Complex Number

Contents

JEE (Advanced) Syllabus

Complex Number : Algebra of complex numbers, addition, multiplication conjugation, polar representation, properties of modulus and principal argument, triangle inequality, cube roots of unity, geometric interpretations.

JEE (Main) Syllabus

Complex Number : Complex numbers as ordered pairs of reals, Representation of complex numbers in the form a + ib and their representation in a plane, Argand diagram, algebra of complex numbers, modulus and argument (or amplitude) of a complex number, square root of a complex number, triangle inequality.

COMPLEX NUMBER

1. INTRODUCTION :

Complex numbers are defined as expressions of the form a + ib where a , b $\in$ R & i = $\sqrt{-1}$. It is denoted by z i.e. z = a + ib. 'a' is called real part of z (Re z) and 'b' is called imaginary part of z (Im z).

Every Complex Number Can Be Regarded As

- Purely real if b = 0
- Imaginary if b $\neq$ 0
  - Purely imaginary if a = 0

Note :

(i) The set R of real numbers is a proper subset of the Complex Numbers. Hence the Complex Number system is N $\subset$ W $\subset$ I $\subset$ Q $\subset$ R $\subset$ C.

(ii) Zero is both purely real as well as purely imaginary but not imaginary.

(iii) i = $\sqrt{-1}$ is called the imaginary unit. Also $i^2 = -1$; $i^3 = -i$; $i^4 = 1$ etc.

In general $i^{4n} = 1$, $i^{4n+1} = i$, $i^{4n+2} = -1$, $i^{4n+3} = -i$, where n $\in$ I

(iv) $\sqrt{a}\,\sqrt{b} = \sqrt{ab}$ only if atleast one of either a or b is non-negative.

SOLVED EXAMPLE

Example 1 : Find the value of $\dfrac{i^{2008}+i^{2010}+i^{2012}+i^{2014}+i^{2016}}{i^{2010}+i^{2012}+i^{2014}+i^{2016}+i^{2018}}$

Solution : $\dfrac{i^{2008}+i^{2010}+i^{2012}+i^{2014}+i^{2016}}{i^{2010}+i^{2012}+i^{2014}+i^{2016}+i^{2018}} = \dfrac{i^{2008}\left(1+i^2+i^4+i^6+i^8\right)}{i^{2010}\left(1+i^2+i^4+i^6+i^8\right)} = \dfrac{1}{i^2} = -1$

Example 2 : If $x = -5 + 2\sqrt{-4}$, find the value of $x^4 + 9x^3 + 35x^2 - x + 4$.

Solution : We have , $x = -5 + 2\sqrt{-4}$

$\Rightarrow$ $x + 5 = 4i$ $\Rightarrow$ $(x + 5)^2 = 16i^2$

$\Rightarrow$ $x^2 + 10x + 25 = -16$ $\Rightarrow$ $x^2 + 10x + 41 = 0$

Now, $x^4 + 9x^3 + 35x^2 - x + 4$

$\Rightarrow$ $x^2(x^2 + 10x + 41) - x(x^2 + 10x + 41) + 4(x^2 + 10x + 41) - 160$

$\Rightarrow$ $x^2(0) - x(0) + 4(0) - 160$ $\Rightarrow$ -160

2. ALGEBRAIC OPERATIONS :

Fundamental operations with complex numbers :

(a) Addition (a + bi) + (c + di) = (a + c) + (b + d)i

(b) Subtraction (a + bi) – (c + di) = (a – c) + (b – d)i

(c) Multiplication (a + bi) (c + di) = (ac – bd) + (ad + bc)i

(d) Division $\frac{a+bi}{c+di} = \frac{a+bi}{c+di}.\frac{c-di}{c-di} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i$

Note :

(i) The algebraic operations on complex numbers are similar to those on real numbers treating i as a polynomial.

(ii) Inequalities in complex numbers (non-real) are not defined. There is no validity if we say that complex number (non-real) is positive or negative.

e.g. $z > 0$, $4 + 2i < 2 + 4i$ are meaningless.

(iii) In real numbers, if $a^2 + b^2 = 0$, then $a = 0 = b$ but in complex numbers, $z_1^2 + z_2^2 = 0$ does not imply $z_1 = z_2 = 0$.

SOLVED EXAMPLE

Example 3 : Find the imaginary part of following complex numbers

(i) $\left(\frac{4i^3-i}{2i+1}\right)^2$ (ii) $(1 + i)^4 + (1 – i)^4$

Solution : (i) $z = \left(\frac{4i^3-i}{2i+1}\right)^2 = \left(\frac{-5i(1-2i)}{5}\right)^2 = (-2-i)^2 = 3 + 4i$

Hence Im (z) = 4

(ii) $z = \{(1 + i)^2\}^2 + \{(1 - i)^2\}^2 = (1 + 2i + i^2)^2 + (1 + i^2 - 2i)^2 = 4i^2 + 4i^2 = -8$, Hence Im (z) = 0

3. EQUALITY IN COMPLEX NUMBER :

Two complex numbers $z_1 = a_1 + ib_1$ & $z_2 = a_2 + ib_2$ are equal if and only if their real and imaginary parts are equal respectively

i.e. $z_1 = z_2 \Leftrightarrow \text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$.

SOLVED EXAMPLE

Example 4 : Find the real values of x and y satisfying the equation $\frac{(1+i)x-2i}{3+i} + \frac{(2-3i)y+i}{3-i} = i$

Solution : $\frac{(1+i)x-2i}{3+i} + \frac{(2-3i)y+i}{3-i} = i \Rightarrow \frac{(1+i)(3-i)x-6i-2+(3+i)(2-3i)y+3i-1}{9+1} = i$

$\Rightarrow (4x + 9y - 3) + i(2x - 7y - 3) = 10i$

$\Rightarrow \quad 4x + 9y - 3 = 0 \quad \ldots (1)$

$2x - 7y - 13 = 0 \quad \ldots (2)$

On solving we get

$x = 3 \quad \& \quad y = -1$

Example 5 : If $z = x + iy$ and $z^{1/3} = a - ib$ then prove that $\frac{x}{a} - \frac{y}{b} = 4\left(a^2 - b^2\right)$

Solution : $z^{1/3} = a - ib \quad \Rightarrow \quad z = (a - ib)^3 \quad \Rightarrow \quad x + iy = (a^3 - 3ab^2) + i(b^3 - 3a^2b)$

$\Rightarrow \quad \frac{x}{a} = a^2 - 3b^2 \quad \& \quad \frac{y}{b} = b^2 - 3a^2$

$\frac{x}{a} - \frac{y}{b} = 4(a^2 - b^2)$

Example 6 : Find the square root of 7 + 24 i.

Solution : Let $\sqrt{7+24i} = a + ib$

Squaring $a^2 - b^2 + 2iab = 7 + 24i$

Compare real & imaginary parts $a^2 - b^2 = 7$ & $2ab = 24$

By solving these two equations

We get $a = \pm 4$, $b = \pm 3$

$\Rightarrow \sqrt{7+24i} = \pm(4 + 3i)$

Problems for Self Practice-01

(1) Write the following as complex number

(i) $\sqrt{-16}$ (ii) $\sqrt{x}$ $(x < 0)$ (iii) roots of $x^2 - (2\cos\theta)x + 1 = 0$

(2) Find the product of the real part of the roots of $z^2 - z = 5 - 5i$

(3) Given that $x, y \in R$, solve : $4x^2 + 3xy + (2xy - 3x^2)i = 4y^2 - (x^2/2) + (3xy - 2y^2)i$

(4) If $a + ib = \frac{c+i}{c-i}$, where c is a real number, then prove that : $a^2 + b^2 = 1$ and $\frac{b}{a} = \frac{2c}{c^2-1}$.

Answers : (1) (i) $0 + 4i$ (ii) $0 + i\sqrt{-x}$ (iii) $\cos\theta + i\sin\theta$, $\cos\theta - i\sin\theta$

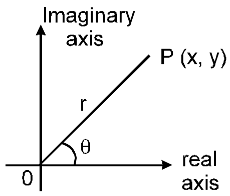
(2) -6

(3) $x = K$, $y = \frac{3K}{2}$ $K \in R$

4. REPRESENTATION OF A COMPLEX NUMBER :

4.1 Cartesian Form (Geometric Representation) :

Every complex number $z = x + iy$ can be represented by a point on the Cartesian plane known as complex plane (Argand diagram) by the ordered pair (x, y).



Length OP is called modulus of the complex number which is denoted by $|z|$ & θ is called the argument or amplitude.

$|z| = \sqrt{x^2 + y^2}$ and $\tan\theta = \left(\frac{y}{x}\right)$ (angle made by OP with positive x–axis)

Note :

(i) **Argument of a complex number :**

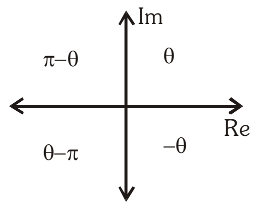
(a) Argument of a complex number is a many valued function. If θ is the argument of a complex number, then $2n\pi + \theta$; $n \in I$ will also be the argument of that complex number. Any two arguments of a complex number differ by $2n\pi$.

(b) The unique value of θ such that $-\pi < \theta \le \pi$ is called **principal value of the argument.**

(c) Principal argument of a complex number z = x + iy can be found out using method given below :

- Find $\theta = \tan^{-1}\left|\frac{y}{x}\right|$ such that $\theta \in \left(0, \frac{\pi}{2}\right)$.

- Use given figure to find out the principal argument according as the point lies in respective quadrant.



(d) Unless otherwise stated, amp z implies principal value of the argument.

(e) The unique value of $\theta = \tan^{-1}\frac{y}{x}$ such that $0 < \theta \le 2\pi$ is called **least positive argument.**

(f) If z is real & negative, $\arg(z) = \pi$.

(g) If z is real & positive, $\arg(z) = 0$

(h) If $\theta = \frac{\pi}{2}$, z lies on the positive side of imaginary axis.

(i) If $\theta = -\frac{\pi}{2}$, z lies on the negative side of imaginary axis.

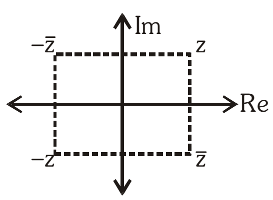
By specifying the modulus & argument a complex number is defined completely. Argument impart direction & modulus impart distance from origin.

For the complex number 0 + 0i the argument is not defined and this is the only complex number which is given by its modulus only.

(ii) **Conjugate of a complex number :**

If z = a + ib then its conjugate complex is obtained by changing the sign of its imaginary part & is denoted by $\bar{z}$. i.e. $\bar{z} = a - ib$.

$\bar{z}$ is the mirror image of z about real axis on Argand's Plane.



SOLVED EXAMPLE

Example 7 : Find the modulus and principal argument of the complex numbers.

(i) $\dfrac{1+3i}{1-2i}$ (ii) $\dfrac{i-1}{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}}$

Solution : (i) Let $z=\dfrac{1+3i}{1-2i}=\dfrac{1+3i}{1-2i}\times\dfrac{1+2i}{1+2i}=-1+i$

$|z|=\sqrt{(-1)^2+1^2}=\sqrt{2}$ $\tan\alpha=\left|\dfrac{1}{-1}\right|=1=\tan\dfrac{\pi}{4}\Rightarrow\alpha=\dfrac{\pi}{4}$

$\because$ Re(z) < 0 and Im(z) > 0 $\Rightarrow$ z lies in second quadrant.

$\therefore$ Principal argument of z = $\pi-\alpha=\pi-\dfrac{\pi}{4}=\dfrac{3\pi}{4}$

(ii) Let $z=\dfrac{i-1}{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}}=\dfrac{i-1}{\frac{1}{2}+\frac{i\sqrt{3}}{2}}=\dfrac{2(i-1)}{(1+i\sqrt{3})}$

$\Rightarrow$ $z=\dfrac{2(i-1)}{(1+i\sqrt{3})}\times\dfrac{(1-i\sqrt{3})}{(1-i\sqrt{3})}\Rightarrow z=\left(\dfrac{\sqrt{3}-1}{2}\right)+i\left(\dfrac{\sqrt{3}+1}{2}\right)$

$\because$ Re(z) > 0 and Im(z) > 0 $\Rightarrow$ z lies in first quadrant.

$\therefore$ $|z|=\sqrt{\left(\dfrac{\sqrt{3}-1}{2}\right)^2+\left(\dfrac{\sqrt{3}+1}{2}\right)^2}=\sqrt{\dfrac{2(3+1)}{4}}=\sqrt{2}$.

$\tan\theta=\left|\dfrac{\sqrt{3}+1}{\sqrt{3}-1}\right|=\tan\dfrac{5\pi}{12}$

$\therefore$ Principal argument of z $=\dfrac{5\pi}{12}$

Example 8 : Find the values of x so that the complex numbers sin x + i cos 2x and cos x – i sin 2x are conjugate to each other

Solution : sinx + i cos2x = cosx + i sin2x $\Rightarrow$ cos2x = sin2x

tanx = 1 & tan 2x = 1

x = $\dfrac{\pi}{4},\dfrac{5\pi}{4},\dfrac{9\pi}{4}$ x = $\dfrac{\pi}{8},\dfrac{5\pi}{8},\dfrac{9\pi}{8}$

$\because$ both equation will not have solution simultaneously.

$\therefore$ $x\in\phi$

Example 9 : Solve for z if $z^2 + |z| = 0$

Solution : Let $z = x + iy$

$\Rightarrow \quad (x + iy)^2 + \sqrt{x^2 + y^2} = 0 \quad \Rightarrow \quad x^2 - y^2 + \sqrt{x^2 + y^2} = 0$ and $2xy = 0$

$\Rightarrow \quad x = 0$ or $y = 0$

when $x = 0 \quad -y^2 + |y| = 0$

$\Rightarrow \quad y = 0, 1, -1 \quad \Rightarrow \quad z = 0, i, -i$

when $y = 0 \quad x^2 + |x| = 0 \quad \Rightarrow \quad x = 0 \quad \Rightarrow \quad z = 0$

Ans. $z = 0, z = i, z = -i$

4.2 Trigonometric / Polar Representation :

$z = r(\cos\theta + i\sin\theta)$ where $|z| = r$; $\arg z = \theta$; $\bar{z} = r(\cos\theta - i\sin\theta)$

Note :

(i) $\cos\theta + i\sin\theta$ is also written as CiS θ.

(ii) Euler's formula :

The formula $e^{ix} = \cos x + i\sin x$ is called Euler's formula.

It was introduced by Euler in 1748, and is used as a method of expressing complex numbers.

Also $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ & $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$ are known as Euler's identities.

4.3 Exponential Representation :

Let z be a complex number such that $|z| = r$ & $\arg z = \theta$, then $z = r.e^{i\theta}$

4.4 Vectorial Representation :

Every complex number can be considered as the position vector of a point. If the point P represents the complex number z then, $\vec{OP} = z$ & $|\vec{OP}| = |z|$.

Note :

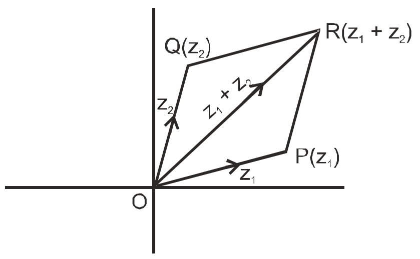


Diagram-1

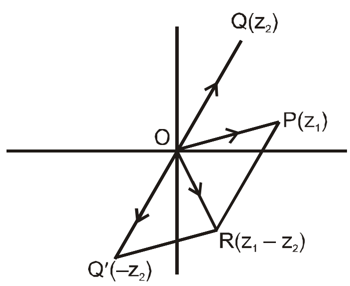


Diagram-2

If two points P and Q represent complex numbers z_1 and z_2 respectively in the Argand plane, then the sum $z_1 + z_2$ is represented by the extremity R of the diagonal OR of parallelogram OPRQ (Diagram-1) having OP and OQ as two adjacent sides and the subtraction $z_1 - z_2$ is represented by the extremity R of the diagonal OR of parallelogram OPRQ' (Diagram-2) having OP and OQ' as two adjacent sides

SOLVED EXAMPLE

Example 10 : Express the following complex numbers in polar and exponential form :

(i) $z = 1 + \cos\frac{18\pi}{25} + i\sin\frac{18\pi}{25}$ (ii) $\frac{i-1}{i\left(1-\cos\frac{2\pi}{5}\right)+\sin\frac{2\pi}{5}}$

Solution : **(i)** $z = 1 + e^{i\frac{18\pi}{25}} = e^{i\frac{9\pi}{25}}\left[e^{i\frac{9\pi}{25}} + e^{-i\frac{9\pi}{25}}\right] \Rightarrow z = 2\cos\left(\frac{9\pi}{25}\right) e^{i\frac{9\pi}{25}}$

Hence polar form is $z = 2\cos\frac{9\pi}{25}\left(\cos\frac{9\pi}{25} + i\sin\frac{9\pi}{25}\right)$

and exponential form is $z = 2\cos\left(\frac{9\pi}{25}\right) e^{i\frac{9\pi}{25}}$

(ii) $z = \frac{(i-1)}{2\sin\left(\frac{\pi}{5}\right)\left[\sin\left(\frac{\pi}{5}\right)i + \cos\left(\frac{\pi}{5}\right)\right]} = \frac{\sqrt{2}e^{i3\pi/4}}{2\sin\left(\frac{\pi}{5}\right)e^{i\pi/5}} = \frac{1}{\sqrt{2}\sin\frac{\pi}{5}}e^{i(11\pi/20)}$

Hence polar form is $z = \frac{1}{\sqrt{2}}\operatorname{cosec}\frac{\pi}{5}\left(\cos\frac{11\pi}{20} + i\sin\frac{11\pi}{20}\right)$

and exponential form is $z = \frac{1}{\sqrt{2}\sin\frac{\pi}{5}}e^{i(11\pi/20)}$

5. DEMOIVRE'S THEOREM:

Case I

Statement : If n is any integer then

(i) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

(ii) $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)\ .....(\cos\theta_n + i\sin\theta_n)$
$= \cos(\theta_1 + \theta_2 + \theta_3 + ...\ \theta_n) + i\sin(\theta_1 + \theta_2 + \theta_3 + ...+ \theta_n)$

Case II

Statement : If $p, q \in Z$ and $q \neq 0$ then

$$(\cos\theta + i\sin\theta)^{p/q} = \cos\left(\frac{2k\pi + p\theta}{q}\right) + i\sin\left(\frac{2k\pi + p\theta}{q}\right)$$

where $k = 0, 1, 2, 3, ..., q - 1$

Note :

Continued product of the roots of a complex quantity should be determined using theory of equations.

SOLVED EXAMPLE

Example 11 : If (cosθ + i sinθ) (cos 2θ + i sin 2θ) ... (cos nθ + i sin nθ) = 1, then find the values of θ

Solution : $e^{i\theta} \cdot e^{i2\theta} \ldots\ldots e^{in\theta} = 1 \Rightarrow e^{i\theta\frac{(n)(n+1)}{2}} = e^{i2m\pi} \Rightarrow \frac{\theta n(n+1)}{2} = 2m\pi \Rightarrow \theta = \frac{4m\pi}{n(n+1)}$ $m \in z$.

Problems for Self Practice-02

(1) Find the set of values of $a \in R$ for which $x^2 + i(a - 1)x + 5 = 0$ will have a pair of conjugate imaginary roots

(2) Find the modulus, argument, principal value of argument, least positive argument of complex numbers

(i) $z = -1 - i\sqrt{3}$

(ii) $z = 1 - \sin\alpha + i\cos\alpha,\ \alpha \in \left(\frac{3\pi}{2}, 2\pi\right)$

(3) Find the value of $e^{i2m\theta}\left(\frac{i\cot\theta + 1}{i\cot\theta - 1}\right)^m$

(4) Prove the identities :

(i) $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$;

(ii) $(\sin 5\theta) / (\sin\theta) = 16\cos^4\theta - 12\cos^2\theta + 1$, if $\theta \neq 0, \pm\pi, \pm 2\pi$ ....

Answer :

(1) $a \in \{1\}$ (2) (i) $|z| = 2$, $\arg(z) = 2n\pi - \frac{2\pi}{3}$, $n \in I$, Least positive argument = $\frac{4\pi}{3}$, amp(z) = $-\frac{2\pi}{3}$

(ii) $|z| = \sqrt{2}\left(\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2}\right)$, arg z = $2n\pi + \frac{\alpha}{2} - \frac{3\pi}{4}$, Least positive argument = $\frac{\alpha}{2} - \frac{3\pi}{4}$,

amp(z) = $\frac{\alpha}{2} - \frac{3\pi}{4}$

(3) 1

6. PROPERTIES OF CONJUGATE / MODULUS / ARGUMANT OF COMPLEX NUMBERS :

6.1 Conjugate :

(i) If z = x + iy, then $x = \frac{z + \bar{z}}{2}$, $y = \frac{z - \bar{z}}{2i}$ (ii) $z = \bar{z} \Leftrightarrow$ z is purely real

(iii) $z + \bar{z} = 0 \Leftrightarrow$ z is purely imaginary (iv) $\overline{(\bar{z})} = z$

(v) $\overline{z_1 \pm z_2} = \overline{z}_1 \pm \overline{z}_2$

(vi) $\overline{z_1 z_2} = \overline{z}_1 . \overline{z}_2$. In general $\overline{z_1 z_2 ....z_n} = \overline{z}_1 . \overline{z}_2 ....\overline{z}_n$

(vii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z}_1}{\overline{z}_2}$; $z_2 \neq 0$

(viii) If $f(\alpha + i\beta) = x + iy \Rightarrow f(\alpha - i\beta) = x - iy$

6.2 Modulus :

(i) $|z| = |\overline{z}| = |-z| = |-\overline{z}|$

(ii) $z\overline{z} = |z|^2$

(iii) $|z_1 z_2| = |z_1|.|z_2|$. In general $|z_1 z_2 ....z_n| = |z_1|.|z_2|.........|z_n|$

(iv) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $z_2 \neq 0$

(v) $|z^n| = |z|^n$, $n \in I$

(vi) $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm 2\,\text{Re}(z_1\overline{z}_2)$ or $|z_1|^2 + |z_2|^2 \pm 2|z_1||z_2| \cos(\alpha - \beta)$, where α, β are $\arg(z_1)$, $\arg(z_2)$ respectively.

Note : Unlike real numbers, $|z| = \begin{cases} z & \text{if } z > 0 \\ -z & \text{if } z < 0 \end{cases}$ is not correct.

SOLVED EXAMPLE

Example 12 : If $x + iy = \sqrt{\frac{a+ib}{c+id}}$ prove that $(x^2 + y^2)^2 = \frac{a^2+b^2}{c^2+d^2}$

Solution : $x + iy = \sqrt{\frac{a+ib}{c+id}}$ ....(i)

$\Rightarrow x - iy = \sqrt{\frac{a-ib}{c-id}}$ ....(ii)

Multiplying (i) & (ii) we get

$x^2 + y^2 = \sqrt{\frac{a^2+b^2}{c^2+d^2}}$

$\Rightarrow (x^2 + y^2)^2 = \frac{a^2+b^2}{c^2+d^2}$ Hence proved

Example 13 : If $\frac{z-1}{z+1}$ is purely imaginary, then prove that $|z| = 1$

Solution : Re $\left(\frac{z-1}{z+1}\right) = 0$ $\Rightarrow$ $\frac{z-1}{z+1} + \left(\overline{\frac{z-1}{z+1}}\right) = 0$

$\Rightarrow \frac{z-1}{z+1} + \frac{\bar{z}-1}{\bar{z}+1} = 0$

$\Rightarrow z\bar{z} - \bar{z} + z - 1 + z\bar{z} - z + \bar{z} - 1 = 0$

$\Rightarrow z\bar{z} = 1$

$\Rightarrow |z|^2 = 1$

$\Rightarrow$ $|z| = 1$ Hence proved

Example 14 : If the complex numbers $z_1, z_2, \ldots\ldots z_n$ lie on the unit circle $|z| = 1$, then show that $|z_1 + z_2 + \ldots\ldots + z_n| = |z_1^{-1} + z_2^{-1} + \ldots\ldots + z_n^{-1}|$.

Solution : Given that $|z_1|^2 = |z_2|^2 = \ldots\ldots |z_n|^2 = 1$

$\Rightarrow z_1\bar{z}_1 = z_2\bar{z}_2 = \ldots\ldots z_n\bar{z}_n = 1$

$\therefore |z_1 + z_2 + \ldots\ldots + z_n| = \left|\overline{z_1 + z_2 + \ldots\ldots + z_n}\right| = \left|\bar{z}_1 + \bar{z}_2 + \ldots\ldots + \bar{z}_n\right|$

$= |z_1^{-1} + z_2^{-1} + \ldots\ldots + z_n^{-1}|$ Hence proved

6.3. Argument :

(i) $\arg(z_1 . z_2) = \arg(z_1) + \arg(z_2) + 2k\pi$; $k \in I$ (ii) $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) + 2k\pi$; $k \in I$

(iii) $\arg(z^n) = n\arg(z) + 2k\pi$; $n,k \in I$ (iv) $\arg(\bar{z}) = -\arg(z) + 2k\pi$; $k \in I$

SOLVED EXAMPLE

Example 15 : If $\arg(z_1) = 170^\circ$ and $\arg(z_2) = 70^\circ$, then find the principal argumant of z_1z_2.

Solution : $\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$

$= 170^\circ + 70^\circ = 240^\circ$

$\therefore$ Principal argument of $z = -(180^\circ - 60^\circ)$

$= -120^\circ$

Example 16 : Let z and ω be two non-zero complex numbers such that $|z| = |\omega|$ and $\arg z = \pi - \arg\omega$, then prove that $z = -\bar{\omega}$

Solution : We have, $|z| = |\omega|$ and $\arg z = \pi - \arg\omega$

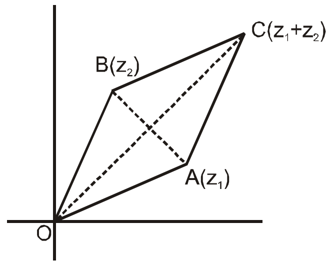
$\Rightarrow \arg(z\omega) = \pi$

$\therefore z\omega = \lambda$; $\lambda < 0$

$\Rightarrow |z\omega| = -\lambda$

Hence $z = \frac{\lambda}{\omega} = \frac{-|z||\omega|}{\omega} = \frac{-\omega\bar{\omega}}{\omega} = -\bar{\omega}$

7. TRIANGULAR INEQUALITY :



In triangle OAC

$OC \le OA + AC$

$OA \le AC + OC$

$AC \le OA + OC$

using these in equalities we have $||z_1| - |z_2|| \le |z_1 + z_2| \le |z_1| + |z_2|$

Similarly from triangle OAB

we have $||z_1| - |z_2|| \le |z_1 - z_2| \le |z_1| + |z_2|$

Note :

(i) $||z_1| - |z_2|| = |z_1 + z_2|$, $|z_1 - z_2| = |z_1| + |z_2|$ iff origin, z_1 and z_2 are collinear and origin lies between z_1 and z_2 .

(ii) $|z_1 + z_2| = |z_1| + |z_2|$, $||z_1| - |z_2|| = |z_1 - z_2|$ iff origin, z_1 and z_2 are collinear and z_1 and z_2 lies on the same side of origin.

SOLVED EXAMPLE

Example 17 : Find the greatest and least value of $|z_1 + z_2|$ if $z_1 = 24 + 7i$ and $|z_2| = 6$

Solution : $\left||z_1| - |z_2|\right| \le |z_1 + z_2| \le |z_1| + |z_2|$

$\Rightarrow \left|\sqrt{24^2 + 7^2} - 6\right| \le |z_1 + z_2| \le \sqrt{24^2 + 7^2} + 6 \Rightarrow 19 \le |z_1 + z_2| \le 31$

$\therefore$ Greatest value of $|z_1 + z_2| = 31$

Least value of $|z_1 + z_2| = 19$

Example 18 : Find the minimum value of $|3z - 3| + |2z - 4|$

Solution : $|3z - 3| \ge |3|z| - 3|$ $\qquad |2z - 4| \ge |2|z| - 4|$

$\Rightarrow |3z - 3| + |2z - 4| \ge |3|z| - 3| + |2|z| - 4| \ge 2$

Hence minimum value of $|3z - 3| + |2z - 4|$ is 2

<u>Problems for Self Practice</u>-03

(1) If $|z| = 1$ and $\omega = \dfrac{z-1}{z+1}$ (where $z \ne -1$), then find $\text{Re}(\omega)$

(2) If z_1 and z_2 are two complex numbers such that $|z_1| < 1 < |z_2|$ then prove that $\left|\dfrac{1 - z_1\bar{z}_2}{z_1 - z_2}\right| < 1$.

(3) If $|z_1| = |z_2|$ and $\arg(z_1/z_2) = \pi$, then find the value of $z_1 + z_2$

(4) If z lies on circle $|z| = 2$, then show that $\left|\dfrac{1}{z^4 - 4z^2 + 3}\right| \le \dfrac{1}{3}$

Answer : (1) 0 (3) 0

8. GEOMETRY USING COMPLEX NUMBERS :

8.1 Distance formula :

If $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, then distance between points z_1, z_2 in argand plane is

$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

8.2 Section formula :

If z_1 and z_2 are affixes of the two points P and Q respectively and point C divides the line segment joining P and Q internally in the ratio m : n then affix z of C is given by

$z = \dfrac{mz_2 + nz_1}{m+n}$; where m, n > 0

If C divides PQ in the ratio m : n externally then $z = \dfrac{mz_2 - nz_1}{m-n}$

Note :

(i) If a, b, c are three real numbers such that $az_1 + bz_2 + cz_3 = 0$; where a + b + c = 0 and a,b,c are not all simultaneously zero, then the complex numbers z_1, z_2 & z_3 are collinear.

(ii) If the vertices A, B, C of a Δ are represented by complex numbers z_1, z_2, z_3 respectively and a, b, c are the length of sides then,

(a) Centroid of the Δ ABC = $\dfrac{z_1 + z_2 + z_3}{3}$:

(b) Orthocentre of the Δ ABC =

$$\frac{(a\sec A)z_1 + (b\sec B)z_2 + (c\sec C)z_3}{a\sec A + b\sec B + c\sec C} \text{ or } \frac{z_1\tan A + z_2\tan B + z_3\tan C}{\tan A + \tan B + \tan C}$$

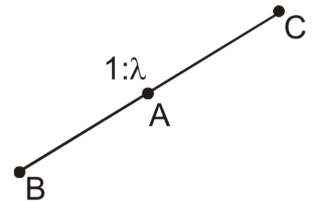
(c) Incentre of the Δ ABC = $\dfrac{az_1 + bz_2 + cz_3}{a+b+c}$

(d) Circumcentre of the Δ ABC = $\dfrac{Z_1 \sin 2A + Z_2 \sin 2B + Z_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}$

SOLVED EXAMPLE

Example 19 : If A, B, C are three points in argand plane representing the complex number z_1, z_2, z_3 such that $z_1 = \dfrac{\lambda z_2 + z_3}{\lambda + 1}$, where $\lambda \in R$, then find the distance of point A from the line joining points B and C.

Solution : $A(z_1)$ divides the line segment joining $B(z_2)$ & $C(z_3)$ in $1 : \lambda$ ratio.
Hence $A(z_1)$, $B(z_2)$ & (z_3) are collinear
$\therefore$ distance of point A is zero.



Example 20 : Let A, B, C represent the complex numbers z_1, z_2, z_3 respectively on the complex plane. If the circumcentre of the triangle ABC lies at the origin, then prove that orthocentre is represented by $z_1 + z_2 + z_3$

Solution : G → Centroid of Δ = $\frac{z_1 + z_2 + z_3}{3}$

H → Orthocentre = z say, O → Circum centre = 0

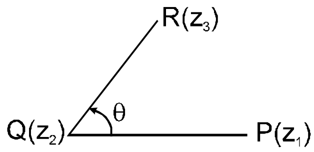
∵ G divides HO in ratio 2 : 1 , therefore

$$\frac{z_1 + z_2 + z_3}{3} = \frac{2 \cdot 0 + 1 \cdot z}{2+1} \quad \Rightarrow \quad z = z_1 + z_2 + z_3$$

8.3 Rotation theorem :

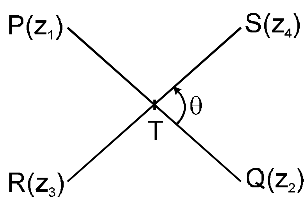
(i) If $P(z_1)$, $Q(z_2)$ and $R(z_3)$ are three complex numbers

and $\angle PQR = \theta$, then $\left(\frac{z_3 - z_2}{z_1 - z_2}\right) = \left|\frac{z_3 - z_2}{z_1 - z_2}\right| e^{i\theta}$



(ii) If $P(z_1)$, $Q(z_2)$, $R(z_3)$ and $S(z_4)$ are four complex numbers

and $\angle STQ = \theta$, then $\frac{z_3 - z_4}{z_1 - z_2} = \left|\frac{z_3 - z_4}{z_1 - z_2}\right| e^{i\theta}$



Note:

If z_1, z_2, z_3 are the vertices of an equilateral triangle where z_0 is its circumcentre then

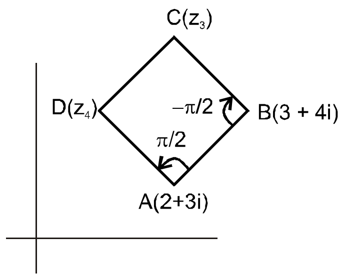
(a) $z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_3z_1 = 0$ (b) $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$

SOLVED EXAMPLE

Example 21 : If A(2 + 3i) and B(3 + 4i) are two vertices of a square ABCD (take in anticlock wise order) then find C and D.

Solution : Let affix of C and D are z_3 and z_4 respectively.

Considering ∠DAB = 90º and AD = AB



we get $\frac{z_4 - (2 + 3i)}{(3 + 4i) - (2 + 3i)} = \frac{AD}{AB} e^{\frac{i\pi}{2}}$

⇒ $z_4 - (2 + 3i) = (1 + i)\, i$

⇒ $z_4 = 2 + 3i + i - 1 = 1 + 4i$

and $\frac{z_3 - (3 + 4i)}{(2 + 3i) - (3 + 4i)} = \frac{CB}{AB} e^{-\frac{i\pi}{2}}$

⇒ $z_3 = 3 + 4i - (1 + i)(-i)$ ⇒ $z_3 = 3 + 4i + i - 1 = 2 + 5i$

Example 22 : Let z_1 and z_2 be two roots of the equation $z^2 + az + b = 0$, z being complex. Further, assume that the origin, z_1 and z_2 form an equilateral triangle. Then show that $a^2 = 3b$.

Solution : If z_1, z_2 and z_3 are vertices of an equilateral triangle. Then, $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$

Since, origin, z_1 and z_2 are the vertices of an equilateral triangle, then $z_1^2 + z_2^2 = z_1z_2$

$\Rightarrow (z_1 + z_2)^2 = 3z_1z_2$ ...(i)

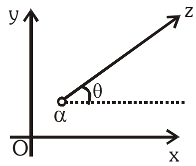
Again z_1, z_2 are the roots of the equation $z^2 + az + b = 0$,

Then, $z_1 + z_2 = -a$ and $z_1z_2 = b$

On putting these values in Eq. (i), we get $(-a)^2 = 3b \Rightarrow a^2 = 3b$.

8.4 Standard Loci in Argand plane :

(i) If $\text{amp}(z-\alpha) = \theta$, then locus of z is a ray emanating from the complex point α (excluding 'α')and inclined at an angle θ to the positive x–axis.

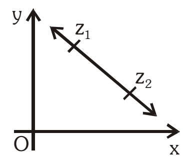


(ii) If $\left|\frac{z-z_1}{z-z_2}\right| = k$, then locus of z is

(a) Perpendicular bisector of the segment joining z_1 and z_2 for k = 1.

(b) Circle for $k \neq 1,0$.

(iii) If $z = z_1 + t(z_1 - z_2)$ where t is a parameter, then locus of z is a line joining z_1 & z_2



Note :

(a) The equation of a line passing through z_1 & z_2 can be expressed in the determinant form as

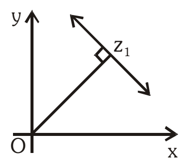
$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$. This is also the condition for three complex numbers z, z_1, z_2 to be collinear.

The above equation on manipulating, takes the form $\bar{\alpha}z + \alpha\bar{z} + r = 0$ where r is real and α is a non zero complex constant.

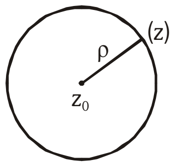
(b) Perpendicular distance of a point z_0 from the line $\bar{\alpha}z + \alpha\bar{z} + r = 0$ is $\frac{|\bar{\alpha}z_0 + \alpha\bar{z}_0 + r|}{2|\alpha|}$

(c) Area of triangle formed by the points z_1, z_2 & z_3 is $\left|\frac{1}{4i}\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}\right|$

(d) The equation of a line passing through the point z_1 & perpendicular to the line joining z_1 to the origin is $z = z_1(1 + it)$ where t is a real parameter



(iv) If $|z - z_0| = \rho$, then locus of z is circle having centre z_0 & radius ρ.



Note :

(a) The above equation on manipulating, takes the form

$z\bar{z} + \bar{\alpha}z + \alpha\bar{z} + r = 0$, r is real centre = $-\alpha$ & radius = $\sqrt{\alpha\bar{\alpha} - r}$.

Circle will be real if $\alpha\bar{\alpha} - r \geq 0$.

(b) The equation of the circle described on the line segment joining z_1 & z_2 as diameter is

$\arg\frac{z - z_2}{z - z_1} = \pm\frac{\pi}{2}$ or $(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$.

(c) Condition for four given points z_1, z_2, z_3 & z_4 to be concyclic is the number

$\frac{z_3 - z_1}{z_3 - z_2} \cdot \frac{z_4 - z_2}{z_4 - z_1}$ should be real. Hence the equation of a circle through 3 non collinear

points z_1, z_2 & z_3 can be taken as $\frac{(z-z_2)(z_3-z_1)}{(z-z_1)(z_3-z_2)}$ is real $\Rightarrow \frac{(z-z_2)(z_3-z_1)}{(z-z_1)(z_3-z_2)} = \frac{(\bar{z}-\bar{z}_2)(\bar{z}_3-\bar{z}_1)}{(\bar{z}-\bar{z}_1)(\bar{z}_3-\bar{z}_2)}$.

(v) If $\text{Arg}\left(\frac{z-z_1}{z-z_2}\right) = \theta$, then locus of z

(a) a line segment if $\theta = \pi$ (b) Pair of ray if $\theta = 0$

(c) Major arc of circle excluding z_1 & z_2 if $0 < \theta < \pi/2$

(d) Minor arc of circle excluding z_1 & z_2 if $\frac{\pi}{2} < \theta < \pi$

(vi) If $|z - z_0| = \left|\frac{\bar{\alpha}z + \alpha\bar{z} + r}{2|\alpha|}\right|$, then locus of z is parabola whose focus is z_0 and directrix is the line $\bar{\alpha}z + \alpha\bar{z} + r = 0$ (Provided $\bar{\alpha}z_0 + \alpha\bar{z}_0 + r \neq 0$)

(vii) If $|z - z_1| + |z - z_2| = K > |z_1 - z_2|$, then locus of z is an ellipse whose focii are z_1 & z_2

(viii) If $\left||z - z_1| - |z - z_2|\right| = K < |z_1 - z_2|$, then locus of z is a hyperbola, whose focii are z_1 & z_2.

SOLVED EXAMPLE

Example 23 : Find the locus of :

(i) $|z - 1|^2 + |z + 1|^2 = 4$ (ii) $\text{Re}(z^2) = 0$ (iii) $|3z - 2| + |3z + 2| = 4$

Solution : (i) Let $z = x + iy$

$\Rightarrow$ $(|x + iy - 1|)^2 + (|x + iy + 1|)^2 = 4$

$\Rightarrow$ $(x - 1)^2 + y^2 + (x + 1)^2 + y^2 = 4$

$\Rightarrow$ $x^2 - 2x + 1 + y^2 + x^2 + 2x + 1 + y^2 = 4$

$\Rightarrow x^2 + y^2 = 1$

Above represents a circle on complex plane with center at origin and radius unity.

(ii) Let z = x + iy $\Rightarrow z^2 = x^2 - y^2 + 2xyi$

$\therefore$ $Re(z^2) = 0 \Rightarrow x^2 - y^2 = 0 \Rightarrow y = \pm x$

Thus $Re(z^2) = 0$ represents a pair of straight lines passing through origin.

(iii) $|3z - 2| + |3z + 2| = 4$

$$\Rightarrow \left|z - \frac{2}{3}\right| + \left|z + \frac{2}{3}\right| = \frac{4}{3} \qquad ....(i)$$

If P(z) be any point, $A \equiv \left(\frac{2}{3}, 0\right)$, $B \equiv \left(\frac{-2}{3}, 0\right)$ then (1) represents PA + PB = $\frac{4}{3}$

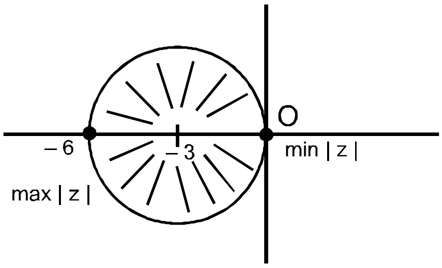
Clearly, AB = $\frac{4}{3} \Rightarrow$ PA + PB = AB

$\Rightarrow$ P lies on the line segment AB.

Example 24 : If $|z + 3| \le 3$ then find minimum and maximum values of

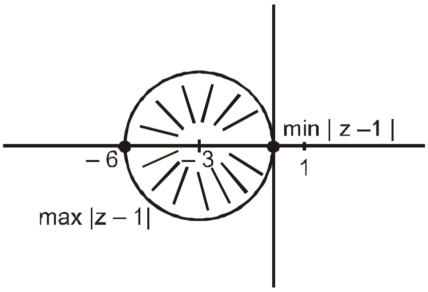
(i) $|z|$ (ii) $|z - 1|$ (iii) $|z + 1|$

Solution : (i)



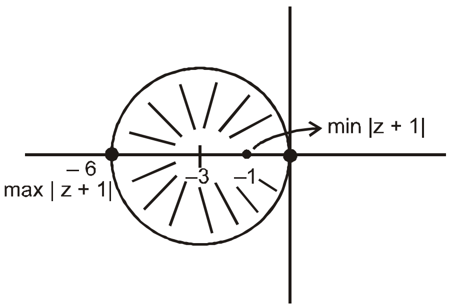
Min. $|z| = 0$ & max. $|z| = 6$

(ii)



Min. $|z - 1| = 1$ & max. $|z - 1| = 7$

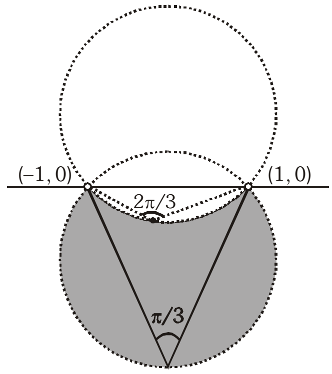
(iii)



Min. $|z + 1| = 0$ & max. $|z + 1| = 5$

Example 25 : Plot the region represented by $\frac{\pi}{3} \le \arg\left(\frac{z+1}{z-1}\right) \le \frac{2\pi}{3}$ in the Argand plane.

Solution : Let us take $\arg\left(\frac{z+1}{z-1}\right) = \frac{2\pi}{3}$, clearly z lies on the minor arc of the circle passing through (1, 0) and (–1, 0). Similarly, $\arg\left(\frac{z+1}{z-1}\right) = \frac{2\pi}{3}$ means that 'z' is lying on the major arc of the circle passing through (1, 0) and (–1, 0). Now if we take any point in the region included between two arcs say $P_1(z_1)$ we get $\frac{\pi}{3} \le \arg\left(\frac{z+1}{z-1}\right) \le \frac{2\pi}{3}$. Thus $\frac{\pi}{3} \le \arg\left(\frac{z+1}{z-1}\right) \le \frac{2\pi}{3}$ represents the shaded region (excluding points (1, 0) and (–1, 0)).



Problems for Self Practice-04

(1) If $A(z_1)$, $B(z_2)$, $C(z_3)$ are vertices of ΔABC in which $\angle ABC = \frac{\pi}{4}$ and $\frac{AB}{BC} = \sqrt{2}$, then find z_2 in terms of z_1 and z_3.

(2) If a, b, c ; u, v, w are complex numbers representing the vertices of two triangles such that $c = (1 - r)\, a + rb$, $w = (1 - r)\, u + rv$ where r is a complex number show that the two triangles are similiar.

(3) A particle starts to travel from a point P on the curve $C_1 : |z - 3 - 4i| = 5$, where $|z|$ is maximum. From P, the particle moves through an angle $\tan^{-1}\frac{3}{4}$ in anticlockwise direction on $|z - 3 - 4i| = 5$ and reaches at point Q. From Q, it comes down parallel to imaginary axis by 2 units and reaches at point R. Find the complex number corresponding to point R in the Argand plane.

(4) Find the complex number z satisfying the equations $|z - 3| = 2$ and $|z| = 2$

Answer : (1) $z_2 = z_3 + i(z_1 - z_3)$ (3) (3 + 7i) (4) $\frac{1}{2}\left(3 \pm i\sqrt{7}\right)$

9. CUBE ROOT OF UNITY :

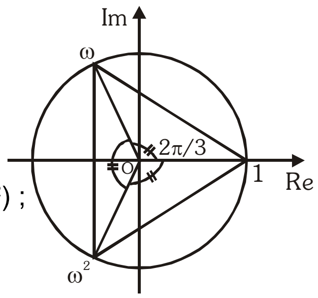
(i) The cube roots of unity are 1 , $\frac{-1 + i\sqrt{3}}{2}(\omega)$, $\frac{-1 - i\sqrt{3}}{2}(\omega^2)$.

(ii) If ω is one of the imaginary cube roots of unity then $1 + \omega + \omega^2 = 0$. In general $1 + \omega^r + \omega^{2r} = 0$; where $r \in I$ but is not the multiple of 3 & $1 + \omega^r + \omega^{2r} = 3$ if $r = 3\lambda$; $\lambda \in I$

(iii) In polar form the cube roots of unity are :

$$1 = \cos 0 + i \sin 0 \ ; \quad \omega = \cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3}, \quad \omega^2 = \cos\frac{4\pi}{3} + i \sin\frac{4\pi}{3}$$

(iv) The three cube roots of unity when plotted on the argand plane constitute the vertices of an equilateral triangle.



(v) The following factorisation should be remembered :

(a, b, c $\in$ R & ω is the cube root of unity)

$a^3 - b^3 = (a - b)(a - \omega b)(a - \omega^2 b)$; $x^2 + x + 1 = (x - \omega)(x - \omega^2)$;

$a^3 + b^3 = (a + b)(a + \omega b)(a + \omega^2 b)$;

$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c)$

SOLVED EXAMPLE

Example 26 : If 1, ω, ω^2 are cube roots of unity, then prove that

(i) $(1 - \omega + \omega^2)(1 + \omega - \omega^2) = 4$

(ii) $(1 - \omega + \omega^2)^5 + (1 + \omega - \omega^2)^5 = 32$

(iii) $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^8) = 9$

(iv) $(1 - \omega + \omega^2)(1 - \omega^2 + \omega^4)(1 - \omega^4 + \omega^8)$ .... to 2n factors $= 2^{2n}$

Solution : (i) $(1 - \omega + \omega^2)(1 + \omega - \omega^2) = (-2\omega)(-2\omega^2) = 4$

(ii) $(1 - \omega + \omega^2) + (1 + \omega - \omega^2)^5$

$= (-2\omega)^5 + (-2\omega^2)^5 = -32\omega^2 - 32\omega = -32(\omega + \omega^2) = 32$

(iii) $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^8)$

$(1 - \omega)(1 - \omega^2)(1 - \omega)(1 - \omega^2)$

$= (1^2 + 1 + 1)^2 = 9$

(iv) $(1 - \omega + \omega^2)(1 - \omega^2 + \omega^4)(1 - \omega^4 + \omega^8)$ ....

$= (-2\omega)(-2\omega^2)(-2\omega)(-2\omega^2)$ ....

$= 2^{2n}$

Example 27 : Let z_1 and z_2 be two non real complex cube roots of unity and $|z - z_1|^2 + |z - z_2|^2 = \lambda$ be the equation of a circle with z_1, z_2 as ends of a diameter then find the value of λ

Solution : $\because |z - z_1|^2 + |z - z_2|^2 = \lambda$, represent a circle whose diagonal extremities are represented by z_1 & z_2

$\therefore \lambda = |w - w^2|^2 = \left|\sqrt{3}\right|^2 = 3$

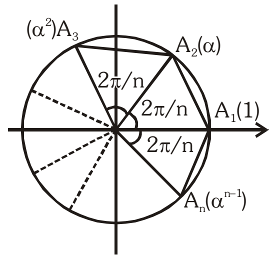
10. nth ROOTS OF UNITY :

If 1, α_1, α_2, α_3.... α_{n-1} are the n, n^{th} root of unity then :

(i) They are in G.P. with common ratio $e^{i(2\pi/n)}$

(ii) Their arguments are in A.P. with common difference $\frac{2\pi}{n}$

(iii) The points represented by n, n^{th} roots of unity are located at the vertices of a regular polygon of n sides inscribed in a unit circle having center at origin, one vertex being on positive real axis.



(iv) $1^p + \alpha_1^p + \alpha_2^p + ... + \alpha_{n-1}^p = 0$ if p is not an integral multiple of n

= n if p is an integral multiple of n

(v) $(1 - \alpha_1)(1 - \alpha_2)......(1 - \alpha_{n-1}) = n$

(vi) $(1 + \alpha_1)(1 + \alpha_2).......(1 + \alpha_{n-1}) = 0$ if n is even and

= 1 if n is odd.

(vii) $1.\ \alpha_1.\ \alpha_2.\ \alpha_3.........\ \alpha_{n-1} = 1$ or -1 according as n is odd or even.

SOLVED EXAMPLE

Example 28 : Solve $(z-1)^4 - 16 = 0$. Find sum of roots and centroid of polygon formed by roots in complex plane.

Solution : $\frac{z-1}{2} = (1)^{\frac{1}{4}}$

$\Rightarrow \quad \frac{z-1}{2} = 1, -1, i, -i \qquad \Rightarrow \qquad z = 3, -1, 1+2i, 1-2i$

Hence, sum of roots = 4 and centroid = $\frac{3-1+1+2i+1-2i}{4} = 1$

Example 29 : If $a = \cos(2\pi/7) + i\sin(2\pi/7)$, then find the quadratic equation whose roots are $\alpha = a + a^2 + a^4$ and $\beta = a^3 + a^5 + a^6$.

Solution : $a = \cos(2\pi/7) + i\sin(2\pi/7)$, which is one of the 7th roots of unity.

Therefore, seventh roots of unity are 1, a, a^2, a^3, a^4, a^5 and a^6.

Also, $a^7 = 1$ ....(1)

Sum of the roots = $1 + a + a^2 + a^3 + a^4 + a^5 + a^6 = 0$

$\therefore S = \alpha + \beta = (a + a^2 + a^4) + (a^3 + a^5 + a^6) = -1$

Product of roots,

$P = \alpha\beta = (a + a^2 + a^4)(a^3 + a^5 + a^6)$

$= a^4 + a^6 + a^7 + a^5 + a^7 + a^8 + a^7 + a^9 + a^{10}$

$= a^4 + a^6 + 1 + a^5 + 1 + a + 1 + a^2 + a^3$ [from Eq. (1)]

$= 3 + (a + a^2 + a^3 + a^4 + a^5 + a^6)$

$= 3 - 1 = 2$

Therefore, required equation is : $x^2 - Sx + P = 0$ or $x^2 + x + 2 = 0$

Example 30 : If $1, \alpha_1, \alpha_2, \alpha_3, ..., \alpha_{n-1}$ be the n^{th} roots of unity, then prove that

(i) $(1 - \alpha_1)(1 - \alpha_2)(1 - \alpha_3)........(1 - \alpha_{n-1}) = n.$

(ii) $(2 - \alpha_1)(2 - \alpha_2)(2 - \alpha_3)........(2 - \alpha_{n-1}) = 2^n - 1$

(iii) $\frac{1}{1-\alpha_1} + \frac{1}{1-\alpha_2} + ... + \frac{1}{1-\alpha_{n-1}} = \frac{n-1}{2}$

Solution : $(z-1)(z-\alpha_1)(z-\alpha_2)\,........\,(z-\alpha_{n-1}) = z^n - 1$ ..(1)

$$\underset{z\to 1}{\text{Limit}}\ (z-\alpha_1)(z-\alpha_2)\,.....\,(z-\alpha_{n-1}) = \underset{z\to 1}{\text{Limit}}\ \frac{z^n-1}{z-1} = n$$

(i) Hence $(1-\alpha_1)(1-\alpha_2)(1-\alpha_3)\,........\,(1-\alpha_{n-1}) = n$.

(ii) Put z = 2 in equation (1) we get $(2-\alpha_1)(2-\alpha_2)(2-\alpha_3)\,........\,(2-\alpha_{n-1}) = 2^n - 1$

(iii) $(z-1)(z-\alpha_1)(z-\alpha_2)\,........\,(z-\alpha_{n-1}) = z^n - 1$

$(z-\alpha_1)(z-\alpha_2)\,........\,(z-\alpha_{n-1}) = 1 + z + z^2 + ... + z^{n-1}$

take log on both sides we get

$\log(z-\alpha_1) + \log(z-\alpha_2) + ... + \log(z-\alpha_n) = \log(1 + z + ... + z^{n-1})$

differentiate and put z = 1, we get $\frac{1}{1-\alpha_1}+\frac{1}{1-\alpha_2}+......+\frac{1}{1-\alpha_n} = \frac{n-1}{2}$

Problems for Self Practice-05

(1) When the polynomial $5x^3 + Mx + N$ is divided by $x^2 + x + 1$, the remainder is 0. Then find M + N.

(2) If ω is an imaginary cube root of unity than prove that

$(a + b\omega + c\omega^2)^3 + (a + b\omega^2 + c\omega)^3 = (2a - b - c)(2b - a - c)(2c - a - b)$

(3) If $1, \alpha_1, \alpha_2, ..., \alpha_{2020}$ are $(2021)^{th}$ roots of unity, then find the value of $\sum_{r=1}^{2020} r(\alpha_r + \alpha_{2021-r})$

(4) Resolve $z^7 - 1$ into linear and quadratic factor with real coefficient.

(5) Find the least positive argument of the 4^{th} root of the complex number $2 - i\sqrt{12}$

Answer : (1) –5

(3) –2021

(4) $(z-1)\left(z^2 - 2\cos\frac{2\pi}{7}z + 1\right).\left(z^2 - 2\cos\frac{4\pi}{7}z + 1\right).\left(z^2 - 2\cos\frac{6\pi}{7}z + 1\right)$

(5) $\frac{5\pi}{12}$

Exercise # 1

PART-I : SUBJECTIVE QUESTIONS

Section (A) : Algebra of Complex Numbers

A-1. Find the value of $x^3 + 7x^2 - x + 16$, where $x = 1 + 2i$.

A-2. Determine least positive value of n for which $\left(\frac{1+i}{1-i}\right)^n = 1$

A-3. Simplify and express the result in the form of a + b*i*

(i) $\left(\frac{1+2i}{2+i}\right)^2$ (ii) $-i(9 + 6i)(2 - i)^{-1}$ (iii) $\frac{(2+i)^2}{2-i} - \frac{(2-i)^2}{2+i}$

A-4. Find the set of values of θ for which $z = \frac{3+2i\sin\theta}{1-2i\sin\theta}$ is

(i) Purely real (ii) Purely imaginary

A-5. Find the real values of x and y for which the following equation is satisfied :

(i) $x^2 - y^2 - i(2x + y) = 2i$ (ii) $\frac{x}{1+2i} + \frac{y}{3+2i} = \frac{5+6i}{8i-1}$

A-6. Find the value of following in the form of a + ib

(i) $\sqrt{-15-8i}$ (ii) $\sqrt{i} + \sqrt{-i}$

A-7. (i) Solve the following equation $z^2 - (3 - 2i)z = (5i - 5)$ expressing your answer in the form of (a + ib).

(ii) If (1 – i) is a root of the equation $z^3 - 2(2 - i)z^2 + (4 - 5i)z - 1 + 3i = 0$, then find the other two roots.

A-8. Prove that , with regard to the quadratic equation $z^2 + (p + ip')z + q + iq' = 0$ where p , p', q , q' are all real.

(i) if the equation has one real root then $q'^2 - pp'q' + qp'^2 = 0$.

(ii) if the equation has two equal roots then $p^2 - p'^2 = 4q$ & $pp' = 2q'$.

Section (B) : Representation of a Complex Number and Demoivre's Theorem

B-1. Find the modulus, argument and principal argument of the complex numbers.

(i) $6(\cos 310° - i\sin 310°)$ (ii) $-2(\cos 30° + i\sin 30°)$ (iii) $\frac{2+i}{4i+(1+i)^2}$

B-2. Find the real values of x & y for which $z_1 = 9y^2 - 4 - 10ix$ and $z_2 = 8y^2 - 20i$ are conjugate complex of each other.

B-3. Express the following complex number in polar form and exponential form :

(i) $-2 + 2i$ (ii) $-1-\sqrt{3}i$

(iii) $\frac{(1+7i)}{(2-i)^2}$ (iv) $(1 - \cos\theta + i\sin\theta)$, $\theta \in (0,\pi)$

B-4. (i) If $iz^3 + z^2 - z + i = 0$, then find $|z|$.

(ii) Find the minimum value of the expression $E = |z|^2 + |z-3|^2 + |z-6i|^2$ (where $z = x + iy$, $x,y \in R$)

B-5. If $(\sqrt{3} + i)^{100} = 2^{99}(a + ib)$, then find

(i) $a^2 + b^2$ (ii) b

B-6. Prove that $\frac{(\cos 2\theta - i\sin 2\theta)^4(\cos 4\theta + i\sin 4\theta)^{-5}}{(i\cos 3\theta + \sin 3\theta)^{-2}(i\cos 3\theta - \sin 3\theta)^{-9}} = -(i\cos 7\theta + \sin 7\theta)$

B-7. If n is a positive integer, prove the following

(i) $(1 + \cos\theta + i\sin\theta)^n + (1 + \cos\theta - i\sin\theta)^n = 2^{n+1}\cos^n\frac{\theta}{2}\cos\frac{n\theta}{2}$.

(ii) $(1 + i)^n + (1 - i)^n = 2^{\frac{n}{2}+1} . \cos\frac{n\pi}{4}$

Section (C) : Argument / Modulus / Conjugate Properties and Triangle Inequality

C-1. If $|z - 2| = 2|z - 1|$, where z is a complex number, prove $|z|^2 = \frac{4}{3}\text{Re}(z)$ using

(i) polar form of z, (ii) $z = x + iy$, (iii) modulus, conjugate properties

C-2. z_1 and z_2 are two complex numbers such that $\frac{z_1 - 2z_2}{2 - z_1\overline{z_2}}$ is unimodular (whose modulus is one), while z_2 is not unimodular. Find $|z_1|$.

C-3. Let z be a complex number such that $z \in c\backslash R$ and $\frac{1+z+z^2}{1-z+z^2} \in R$, then prove that $|z| = 1$.

C-4. If $k > 0$, $|z| = |w| = k$ and $\alpha = \frac{z-\overline{w}}{k^2 + z\overline{w}}$, then find $\text{Re}(\alpha)$.

C-5. If $a = e^{i\alpha}$, $b = e^{i\beta}$, $c = e^{i\gamma}$ and $\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma$, then prove the following

(i) $a + b + c = 0$ (ii) $ab + bc + ca = 0$

(iii) $a^2 + b^2 + c^2 = 0$ (iv) $\Sigma\cos 2\alpha = 0 = \Sigma\sin 2\alpha$

(v) $\Sigma\sin 3\alpha = 3\sin(\alpha + \beta + \gamma)$ (vi) $\Sigma\cos 3\alpha = 3\cos(\alpha + \beta + \gamma)$

C-6. If $|z_1 + z_2|^2 = |z_1|^2+|z_2|^2$ then prove that $\left(\frac{z_1}{z_2}\right)$ is purely imaginary

C-7. (i) If z_1 and z_2 are conjugate to each other, then find arg $(-z_1z_2)$.

(ii) If $z = \frac{\left(\sqrt{3}+i\right)^{17}}{(1-i)^{50}}$, then find principal argument of z.

C-8. (i) If $|z_1| = 1$ and $|z_2| = 2$ then find the maximum value of $|z_1 - 2z_2|$

(ii) Find the minimum value of $|z - 1|$ if $\left||z-3|-|z+1|\right| = 2$

(iii) Find the range of values of $|z-4|$ If $|z-1|+|z+3| \le 8$

Section (D) : Rotation Theorem and Geometry of Complex Number

D-1. A complex number z = 3 + 4i is rotated about another fixed complex number z_1 = 1 + 2i in anticlockwise direction by 45° angle. Find the complex number represented by new position of z in argand plane.

D-2. If O is origin and affixes of P, Q, R are respectively z, iz, z + iz. Locate the points on complex plane. If area of ΔPQR = 200 then find (i) |z| (ii) sides of quadrilateral OPRQ

D-3. (i) If a & b are real numbers between 0 & 1 such that the points $z_1 = a + i$, $z_2 = 1 + bi$ & $z_3 = 0$ form an equilateral triangle, then find the values of 'a' and 'b'.

(ii) Let $z_1 = 1 + i$ and $z_2 = -1 - i$. Find $z_3 \in C$ such that triangle $z_1z_2z_3$ is equilateral.

D-4. Let z_1 , z_2 , z_3 are three pair wise distinct complex numbers and t_1, t_2, t_3 are non-negative real numbers such that $t_1 + t_2 + t_3 = 1$. Prove that the complex number $z = t_1z_1 + t_2z_2 + t_3z_3$ lies inside a triangle with vertices z_1, z_2, z_3 or on its boundary.

D-5. Interpret the following loci in $z \in C$.

(i) $1 < |z - 2i| < 3$ (ii) Re $\left(\frac{z+2i}{iz+2}\right) \le 4$ $(z \ne 2i)$

(iii) Arg (z + i) – Arg (z – i) = $\pi/2$ (iv) Arg (z – a) = $\pi/3$ where a = 3 + 4i.

D-6. If |z – 1 –i| = 1, then prove that locus of a point represented by the complex number 5(z – i) – 6 is a circle. Also find the centre and radius of the circle.

D-7. If $|z + 3 - \sqrt{3}\,i| = \sqrt{3}$, then find the complex number having

(i) Greatest and least value of |z|

(ii) Greatest and least value of princpal arg (z)

D-8. (i) Find the length of arc described by the locus of a complex number z satisfying $\arg\left(\frac{z-5i}{z+5i}\right) = \frac{\pi}{4}$

(ii) Find the area of region bounded by the locus of a complex number Z satisfying $\arg\left(\frac{z+5i}{z-5i}\right) = \pm\frac{\pi}{4}$.

D-9. Show that $z\bar{z} + (4 - 3i)z + (4 + 3i)\bar{z} + 5 = 0$ represents circle. Hence find centre and radius.

D-10. Find the Cartesian equation of the locus of 'z' in the complex plane satisfying, $|z - 4| + |z + 4| = 16$.

Section (E) : Cube Root and n^{th} Root of Unity.

E-1. If $\omega \neq 1$ is a cube root of unity and $a + b = 21$, $a^3 + b^3 = 105$, then find the value of $(a\omega^2 + b\omega)(a\omega + b\omega^2)$.

E-2. Find the sum of series $(1 + \omega)(1 + \omega^2) + (2 + \omega)(2 + \omega^2) + ... + (n + \omega)(n + \omega^2)$ where ω is one of the imaginary cube root of unity.

E-3. If $x = 1 + i\sqrt{3}$; $y = 1 - i\sqrt{3}$ and $z = 2$, then prove that $x^p + y^p = z^p$ for every prime $p > 3$.

E-4. Let ω is non-real root of $x^3 = 1$

(i) If $P = \omega^n$, $(n \in N)$ and $Q = ({}^{2n}C_0 + {}^{2n}C_3 + ...) + ({}^{2n}C_1 + {}^{2n}C_4 + ...)\omega + ({}^{2n}C_2 + {}^{2n}C_5 + ...)\omega^2$ then find $\frac{P}{Q}$.

(ii) If $P = 1 - \frac{\omega}{2} + \frac{\omega^2}{4} - \frac{\omega^3}{8}$ ....upto ∞ terms and $Q = \frac{1-\omega^2}{2}$ then find value of PQ.

E-5. Find the complex number satisfying the equation $z^3 = 8i$

E-6. Find the roots of the equation $z^6 + 64 = 0$ where real part is positive.

E-7. If $1, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the roots of $x^5 - 1 = 0$, then find the value of $\frac{\omega - \alpha_1}{\omega^2 - \alpha_1} \cdot \frac{\omega - \alpha_2}{\omega^2 - \alpha_2} \cdot \frac{\omega - \alpha_3}{\omega^2 - \alpha_3} \cdot \frac{\omega - \alpha_4}{\omega^2 - \alpha_4}$

(where ω is imaginary cube root of unity.)

E-8. If $\alpha = e^{i8\pi/11}$ then find $\text{Re}(\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5)$

PART-II : OBJECTIVE QUESTIONS

Section (A) : Algebra of Complex Numbers

A-1. The value of $\sum_{n=0}^{100} i^{n!}$ equals

(A) – 1 (B) i (C) 2i + 95 (D) 97 + i

A-2. The values of x and y satisfying the equation $\frac{(1+i)x - 2i}{3+i} + \frac{(2-3i)y + i}{3-i} = i$ are

(A) x = –1, y =3 (B) x = 3, y = –1 (C) x = 0, y = 1 (D) x = 1, y = 0

A-3. Let $z = 9 + bi$ where b is non zero real and $i^2 = -1$. If the imaginary part of z^2 and z^3 are equal, then b^2 equals

(A) 261 (B) 225 (C) 125 (D) 361

A-4. Let Z is complex satisfying the equation $z^2 - (3 + i)z + m + 2i = 0$, where $m \in R$. Suppose the equation has a real root. The additive inverse of non real root, is

(A) $1 - i$ (B) $1 + i$ (C) $-1 - i$ (D) -2

A-5. Consider the equation $10z^2 - 3iz - k = 0$, where z is a complex variable and $i^2 = -1$. Which of the following statements is True?

(A) For all real positive numbers k, both roots are pure imaginary.
(B) For negative real numbers k, both roots are pure imaginary.
(C) For all pure imaginary numbers k, both roots are real and irrational.
(D) For all complex numbers k, neither root is real.

A-6. Suppose three real number a, b, c are in GP. Let $z = \frac{a+ib}{c-ib}$. Then

(A) $z = \frac{ib}{c}$ (B) $z = \frac{ic}{b}$ (C) $z = \frac{ia}{c}$ (D) $z = 0$

Section (B) : Representation of a Complex Number and Demoivre's Theorem

B-1. If z is a complex number such that $z^2 = (\overline{z})^2$, then

(A) z is purely real (B) z is purely imaginary
(C) either z is purely real or purely imaginary (D) none of these

B-2. If $z = (3 + 7i)(p + iq)$, where $p, q \in I - \{0\}$, is purely imaginary, then minimum value of $|z|^2$ is

(A) 0 (B) 58 (C) $\frac{3364}{3}$ (D) 3364

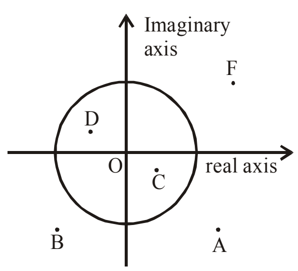
B-3. If $z = \frac{\pi}{4}(1+i)^4\left(\frac{1-\sqrt{\pi}\,i}{\sqrt{\pi}+i}+\frac{\sqrt{\pi}-i}{1+\sqrt{\pi}\,i}\right)$, then $\left(\frac{|z|}{amp(z)}\right)$ equals

(A) 1 (B) π (C) 3π (D) 4

B-4. The complex number z satisfying $z + |z| = 1 + 7i$ then the value of $|z|$ equals

(A) 625 (B) 169 (C) 49 (D) 25

B-5. The diagram shows several numbers in the complex plane. The circle is the unit circle centered at the origin. One of these numbers is the reciprocal of F, which is



(A) A (B) B (C) C (D) D

B-6. For $Z_1 = \sqrt[6]{\frac{1-i}{1+i\sqrt{3}}}$; $Z_2 = \sqrt[6]{\frac{1-i}{\sqrt{3}+i}}$; $Z_3 = \sqrt[6]{\frac{1+i}{\sqrt{3}-i}}$ which of the following holds good?

(A) $\sum |Z_1|^2 = \frac{3}{2}$ (B) $|Z_1|^4 + |Z_2|^4 = |Z_3|^{-8}$

(C) $\sum |Z_1|^3 + |Z_2|^3 = |Z_3|^{-6}$ (D) $|Z_1|^4 + |Z_2|^4 = |Z_3|^8$

B-7. The expression $\left(\frac{1+i\tan\alpha}{1-i\tan\alpha}\right)^n - \frac{1+i\tan n\alpha}{1-i\tan n\alpha}$ when simplified reduces to :

(A) zero (B) 2 sin n α (C) 2 cos n α (D) none

B-8. If $x_n = \cos\left(\frac{\pi}{2^n}\right) + i\sin\left(\frac{\pi}{2^n}\right)$, then $x_1x_2x_3$.........∞ is equal to -

(A) –1 (B) 1 (C) 0 (D) ∞

Section (C) : Argument / Modulus / Conjugate Properties and Triangle Inequality

C-1. Number of complex numbers z satisfying $z^3 = \bar{z}$ is

(A) 1 (B) 2 (C) 4 (D) 5

C-2. If (2 + i)(2 + 2i) (2 + 3i) ...... (2 + ni) = x + iy, then the value of 5.8.13. ....$(4 + n^2)$

(A) $(x^2 + y^2)$ (B) $\sqrt{(x^2 + y^2)}$ (C) $2(x^2 + y^2)$ (A) (x + y)

C-3. If z_1, z_2, z_3 are 3 distinct complex numbers such that $\frac{3}{|z_2 - z_3|} = \frac{4}{|z_3 - z_1|} = \frac{5}{|z_1 - z_2|}$,

then the value of $\frac{9}{z_2 - z_3} + \frac{16}{z_3 - z_1} + \frac{25}{z_1 - z_2}$ equals

(A) 0 (B) 3 (C) 4 (D) 5

C-4. If $(a + ib)^5 = \alpha + i\beta$, then $(b + ia)^5$ is equal to

(A) $\beta + i\alpha$ (B) $\alpha - i\beta$ (C) $\beta - i\alpha$ (D) $-\alpha - i\beta$

C-5. If z = x + iy satisfies amp (z – 1) = amp (z + 3) then the value of (x – 1) : y is equal to

(A) 2 : 1 (B) 1 : 3 (C) – 1 : 3 (D) does not exist

C-6. If z_1 and z_2 are two non-zero complex numbers such that $\left|\frac{z_1}{z_2}\right| = 2$ and $\arg(z_1z_2) = \frac{3\pi}{2}$, then $\frac{\bar{z}_1}{z_2}$ is equal to

(A) 2 (B) –2 (C) –2i (D) 2i

C-7. Number of complex numbers z such that |z| = 1 and $|z/\bar{z} + \bar{z}/z| = 1$ is (arg(z) $\in [0, 2\pi]$)

(A) 4 (B) 6 (C) 8 (D) more than 8

C-8. If $|z| = 1$ and $z \neq \pm1$, then one of the possible values of $\arg(z) - \arg(z + 1) - \arg(z - 1)$ is

(A) $-\frac{\pi}{6}$ (B) $\frac{\pi}{3}$ (C) $-\frac{\pi}{2}$ (D) $\frac{\pi}{4}$

C-9. If z_1 & z_2 are two non-zero complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then $\text{Arg}\, z_1 - \text{Arg}\, z_2$ is equal to

(A) $-\pi$ (B) $-\pi/2$ (C) 0 (D) $\pi/2$

C-10. The minimum value of $|z - 1 + 2i| + |4i - 3 - z|$ is

(A) $\sqrt{5}$ (B) 5 (C) $2\sqrt{13}$ (D) $\sqrt{15}$

Section (D) : Rotation Theorem and Geometry of Complex Number

D-1. Complex numbers z_1, z_2, z_3 are the vertices A, B, C respectively of an isosceles right angled triangle with right angle at C and $(z_1 - z_2)^2 = k(z_1 - z_3)(z_3 - z_2)$, then the value of 'k' is

(A) 1 (B) 2 (C) 3 (D) –2

D-2. If z_1, z_2, z_3 are vertices of an equilateral triangle inscribed in the circle $|z| = 2$ and if $z_1 = 1 + i\sqrt{3}$, then

(A) $z_2 = -2$, $z_3 = 1 + i\sqrt{3}$ (B) $z_2 = 2$, $z_3 = 1 - i\sqrt{3}$

(C) $z_2 = -2$, $z_3 = 1 - i\sqrt{3}$ (D) $z_2 = 1 - i\sqrt{3}$, $z_3 = -1 - i\sqrt{3}$

D-3. If z_1, z_2, z_3 are the vertices of the ΔABC on the complex plane which are also the roots of the equation, $z^3 - 3\alpha z^2 + 3\beta z + x = 0$, then the condition for the ΔABC to be equilateral triangle is

(A) $\alpha^2 = \beta$ (B) $\alpha = \beta^2$ (C) $\alpha^2 = 3\beta$ (D) $\alpha = 3\beta^2$

D-4. The points z_1, z_2, z_3, z_4 in the complex plane are the vertices of a parallelogram taken in order if and only if :

(A) $z_1 + z_4 = z_2 + z_3$ (B) $z_1 + z_3 = z_2 + z_4$ (C) $z_1 + z_2 = z_3 + z_4$ (D) $z_1 z_3 = z_2 z_4$

D-5. The area of the triangle whose vertices are the roots $z^3 + iz^2 + 2i = 0$ is

(A) 2 (B) $\frac{3}{2}\sqrt{7}$ (C) $\frac{3}{4}\sqrt{7}$ (D) $\sqrt{7}$

D-6. The inequality $|z - 4| < |z - 2|$ represents :

(A) Re(z) > 0 (B) Re(z) < 0 (C) Re (z) > 2 (D) Re(z) > 3

D-7. The locus of z, for $\arg z = -\pi/3$ is

(A) same as the locus of z for $\arg z = 2\pi/3$

(B) same as the locus of z for $\arg z = \pi/3$

(C) the part of the straight line $\sqrt{3}x + y = 0$ with $(y < 0, x > 0)$

(D) the part of the straight line $\sqrt{3}x + y = 0$ with $(y > 0, x < 0)$

D-8. The maximum & minimum values of $|z + 1|$ when $|z + 3| \le 3$ are :

(A) (5, 0) (B) (6, 0) (C) (7, 1) (D) (5, 1)

D-9. The equation of the radical axis of the two circles represented by the equations, $|z - 2| = 3$ and $|z - 2 - 3i| = 4$ on the complex plane is :

(A) $3y + 1 = 0$ (B) $3y - 1 = 0$ (C) $2y - 1 = 0$ (D) none

D-10. If z is a complex number satisfying the equation $|z - (1 + i)|^2 = 2$ and $\omega = \frac{2}{z}$, then the locus traced by 'ω' in the complex plane is

(A) $x - y - 1 = 0$ (B) $x + y - 1 = 0$ (C) $x - y + 1 = 0$ (D) $x + y + 1 = 0$

D-11. If z is a complex number satisfying the equation $|z + i| + |z - i| = 8$, on the complex plane then maximum value of $|z|$ is

(A) 2 (B) 4 (C) 6 (D) 8

Section (E) : Cube Root and nth Root of Unity.

E-1. If α & β are imaginary cube roots of unity then $\alpha^n + \beta^n$ is equal to (where $n \in I$) -

(A) $2\cos\frac{2n\pi}{3}$ (B) $\cos\frac{2n\pi}{3}$ (C) $2i\sin\frac{2n\pi}{3}$ (D) $i\sin\frac{2n\pi}{3}$

E-2. If $(w \neq 1)$ is a cube root of unity then $\begin{vmatrix} 1 & 1+i+w^2 & w^2 \\ 1-i & -1 & w^2-1 \\ -i & -i+w-1 & -1 \end{vmatrix} =$

(A) 0 (B) 1 (C) i (D) w

E-3. If $x = a + b + c$, $y = a\alpha + b\beta + c$ and $z = a\beta + b\alpha + c$, where α and β are imaginary cube roots of unity, then xyz =

(A) $2(a^3 + b^3 + c^3)$ (B) $2(a^3 - b^3 - c^3)$ (C) $a^3 + b^3 + c^3 - 3abc$ (D) $a^3 - b^3 - c^3$

E-4. If ω is imaginary cube root of unity then the value of $\sum_{r=0}^{54}(1 + \omega^r + \omega^{2r})$ equals to

(A) 54 (B) 55 (C) 57 (D) 0

E-5. If α is non real and $\alpha = \sqrt[5]{1}$ then the value of $2^{|1 + \alpha + \alpha^2 + \alpha^{-2} - \alpha^{-1}|}$ is equal to

(A) 4 (B) 2 (C) 1 (D) 8

E-6. If 1, $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots\ldots \alpha_8$, are ninth roots of unity (taken in counter-clockwise sequence in the Argand plane), then the value of $|(2 - \alpha_1)(2 - \alpha_3)(2 - \alpha_5)(2 - \alpha_7)|$ equals to

(A) 1 (B) $\sqrt{15}$ (C) $\sqrt{511}$ (D) $\sqrt{512}$

E-7. The number of roots of the equation $z^{15} = 1$ satisfying $|\arg(z)| < \pi/2$ are

(A) 5 (B) 6 (C) 7 (D) 8

PART-III : MATCH THE COLUMN

1. Let $\alpha, \beta, \gamma \in R$ such that $\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha) = -\frac{3}{2}$

	Column–I		**Column – II**
(A)	$\Sigma \sin(\alpha + \beta) = \Sigma \cos(\alpha + \beta) =$	(P)	0
(B)	$\Sigma \cos(2\alpha - \beta - \gamma)$	(Q)	$3\cos\alpha \cos\beta \cos\gamma$
(C)	$\Sigma \cos 3\alpha =$	(R)	$3\cos(\alpha + \beta + \gamma)$
(D)	If $\theta \in R$ then $\frac{\Sigma \cos^3(\theta + \alpha)}{\Pi \cos(\theta + \alpha)} =$	(S)	3

2. Match the equation in z, in **Column-I** with the corresponding values of arg(z) in **Column-II**.

	Column-I (equations in z)		**Column-II** (principal value of arg (z))
(A)	$z^2 - z + 1 = 0$	(P)	$-2\pi/3$
(B)	$z^2 + z + 1 = 0$	(Q)	$-\pi/3$
(C)	$2z^2 + 1 + i\sqrt{3} = 0$	(R)	$\pi/3$
(D)	$2z^2 + 1 - i\sqrt{3} = 0$	(S)	$2\pi/3$

3. Which of the condition/ conditions in column II are satisfied by the quadrilateral formed by z_1, z_2, z_3, z_4 in order given in column I?

	Column - I		**Column-II**
(A)	Parallelogram	(P)	$z_1 - z_4 = z_2 - z_3$
(B)	Rectangle	(Q)	$\|z_1 - z_3\| = \|z_2 - z_4\|$
(C)	Rhombus	(R)	$\frac{z_1 - z_2}{z_3 - z_4}$ is real
(D)	Square	(S)	$\frac{z_1 - z_3}{z_2 - z_4}$ is purely imaginary
		(T)	$\frac{z_1 - z_2}{z_3 - z_2}$ is purely imaginary

Exercise # 2

PART-I : OBJECTIVE QUESTIONS

1. Let $Z_1 = (8 + i)\sin\theta + (7 + 4i)\cos\theta$ and $Z_2 = (1 + 8i)\sin\theta + (4 + 7i)\cos\theta$ are two complex numbers. If $Z_1 \cdot Z_2 = a + ib$ where $a, b \in R$ then the largest value of $(a + b)\ \forall\ \theta \in R$, is

(A) 75 (B) 100 (C) 125 (D) 130

2. If $|z|^2 - 2iz + 2c(1 + i) = 0$, then the value of z is, where c is real.

(A) $z = c + 1\ i(-1 \pm \sqrt{1-2c-c^2})$, where $c \in [-1 - \sqrt{2}, -1 + \sqrt{2}]$

(B) $z = c - 1\ i(-1 \pm \sqrt{1-2c-c^2})$, where $c \in [-1 - \sqrt{2}, -1 + \sqrt{2}]$

(C) $z = 2c + 1\ i(-1 \pm \sqrt{1-2c-c^2})$, where $c \in [-1 - \sqrt{2}, -1 + \sqrt{2}]$

(D) $z = c + 1\ i(-1 \pm \sqrt{1-2c-c^2})$, where $c \in [-1 - \sqrt{2}, 1 + \sqrt{2}]$

3. If $z_1 = -3 + 5i$; $z_2 = -5 - 3i$ and z is a complex number lying on the line segment joining z_1 & z_2, then arg(z) can be :

(A) $-\frac{3\pi}{4}$ (B) $-\frac{\pi}{4}$ (C) $\frac{\pi}{6}$ (D) $\frac{5\pi}{6}$

4. If z is a point on the Argand plane such that $|z - 1| = 1$, then $\frac{z-2}{z}$ is equal to -

(A) tan (arg z) (B) cot (arg z) (C) i tan (arg z) (D) none of these

5. If $\cos\theta + i\sin\theta$ is a root of the equation $x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n = 0$ then the value of $\sum_{r=1}^{n} a_r \cos r\theta$ equals (where all coefficient are real)

(A) 0 (B) 1 (C) –1 (D) none

6. If $|z_1| = 2$, $|z_2| = 3$, $|z_3| = 4$ and $|2z_1 + 3z_2 + 4z_3| = 4$, then the value of $|8z_2z_3 + 27z_3z_1 + 64z_1z_2|$ is

(A) 16 (B) 24 (C) 48 (D) 96

7. Let z,w be complex numbers such that $\bar{z} + i\bar{w} = 0$ and arg zw = π. If Re(z) < 0, then principal arg z =

(A) $\frac{\pi}{4}$ (B) $\frac{2\pi}{3}$ (C) $\frac{3\pi}{4}$ (D) $\frac{5\pi}{6}$

8. If $|z_1 - 1| < 1$, $|z_2 - 2| < 2$, $|z_3 - 3| < 3$, then $|z_1 + z_2 + z_3|$

(A) is less than 6 (B) is more than 3 (C) is less than 12 (D) lies between 6 and 12

9. Let O ≡ (0, 0); A ≡ (3, 0); B ≡ (0, –1) and C = (3, 2), then minimum value of $|z| + |z - 3| + |z + i| + |z - 3 - 2i|$ occur at

(A) intersection point of AB and CO (B) intersection point of AC and BO

(C) intersection point of CB and AO (D) mean of O, A, B, C

10. A particle starts from a point $z_0 = 1 + i$, where $i = \sqrt{-1}$. It moves horizontally away from origin by 2 units and then vertically away from origin by 3 units to reach a point z_1. From z_1 particle moves $\sqrt{5}$ units in the direction of $2\hat{i} + \hat{j}$ and then it moves through an angle of $\text{cosec}^{-1}\sqrt{2}$ in anticlockwise direction of a circle with centre at origin to reach a point z_2. The arg z_2 is given by

(A) $\sec^{-1}2$ (B) $\cot^{-1}0$ (C) $\sin^{-1}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)$ (D) $\cos^{-1}\left(\frac{-1}{2}\right)$

11. Let P denotes a complex number z on the Argand's plane, and Q denotes a complex number $\sqrt{2|z|^2}\ \text{CiS}\left(\frac{\pi}{4}+\theta\right)$ where θ = amp z. If 'O' is the origin, then the Δ OPQ is :

(A) isosceles but not right angled (B) right angled but not isosceles
(C) right isosceles (D) equilateral.

12. If P and Q are respectively by the complex numbers z_1 and z_2 such that $\left|\frac{1}{z_1}+\frac{1}{z_2}\right| = \left|\frac{1}{z_1}-\frac{1}{z_2}\right|$, then the circumcentre of ΔOPQ (where O is the origin) is

(A) $\frac{z_1 - z_2}{2}$ (B) $\frac{z_1 + z_2}{2}$ (C) $\frac{z_1 + z_2}{3}$ (D) $z_1 + z_2$

13. The real values of the parameter 'a' for which at least one complex number z = x + iy satisfies both the equality $|z - ai| = a + 4$ and the inequality $|z - 2| < 1$.

(A) $\left(-\frac{21}{10}, -\frac{5}{6}\right)$ (B) $\left(-\frac{7}{2}, -\frac{5}{6}\right)$ (A) $\left(\frac{5}{6}, \frac{7}{2}\right)$ (A) $\left(-\frac{21}{10}, \frac{7}{2}\right)$

14. The number of solution(s) of the system of the equations $\left||z+4| - |z-3i|\right| = 5$ & |z| = 4 is/are

(A) 0 (B) 1 (C) 2 (D) 4

15. Let z is a complex number satisfying the equation $Z^6 + Z^3 + 1 = 0$. If this equation has a root $re^{i\theta}$ with 90° < θ < 180° then the value of 'θ' is

(A) 100° (B) 110° (C) 160° (D) 170°

16. If ω and ω^2 are the non-real cube roots of unity and a, b, c $\in$ R such that $\frac{1}{a+\omega}+\frac{1}{b+\omega}+\frac{1}{c+\omega} = 2\omega^2$ and $\frac{1}{a+\omega^2}+\frac{1}{b+\omega^2}+\frac{1}{c+\omega^2} = 2\omega$. Then the value of $\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}$ is

(A) –2 (B) 2 (C) 0 (D) 16

17. If Z_r ; r = 1, 2, 3,..., 50 are the roots of the equation $\sum_{r=0}^{50} (Z)^r = 0$, then the value of $\sum_{r=1}^{50} \frac{1}{Z_r - 1}$ is

(A) – 85 (B) – 25 (C) 25 (D) 75

18. If $z^4 + 1 = \sqrt{3}\,i$

(A) z^3 is purely real
(B) z represents the vertices of a square of side $2^{1/4}$
(C) z^9 is purely imaginary
(D) z represents the vertices of a square of side $2^{3/4}$.

PART-II : NUMERICAL QUESTIONS

1. If $a_1, a_2, a_3 \ldots a_n, A_1, A_2, A_3 \ldots A_n$, k are all real numbers and number of imaginary roots of the equation $\frac{A_1^2}{x-a_1} + \frac{A_2^2}{x-a_2} + \ldots\ldots + \frac{A_n^2}{x-a_n} = k$ is α. Then the value of α is equal to

2. If z_1, z_2 are complex numbers such that $\text{Re}(z_1) = |z_1 - 2|$, $\text{Re}(z_2) = |z_2 - 2|$ and $\arg(z_1 - z_2) = \frac{\pi}{3}$, then $\left(\text{Im}(z_1 + z_2)\right)^2$ is equal to

3. If $x = 9^{1/3}\, 9^{1/9}\, 9^{1/27} \ldots\infty$, $y = 4^{1/3}\, 4^{-1/9}\, 4^{1/27} \ldots \infty$, and $z = \sum_{r=1}^{\infty} (1+i)^{-r}$ and principal argument of $P = (x + yz)$ is $-\tan^{-1}\left(\frac{\sqrt{a}}{b}\right)$ then $a^3 + b^3$ is equal to (where a & b are co-prime natural numbers)

4. A function 'f' is defined by $f(z) = (4 + i)z^2 + \alpha z + \gamma$ for all complex number z, where α and γ are complex numbers if f(1) and f(i) are both real then the smallest possible values of $|\alpha| + |\gamma|$ is (take $\sqrt{2} = 1.41$)

5. The value of $\left(\sqrt{6-2\sqrt{5}} + i\sqrt{2\sqrt{5}+10}\right)^5 + \left(\sqrt{6-2\sqrt{5}} - i\sqrt{2\sqrt{5}+10}\right)^5$ is equal to

6. If $x_1^2 + y_1^2 = x_2^2 + y_2^2 = x_3^2 + y_3^2 = 4$ where $x_i, y_i \in R$, i = 1, 2, 3 then the maximum value of $(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2 + (y_1 - y_2)^2 + (y_2 - y_3)^2 + (y_3 - y_1)^2$ is equal to

7. If a complex number z satisfies $|z|^2 + \frac{4}{|z|^2} - 2\left(\frac{z}{\bar{z}} + \frac{\bar{z}}{z}\right) - 16 = 0$, then the maximum value of |z| is (Take $\sqrt{6} = 2.45$)

8. $z_1, z_2 \in c$ and $z_1^2 + z_2^2 \in R$,
$z_1(z_1^2 - 3z_2^2) = 2$, $z_2(3z_1^2 - z_2^2) = 11$
If $z_1^2 + z_2^2 = \lambda$ then the value of λ is

9. If z and ω are two non-zero complex numbers such that $|z\omega| = 1$, and $\arg(z) - \arg(\omega) = \frac{\pi}{2}$, then the value of $\frac{i\bar{z}\omega}{4}$ is

10. How many complex number z such that $|z| < \frac{1}{3}$ and $\sum_{r=1}^{n} a_r z^r = 1$ where $|a_r| < 2$.

11. If $|z_2 + iz_1| = |z_1| + |z_2|$ and $|z_1| = 3$ & $|z_2| = 4$, if affix of A, B, C are z_1, z_2, $\left(\frac{z_2 - iz_1}{1-i}\right)$ respectively. Then area of ΔABC is equal

12. If ω is any complex number such that $z\omega = |z|^2$ and $|z - \bar{z}| + |\omega + \bar{\omega}| = 4$, if ω varies, then the area (in sq. units) bounded by the locus of z is

13. If a variable circle S touches $S_1 : |z - z_1| = 7$ internally and $S_2 : |z - z_2| = 4$ externally while the curves S_1 & S_2 touch internally to each other, $(z_1 \neq z_2)$, then eccentricity of the locus of the centre of the curve S is

14. Let z_1, z_2 are complex numbers and if $|z_1| = 2$ and $(1 - i)z_2 + (1 + i)\bar{z}_2 = K\sqrt{2}$, K > 0 such that the minimum value of $|z_1 - z_2|$ equals 2 then the value of K is

15. Let 1, ω and ω^2 be the cube roots of unity. The least possible degree of a polynomial, with real coefficients having $2\omega^2$, $3 + 4\omega$, $3 + 4\omega^2$ and $5 - \omega - \omega^2$ as root is equal to

16. If $L = \lim_{n\to\infty} \left[\frac{n}{(1-n\omega)(1-n\omega^2)} + \frac{n}{(2-n\omega)(2-n\omega^2)} + \ldots\ldots + \frac{n}{(n-n\omega)(n-n\omega^2)}\right]$,

then the value of L^2 is (where w is non real cube root of unity and assume $\pi^2 = 10$)

17. Let 1, Z_1, Z_2, ... , Z_{14} are 15 roots of unity, then principal argument (in degrees) of the complex number

$\left(\frac{1 + Z_1 + Z_2 + Z_3 + \ldots\ldots\ldots + Z_7}{1 + Z_8 + Z_9 + Z_{10} + \ldots\ldots. + Z_{14}}\right)$ is equal to

18. Let $z = \sum_{p=1}^{32} (3p + 2)\left(\sum_{q=1}^{10}\left(\sin\frac{2q\pi}{11} - i\cos\frac{2q\pi}{11}\right)\right)^p$, then $|z|$ is equal to (Take $\sqrt{2}$ = 1.41)

PART - III : ONE OR MORE THAN ONE CORRECT QUESTION

1. If all the three roots of $az^3 + bz^2 + cz + d = 0$ have negative real parts (a, b, c $\in$ R), then

(A) ab > 0 (B) bc > 0 (C) ad > 0 (D) bc – ad > 0

2. If the biquadratic $x^4 + ax^3 + bx^2 + cx + d = 0$ (a, b, c, d $\in$ R) has 4 non real roots, two with sum 3 + 4i and the other two with product 13 + i.

(A) b = 51 (B) a = –6 (C) c = –70 (D) d = 170

3. Let $i = \sqrt{-1}$. Define a sequence of complex number by $z_1 = 0$, $z_{n+1} = z_n^{\,2} + i$ for $n \geq 1$. Then which of the following are true.

(A) $|z_{2050}| = \sqrt{3}$ (B) $|z_{2017}| = \sqrt{2}$ (C) $|z_{2016}| = 1$ (D) $|z_{2111}| = \sqrt{2}$

4. The value of $i^n + i^{-n}$, for $i = \sqrt{-1}$ and $n \in I$ is :

(A) $\frac{2^n}{(1-i)^{2n}} + \frac{(1+i)^{2n}}{2^n}$ (B) $\frac{(1+i)^{2n}}{2^n} + \frac{(1-i)^{2n}}{2^n}$

(C) $\frac{(1+i)^{2n}}{2^n} + \frac{2^n}{(1-i)^{2n}}$ (D) $\frac{2^n}{(1+i)^{2n}} + \frac{2^n}{(1-i)^{2n}}$

5. If $2\cos\theta = x + \frac{1}{x}$ and $2\cos\varphi = y + \frac{1}{y}$, then

(A) $x^n + \frac{1}{x^n} = 2\cos(n\theta)$, $n \in z$ (B) $\frac{x}{y} + \frac{y}{x} = 2\cos(\theta - \varphi)$

(C) $xy + \frac{1}{xy} = 2\cos(\theta + \varphi)$ (D) $x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\theta + n\varphi)$, $m, n \in z$

6. Which of the following are true.

(A) $\cos x + {}^nC_1 \cos 2x + {}^nC_2 \cos 3x + \ldots + {}^nC_n \cos(n+1)x = 2^n . \cos^n \frac{x}{2} . \cos\left(\frac{n+2}{2}\right)x$

(B) $\sin x + {}^nC_1 \sin 2x + {}^nC_2 \sin 3x + \ldots + {}^nC_n \sin(n+1)x = 2^n . \cos^n \frac{x}{2} . \sin\left(\frac{n+2}{2}\right)x$

(C) $1 + {}^nC_1 \cos x + {}^nC_2 \cos 2x + \ldots + {}^nC_n \cos nx = 2^n . \cos^n \frac{x}{2} . \cos\left(\frac{nx}{2}\right)x$

(D) ${}^nC_1 \sin x + {}^nC_2 \sin 2x + \ldots + {}^nC_n \sin nx = 2^n . \cos^n \frac{x}{2} . \sin\left(\frac{nx}{2}\right)$

7. Let $Z_1 = x_1 + iy_1$, $Z_2 = x_2 + iy_2$ be complex numbers in fourth quadrant of argand plane and $|Z_1| = |Z_2| = 1$, $\text{Re}(Z_1Z_2) = 0$. The complex numbers $Z_3 = x_1 + ix_2$, $Z_4 = y_1 + iy_2 - Z_5 = x_1 + iy_2$, $Z_6 = x_2 + iy_1$, will always satisfy

(A) $|Z_4| = 1$ (B) $\arg(Z_1Z_4) = \frac{\pi}{2}$

(C) $\frac{Z_5}{\cos(\arg Z_1)} + \frac{Z_6}{\sin(\arg Z_1)}$ is purely real (D) $Z_5^2 + (\overline{Z}_6)^2$ is purely imaginary

8. Let z_1 and z_2 are two complex numbers such that $(1 - i)z_1 = 2z_2$ and $\arg(z_1z_2) = \frac{\pi}{2}$, then $\arg(z_2)$ is equal to

(A) $3\pi/8$ (B) $\pi/8$ (C) $5\pi/8$ (D) $-7\pi/8$

9. If $Z = \frac{(1+i)(1+2i)(1+3i).....(1+ni)}{(1-i)(2-i)(3-i).....(n-i)}$, $n \in N$ then principal argument of Z can be

(A) 0 (B) $\frac{\pi}{2}$ (C) $-\frac{\pi}{2}$ (D) π

10. Let $\left|z - \frac{4}{z}\right| = 2$, then

(A) Greatest value of |z| is $\sqrt{5}$ + 1 (B) Greatest value of |z| is $\frac{\sqrt{5}+1}{2}$

(C) Least value of |z| is $\sqrt{5}$ – 1 (D) Least value of |z| is $\frac{\sqrt{5}-1}{2}$

11. Let a, b, c be distinct complex numbers with |a| = |b| =|c| = 1 and z_1, z_2 be the roots of the equation $az^2 + bz + c = 0$ with $|z_1| = 1$.Let P and Q represent the complex numbers z_1 and z_2 in the Argand plane with $\angle POQ = \theta$, $0° < \theta < 180°$ (where O being the origin). Then

(A) $b^2 = ac$; $\theta = \frac{2\pi}{3}$ (B) $\theta = \frac{2\pi}{3}$; PQ = $\sqrt{3}$ (C) PQ = $2\sqrt{3}$; $b^2 = ac$ (D) $\theta = \frac{\pi}{3}$; $b^2 = ac$

12. Let z_1, z_2, z_3, are the vertices of ΔABC, respectively, such that $\frac{z_3 - z_2}{z_1 - z_2}$ is purely imaginary number. A square on side AC is drawn outwardly. P(z_4) is the centre of square, then

(A) $|z_1 - z_2| = |z_2 - z_4|$ (B) $\arg\left(\frac{z_1 - z_2}{z_4 - z_2}\right) + \arg\left(\frac{z_3 - z_2}{z_4 - z_2}\right) = \frac{\pi}{2}$

(C) $\arg\left(\frac{z_1 - z_2}{z_4 - z_2}\right) + \arg\left(\frac{z_3 - z_2}{z_4 - z_2}\right) = 0$ (D) z_1, z_2, z_3 and z_4 lie on a cricle.

13. Let A and B be two distinct points denoting the complex numbers α and β respectively. A complex number z lies between A and B where $z \neq \alpha$, $z \neq \beta$. Which of the following relation(s) hold good?

(A) $|\alpha - z| + |z - \beta| = |\alpha - \beta|$

(B) $\exists$ a positive real number 't' such that $z = (1 - t)\alpha + t\beta$

(C) $\begin{vmatrix} z-\alpha & \bar{z}-\bar{\alpha} \\ \beta-\alpha & \bar{\beta}-\bar{\alpha} \end{vmatrix} = 0$ (D) $\begin{vmatrix} z & \bar{z} & 1 \\ \alpha & \bar{\alpha} & 1 \\ \beta & \bar{\beta} & 1 \end{vmatrix} = 0$

14. If z is a complex number which simultaneously satisfies the equations $3|z - 12| = 5|z - 8i|$ and $|z - 4| = |z - 8|$ then the Im(z) can be

(A) 15 (B) 16 (C) 17 (D) 8

15. The equation $||z + i| - |z - i|| = k$ represents

(A) a hyperbola if $0 < k < 2$ (B) no locus if $k > 2$

(C) a straight line if $k = 0$ (D) a pair of ray if $k = 2$

16. If $\left|\frac{z-\alpha}{z-\beta}\right| = k, k > 0$ where, $z = x + iy$ and $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$ are fixed complex numbers. Then which of the following are true

(A) if $k \neq 1$ then locus is a circle whose centre is $\left(\frac{k^2\beta-\alpha}{k^2-1}\right)$

(B) if $k \neq 1$ then locus is a circle whose radius is $\left|\frac{k(\alpha-\beta)}{1-k^2}\right|$

(C) if $k = 1$ then locus is perpendicular bisector of line joining $\alpha = \alpha_1 + i\alpha_2$ and $\beta = \beta_1 + i\beta_2$

(D) if $k \neq 1$ then locus is a circle whose centre is $\left(\frac{k^2\alpha-\beta}{k^2-1}\right)$

17. Let z_1, z_2, z_3 are the coordinates of the vertices of the triangle $A_1A_2A_3$. Which of the following statements are equivalent.

(A) $A_1A_2A_3$ is an equilateral triangle.

(B) $(z_1 + \omega z_2 + \omega^2 z_3)(z_1 + \omega^2 z_2 + \omega z_3) = 0$, where ω is the cube root of unity.

(C) $\frac{z_2-z_1}{z_3-z_2} = \frac{z_3-z_2}{z_1-z_3}$

(D) $\begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ z_2 & z_3 & z_1 \end{vmatrix} = 0$

18. If α, β, γ are distinct roots of $x^3 - 3x^2 + 3x + 7 = 0$ (and ω is imaginary cube root of unity), then the value of $\frac{\alpha-1}{\beta-1} + \frac{\beta-1}{\gamma-1} + \frac{\gamma-1}{\alpha-1}$ can be equal to

(A) ω^2 (B) $2\omega^2$ (C) $3\omega^2$ (D) 3ω

19. Let P(x) and Q(x) be two polynomials. Suppose that $f(x) = P(x^3) + x\,Q(x^3)$ is divisible by $x^2 + x + 1$, then

(A) P(x) is divisible by (x–1), but Q(x) is not divisible by (x–1)

(B) Q(x) is divisible by (x–1), but P(x) is not divisible by (x–1)

(C) Both P(x) and Q(x) are divisible by (x–1)

(D) f(x) is divisible by (x –1)

20. If $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_{n-1}$ are the imaginary n^{th} roots of unity then the product $\prod_{r=1}^{n-1}(i-\alpha_r)$ (where $i = \sqrt{-1}$) can take the value equal to

(A) 0 (B) 1 (C) i (D) (1 + i)

21. If α is imaginary n^{th} $(n \geq 3)$ root of unity. Which of the following are true.

(A) $\sum_{r=1}^{n-1}(n-r)\,\alpha^r = \frac{n\alpha}{1-\alpha}$

(B) $\sum_{r=1}^{n-1}(n-r)\,\sin\frac{2r\pi}{n} = \frac{n}{2}\cot\frac{\pi}{n}$.

(C) $\sum_{r=1}^{n-1}(n-r)\,\cos\frac{2r\pi}{n} = -\frac{n}{2}$

(D) $\sum_{r=1}^{n-1}(n-r)\,\alpha^r = \frac{n}{1-\alpha}$

22. Which of the following is true

(A) roots of the equation $z^{10} - z^5 - 992 = 0$ with real part positive = 5

(B) roots of the equation $z^{10} - z^5 - 992 = 0$ with real part negative = 5

(C) roots of the equation $z^{10} - z^5 - 992 = 0$ with imaginary part non-negative = 6

(D) roots of the equation $z^{10} - z^5 - 992 = 0$ with imaginary part negative = 4

23. Which of the following is true?

(A) The number of common roots of the equations $z^{17} = 1$ and $z^{13} = 1$ is 1

(B) The number of common roots of the equations $z^{40} = 1$ and $z^{36} = 1$ is 4

(C) The number of common roots of the equations $z^{40} = 1$, $z^{36} = 1$ and $z^{20} = 1$ is 1

(D) The number of common roots of the equations $z^{24} = 1$, $z^{40} = 1$ and $z^{64} = 1$ is 8

PART - IV : COMPREHENSION

Comprehension # 1 (Q.No.1 to Q.No.2)

Let $(1 + x)^n = C_0 + C_1x + C_2x^2 + ...+ C_nx^n$. For sum of series $C_0 + C_1 + C_2 + ...$, put x = 1. For sum of series $C_0 + C_2 + C_4 + C_6 + ...$, or $C_1 + C_3 + C_5 + ...$ add or substract equations obtained by putting x = 1 and x = – 1.

For sum of series $C_0 + C_3 + C_6 + ...$ or $C_1 + C_4 + C_7 + ...$ or $C_2 + C_5 + C_8 + ...$ we subsitute x = 1, $x = \omega$, $x = \omega^2$ and add or manupulate results.

Similarly, if suffixes differe by 'p' then we substitute p^{th} roots of unity and add.

1. $C_0 + C_3 + C_6 + C_9 +........ =$

(A) $\frac{1}{3}\left[2^n - 2\cos\frac{n\pi}{3}\right]$ (B) $\frac{1}{3}\left[2^n + 2\cos\frac{n\pi}{3}\right]$ (C) $\frac{1}{3}\left[2^n - 2\sin\frac{n\pi}{3}\right]$ (D) $\frac{1}{3}\left[2^n + 2\sin\frac{n\pi}{3}\right]$

2. $C_1 + C_5 + C_9 +.... =$

(A) $\frac{1}{4}\left[2^n - 2^{n/2}2\cos\frac{n\pi}{4}\right]$

(B) $\frac{1}{4}\left[2^n + 2^{n/2}2\cos\frac{n\pi}{4}\right]$

(C) $\frac{1}{4}\left[2^n - 2^{n/2}2\sin\frac{n\pi}{4}\right]$

(D) $\frac{1}{4}\left[2^n + 2^{n/2}2\sin\frac{n\pi}{4}\right]$

Comprehension # 2 (Q.No.3 to Q.No.5)

Let A, B, C be three sets of complex numbers as defined below.

A = {z : | z +1 | $\le$ 2 + Re(z)}, B = {z : | z –1 | $\ge$ 1} and C = $\left\{z : \left|\frac{z-1}{z+1}\right| \ge 1\right\}$

3. The number of point(s) having integral coordinates in the region $A \cap B \cap C$ is

(A) 4 (B) 5 (C) 6 (D) 10

4. The area of region bounded by $A \cap B \cap C$ is

(A) $2\sqrt{3}$ (B) $\sqrt{3}$ (C) $4\sqrt{3}$ (D) 2

5. The real part of the complex number in the region $A \cap B \cap C$ and having maximum amplitude is

(A) –1 (B) $\frac{-3}{2}$ (C) $\frac{1}{2}$ (D) – 2

Comprehension # 3 (Q.No. 6 to Q.No.7)

Logarithm of a complex number is given by

$\log_e(x + iy) = \log_e(|z|e^{i\theta})$

$= \log_e|z| + \log_e e^{i\theta}$

$= \log_e |z| + i\theta$

$= \log_e \sqrt{(x^2 + y^2)} + i \arg(z)$

$\therefore$ $\log_e(z) = \log_e|z| + i \arg(z)$

In general $\log_e (x + iy) = \frac{1}{2} \log_e (x^2 + y^2) + i\left(2n\pi + \tan^{-1}\frac{y}{x}\right)$ where $n \in I$.

6. Write $\log_e (1 + \sqrt{3}\, i)$ in (a + ib) form

(A) $\log_e 2 + i(2n\pi + \frac{\pi}{3})$ (B) $\log_e 3 + i(n\pi + \frac{\pi}{3})$

(C) $\log_e 2 + i(2n\pi + \frac{\pi}{6})$ (D) $\log_e 2 + i(2n\pi - \frac{\pi}{3})$

7. Find the real part of $(1 - i)^{-i}$.

(A) $e^{\pi/4 + 2n\pi} \cos\left(\frac{1}{2}\log_e 2\right)$ (B) $e^{-\pi/4 + 2n\pi} \cos\left(\frac{1}{2}\log_e 2\right)$

(C) $e^{-\pi/4 + 2n\pi} \cos(\log_e 2)$ (D) $e^{-\pi/2 + 2n\pi} \cos\left(\frac{1}{2}\log_e 2\right)$

Exercise # 3

PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

*** Marked Questions may have more than one correct option.**

1*. Let z_1 and z_2 be two distinct complex numbers and let $z = (1 - t)\, z_1 + tz_2$ for some real number t with $0 < t < 1$. If Arg(w) denotes the principal argument of a nonzero complex number w, then

[IIT-JEE-2010, Paper-1, (3, 0)/84]

(A) $|z - z_1| + |z - z_2| = |z_1 - z_2|$ (B) $\text{Arg}\,(z - z_1) = \text{Arg}\,(z - z_2)$

(C) $\begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$ (D) $\text{Arg}\,(z - z_1) = \text{Arg}\,(z_2 - z_1)$

2. Let ω be the complex number $\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$. Then the number of distinct complex numbers z satisfying

$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$ is equal to **[IIT-JEE-2010, Paper-1, (3, 0)/84]**

3. Match the statements in **Column-I** with those in **Column-II.** **[IIT-JEE-2010, Paper-2, (8, 0)/79]**

[Note : Here z takes values in the complex plane and Im z and Re z denote, respectively, the imaginary part and the real part of z.]

Column-I	**Column-II**
(A) The set of points z satisfying $|z - i|z|| = |z + i|z||$ is contained in or equal to	(p) an ellipse with eccentricity $\frac{4}{5}$
(B) The set of points z satisfying $|z + 4| + |z - 4| = 10$ is contained in or equal to	(q) the set of points z satisfying Im z = 0
(C) If $|w| = 2$, then the set of points $z = w - \frac{1}{w}$ is contained in or equal to	(r) the set of point z satisfying $|\text{Im } z| \le 1$
(D) If $|w| = 1$, then the set of points $z = w + \frac{1}{w}$ is contained in or equal to	(s) the set of points z satisfying $|\text{Re } z| \le 2$
	(t) the set of points z satisfying $|z| \le 3$

4. If z is any complex number satisfying $|z - 3 - 2i| \le 2$, then the minimum value of $|2z - 6 + 5i|$ is

[IIT-JEE 2011, Paper-1, (4, 0), 80]

5. Let $\omega = e^{\frac{2\pi i}{3}}$, and a, b, c, x, y, z be non-zero complex numbers such that

$$a + b + c = x$$
$$a + b\omega + c\omega^2 = y$$
$$a + b\omega^2 + c\omega = z.$$

Then the value of $\frac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2}$ is **[IIT-JEE 2011, Paper-2, (4, 0), 80]**

6. Let z be a complex number such that the imaginary part of z is non zero and $a = z^2 + z + 1$ is real. Then a cannot take the value **[IIT-JEE 2012, Paper-1, (3, –1), 70]**

(A) –1 (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{3}{4}$

7. Let complex numbers α and $\frac{1}{\bar{\alpha}}$ lies on circles $(x - x_0)^2 + (y - y_0)^2 = r^2$ and $(x - x_0)^2 + (y - y_0)^2 = 4r^2$, respectively. If $z_0 = x_0 + iy_0$ satisfies the equation $2|z_0|^2 = r^2 + 2$, then $|\alpha|$ =

[JEE (Advanced) 2013, Paper-1, (2, 0)/60]

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{2}$ (C) $\frac{1}{\sqrt{7}}$ (D) $\frac{1}{3}$

8.* Let $w = \frac{\sqrt{3} + i}{2}$ and $P = \{w^n : n = 1,2,3,....\}$. Further $H_1 = \left\{z \in C : \text{Re } z > \frac{1}{2}\right\}$ and $H_2 = \left\{z \in C : \text{Re } z < -\frac{1}{2}\right\}$, where C is the set of all complex numbers. If $z_1 \in P \cap H_1$, $z_2 \in P \cap H_2$ and O represents the origin, then $\angle z_1 O z_2$ = **[JEE (Advanced) 2013, Paper-2, (3, –1)/60]**

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{6}$ (C) $\frac{2\pi}{3}$ (D) $\frac{5\pi}{6}$

9.* Let ω be a complex cube root of unity with $\omega \neq 1$ and $P = [p_{ij}]$ be a n × n matrix with $p_{ij} = \omega^{i+j}$. Then $P^2 \neq 0$, when n = **[JEE (Advanced) 2013, Paper-2, (3, –1)/60]**

(A) 57 (B) 55 (C) 58 (D) 56

Paragraph for Question Nos. 10 to 11

Let $S = S_1 \cap S_2 \cap S_3$, where $S_1 = \{z \in C : |z| < 4\}$, $S_2 = \left\{z \in C : \text{Im}\left[\frac{z - 1 + \sqrt{3}i}{1 - \sqrt{3}i}\right] > 0\right\}$

and $S_3 : \{z \in C : \text{Re } z > 0\}$.

10. Area of S = **[JEE (Advanced) 2013, Paper-2, (3, –1)/60]**

(A) $\frac{10\pi}{3}$ (B) $\frac{20\pi}{3}$ (C) $\frac{16\pi}{3}$ (D) $\frac{32\pi}{3}$

11. $\min_{z \in S} |1 - 3i - z|$ = **[JEE (Advanced) 2013, Paper-2, (3, –1)/60]**

(A) $\frac{2 - \sqrt{3}}{2}$ (B) $\frac{2 + \sqrt{3}}{2}$ (C) $\frac{3 - \sqrt{3}}{2}$ (D) $\frac{3 + \sqrt{3}}{2}$

12. Let $z_k = \cos\left(\frac{2k\pi}{10}\right) + i\sin\left(\frac{2k\pi}{10}\right)$; $k = 1, 2, ...9$. **[JEE (Advanced) 2014, Paper-2, (3, –1)/60]**

List I		**List II**	
P.	For each z_k there exists a z_j such that $z_k . z_j = 1$	**1.**	True
Q.	There exists a $k \in \{1,2,....,9\}$ such that $z_1 . z = z_k$ has no solution z in the set of complex numbers.	**2.**	False
R.	$\frac{\vert 1-z_1 \vert \vert 1-z_2 \vert ..... \vert 1-z_9 \vert}{10}$ equals	**3.**	1
S.	$1 - \sum_{k=1}^{9} \cos\left(\frac{2k\pi}{10}\right)$ equals	**4.**	2

	P	Q	R	S
(A)	1	2	4	3
(B)	2	1	3	4
(C)	1	2	3	4
(D)	2	1	4	3

13. For any integer k, let $\alpha_k = \cos\left(\frac{k\pi}{7}\right) + i\sin\left(\frac{k\pi}{7}\right)$, where $i = \sqrt{-1}$. The value of the expression

$$\frac{\sum_{k=1}^{12} \left|\alpha_{k+1} - \alpha_k\right|}{\sum_{k=1}^{3} \left|\alpha_{4k-1} - \alpha_{4k-2}\right|}$$ is **[JEE (Advanced) 2015, P-2 (4, 0) / 80]**

14. Let $z = \frac{-1+\sqrt{3}i}{2}$, where $i = \sqrt{-1}$, and $r, s \in \{1, 2, 3\}$. Let $P = \begin{bmatrix} (-z)^r & z^{2s} \\ z^{2s} & z^r \end{bmatrix}$ and I be the identity matrix of order 2. Then the total number of ordered pairs (r, s) for which $P^2 = -I$ is
[JEE (Advanced) 2016, P-2 (3, 0) / 62]]

15*. Let $a,b \in \mathbb{R}$ and $a^2 + b^2 \neq 0$. Suppose $S = \left\{z \in \mathbb{C} : z = \frac{1}{a+ibt}, t \in \mathbb{R}, t \neq 0\right\}$, where $i = \sqrt{-1}$.

If $z = x + iy$ and $z \in S$, then (x,y) lies on **[JEE (Advanced) 2016, P-2 (4, –2) / 62]**

(A) the circle with radius $\frac{1}{2a}$ and centre $\left(\frac{1}{2a}, 0\right)$ for $a > 0, b \neq 0$

(B) the circle with radius $-\frac{1}{2a}$ and centre $\left(-\frac{1}{2a}, 0\right)$ for $a < 0, b \neq 0$

(C) the x-axis for $a \neq 0, b = 0$

(D) the y-axis for $a = 0, b \neq 0$

16*. Let a, b, x and y be real numbers such that a – b = 1 and y ≠ 0. If the complex number z = x + iy satisfies $\text{Im}\left(\frac{az+b}{z+1}\right) = y$, then which of the following is(are) possible value(s) of x ?

[JEE (Advanced) 2017, P-1 (4, –2) / 61]

(A) $-1-\sqrt{1-y^2}$ (B) $1+\sqrt{1+y^2}$ (C) $1-\sqrt{1+y^2}$ (D) $-1+\sqrt{1-y^2}$

17*. For a non-zero complex number z, let arg(z) denotes the principal argument with $-\pi < \arg(z) \le \pi$. Then, which of the following statement(s) is (are) **FALSE** ?

[JEE (Advanced) 2018, P-1 (4, –2) / 60]

(A) arg(–1 – *i*) = $\frac{\pi}{4}$, where *i* = $\sqrt{-1}$

(B) The function *f* : ℝ → (–π, π], defined by *f*(t) = arg(–1 + *it*) for all *t* ∈ ℝ, is continuous at all points of ℝ, where *i* = $\sqrt{-1}$

(C) For any two non-zero complex numbers z_1 and z_2, $\arg\left(\frac{z_1}{z_2}\right) - \arg(z_1) + \arg(z_2)$ is an integer multiple of 2π

(D) For any three given distinct complex numbers z_1, z_2 and z_3, the locus of the point z satisfying the condition $\arg\left(\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}\right) = \pi$, lies on a straight line

18*. Let s, t, r be the non-zero complex numbers and L be the set of solutions z = x + iy $\left(x, y \in \mathbb{R}, i = \sqrt{-1}\right)$ of the equation $sz + t\bar{z} + r = 0$, where $\bar{z} = x - iy$. Then, which of the following statement(s) is (are) TRUE ?

[JEE (Advanced) 2018, P-2 (4, –2) / 60]

(A) If L has exactly one element, then |s| ≠ |t|

(B) If |s| = |t|, then L has infinitely many elements

(C) The number of elements in $L \cap \{z : |z-1+i| = 5\}$ is at most 2

(D) If L has more than one element, then L has infinitely many elements

19. Let S be the set of all complex numbers z satisfying $|z-2+i| \ge \sqrt{5}$. If the complex number z_0 is such that $\frac{1}{|z_0-1|}$ is the maximum of the set $\left\{\frac{1}{|z-1|} : z \in S\right\}$, then the principal argument of $\frac{4-z_0-\bar{z}_0}{z_0-\bar{z}_0+2i}$ is

[JEE (Advanced) 2019, P-1 (3, –1) / 62]

(1) $\frac{\pi}{4}$ (2) $-\frac{\pi}{2}$ (3) $\frac{3\pi}{4}$ (4) $\frac{\pi}{2}$

20. Let ω ≠ 1 be a cube root of unity. Then the minimum of the set $\{|a + b\omega + c\omega^2|^2$: a, b, c distinct non-zero integers$\}$ equals ____

[JEE (Advanced) 2019, P-1 (3, 0) / 62]

PART - II : JEE(MAIN) / AIEEE PROBLEMS (PREVIOUS YEARS)

1. If α and β are the roots of the equation $x^2 - x + 1 = 0$, then $\alpha^{2009} + \beta^{2009} =$ **[AIEEE 2010, (4, –1), 144]**

(1) -1 (2) 1 (3) 2 (4) -2

2. The number of complex numbers z such that $|z - 1| = |z + 1| = |z - i|$ equals **[AIEEE 2010, (4, –1), 120]**

(1) 1 (2) 2 (3) ∞ (4) 0

3. If $\omega(\neq 1)$ is a cube root of unity, and $(1 + \omega)^7 = A + B\omega$. Then (A, B) equals **[AIEEE 2011, I, (4, –1), 120]**

(1) (0, 1) (2) (1, 1) (3) (1, 0) (4) (–1, 1)

4. Let α, β be real and z be a complex number. If $z^2 + \alpha z + \beta = 0$ has two distinct roots on the line Re z = 1, then it is necessary that : **[AIEEE- 2011, I, (4, –1), 120]**

(1) $\beta \in (0, 1)$ (2) $\beta \in (-1, 0)$ (3) $|\beta| = 1$ (4) $\beta \in (1, \infty)$

5. If z is a complex number of unit modulus and argument θ, then arg $\left(\frac{1+z}{1+\bar{z}}\right)$ equals :

[AIEEE - 2013, (4, –¼), 120]

(1) $-\theta$ (2) $\frac{\pi}{2} - \theta$ (3) θ (4) $\pi - \theta$

6. If z a complex number such that $|z| \geq 2$, then the minimum value of $\left|z + \frac{1}{2}\right|$: **[JEE(Main) 2014, (4, – ¼), 120]**

(1) is strictly greater than 5/2 (2) is strictly greater than 3/2 but less than 5/2

(3) is equal to 5/2 (4) lie in the interval (1, 2)

7. A complex number z is said to be unimodular if $|z| = 1$. Suppose z_1 and z_2 are complex numbers such that $\frac{z_1 - 2z_2}{2 - z_1\bar{z}_2}$ is unimodular and z_2 is not unimodular. Then the point z_1 lies on a :

[JEE(Main) 2015, (4, – ¼), 120]

(1) straight line parallel to x-axis (2) straight line parallel to y-axis

(3) circle of radius 2 (4) circle of radius $\sqrt{2}$

8. A value of θ for which $\frac{2+3i\sin\theta}{1-2i\sin\theta}$ is purely imaginary, is : **[JEE(Main) 2016, (4, – 1), 120]**

(1) $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (2) $\frac{\pi}{3}$ (3) $\frac{\pi}{6}$ (4) $\sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$

9. Let ω be a complex number such that $2\omega + 1 = z$ where $z = \sqrt{-3}$. If $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2 - 1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix} = 3k$, then k is equal to :- **[JEE(Main) 2017, (4, – 1), 120]**

(1) 1 (2) $-z$ (3) z (4) -1

10. If $\alpha, \beta \in C$ are the distinct roots of the equation $x^2 - x + 1 = 0$, then $\alpha^{101} + \beta^{107}$ is equal to-

[JEE(Main) 2018, (4, – 1), 120]

(1) 0 (2) 1 (3) 2 (4) –1

11. Let z_1 and z_2 be any two non-zero complex numbers such that $3|z_1| = 4|z_2|$. If $z = \frac{3z_1}{2z_2} + \frac{2z_2}{3z_1}$ then: **[JEE(Main) 2019, Online (10-01-19) P-2 (4, – 1), 120]**

(1) $|z| = \frac{1}{2}\sqrt{\frac{17}{2}}$ (2) Re(z) = 0 (3) $|z| = \sqrt{\frac{5}{2}}$ (4) Im(z) = 0

12. Let Z_1 and Z_2 be two complex numbers satisfying $|Z_1| = 9$ and $|Z_2 - 3 - 4i| = 4$. Then the minimum value of $|Z_1 - Z_2|$ is : **[JEE(Main) 2019, Online (12-01-19) P-2 (4, – 1), 120]**

(1) 0 (2) 1 (3) $\sqrt{2}$ (4) 2

13. If $z = \frac{\sqrt{3}}{2} + \frac{i}{2}\left(i = \sqrt{-1}\right)$, then $\left(1 + iz + z^5 + iz^8\right)^9$ is equal to

[JEE(Main) 2019, Online (08-04-19) P-2 (4, – 1), 120]

(1) –1 (2) 1 (3) 0 (4) $\left(-1+2i\right)^9$

14. Let $z \in C$ be such that $|z| < 1$. If $\omega = \frac{5+3z}{5(1-z)}$, then :- **[JEE(Main) 2019, Online (09-04-19) P-2 (4, – 1), 120]**

(1) $5\text{Im}(\omega) < 1$ (2) $4\text{Im}(\omega) > 5$ (3) $5\text{Re}(\omega) > 1$ (4) $5\text{Re}(\omega) > 4$

15. The equation $|z-i| = |z-1|$, $i = \sqrt{-1}$, represents: **[JEE(Main) 2019, Online (12-04-19) P-1 (4, – 1), 120]**

(1) the line through the origin with slope –1 (2) a circle of radius 1.

(3) a circle of radius $\frac{1}{2}$. (4) the line through the origin with slope 1.

16. If $\text{Re}\left(\frac{z-1}{2z+i}\right) = 1$, where $z = x + iy$, then the point (x,y) lies on a :

[JEE(Main) 2020, Online (07-01-20) P-1 (4, – 1), 100]

(1) circle whose centre is at $\left(-\frac{1}{2}, -\frac{3}{2}\right)$ (2) circle whose diameter is $\frac{\sqrt{5}}{2}$

(3) straight line whose slope is $\frac{3}{2}$ (4) straight line whose slope is $-\frac{2}{3}$

17. If z be a complex number satisfying $|\text{Re}(z)| + |\text{Im}(z)| = 4$, then $|z|$ cannot be

[JEE(Main) 2020, Online (09-01-20) P-2 (4, – 1), 100]

(1) $\sqrt{\frac{17}{2}}$ (2) $\sqrt{10}$ (3) $\sqrt{8}$ (4) $\sqrt{7}$

Answers

Exercise # 1

PART-I

Section (A) :

A-1. $-17 + 24i$

A-2. 4

A-3. (i) $\frac{7}{25}+\frac{24}{25}$ i; (ii) $\frac{21}{5}-\frac{12}{5}$i; (iii) $\frac{22}{5}$ i

A-4. (i) $n\pi, n \in I$ (ii) $n\pi \pm \frac{\pi}{3}, n \in I$

A-5. (i) $(-2, 2)$ or $\left(-\frac{2}{3}, -\frac{2}{3}\right)$

(ii) x = 1 and y = 2

A-6. (i) $\pm(1 - 4i)$ (ii) $\pm\sqrt{2} + 0i$ or $0 \pm \sqrt{2}\, i$

A-7. (i) z = (2 + i) or (1 – 3i); (ii) z = 1 or 2 – i

Section (B) :

B-1. (i) Modulus = 6, Arg = $2k\pi + \frac{5\pi}{18}$ (k ∈ I),

Principal Arg = $\frac{5\pi}{18}$

(ii) Modulus = 2, Arg = $2k\pi + \frac{7\pi}{6}$ (k ∈ I),

Principal Arg = $-\frac{5\pi}{6}$

(iii) Modulus = $\frac{\sqrt{5}}{6}$, Arg = $2k\pi - \tan^{-1}2$ (k ∈ I),

Principal Arg = $-\tan^{-1}2$

B-2. [(–2, 2) ; (–2, –2)]

B-3. (i) $2\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right);\ 2\sqrt{2}e^{i\left(\frac{3\pi}{4}\right)}$

(ii) $2\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right);\ 2e^{i\left(\frac{4\pi}{3}\right)}$

(iii) $\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right);\sqrt{2}e^{i\left(\frac{3\pi}{4}\right)}$

(iv) $2\sin\left(\frac{\theta}{2}\right)\left(\cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right)+i\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right);\ 2\sin\left(\frac{\theta}{2}\right)e^{i\left(\frac{\pi}{2}-\frac{\theta}{2}\right)}$

B-4. (i) 1, (ii) 30

B-5. (i) 4 (ii) $\sqrt{3}$

Section (C) :

C-2. **2**

C-4. 0

C-7. (i) $\pi + 2k\pi, k \in I$ (ii) $-2\pi/3$

C-8. (i) 5 (ii) 1 (iii) [0, 9]

Section (D) :

D-1. $1+(2+2\sqrt{2})i$

D-2. (i) |z| = 20

(ii) OP = OQ = PR = QR = 20

D-3. (i) $a = b = 2 - \sqrt{3}$

(ii) $\sqrt{3}(1-i)$ or $\sqrt{3}(-1+i)$

D-5. (i) The region between the concentric circles with centre at (0 , 2) & radii 1 & 3 units

(ii) region outside or on the circle with centre $\frac{1}{2}$ + 2i and radius $\frac{1}{2}$.

(iii) semicircle (in the 1st & 4th quadrant) $x^2 + y^2 = 1$

(iv) a ray emanating from the point (3 + 4i) directed away from the origin & having equation

$$\sqrt{3}x - y + 4 - 3\sqrt{3} = 0$$

D-6. Centre (–1, 0) and radius 5

D-7. (i) $-\frac{9}{2}+\frac{3\sqrt{3}}{2}i, \frac{-3}{2}+i\frac{\sqrt{3}}{2}$

(ii) $-3 + 0i$, $\frac{-3}{2}+\frac{3\sqrt{3}}{2}i$

D-8. (i) $\frac{15\pi}{\sqrt{2}}$

(ii) $75\pi + 50$

D-9. $-4 - 3i$, $2\sqrt{5}$

D-10. $\frac{x^2}{64}+\frac{y^2}{48}=1$

Section (E) :

E-1. 5

E-2. $\frac{n}{3}(n^2+2)$

E-4. (i) 1 (ii) 1

E-5. $-2i$, $i + \sqrt{3}$ & $i - \sqrt{3}$

E-6. $2e^{\frac{i\pi}{6}}$, $2e^{i\frac{11\pi}{6}}$

E-7. ω

E-8. –1/2

PART-II

Section (A) :

A-1.	(C)	**A-2.**	(B)
A-3.	(B)	**A-4.**	(C)
A-5.	(B)	**A-6.**	(A)

Section (B) :

B-1.	(C)	**B-2.**	(D)
B-3.	(D)	**B-4.**	(D)
B-5.	(C)	**B-6.**	(B)
B-7.	(A)	**B-8.**	(A)

Section (C) :

C-1.	(D)	**C-2.**	(A)
C-3.	(A)	**C-4.**	(A)
C-5.	(D)	**C-6.**	(D)
C-7.	(C)	**C-8.**	(C)
C-9.	(C)	**C-10.**	(C)

Section (D) :

D-1.	(B)	**D-2.**	(C)
D-3.	**(A)**	**D-4.**	(B)
D-5.	(A)	**D-6.**	(D)
D-7.	(C)	**D-8.**	(A)
D-9.	(B)	**D-10.**	(A)
D-11.	(B)		

Section (E) :

E-1.	(A)	**E-2.**	(A)
E-3.	(C)	**E-4.**	(C)
E-5.	(A)	**E-6.**	(C)
E-7	(C)		

PART-III

1. (A) $\to$ P; (B) $\to$ S; (C) $\to$ R; (D) $\to$ S

2. (A) $\to$ Q,R; (B) $\to$P, S; (C) $\to$ Q, S; (D) $\to$ P, R

3. (A) $\to$ P,R; (B) $\to$ P,Q,R,T; (C) $\to$ P,R,S; (D) $\to$ P,Q,R,S,T.

Exercise # 2

PART-I

1.	(C)	**2.**	(A)
3.	(D)	**4.**	(C)
5.	(C)	**6.**	(D)
7.	(C)	**8.**	(C)
9.	(C)	**10.**	(B)
11.	(C)	**12.**	(B)
13.	(A)	**14.**	(C)
15.	(C)	**16.**	(B)
17.	(B)	**18.**	(D)

PART-II

1.	0	**2.**	5.33
3.	35	**4.**	1.41
5.	2048	**6.**	36
7.	4.45	**8.**	5
9.	0.25	**10.**	0
11.	6.25	**12.**	8
13.	0.27	**14.**	8
15.	5	**16.**	0.37
17.	168	**18.**	67.68

PART - III

1.	(A),(B),(C)	**2.**	(A),(B),(C),(D)
3.	(B),(C),(D)	**4.**	(B),(D)
5.	(A),(B),(C),(D)	**6.**	(A),(B),(C),(D)
7.	(A),(B),(C),(D)	**8.**	(B),(D)
9.	(A),(B),(C),(D)	**10.**	(A),(C)
11.	(A),(B)	**12.**	(C),(D)
13.	(A), (B),(C),(D)	**14.**	(C), (D)
15.	(A),(B),(C),(D)	**16.**	(A), (B), (C)
17.	(A),(B),(C),(D)	**18.**	(C),(D)
19.	(C),(D)	**20.**	(A),(B),(C),(D)
21.	(A),(B),(C)	**22.**	(A),(B),(C),(D)
23.	(A), (B), (D)		

PART - IV

1.	(B)	**2.**	(D)
3.	(B)	**4**	(A)
5.	(B)	**6.**	(A)
7.	(B)		

Exercise # 3

PART - I

1. (A), (C), (D) **2.** 1

3. (A) - (q,r), (B)-(p), (C) - (p,s,t), (D) - (q,r,s,t)

4. (5)

5. 3, Bonus (w = $e^{i\pi/3}$ is a typographical error, because of this the answer cannot be an integer.)

6. (D)

7.	(C)	**8.**	(C), (D)
9.	(B), (C), (D)	**10.**	(B)
11.	(C)	**12.**	(C)
13.	4	**14.**	1
15.	(A), (C), (D)	**16.**	(A), (D)
17.	(A), (B), (D)	**18.**	(A), (C), (D)
19.	(2)	**20.**	3.00

PART - II

1.	(2)	**2.**	(1)
3.	(2)	**4.**	(4)
5.	(3)	**6.**	(4)
7.	(3)	**8.**	(1)
9.	(2)	**10.**	(2)
11.	Bonus	**12.**	(1)
13.	(1)	**14.**	(3)
15.	(4)	**16.**	2
17.	4		

SUBJECTIVE QUESTIONS

1. If the equation $z^3 + (3 + i) z^2 + 3z - (m - i) = 0$, where $m \in R$ has all purely imaginary roots, then find the sum of all possible values of m.

2. Show that the product,

$$\left[1+\left(\frac{1+i}{2}\right)\right]\left[1+\left(\frac{1+i}{2}\right)^2\right]\left[1+\left(\frac{1+i}{2}\right)^{2^2}\right]\ldots\ldots\left[1+\left(\frac{1+i}{2}\right)^{2^n}\right] \text{ is equal to } \left(1-\frac{1}{2^{2^n}}\right)(1+i) \text{ where } n \geq 2.$$

3. Let z_r $(1 \leq r \leq 4)$ be complex numbers such that $|z_r| = \sqrt{r+1}$ and $|30 z_1 + 20 z_2 + 15 z_3 + 12 z_4| = k\,|z_1z_2z_3 + z_2z_3z_4 + z_3z_4z_1 + z_4z_1z_2|$, then find the value of k.

4. If $|z|^2 + \bar{A}z^2 + A\bar{z}^2 + B\bar{z} + \bar{B}z + c = 0$ represents a pair of intersecting lines with angle of intersection 'θ' then find the value of |A|

5. If $z^2 + \alpha z + \beta = 0$ (α, β are complex numbers) has a real root then prove that

$$(\bar{\alpha} - \alpha)(\alpha\bar{\beta} - \bar{\alpha}\beta) = (\beta - \bar{\beta})^2$$

6. If z_1, z_2, z_3 be three complex number such that

$|z_1| = |z_2| = |z_3| = 1$ and $\frac{z_1^2}{z_2z_3} + \frac{z_2^2}{z_1z_3} + \frac{z_3^2}{z_1z_2} + 1 = 0$

then sum of all the possible values of $|z_1 + z_2 + z_3|$

7. Number of complex number (z) satisfying $|z|^2 = |z|^{n-2}z^2 + |z|^{n-2}z + 1$ such that $\text{Re}(z) \neq -\frac{1}{2}$.

8. Let z_1 & z_2 be any two arbitrary complex numbers then prove that

(i) $|z_1 + z_2| = \left|\frac{z_1}{|z_1|}|z_2| + \frac{z_2}{|z_2|}|z_1|\right|$ (ii) $|z_1 + z_2| \geq \frac{1}{2}\left(|z_1| + |z_2|\right)\left|\frac{z_1}{|z_1|} + \frac{z_2}{|z_2|}\right|$.

9. Prove that

(i) $\left|\frac{z}{|z|} - 1\right| \leq |\arg z|$. (ii) $|z - 1| \leq ||z| - 1| + |z|\,|\arg z|$.

10. Prove that

$|\text{Img}(z^n)| \leq n\,|\text{Img}(z)||z|^{n-1}$, $n \in I^{\oplus}$

11. If $\theta \in [\pi/6, \pi/3]$, i = 1, 2, 3, 4, 5 and $z^4 \cos\theta_1 + z^3 \cos\theta_2 + z^2 \cos\theta_3 + z \cos\theta_4 + \cos\theta_5 = 2\sqrt{3}$,

then show that $|z| > \frac{3}{4}$

12. If z_1, z_2, z_3 are complex numbers such that $\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3}$, show that the points represented by z_1, z_2, z_3 lie on a circle passing through the origin.

13. P is a point on the argand diagram on the circle with OP as diameter, two point Q and R are taken such that $\angle POQ = \angle QOR = \theta$. If O is the origin and P, Q, R are represented by complex z_1, z_2, z_3 respectively then show that $z_2^2 \cos 2\theta = z_1 z_3 \cos^2\theta$

14. Consider the locus of the complex number z in the Argand plane is given by $\text{Re}(z) - 2 = |z - 7 + 2i|$. Let $P(z_1)$ and $Q(z_2)$ be two complex number satisfying the given locus and also satisfying

$\arg\left(\frac{z_1 - (2+\alpha i)}{z_2 - (2+\alpha i)}\right) = \frac{\pi}{2} (\alpha \in R)$ then find the minimum value of PQ

15. Find the mirror image of the curve $\left|\frac{z - z_1}{z - z_2}\right| = a$, $a \in R^+$ $a \neq 1$ about the line $|z - z_1| = |z - z_2|$.

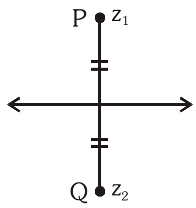
16. Let z_1 and z_2 are the two compelx numbers satisfying $|z - 3 - 4i| = 3$. Such that $\text{Arg}\left(\frac{z_1}{z_2}\right)$ is maximum then find the value of $|z_1 - z_2|$.

17. If z_1 and z_2 are the two complex numbers satisfying $|z - 3 - 4i| = 8$ and $\text{Arg}\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2}$ then find the range of the values of $|z_1 - z_2|$.

18. If $|z - z_1| = |z_1|$ and $|z - z_2| = |z_2|$ be the two circles and the two circles touch each other then prove that $\text{Img}\left(\frac{z_1}{z_2}\right) = 0$

19. If $\begin{vmatrix} p & q & r \\ q & r & p \\ r & p & q \end{vmatrix} = 0$; where p, q, r are the modulus of non-zero complex numbers u, v, w respectively, prove that, $\arg \frac{w}{v} = \arg \left(\frac{w-u}{v-u}\right)^2$.

20. Two given points P & Q are the reflection points w.r.t. a given straight line if the given line is the right bisector of the segment PQ. Prove that the two points denoted by the complex numbers z_1 & z_2 will be the reflection points for the straight line $\bar{\alpha}z + \alpha\bar{z} + r = 0$ if and only if ; $\bar{\alpha}z_1 + \alpha\bar{z}_2 + r = 0$, where r is real and α is non zero complex constant.



21. The points represented by the complex numbers a, b, c lie on a circle with centre O and radius r. The tangent at c cuts the chord joining the points a, b at z. Show that z = $\dfrac{a^{-1}+b^{-1}-2c^{-1}}{a^{-1}b^{-1}-c^{-2}}$

22. Show that for the given complex numbers z_1 and z_2 and for a real c the equation

$$(z_1+\lambda z_2)\bar{z}+(\bar{z}_1+\lambda\bar{z}_2)z+c=0$$

represents a family of concurrent lines and and also find the fixed point of the family.

23. Find the locus of mid-point of line segment intercepted between real and imaginary axes, by the line $a\bar{z}+\bar{a}z+b=0$, where 'b' is real parameter and 'a' is a fixed complex number such that Re(a) $\neq$ 0, Im(a) $\neq$ 0.

24. Given $z_1 + z_2 + z_3 = A$, $z_1 + z_2\,\omega + z_3\,\omega^2 = B$, $z_1 + z_2\,\omega^2 + z_3\,\omega = C$, where ω is cube root of unity,

(a) express z_1, z_2, z_3 in terms of A, B, C.

(b) prove that, $|A|^2 + |B|^2 + |C|^2 = 3\left(|z_1|^2 + |z_2|^2 + |z_3|^2\right)$.

(c) prove that $A^3 + B^3 + C^3 - 3ABC = 27z_1z_2z_3$

25. If w $\neq$ 1 is n^{th} root of unity, then find the value of $\sum_{k=0}^{n-1}|z_1+w^k z_2|^2$

26. Let a, b, c be distinct complex numbers such that $\dfrac{a}{1-b} = \dfrac{b}{1-c} = \dfrac{c}{1-a}$ = k, (a, b, c $\neq$ 1). Find the value of k.

27. If $\alpha = e^{\frac{2\pi i}{7}}$ and $f(x) = A_0 + \sum_{k=1}^{20} A_k x^k$, then find the value of,

$f(x) + f(\alpha x) + \ldots\ldots + f(\alpha^6 x)$ independent of α.

28. Given, $z = \cos\frac{2\pi}{2n+1} + i\sin\frac{2\pi}{2n+1}$, 'n' a positive integer, find the equation whose roots are,

$\alpha = z + z^3 + ... + z^{2n-1}$ and $\beta = z^2 + z^4 + ... + z^{2n}$.

29. Prove that $\cos\left(\frac{2\pi}{2n+1}\right) + \cos\left(\frac{4\pi}{2n+1}\right) + \cos\left(\frac{6\pi}{2n+1}\right) + ... + \cos\left(\frac{2n\pi}{2n+1}\right) = -\frac{1}{2}$ When $n \in N$.

30. If $1, \alpha_1, \alpha_2, \alpha_3, ..., \alpha_{n-1}$ be the n^{th} roots of unity, then prove that

$\sin\frac{\pi}{n} . \sin\frac{2\pi}{n} . \sin\frac{3\pi}{n} .... \sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}$

Answers

1. -21 **3.** $\sqrt{30}$ **4.** $\frac{\sec\theta}{2}$

6. 3 **7.** 1 **14.** 10

15. $\left|\frac{z - z_2}{z - z_1}\right| = a$ **16.** $\frac{24}{5}$ **17.** $|(z_1 - z_2)| \in [3\sqrt{2}, 13\sqrt{2}]$

22. $z = \frac{Cz_2}{z_1\bar{z}_2 - z_2\bar{z}_1}$ **23.** $\bar{a}\bar{z} + az = 0$

24. (a) $z_1 = \frac{A + B + C}{3}$, $z_2 = \frac{A + B\omega^2 + C\omega}{3}$, $z_3 = \frac{A + B\omega + C\omega^2}{3}$

25. $n(|z_1|^2 + |z_2|^2$ **26.** $-\omega$ or $-\omega^2$ **27.** $7A_0 + 7A_7x^7 + 7A_{14}x^{14}$

28. $z^2 + z + \frac{\sin^2 n\theta}{\sin^2\theta} = 0$, where $\theta = \frac{2\pi}{2n+1}$

Self Assessment Paper

JEE ADVANCED

Maximum Marks : 62 **Total Time : 1:00 Hr**

SECTION-1 : ONE OPTION CORRECT (Marks - 12)

1. The complex number z which satisfies the equation |z| = 1 and $\left|\frac{z-\sqrt{2}(1+i)}{z}\right|$ = 1 is

(A) 1 (B) 1 + i (C) $\frac{1+i}{\sqrt{2}}$ (D) $\frac{-1-i}{\sqrt{2}}$

2. Let z_1 and z_2 be non-zero complex numbers satisfying $z_1^2 + 2z_2^2 = 2z_1z_2$, then the triangle with vertices at origin, z_1 and z_2 is

(A) an isoceless, non right angled triangle (B) a non-isoceles, right angled triangle

(C) an equilateral triangle (D) a right angled isocless triangle

3. If z = x + iy; $x \neq -\frac{1}{2}$, then number of values of z satisfying $|z|^n = z^2|z|^{n-2} + z|z|^{n-2} + 1 \; \forall \; n \in N; n > 1$ is

(A) 0 (B) 1 (C) 2 (D) 3

4. If $z_1, z_2, z_3 \in C$ satisfying $|z_1| = |z_3| = |z_3| = 1$, $z_1 + z_2 + z_3 = 1$ and $z_1 z_2 z_3 = 1$. Also $Im(z_1) < Im(z_2) < Im(z_3)$, then $|z_1, + z_2^2 + z_3^3|$ is

(A) 1 (B) $\sqrt{3}$ (C) $\sqrt{5}$ (D) $\sqrt{7}$

SECTION-2 : ONE OR MORE THAN ONE CORRECT (Marks - 32)

5. If $k \in R - \{0\}$ and $k + |k + z^2| = |z|^2$, then arg z can be

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$ (C) $-\frac{\pi}{2}$ (D) π

6. If z_1, z_2, z_3 are the vertices of a triangle such that $|z_1 - z_2| = |z_1 - z_3|$, then arg $\left(\frac{2z_1 - z_2 - z_3}{z_3 - z_2}\right)$

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$ (C) $\frac{5\pi}{6}$ (D) $-\frac{\pi}{2}$

7. Let $x_1, x_2, \ldots\ldots\ldots, x_6$ be the roots of the equation $x^6 + 2x^5 + 4x^4 + 8x^3 + 16x^2 + 32x + 64 = 0$, then

(A) $\left(\frac{x_i}{2}\right)^7 = 1 \; \forall \; i$ (B) $\frac{x_i}{2} + \left(\frac{x_i}{2}\right)^6$ is real (C) $|x_i| = 2 \; \forall \; i$ (D) $\arg\left(\frac{x_1}{x_2}\right) = \frac{2\pi}{7}$

8. $\sum_{r=1}^{1006} |z^{2r+1} - z^{2r-1}| = \sum_{r=1}^{1006} |z^{2r} - z^{2r-2}| = 2012$

where $z \in C$, then

(A) $|z| = 1$ (B) $|z^2 - 1| = 2$ (C) $z = \pm i$ (D) $z = \frac{1 \pm i}{\sqrt{2}}$

9. Consider a complex number $z = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$ and a set $A = \{z_k\}$, $z_k = z^k$; k = 0, 1, 2, ...9}, then

(A) all elements of A lies on a unit circle centred at origin

(B) $\arg(z_p) - \arg(z_q) = \frac{2\pi}{5}$ for exactly 8 ordered pairs (z_p, z_q); $z_p, z_q \in A$ and $\arg(z) \in [0, 2\pi)$

(C) number of elements in A for which $\text{Re}(z_k) > \sin 18°$ is 3

(D) number of elements in A for which $\text{Im}(z_k) + \sin 36° > 0$ is 6

10. Let z_1 and z_2 are two non-zero complex numbers such that $|z_1 + z_2| = |z_1| = |z_2|$, then $\frac{z_1}{z_2}$ may be

(A) $1 + \omega$ (B) $1 + \omega^2$ (C) ω (D) ω^2

11. If $z_1 = 5 + 12i$ and $|z_2| = 4$, then

(A) maximum $(|z_1 + iz_2|) = 17$

(B) minimum $(|z_1 + (1 + i) z_2| = 13 - 4\sqrt{2}$

(C) minimum $\left|\frac{z_1}{z_2 + \frac{4}{z_2}}\right| = \frac{13}{4}$

(D) maximum $\left|\frac{z_1}{z_2 + \frac{4}{z_2}}\right| = \frac{13}{3}$

12. The value of $z \in c$ such that the expression. $S = \frac{\text{Im}(z^5)}{(\text{Im}(z)^5)}$ is minimum is/are

(A) $z = \lambda(1 + i)$ (B) $z = \lambda(1 - i)$ (C) $z = \lambda(2 + i)$ (D) $z = \lambda(1 + 2i)$

SECTION-3 : NUMERICAL VALUE TYPE (Marks - 18)

13. If $|z_0| = |z_0 - w| = |z_0 - w^2|$ where w and w^2 are non-real cube of unity and $|z - z_0| \le 2$ the maximum value of $\left|\frac{3z}{2}\right|$ is

14. If $\lambda \in R$ so that the origin and the non-real roots of the equation $2z^2 + 2z + \lambda = 0$ form the vertices of an equilateral triangle in the argand plane, then $\frac{1}{\lambda}$ equal

15. If 1, α_1, α_2, ... α_{2016} are $(2017)^{th}$ roots of unity, then the absolute value of $\sum_{r=1}^{2016} \frac{r(\alpha_r + \alpha_{2017-r})}{50}$ equal

16. If z is any point on the circle |z + 1| = 3, then three-fourth of the reciprocal of radius of circle represented by $w = (4 + i - z)^{-1}$ is.

17. Let A, B, C be three sets of complex numbers as defined below

$A = \{z : |z + 1| \le 2 + \text{Re}(z)\}$ $B = \{z : |z - 1| \ge 1\}$

and $C = \left\{z : \left|\frac{z+1}{z-1}\right| \ge 1\right\}$, then number of point (s) having integral coordinate in the region $A \cap B \cap C$ is

18. Consider curves C_1: $|\arg(z - 2) = \frac{\pi}{4}$, $C_2 : |\arg(z + 2)| = \frac{3\pi}{4}$ and $C_3 : |z - \alpha_i| = 4\sqrt{2}$, $\alpha \in R$ on the complex plane such that C_3 touches both C_1 and C_2, where sum of absolute values of α is λ, then $\sqrt{\lambda + \frac{1}{4}}$ equals.

Answers

1.	(C)	**2.**	(D)	**3.**	(B)	**4.**	(C)
5.	(A),(C)	**6.**	(A),(D)	**7.**	(A),(B),(C)	**8.**	(A),(B), (C)
9.	(A),(B),(C),(D)	**10.**	(C), (D)	**11.**	(A), (B), (D)	**12.**	(A), (B)
13.	4.50	**14.**	1.50	**15.**	40.34	**16.**	4.25
17.	5	**18.**	3.50				