4px + P + 2 = 0$$

$$a = 4$$

$$b = 4p$$

$$c = p + 2$$

$$\boxed{\boldsymbol{\Delta = b^2 - 4ac}}$$

$$= (4p)^2 - 4\,(4)\,(p + 2)$$

$$= 16p^2 - 16p - 32$$

$$= 16\,(p^2 - p - 2)$$

$$= 16\,(p + 1)\,(p - 2)$$

$$= 16\,(p + 1)\,(p - 2)$$

$\therefore\ \Delta < 0$ if $-1 < p < 2$

$\Delta = 0$ if $p = -1$ (or) $p = 2$

$\Delta = 0$ if $-\infty < p < -1$ or $2 < p < \infty$

$\therefore$ Imaginary roots if $-1 < p < 2$;

equal real roots if $p = -1$ or $p = 2$

distinct real roots if $-\infty < p < -1$ or $2 < p < \infty$

9. If p and q are the roots of the equation $lx^2 + nx + n = 0$, show that $\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{n}{l}} = 0$.

$lx^2 + nx + n = 0$ Given roots are p and q

$a = l\ ;\ b = n\ ;\ c = n$

sum of roots $= \frac{-b}{a}$

$p + q = \frac{-n}{l}$

Product of roots $= \frac{c}{a}$

$$pq = \frac{n}{l}$$

Now,

$$\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}}$$

$$= \frac{\sqrt{p}}{\sqrt{q}} + \frac{\sqrt{q}}{\sqrt{p}}$$

$$= \frac{p+q}{\sqrt{pq}}$$

$$= \frac{\frac{-n}{l}}{\sqrt{\frac{n}{l}}}$$

$$\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}=-\sqrt{\frac{n}{l}}$$

$$\therefore \sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}+\sqrt{\frac{n}{l}}=0 \text{ proved.}$$

10. If the equations $x^2+px+q=0$ and $x^2+p'x+q'=0$ have a common root, show that it be equal to $\frac{pq^1-p^1q}{q-q^1}$ or $\frac{q-q^1}{p^1-p}$

$x^2+px+q=0$	$x^2+p^1x+q^1=0$
Let roots α and β_1	Let roots α and β_2
$\alpha+\beta_1=-p$ ...(1)	$\alpha+\beta_2=-p^1$ ...(1)
$\alpha\beta_1=q$ (2)	$\alpha\beta=q^1$
(1) $\Rightarrow \alpha=-p-\beta_1$	(1) $\Rightarrow \alpha=-p'-\beta_2$
(2) $\Rightarrow \alpha=\frac{q}{\beta_1}$	(2) $\Rightarrow \alpha=\frac{q'}{\beta_2}$

From (1)

$$-p-\beta_1=-p^1-\beta_2$$

$$p^1-p=\beta_1-\beta_2 \quad ...(3)$$

From (2)

$$\frac{q}{\beta_1}=\frac{q^1}{\beta_2}$$

$$q\beta_2=q^1\beta_1$$

$$\beta_1q^1-\beta_2q=0 \quad ...(4)$$

$$\beta_1-\beta_2=p^1-p$$

$$p_1q'-\beta_2q=0$$

$$\begin{bmatrix}1 & -1\\ q^1 & -q\end{bmatrix}\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix}=\begin{bmatrix}p^1-p\\ 0\end{bmatrix}$$

$$\Delta = \begin{vmatrix} 1 & -1 \\ q^1 & -q \end{vmatrix}$$

$$= -q + q^1$$

$$\Delta \beta_1 = \begin{vmatrix} p^1 - p & -1 \\ 0 & -q \end{vmatrix}$$

$$= pq - pq^1$$

$$\therefore \beta_1 = \frac{\Delta \beta_1}{\Delta}$$

$$= \frac{pq - p^1 q}{q^1 - q}$$

$$\Delta \beta_2 = \begin{vmatrix} 1 & p^1 - p \\ q^1 & 0 \end{vmatrix}$$

$$= 0 - (q_1 p' - pq')$$

$$= pq^1 - q_1 p^1$$

$$\therefore \beta_2 = \frac{\Delta \beta_2}{\Delta}$$

$$= \frac{pq^1 - q_1 p^1}{q^1 - q}$$

Put $\beta_1 \beta_2$ values in (2)

$$\alpha = \frac{q}{\beta_1}$$

$$= q\left(\frac{q^1 - q}{pq - p^1 q}\right)$$

$$= q\left(\frac{q^1 - q}{q\,(p - p^1}\right)$$

$$\alpha = \frac{q^1 - q}{p - p^1}$$

$$\alpha = \frac{q - q^1}{p - p^1} \text{ proved}$$

$$\alpha = -p - \beta_1$$

$$= -p - \left(\frac{pq - p^1 q}{q^1 - q}\right)$$

$$= \frac{-pq^1 + pq \_ pq + p^1 q}{q^1 - q}$$

$$= \frac{p^1 q - pq^1}{q^1 - q}$$

$$= \frac{pq^1 - p^1 q}{q^1 - q} \text{ proved}$$

(Similarly we can put β_2 value also)

Type II: [cubic polynomial based sums

Q.No. 1, 2, (i) (ii) (iii), 3, 4, 6, 7, 11, 12. example 3,3, 3.5, 3.6.

1. If the sides of a cubic box are increased by 1,2, 3 units respectively to form a cuboid, then the volume is increased by 52 cubic units, Find the volume of the cuboid.

Let 'x' be the side of the cube given sides are increased by 1, 2, 3 units i.e., $x + 1, x + 2$ and $x + 3$

$\therefore$ given data

$$(x + 1)(x + 2)(x + 3) = x^3 + 52$$

$$(x + 1)(x^2 + 5x + 6) = x^3 + 52$$

$$x^3 + 5x^2 + 6x + x^2 + 5x + 6 = x^3 + 52$$

$$6x^2 + 11x + 6 - 52 = 0$$

$$6x^2 + 11x - 46 = 0$$

$$\left(x + \frac{23}{6}\right)(x - 2) = 0$$

$x + \frac{23}{6} = 0$	$x - 2 = 0$
$x = \frac{-23}{6}$	$x = 2$
not possible	

$\therefore$ side of the given cube is 2 units volume of cube $= 2^3 = 8$ cubic units

volume of cuboid $= (x + 1)(x + 2)(x + 3)$

$= (2 + 1)(2 + 2)(2 + 3)$

$= 3 \times 4 \times 5$

$= 60$ Cubic units

2. Construct a cubic equation with roots (i) 1, 2 and 3

$\alpha = 1, \beta = 2, \gamma = 3$

Equation

$$x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma = 0$$

$$x^3 - (1 + 2 + 3)x^2 + (2 + 6 + 3)x - (1)(2)(3) = 0$$

$$x^3 - 6x^2 + 11x - 6 = 0$$

(ii) 1, 1 and − 2

$$\alpha = 1, \beta = 1, \gamma = -2$$

Equation

$$x^3 - (\alpha + \beta + \gamma)\, x^2 + (\alpha\,\beta + \beta\,\gamma + \gamma\,\alpha)\, x - \alpha\,\beta\,\gamma = 0$$
$$x^3 - (1 + 1 - 2)\, x^2 + (1 - 2 - 2)\, x - (1)\,(1)\,(-2) = 0$$
$$x^3 - 3x + 2 = 0$$

(iii) 2, – 2 and 4

$$\alpha = 2,\ \beta = -2,\ \gamma = 4$$

Equation

$$x^3 - (\alpha + \beta + \gamma)\, x^2 + (\alpha\,\beta + \beta\,\gamma + \gamma\,\alpha)\, x - \alpha\,\beta\,\gamma = 0$$
$$x^3 - (2 - 2 + 4)\, x^2 + (-4 - 8 + 8)\, x - (2)\,(-2)\,(4) = 0$$
$$x^3 - 4x^2 - 4x + 16 = 0$$

3. **If α, β and γ are the roots of the cubic equation $x^3 + 2x^2 + 3x + 4 = 0$, form a cubic equation whose roots are (i) $2\,\alpha, 2\,\beta, 2\,\gamma$ (ii) $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ (iii) $-\alpha - \beta - \gamma$**

$$x^3 + 2x^2 + 3x + 4 = 0$$
$$a = 1\ ;\ b = 2\ ;\ c = 3\ ;\ d = 4$$

- $\alpha + \beta + \gamma = \frac{-b}{a}$

  $\alpha + \beta + \gamma = -2$

- $\alpha\,\beta + \beta\,\gamma + \gamma\,\alpha = \frac{c}{a}$

  $\alpha\,\beta + \beta\,\gamma + \gamma\,\alpha = 3$

- $\alpha\,\beta\,\gamma = \frac{-d}{a}$

  $\alpha\,\beta\,\gamma = -4$

(i) $2\,\alpha, 2\,\beta, 2\gamma$

$2\,\alpha + 2\,\beta + 2\,\gamma$
$= 2\,(\alpha + \beta + \gamma)$
$= 2\,(-2)$
$= -4$

$(2\,\alpha)\,(2\,\beta) + (2\,\beta)\,(2\,\gamma) + (2\,\gamma)\,(2\,\alpha)$
$= 4\,\alpha\,\beta + 4\,\beta\,\gamma + 4\,\gamma\,\alpha$
$= 4\,(\alpha\,\beta + \beta\,\gamma + \gamma\,\alpha)$
$= 4\,(3)$
$= 12$

- $(2\,\alpha)\,(2\,\beta)\,(2\,\gamma) = 8\,\alpha\,\beta\,\gamma$

  $= 8\,(-4)$

  $= -32$

∴ **Equation**

$$x^3 - (2\alpha + 2\beta + 2\gamma)\,x^2 + [\,(2\alpha)(2\beta) + (2\beta)(2\gamma) + (2\gamma)(2\alpha)\,]\,x$$

$$- (2\alpha)(2\beta)(2\gamma) = 0$$

$$x^3 - (-4)\,x^2 + 12x - (-32) = 0$$

$$x^3 + 4x^2 + 12x + 32 = 0$$

(ii) $\mathbf{\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}}$

- $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$

  $= \frac{3}{-4}$

- $\left(\frac{1}{\alpha}\right)\left(\frac{1}{\beta}\right) + \left(\frac{1}{\beta}\right)\left(\frac{1}{\gamma}\right) + \left(\frac{1}{\gamma}\right)\left(\frac{1}{\alpha}\right)$

  $= \frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha}$

  $= \frac{\gamma + \alpha + \beta}{\alpha\beta\gamma}$

  $= \frac{-2}{-4}$

  $= \frac{1}{2}$

- $\left(\frac{1}{\alpha}\right)\left(\frac{1}{\beta}\right)\left(\frac{1}{\gamma}\right) = \frac{1}{\alpha\beta\gamma}$

  $= \frac{1}{-4}$

∴ **equation**

$$x^3 - \left(\frac{-3}{4}\right)x^2 + \left(\frac{1}{2}\right)x - \left(\frac{-1}{4}\right) = 0$$

$$x^3 + \frac{3x^2}{4} + \frac{x}{2} + \frac{1}{4} = 0$$

$$4x^3 + 3x^2 + 2x + 1 = 0$$

3. Zero sum of all co-efficients

The sum of the co-efficients of the polynomial is 0, then 1 is a root of $p(x)$

4. Equal sums of co-efficients of odd and even powers

The sum of the co-efficients of the odd and even powers are equal, then -1 is a root of $p(x)$

5. Roots in progressions

Sometimes the roots are in "arithmetic progression" and the roots are in geometric progression.

Type I: [solve the equation]
Q.No. 1, 6, (i),(ii), Example 3.17, 3.18, 4.

1. Solve the cubic equation $2x^3 - x^2 - 18x + 9 = 0$ if sum of its roots vanishes.

Let α, β and γ are the roots of $2x^3 - x^2 - 18x + 9 = 0$

Given: $\alpha + \beta = 0$

Now

$$\alpha + \beta + \gamma = \frac{-b}{a}$$

$$0 + \gamma = \frac{1}{2}$$

$$\boxed{\gamma = \frac{1}{2}}$$

using synthetic division

$$\begin{array}{c|cccc} \frac{1}{2} & 2 & -1 & -18 & 9 \\ & 0 & 1 & 0 & -9 \\ \hline & 2 & 0 & -18 & 0 \end{array}$$

$$2x^2 - 18 = 0$$

$$2(x^2 - 9) = 0$$

$$x^2 - 9 = 0$$

$$x^2 = 9$$

$$x = \pm 3$$

$\therefore$ roots are $3, -3$ and $\frac{1}{2}$

6. Solve the cubic equations: (i) $2x^3 - 9x^2 + 10x = 3$
(ii) $8x^3 - 2x^2 - 7x + 3 = 0$

(i) $2x^3 - 9x^2 + 10x - 3 = 0$

Here sum of co-efficients is '0'

$(2 - 9 + 10 - 3 = 12 - 12 = 0)$

$\therefore$ 1 is a root

$$\begin{array}{r|rrrr} 1 & 2 & -9 & 10 & -3 \\ & 0 & 2 & -7 & 3 \\ \hline & 2 & -7 & 3 & \multicolumn{1}{|r}{0} \end{array}$$

$$2x^2 - 7x + 3 = 0$$

$$(x - 3)\left(x - \frac{1}{2}\right) = 0$$

$$x - 3 = 0 \qquad x - \frac{1}{2} = 0$$

$$\boldsymbol{x = 3} \qquad \boldsymbol{x = \frac{1}{2}}$$

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$-6 \quad -1$
$\frac{-6}{2}, \quad \frac{-1}{2}$
$-3, \quad -\frac{1}{2}$

$\therefore$ roots are 1, 3 and $\frac{1}{2}$

(ii) $8x^3 - 2x^2 - 7x + 3 = 0$

Here sum of alternate terms are equal (i.e) $8 - 7 = -2 + 3 = 1$]

$\therefore -1$ is a root

$$\begin{array}{r|rrrr} -1 & 8 & -12 & -7 & 3 \\ & 0 & -8 & 10 & -3 \\ \hline & 8 & -10 & 3 & \multicolumn{1}{|r}{0} \end{array}$$

$$\therefore\ 8x^2 - 10x + 3 = 0$$

$$(x - 3/4)\,(x - \tfrac{1}{2}) = 0$$

$$\therefore\ x = \frac{3}{4},\ x = \frac{1}{2}$$

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$-6 \quad -4$

$\frac{-6}{8}, \quad \frac{-4}{8}$

$\frac{-3}{4}, \quad \frac{-1}{2}$

$\therefore$ roots are $-1, \frac{3}{4}$ and $\frac{1}{2}$

Example 3.17

Solve the equation $\boldsymbol{x^3 - 3x^2 - 33x + 35 = 0}$

Here sum of co-efficients are zero.

$$(1 - 3 - 33 + 35 = 36 - 36 = 0)$$

$\therefore$ 1 is a root

$$\begin{array}{c|cccc} 1 & 1 & -3 & -33 & 35 \\ & 0 & 1 & -2 & -35 \\ \hline & 1 & -2 & -35 & \boxed{0} \end{array}$$

$$x^2 - 2x - 35 = 0$$

$$(x - 7)\,(x + 5) = 0$$

$$\begin{array}{r|r} x - 7 = 0 & x + 5 = 0 \\ x = 7 & x = -5 \end{array}$$

$\therefore$ roots are 1, 7 and -5

Example 3.18

Solve the equation $\boldsymbol{2x^3 + 11x^2 - 9x - 18 = 0.}$

Here sum of alternate terms are equal [i.e $2 - 9 = 11 - 18 = -7$]

$\therefore -1$ is a root.

$$\begin{array}{c|cccc} -1 & 2 & 11 & -9 & -18 \\ & 0 & -2 & -9 & 18 \\ \hline & 2 & 9 & -18 & \boxed{0} \end{array}$$

$$\therefore\ 2x^2 + 9x - 18 = 0$$

$$(x + 6)\,(x - 3/2) = 0$$

$$x = -6, \frac{3}{2}$$

-36

$12 \quad -3$

$\frac{12}{2}, \quad \frac{-3}{2}$

$6, \quad \frac{-3}{2}$

$\therefore$ Roots are $-1, -6$ and $\frac{3}{2}$

4. Determine k and solve the equation $2x^3 - 6x^2 + 3x + k = 0$ if one of its roots is twice the sum of the other two roots.

Let α, β and γ are the roots of

$2x^3 - 6x^2 + 3x + k = 0$

Given: $\alpha = 2(\beta + \gamma)$

$$\frac{\alpha}{2} = \beta + \gamma$$

Now

$$\alpha + \beta + \gamma = \frac{-b}{a}$$

$$\alpha + \frac{\alpha}{2} = \frac{6}{2}$$

$$\frac{3\alpha}{2} = \frac{6}{2}$$

$$3\alpha = 6$$

$$\alpha = \frac{6}{3}$$

$$\boxed{\alpha = 2 \text{ one root}}$$

Now,

$$\begin{array}{c|cccc} 2 & 2 & -6 & 3 & k \\ & 0 & 4 & -4 & -2 \\ \hline & 2 & -2 & -1 & \boxed{0} \end{array}$$

Here $k - 2 = 0$

$$\boxed{k = 2}$$

Also

$$2x^2 - 2x - 1 = 0$$

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

$$= \frac{2 \pm \sqrt{4 - 4(2)(-1)}}{2(2)}$$

$$= \frac{2 \pm \sqrt{4+8}}{4}$$

$$= \frac{2 \pm \sqrt{12}}{4}$$

$$= \frac{2 \pm 2\sqrt{3}}{4}$$

$$= \frac{2(1 \pm \sqrt{3})}{4}$$

$$= \frac{1 \pm \sqrt{3}}{2}$$

$\therefore$ roots are 2, $\frac{1+\sqrt{3}}{2}$ and $\frac{1-\sqrt{3}}{2}$

Type II: [Solve even powers polynomial]

Q.No.7, Example 3.16.

7. Solve the equation: $x^4 - 14x^2 + 45 = 0$

$$x^4 - 14x^2 + 45 = 0$$

Let $y = x^2$ then we get

$$y^2 - 14y + 45 = 0$$

$$(y-5)(y-9) = 0$$

$y = 5 = 0$	$y - 9 = 0$
$y = 5$	$y = 9$
put $y = x^2$	put $y = x^2$
$x^2 = 5$	$x^2 = 9$
$x = \pm\sqrt{5}$	$x = \pm 3$

$\therefore$ roots are 3 ; -3, $\sqrt{5}$ and $-\sqrt{5}$

Example 3.16

Solve the equation $x^4 - 9x^2 + 20 = 0$

$$x^4 - 9x^2 + 20 = 0$$

Put $y = x^2$ then we get

$$y^2 - 9y + 20 = 0$$

$$(y-4)(y-5) = 0$$

$y - 4 = 0$	$y - 5 = 0$
$y = 4$	$y = 5$
put $y = x^2$	put $y = x^2$
$x^2 = 4$	$x^2 = 5$
$x = \pm 2$	$x = \pm\sqrt{5}$

$\therefore$ roots are $2, -2, \sqrt{5}$ and $-\sqrt{5}$

Type II: [Find all zeros of the given polynomial]

Q.No.5 Example 3.15

5. Find all zeros of the polynomial $x^6 - 3x^5 - 5x^4 + 22x^3 - 39x^2 - 39x + 135$, if it is known that $1 + 2i$ and $\sqrt{3}$ are two of its zeros.

$x^6 - 3x^5 - 5x^4 + 22x^3 - 39x^2 - 39x + 135$

Co-efficient of the equations are all rational numbers.

Given: $1 + 2i$ is a root

$1 - 2i$ also another root

sum of roots $= (1 + 2i) + (1 - 2i) = 2$

product of roots $= (1 + 2i)(1 - 2i)$

$= 1^2 + 2^2$

$= 5$

$\therefore$ **equation**

$x^2 -$ (sum) $x +$ product $= 0$

$x^2 - 2x + 5 = 0$

Also

Given: $\sqrt{3}$ as root another root is $-\sqrt{3}$

sum of roots $= \sqrt{3} - \sqrt{3} = 0$

product of roots $= (\sqrt{3})(-\sqrt{3}) = -3$

∴ **equation**

$$x^2 - \text{(sum) } x + \text{ product } = 0$$

$$x^2 - 0x - 3 = 0$$

$$x^2 - 3 = 0$$

∴ $(x^2 - 2x + 5)\,(x^2 - 3)$ is a factor of the given polynomial

divided the polynomial by this factor

$$(x^2 - 2x + 5)\,(x^2 - 3) = x^4 - 3x^2 - 2x^3 + 6x + 5x^2 - 15$$

$$= x^4 - 2x^3 + 3x^2 + 6x - 15$$

$$\begin{array}{r|rrrrrrr}
 & 1 & -1 & -9 & & & & \\
\hline
1\quad -2\quad 2\quad 6\quad -15 & 1 & -3 & -5 & 22 & -39 & -39 & 135 \\
 & (-) & (+) & (-) & (-) & (+) & & \\
 & 1 & -2 & 2 & 6 & -15 & & \\
\hline
 & & -1 & -7 & 16 & -24 & -39 & \\
 & & (+) & (-) & (+) & (+) & (-) & \\
 & & -1 & 2 & -2 & -6 & 15 & \\
\hline
 & & & -9 & 18 & -18 & -54 & 135 \\
 & & & (+) & (-) & (+) & (+) & (-) \\
 & & & -9 & 18 & -18 & -54 & 135 \\
\hline
 & & & & & 0 & & \\
\hline
\end{array}$$

$$Q \Rightarrow x^2 - x - 9 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{1 \pm \sqrt{1 - 4\,(1)\,(-9)}}{2}$$

$$= \frac{1 \pm \sqrt{1 + 36}}{2}$$

$$= \frac{1 \pm \sqrt{37}}{2}$$

∴ Roots are

$1 + 2i,\ 1 - 2i,\ \sqrt{3},\ -\sqrt{3},\ \frac{1 + \sqrt{37}}{2}$ and $\frac{1 - \sqrt{37}}{2}$

Example 3.15

***If* $2+i$ *and* $3-\sqrt{2}$ *are roots of the equation* $x^6 - 13x^5 + 62x^4 - 126x^3 + 65x^2 + 127x - 140 = 0$**

Since the co-efficient of the equations are all rational numbers, and $2+i$ and $3-\sqrt{2}$ are roots, we get $2-i$ and $3+\sqrt{2}$ are also roots of the give equations.

Thus their product

$$= [x-(2+i)]\,[x-(2-i)]\,[x-(3-\sqrt{2})]\,[x-(3+\sqrt{2})]$$

$$= [(x-2)-i]\,[(x-2)+i]\,[(x-3+\sqrt{2}]\,[(x-3)-\sqrt{2}]$$

$$= [(x-2)^2+1^2]\,[(x-3)^2-(\sqrt{2})^2]$$

$$= (x^2-4x+4+1)\,(x^2-6x+9-2)$$

$$= (x^2-4x+5)\,(x^2-6x+7) \text{ is a factor}$$

Divide the Given polynomial by this factor.

i.e $(x^2-4x+5)\,(x^2-6x+7)$

$$= x^4 - 6x^3 + 7x^2 - 4x^3 + 24x^2 - 28x + 5x^2 - 30x + 35$$

$$= x^4 - 10x^3 + 36x^2 - 58x + 35$$

					1	− 3	− 4				
1	− 10	36	− 58	35	1	− 13	62	− 126	65	127	− 140
					(−)	(+)	(−)	(+)	(−)		
					1	− 10	36	− 58	35		
						− 3	26	− 68	30	127	
						(+)	(−)	(+)	(−)	(+)	
						− 3	30	− 108	174	− 105	
							− 4	40	− 144	232	− 140
							(+)	(−)	(+)	(−)	(+)
							− 4	40	− 144	232	− 140
									0		

x^2-3x-4 is another factor.

$x^2-3x-4=0$

$(x-4)\,(x+1)=0$

$$\begin{array}{r|r} x-4=0 & x+1=0 \\ x=4 & x=-1 \end{array}$$

$\therefore$ Roots are

$2+i, 2-i, 3+\sqrt{2}, 3-\sqrt{2}, -1$ and 4.

Type IV: [Roots in progressions A.P,G.P and H.P)
Q.No. 2,3, Example 3.19, 3.20, 3.21 Example 3.22]

2. **Solve the equation $9x^3 - 36x^2 + 44x - 16 = 0$ if the roots form an arithmetic progressive**

$9x^3 - 36x^2 + 44x - 16 = 0$

There are 3 roots which are in A.P.

$\therefore \alpha = a - d$

$\beta = a$

$\gamma = a + d$

Now

$$\alpha + \beta + \gamma = \frac{-b}{a}$$

$$a - d + a + a + d = \frac{36}{9}$$

$$3a = 4$$

$$\boxed{a = \frac{4}{3}}$$

$$\alpha\beta\gamma = \frac{-d}{a}$$

$$(a-d)(a)(a+d) = \frac{16}{9}$$

$$(a^2 - d^2)\,a = \frac{16}{9}$$

$$\left(\frac{16}{9} - d^2\right)\frac{4}{3} = \frac{16}{9}$$

$$\frac{16}{9} - d^2 = \frac{16}{9} \times \frac{3}{4}$$

$$\frac{16}{9} - d^2 = \frac{12}{9}$$

$$\frac{16}{9} - \frac{12}{9} = d^2$$

$$\frac{4}{9} = d^2$$

$$\boxed{\pm\frac{2}{3} = d}$$

$\therefore$ roots are $\left(\text{Let } a = \frac{4}{3}, d = \frac{2}{3}\right)$

$$a - d = \frac{4}{3} - \frac{2}{3} = \frac{2}{3}$$

$$a = \frac{4}{3}$$

$$a + d = \frac{4}{3} + \frac{2}{3} = \frac{6}{3} = 2$$

$\therefore \frac{2}{3}, \frac{4}{3}, 2$ are the roots of the given equation.

3. Solve the equation $3x^3 - 26x^2 - 52x - 24 = 0$ if its roots form a geometric progression.

$$3x^3 - 26x^2 + 52x - 24 = 0$$

Given: roots are in G.P

$$\therefore \alpha = \frac{a}{r}$$

$$\beta = a$$

$$\gamma = ar$$

Now

$$\alpha + \beta + \gamma = \frac{-b}{a}$$

$$\frac{a}{r} + a + ar = \frac{26}{3}$$

$$a\left(\frac{1}{r} + 1 + r\right) = \frac{26}{3} \qquad \text{...(1)}$$

$$\alpha\,\beta\,\gamma = \frac{-d}{a}$$

$$\frac{a}{r} \times a \times ar = \frac{24}{3}$$

$$a^3 = 8$$

$$\boxed{\boldsymbol{a = 2}}$$

$$(1) \;\Rightarrow 2\left(\frac{1}{r} + 1 + r\right) = \frac{26}{3}$$

$$\frac{1 + r + r^2}{r} = \frac{26}{3 \times 2}$$

$$\frac{1 + r + r^2}{r} = \frac{13}{3}$$

$$3 + 3r + 3r^2 = 13r$$

$$3r^2 + 3r - 13r + 3 = 0$$

$$3r^2 - 10r + 3 = 0$$

$$(r - 3)\,(r - 1/3) = 0$$

$$r - 3 = 0 \quad \Big| \quad \text{or } r - \frac{1}{3} = 0$$

$$\boldsymbol{r = 3} \quad \Big| \quad \boldsymbol{r = \frac{1}{3}}$$

$a = 2$ and $r = 3$

$$r = \frac{a}{r} = \frac{2}{3}$$

$$\beta = a = 2$$

$$\gamma = ar = 2 \times 3 = 6$$

$\therefore$ roots are 2,6 and $\frac{2}{3}$

9

$-9 \qquad -1$

$\frac{-9}{3}, \qquad \frac{-1}{3}$

$-3 \qquad \frac{-1}{3}$

Example 3.19

Obtain the condition that the roots of $x^3 + px^2 + qx + 8 = 0$ are in A.P.

Let roots are $a - d, a, a + d$

Now

$$\alpha + \beta + \gamma = \frac{-b}{a}$$

$$a - d + a + a + d = -p$$

$$3a = -p$$

$$\boxed{a = \frac{-p}{3}}$$

a is a root of the given equation.

$$x^3 + px^2 + qx + r = 0$$

$$\left(\frac{-p}{3}\right)^3 + p\left(\frac{-p}{3}\right)^2 + q\left(\frac{-p}{3}\right) + r = 0$$

$$\frac{-p^3}{27} + \frac{p^3}{9} - \frac{pq}{3} + r = 0$$

$$\frac{-p^3 + 3p^3 - 9pq + 27r}{27} = 0$$

$$2p^3 - 9pq + 27r = 0$$

$9pq = 2p^3 + 27r$ is the required condition.

Example 3.20

Find the condition that the roots of $ax^3 + bx^2 + cx + d = 0$ are in G.P. Assume $a, b, c, d \neq 0$

Let the roots be in G.P.

$\alpha = \frac{\alpha}{r}$; $\beta = \alpha$; $\gamma = \alpha\gamma$

Now,

- $\alpha + \beta + \gamma = \frac{-b}{a}$

$$\frac{\alpha}{r} + \alpha + \alpha\gamma = \frac{-b}{a}$$

$$\alpha\left(\frac{1}{r} + 1 + r\right) = \frac{-b}{a} \qquad \text{...(1)}$$

- $\alpha\beta + \beta\gamma + \gamma\alpha = \dfrac{c}{a}$

$$\left(\frac{\alpha}{r}\alpha\right) + (\alpha)(\alpha r) + (\alpha r)\left(\frac{\alpha}{r}\right) = \frac{c}{a}$$

$$\alpha^2\left(\frac{1}{r} + r + 1\right) = \frac{c}{a} \qquad \text{...(2)}$$

- $\alpha\beta\gamma = -\dfrac{d}{a}$

$$\frac{\alpha}{r}\alpha\,\alpha r = \frac{-d}{a}$$

$$\alpha^3 = -\frac{d}{a} \qquad \text{...(3)}$$

Dividing (2) by (1)

$$\alpha = -\frac{c}{b}$$

put in (3)

$$\left(-\frac{c}{b}\right)^3 = -\frac{d}{a}$$

$$-\frac{c^3}{b^3} = -\frac{d}{a}$$

$ac^3 = db^3$ is the required condition

Example 3.21

If the roots of $x^3 + px^2 + qx + r = 0$ are in H.P. Prove that $9pqr = 27r^2 + 2p$

✍ **Solution:**

Let the roots be in H.P then their reciprocals are in A.P and roots of the equation

$$\left(\frac{1}{x}\right)^3 + p\left(\frac{1}{x}\right)^2 + q\left(\frac{1}{x}\right) + r = 0$$

$$\frac{1}{x^3} + \frac{p}{x^2} + \frac{q}{x} + r = 0$$

$$\frac{1 + px + qx^2 + rx^3}{x^3} = 0$$

i.e $$rx^3 + qx^2 + px + 1 = 0 \qquad ...(1)$$

Given: roots are in A.P

$$\therefore \alpha - d, \alpha, \alpha + d$$

Now,

$$\alpha + \beta + \gamma = \frac{-b}{a}$$

$$(\alpha - d) + (d) + (\alpha + d) = \frac{-q}{r}$$

$$3\alpha = \frac{-q}{r}$$

$$\boxed{\alpha = \frac{-q}{3r}}$$

α is one of the root of equation (1) so we get

$$r\left(\frac{-q}{3r}\right)^3 + q\left(\frac{-q}{3r}\right)^2 + p\left(\frac{-q}{3r}\right) + 1 = 0$$

$$r\left(\frac{-q^3}{26\,r^3}\right) + q\left(\frac{q^2}{9r^2}\right) - \frac{pq}{3r} + 1 = 0$$

$$\frac{-q^3}{27r^2} + \frac{q^3}{9r^2} - \frac{pq}{3r} + 1 = 0$$

$$\frac{-q^3 + 3q^3 - 9pqr + 27r^2}{27r^2} = 0$$

$$2q^3 - 9pqr + 27r^2 = 0$$

$$\boxed{\mathbf{9pqr = 2q^3 + 27r^2}}$$

is the required condition.

Exercise 3.4

KEY POINTS

Partly Factored Polynomials

Quadric polynomial equations of the form

$(ax + b)\ (cx + d)\ (px + q)\ (rx + s) + k = 0, k \neq 0$ which can be rewritten in the form $(\alpha x^2 + \beta x + \lambda)\ (\alpha x^2 + \beta x + \mu) + k = 0$

Type I: [Solve]

Q.No. 1 (i), (ii), 2, Example 3.23, 3.24

1. Solve: (i) $(x - 5)\ (x - 7)\ (x + 6)\ (x + 4) = 504.$

$(x - 5)\ (x - 7)\ (x + 6)\ (x + 4) = 504$

Rewriting the equation as

$(x - 5)\ (x + 4)\ (x - 7)\ (x + 6) = 504$

$(x^2 - x - 20)\ (x^2 - x - 42) = 504$

Let $y = x^2 - x$, then we get

$$(y - 20)\ (y - 42) = 504$$

$$y^2 - 62y + 840 - 504 = 0$$

$$y^2 - 62y + 336 = 0$$

$$(y - 6)\ (y - 56) = 0$$

336 → -6, -56

$y - 6 = 0$

put $y = x^2 - x$

$x^2 - x - 6 = 0$

$(x - 3)\ (x + 2) = 0$

$x - 3 = 0 \quad x + 2 = 0$

$\boldsymbol{x = 3} \quad \boldsymbol{x = -2}$

$y - 56 = 0$

put $y = x^2 - x$

$x^2 - x - 56 = 0$

$(x - 8)\ (x + 7) = 0$

$x - 8 = 0 \quad x + 7 = 0$

$\boldsymbol{x = 8} \quad \boldsymbol{x = -7}$

(ii) $(x - 4)\ (x - 7)\ (x - 2)\ (x + 1) = 16$

$(x - 4)\ (x - 7)\ (x - 2)\ (x + 1) = 16$

Rewriting the equation as

$$(x - 4)\ (x - 2)\ (x - 7)\ (x + 1) = 16$$

$$[(x-4)(x-2)][(x-7)(x+1)] = 16$$

$$(x^2-6x+8)(x^2-6x-7) = 16$$

Put $y = x^2 - 6x$, we get

$$(y+8)(y-7) = 16$$

$$y^2 + y - 56 - 16 = 0$$

$$y^2 + y - 72 = 0$$

$$(y+9)(y-8) = 0$$

$y + 9 = 0$

put $\boldsymbol{y = x^2 - 6x}$

$$x^2 - 6x + 9 = 0$$

$$(x-3)(x-3) = 0$$

$$x - 3 = 0; \quad x - 3 = 0$$

$$x = 3, x = 3$$

$y - 8 = 0$

put $y = x^2 - 6x$

$$x^2 - 6x - 8 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{6 \pm \sqrt{36 - 4(-8)}}{2}$$

$$= \frac{b \pm \sqrt{36 + 32}}{2}$$

$$= \frac{6 \pm \sqrt{68}}{2}$$

$$= \frac{6 \pm 2\sqrt{17}}{2}$$

$$\boxed{\boldsymbol{x = 3 + \sqrt{17}}}$$

$\therefore$ roots are 3,3, $3 + \sqrt{17}$ and $3 - \sqrt{17}$

2. Solve; $\boldsymbol{(2x-1)(x+3)(x-2)(2x+3) + 20 = 0}$

$$(2x-1)(x+3)(x-2)(2x+3) + 20 = 0$$

Rewriting the equation as

$$(2x-1)(2x+3)(x+3)(x-2) + 20 = 0$$

$$\left[2\left(x - \frac{1}{2}\right)2\left(x + \frac{3}{2}\right)\right][(x+3)(x-2)] + 20 = 0$$

$$4\left[x^2+x-\frac{3}{4}\right][x^2+x-6]+20=0$$

Put $y=x^2+x$

$$4\left(y-\frac{3}{4}\right)(y-6)+20=0$$

$$(4y-3)\,(y-6)+20=0$$

$$4y^2-24y-3y+18+20=0$$

$$4y^2-27y+38=0$$

$$(y-2)\left(y-\frac{19}{4}\right)=0$$

152

$-19 \qquad -8$

$\frac{-19}{4}, \qquad \frac{-8}{4}$

$\frac{-19}{4}, \qquad -2$

* $y-2=0$

put $y=x^2+x$

$x^2+x-2=0$

$(x+2)\,(x+1)=0$

$x+2=0 \quad x-1=0$

$\boldsymbol{x=-2} \qquad \boldsymbol{x=1}$

* $y-\frac{19}{4}=0$

put $y=x^2+x$

$x^2+x-\frac{19}{4}=0$

$4x^2+4x-19=0$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-4\pm\sqrt{16-4\,(4)\,(-19)}}{2\,(4)}$$

$$=\frac{-4\pm\sqrt{16+304}}{8}$$

$$=\frac{-4\pm\sqrt{320}}{8}$$

$$=\frac{-4\pm 8\sqrt{5}}{8}$$

$$=\frac{4\,(-1\pm 2\sqrt{5})}{8}$$

$$x=\frac{-1\pm 2\sqrt{5}}{2}$$

$\therefore$ roots are

$$1,\,-2,\,\frac{-1+2\sqrt{5}}{2}\text{ and }\frac{-1-2\sqrt{5}}{2}$$

Example 3.23

***Solve the equation* $(x-2)(x-7)(x-3)(x+2)+19=0$**

Rewriting the equation as,

$$(x-2)(x-3)(x-7)(x+2)+19=0$$

$$[(x-2)(x-3)][(x-7)(x+2)]+19=0$$

$$(x^2-5x+6)(x^2-5x-14)+19=0$$

Put $y=x^2-5x$

$$(y+6)(y-14)+19=0$$

$$y^2-8y-84+19=0$$

$$y^2-8y-65=0$$

$$(y+5)(y-13)=0$$

-65 → 5, -13

* $y+5=0$

put $y=x^2-5x$

$$x^2-5x+5=0$$

$$x=-b\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$=\frac{5\pm\sqrt{25-34\,(5)}}{2}$$

$$=\frac{5\pm\sqrt{25-20}}{2}$$

$$=\frac{5\pm\sqrt{5}}{2}$$

* $y-13=0$

put $y=x^2-5x$

$$x^2-5x-13=0$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{5\pm\sqrt{25-4\,(-13)}}{2}$$

$$=\frac{5\pm\sqrt{25+52}}{2}$$

$$=\frac{5\pm\sqrt{77}}{2}$$

$\therefore$ Roots are $\frac{5+\sqrt{5}}{2}, \frac{5-\sqrt{5}}{2}, \frac{5+\sqrt{77}}{2}$ and $\frac{5-\sqrt{77}}{2}$

Example 3.4

***Solve the equation* $(2x - 3)(6x - 1)(3x - 12)(x - 12) - 7 = 0$.**

Rewriting the equation as

$$(2x - 3)(6x - 1)(3x - 2)(x - 12) - 7 = 0$$

$$[(2x - 3)(3x - 2)][(6x - 1)(x - 12)] - 7 = 0$$

$$(6x^2 - 4x - 9x + 6)(6x^2 - 12x - x + 12) - 7 = 0$$

$$(6x^2 - 13x + 6)(6x^2 - 13x + 12) - 7 = 0$$

Put $y = 6x^2 - 13x$

$$(y + 6)(y + 12) - 7 = 0$$

$$y^2 + 18y + 72 - 7 = 0$$

$$y^2 + 18y + 65 = 0$$

$$(y + 5)(y + 13) = 0$$

* $y + 5 = 0$

put $y = 6x^2 - 13x$

$$6x^2 - 13x + 5 = 0$$

$$\left(x - \frac{1}{2}\right)\left(x - \frac{5}{3}\right) = 0$$

$$x - \frac{1}{2} = 0 \qquad x - \frac{5}{3} = 0$$

$$x = \frac{1}{2} \qquad x = \frac{5}{3}$$

* $y + 13 = 0$

put $y = 6x^2 - 13x$

$$6x^2 - 13x + 13 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{13 \pm \sqrt{169 - 4(6)(13)}}{2(6)}$$

$$= \frac{13 \pm \sqrt{169 - 312}}{12}$$

$$= \frac{13 \pm \sqrt{-143}}{12}$$

30

$-3 \quad -10$

$\frac{-3}{6}, \quad \frac{-10}{6}$

$\frac{-1}{2}, \quad \frac{-5}{3}$

$\therefore$ Solution

$$\frac{1}{2}, \frac{5}{3}, \frac{13 + i\sqrt{143}}{12} \text{ and } \frac{13 - i\sqrt{143}}{12}$$